ON SYMMETRIC SIEGEL DOMAINS

Dedicated to the memory of Taira Honda

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Introduction

Let V be a convex cone in a real vector space X, $F: Y \times Y \to X^c$ a V-positive hermitian map on a complex vector space Y, and

$$D(V, F) = \{(z, u) \in X^c \times Y ; \mathscr{I}_m z - F(u, u) \in V\}$$

the Siegel domain associated to V and F. $D(V, F)$ is said to be symmetric, if for each point $p \in D(V, F)$ there exists an involutive holomorphic automorphism σ_p of $D(V, F)$ such that the fixed point set of σ_p consists of only the point p. Satake [6] showed that the symmetric Siegel domain $D(V, F)$ is characterized by the following three conditions (i), (ii) and (iii).

(i) *V is a self-dual homogeneous convex cone in X.*

Let $G(V)$ denote the group of linear automorphisms of V and take a point $e \in V$. Then the Lie algebra $\mathfrak{g}(V)$ of $G(V)$ is a reductive algebraic Lie algebra in $\mathfrak{gl}(X)$ and the stabilizer

$$\mathfrak{k}(V) = \{A \in \mathfrak{g}(V) ; Ae = 0\}$$

of e is a maximal compact subalgebra of $\mathfrak{g}(V)$. Let

(C) $$\mathfrak{g}(V) = \mathfrak{k}(V) + \mathfrak{p}(V)$$

be the associated Cartan decomposition of $\mathfrak{g}(V)$. Then, for each $x \in X$ there exists a unique element $T(x) \in \mathfrak{p}(V)$ satisfying $T(x)e = x$. Choose an inner product $(\ ,\ )$ on X such that if ${}^t A$ denotes the transpose of $A \in \mathfrak{gl}(X)$ with respect to this inner product, the Cartan involution associated to (C) is given by $A \mapsto -{}^t A$. We define a hermitian inner product $\langle\ ,\ \rangle$ on Y by

$$\langle u, v \rangle = (e, F(u, v)) \qquad \text{for } u, v \in Y ,$$

denoting the C-linear extension to X^C of $(\ ,\ )$ by the same notation $(\ ,\ )$. Let Herm (Y) be the real subspace of the space End (Y) of C-linear endomorphisms of Y, consisting of those endomorphisms which are self-adjoint with respect to this hermitian inner product $\langle\ ,\ \rangle$. Then the second condition is stated as

(ii) *For each $x \in X$ there exists $R(x) \in$ Herm (Y) such that*

$$F(R(x)u, v) + F(u, R(x)v) = T(x)F(u, v) \qquad for \ \ u, v \in Y \ .$$

The above $R(x)$ is uniquely determined for $x \in X$, and so we have an R-linear map $R: X \to$ Herm (Y) by the correspondence $x \mapsto R(x)$. The C-linear extension of the map R will be also denoted by $R: X^C \to$ End (Y). Then the third condition is stated as

(iii) *For each $u, v \in Y$, $x \in X$, we have*

$$R(F(u, v))R(x)u = R(F(u, R(x)v))u \ .$$

In this note, we shall classify Siegel domains satisfying the conditions (i) and (ii), and then determine the ones with the condition (iii) among these Siegel domains. Our problem reduces to the classification in the case where V is an indecomposable self-dual homogeneous convex cone, and so we shall restrict ourselves to consider $D(V, F)$ for such convex cone V.

Take a maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}(V)$ and put $m = \dim \mathfrak{a}$. Then the multiplicities of $\mathfrak{a}$-roots of $\mathfrak{g}(V)$ are the same for all $\mathfrak{a}$-roots. We define a positive integer ν to be this multiplicity if $m \geqq 2$, and $\nu = 1$ if $m = 1$. Then the pair (m, ν) is a complete invariant for the linear isomorphism classes of indecomposable self-dual homogeneous convex cones, and the possibilities of (m, ν) are $m = 1$, $\nu = 1$; $m = 2$, $\nu \geqq 1$; $m = 3$, $\nu = 1, 2, 4$ or 8; $m \geqq 4$, $\nu = 1, 2$ or 4. The cone with the invariant (m, ν) will be denoted by V_m^ν. We associate to such (m, ν) an algebra F_m^ν over R in the following way: $F_m^\nu =$ the Clifford algebra $C_{\nu-1}$ associated to the negative definite quadratic form on $R^{\nu-1}$, if $m = 2$; $F_m^\nu =$ the division algebra over R of dimension ν, if $m \neq 2$. Let $\lambda \mapsto \bar{\lambda}$ denote the canonical involution of F_m^ν. We define a subspace P_m^ν of F_m^ν by $P_m^\nu = R1 + R^{\nu-1} \subset C_{\nu-1}$ if $m = 2$; $P_m^\nu = F_m^\nu$ if $m \neq 2$, and denote by $\mathscr{P}: F_m^\nu \to P_m^\nu$ the orthogonal projection to P_m^ν relative to the standard inner product of F_m^ν. Let $c_0 = 1$, $c_1, \cdots, c_{\nu-1}$ be the standard basis of P_m^ν with relations

$$c_i^2 = -1 \ (1 \leqq i \leqq \nu - 1) \ , \ c_i c_k + c_k c_i = 0 \ (1 \leqq i < k \leqq \nu - 1) \ .$$

We denote by Aut* $(F_m^\nu, \mathscr{P})$ the group of those automorphisms of the algebra F_m^ν commuting with $\lambda \mapsto \bar\lambda$ and $\mathscr{P}$, and by $\mathfrak{M}^*(F_m^\nu)$ the set of unitary equivalence classes of F_m^ν-modules W over C with a hermitian inner product $\langle \ , \ \rangle$ satisfying the condition

(∗) $\langle \lambda u, v \rangle = \langle u, \bar\lambda v \rangle$ for $\lambda \in F_m^\nu$, $u, v \in W$.

We define $\mathfrak{M}(m, \nu)$ to be the orbit space Aut* $(F_m^\nu, \mathscr{P}) \backslash \mathfrak{M}^*(F_m^\nu)$ relative to the natural action of Aut* $(F_m^\nu, \mathscr{P})$ on $\mathfrak{M}^*(F_m^\nu)$. Then our results are stated as follows.

THEOREM A. *The set $\mathfrak{M}(m, \nu)$ is in a bijective correspondence with the set of affine isomorphism classes of those Siegel domains $D(V, F)$ satisfying $V = V_m^\nu$ and the condition* (ii).

THEOREM B.[1] *Let W be an F_m^ν-module with a hermitian inner product $\langle \ , \ \rangle$ satisfying the condition* (∗), *and $D(V_m^\nu, F)$ the Siegel domain corresponding to W. Then $D(V_m^\nu, F)$ satisfies the condition* (iii) *if and only if $m = 1$, or $m \geqq 2$ and*

(♯) $$\sum_{i=0}^{\nu-1} \langle c_i u, v \rangle \langle c_i u, w \rangle = 0   \text{ for } u, v, w \in W \ .$$

By means of Theorems A and B, we get a new classification of symmetric bounded domains without the use of the classification of semi-simple Lie algebras of hermitian type.

Finally we note (cf. Tsuji [8]) that if an F_m^ν-module W $(m \geqq 2)$ does not satisfy the condition (♯), then the corresponding Siegel domain $D(V_m^\nu, F)$ has no infinitesimal automorphisms other than infinitesimal affine automorphisms.

1. Homogeneous Siegel domains

In this section we recall definitions and fundamental results on homogeneous Siegel domains. For homogeneous cones and T-algebras, see [9], [10], [11].

Let X be a finite dimensional real vector space. An open subset V of X is called a *convex cone*, if (1) $x, y \in V$, $0 < \lambda < 1 \Rightarrow \lambda x + (1 - \lambda)y \in V$, (2) $x \in V$, $\lambda > 0 \Rightarrow \lambda x \in V$, and (3) V does not contain any straight

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1) In the case of F_2^ν, the same result is obtained by T. Tsuji (unpublished).

line. We denote by $G(V)$ the subgroup of $GL(X)$ consisting of those $\varphi \in GL(X)$ satisfying $\varphi V = V$. V is said to be *homogeneous* if $G(V)$ acts transitively on V. Two convex cones $V \subset X$ and $V' \subset X'$ are said to be *linearly isomorphic*, if there exists a linear isomorphism $\varphi : X \to X'$ satisfying $\varphi V = V'$. Let $V_i \subset X_i$ $(1 \le i \le r)$ be convex cones. Put $X = X_1 + \cdots + X_r$ (direct sum) and $V = V_1 + \cdots + V_r$. Then V is a convex cone in X. This cone is called the *direct product* of the V_i $(1 \le i \le r)$ and denoted by $V = V_1 \times \cdots \times V_r$. A convex cone V is said to be *decomposable* if it is linearly isomorphic to the direct product $V_1 \times V_2$ of non-trivial convex cones V_1 and V_2; otherwise it is said to be *indecomposable*. For a convex cone $V \subset X$, the subset V^* of the dual space X^* of X defined by

$$V^* = \{\xi \in X^* \, ; \, \xi(x) > 0 \text{ for any } x \in \overline{V} - \{0\}\} \, ,$$

is also a convex cone. V is said to be *self-dual* if there exists an inner product $(\, , \,)$ on X such that the canonical isomorphism $\varphi : X \to X^*$ defined by $\varphi(x)(y) = (x, y)$ satisfies $\varphi V = V^*$. Each (self-dual) homogeneous convex cone is linearly isomorphic to the direct product $V_1 \times \cdots \times V_r$ of indecomposable (self-dual) homogeneous convex cones V_i $(1 \le i \le r)$, where the V_i are unique up to linear isomorphism and permutation (Vinberg [9], [10]).

Let $V \subset X$ be a convex cone and Y a finite dimensional complex vector space. A map $F : Y \times Y \to X^c$, where X^c denotes the complexification of X, is called a *hermitian map*, if (1) $F(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 F(u_1, v) + \lambda_2 F(u_2, v)$ for $\lambda_1, \lambda_2 \in C$, $u_1, u_2, v \in Y$, and (2) $\overline{F(u, v)} = F(v, u)$ for $u, v \in Y$. It is said to be *V-positive*, if (1) $F(u, u)$ belongs to the closure $\overline{V}$ of V for each $u \in Y$, and (2) $F(u, u) = 0 \Rightarrow u = 0$. For a V-positive hermitian map F, the domain $D(V, F)$ in $X^c \times Y$ defined by

$$D(V, F) = \{(z, u) \in X^c \times Y \, ; \, \mathscr{I}m \, z - F(u, u) \in V\} \, ,$$

is called the *Siegel domain* associated to V and F. It is known (Pjateckii-Šapiro [5]) that $D(V, F)$ is holomorphically isomorphic to a bounded domain in $X^c \times Y$. We denote by $A(D(V, F))$ the group of those affine automorphisms φ of $X^c \times Y$ satisfying $\varphi(D(V, F)) = D(V, F)$. $D(V, F)$ is said to be *affinely homogeneous* if $A(D(V, F))$ acts transitively on $D(V, F)$. A Siegel domain satisfying the Satake's conditions (i) and (ii) is always affinely homogeneous (cf. § 3). If $D(V, F)$ is affinely homogeneous, then V is homogeneous (Pjateckii-Šapiro [5]). Two Siegel

domains $D(V, F) \subset X^c \times Y$ and $D(V', F') \subset X'^c \times Y'$ are said to be *affinely isomorphic* if there exists an affine isomorphism $\varphi \colon X^c \times Y \to X'^c \times Y'$ satisfying $\varphi(D(V, F)) = D(V', F')$. Let $D(V_i, F_i) \subset X_i^c \times Y_i$ $(1 \leq i \leq r)$ be Siegel domains. Put $V = V_1 \times \cdots \times V_r \subset X = X_1 + \cdots + X_r$ and $Y = Y_1 + \cdots + Y_r$. We define a V-positive hermitian map $F \colon Y \times Y \to X^c$ by

$$F\left(\sum_i u_i, \sum_i v_i\right) = \sum_i F_i(u_i, v_i) \qquad \text{for } u_i, v_i \in Y_i \ (1 \leq i \leq r) \,.$$

The Siegel domain $D(V, F)$ associated to V and F is called the *direct product* of the $D(V_i, F_i)$ $(1 \leq i \leq r)$ and denoted by $D(V, F) = D(V_1, F_1) \times \cdots \times D(V_r, F_r)$. A Siegel domain $D(V, F)$ is said to be *decomposable* if it is affinely isomorphic to the direct product $D(V_1, V_1) \times D(V_2, F_2)$ of non-trivial Siegel domains $D(V_1, F_1)$ and $D(V_2, F_2)$; otherwise it is said to be *indecomposable*.

In general, a complex manifold D is said to be *homogeneous* if the group of holomorphic automorphisms of D acts transitively on D. D is said to be *symmetric*, if for each $p \in D$ there exists an involutive holomorphic automorphism σ_p of D such that the fixed point set of σ_p consists of only the point p. It is known (Vinberg-Gindikin-Pjateckii-Šapiro [12]) that the set of affine isomorphism classes of affinely homogeneous Siegel domains is in the bijective correspondence with the set of holomorphic isomorphism classes of homogeneous bounded domains. Each symmetric Siegel domain is affinely homogeneous, since the Bergman metric of a Siegel domain is complete (Nakajima [3]), and a homogeneous Siegel domain is affinely homogeneous (Kaup-Matsushima-Ochiai [2]). Thus we have the bijective correspondence between the set of affine isomorphism classes of symmetric Siegel domains and the set of holomorphic isomorphism classes of symmetric homogeneous bounded domains. Furthermore it is known (Kaneyuki [1]) that each affinely homogeneous Siegel domain $D(V, F)$ is affinely isomorphic to the direct product $D(V_1, F_1) \times \cdots \times D(V_r, F_r)$ of indecomposable affinely homogeneous Siegel domains $D(V_i, F_i)$ $(1 \leq i \leq r)$, where the $D(V_i, F_i)$ are unique up to affine isomorphism and permutation, and that $D(V, F)$ is indecomposable if and only if V is indecomposable. Thus our problem reduces to the case where V is indecomposable.

Now we recall the notion of T-algebras. Let

$$A = \sum_{1 \leq i,\, k \leq m} A_{ik}$$

be a finite dimensional bigraded distributive algebra over R and $a \mapsto a^*$ be an involutive anti-automorphism of A. A is called a *matrix algebra with involution* of rank m, if (1) $A_{ik}A_{k\ell} \subset A_{i\ell}$, and $A_{ik}A_{pq} = \{0\}$ if $k \neq p$, (2) $A_{ik}^* = A_{ki}$, and (3) A_{ii} is a subalgebra of A with the unit element e_i, for each i. An element

$$a = \sum_{1 \leq i,\, k \leq m} a_{ik}, \qquad a_{ik} \in A_{ik}$$

of A will be often denoted by the matricial form $a = (a_{ik})$. The general element of A_{ik} (resp. of A) will be denoted by $a_{ik}, b_{ik}, c_{ik}, \cdots$ (resp. by $a, b, c, \cdots$). A matrix algebra A with involution of rank m is called a *T-algebra* of rank m, if

(I)   There exists an algebra isomorphism $\theta : A_{ii} \to R$ for each i.

(II) $e_i a_{ik} = a_{ik}$.

(III)   There exist $n_i > 0$ $(1 \leq i \leq m)$ such that

$$n_i \theta(a_{ik}b_{ki}) = n_k \theta(b_{ki}a_{ik}).$$

(IV) $\theta(a_{ik}a_{ik}^*) > 0$ if $a_{ik} \neq 0$.

(V) $a_{ik}(b_{k\ell}c_{\ell i}) = (a_{ik}b_{k\ell})c_{\ell i}$.

(VI) $a_{ik}(b_{k\ell}c_{\ell p}) = (a_{ik}b_{k\ell})c_{\ell p}$ if $i < k < \ell$ and $k < p$.

(VII) $a_{ik}(b_{k\ell}b_{k\ell}^*) = (a_{ik}b_{k\ell})b_{k\ell}^*$ if $i < k < \ell$.

For a T-algebra A of rank m, we define

$$n_{ik} = \dim A_{ik} \qquad \text{for } 1 \leq i,\, k \leq m,$$

$$\operatorname{tr} a = \sum_{1 \leq i \leq m} n_i \theta(a_{ii}) \qquad \text{for } a = (a_{ik}) \in A,$$

$$(a, b) = \operatorname{tr} ab^* \qquad \text{for } a, b \in A.$$

Then $(\, , \,)$ is an inner product on A satisfying

$$(a^*, b^*) = (a, b),$$

$$(ab, c) = (b, a^*c) = (a, cb^*).$$

It follows from (II) that $e = \sum_{1 \leq i \leq m} e_i$ is the unit element of the algebra A. A permutation $i \mapsto \tilde{i}$ of the set $\{1, 2, \cdots, m\}$ of indices is said to be *admissible for* A, if $i < k$, $\tilde{i} > \tilde{k}$ imply $n_{ik} = 0$. The renaming $A_{ik} \rightsquigarrow A_{\tilde{i}\tilde{k}}$ of the grading of A by an admissible permutation $i \mapsto \tilde{i}$, is called an *inessential change of the grading* of A. T-algebras A of rank

m and A' of rank m' are said to be *isomorphic*, if $m = m'$ and there exists an algebra isomorphism of A onto A' such that it becomes an isomorphism as bigraded involutive algebras (i.e., it preserves gradings and commutes with involutions), after an inessential change of the grading of A.

For a T-algebra A of rank m, we put

$$X(A) = \{a \in A\,;\, a^* = a\}\,,$$

$$\mathscr{T}(A) = \{(a_{ik}) \in A\,;\, a_{ik} = 0 \ (i > k),\ \theta(a_{ii}) > 0 \ (1 \le i \le m)\}\,,$$

$$V(A) = \{tt^*\,;\, t \in \mathscr{T}(A)\} \subset X(A)\,.$$

Then (Vinberg [10]) $V(A)$ is a homogeneous convex cone in $X(A)$ and the correspondence $A \rightsquigarrow V(A)$ induces a bijection from the set of isomorphism classes of T-algebras onto the set of linear isomorphism classes of homogeneous convex cones. Furthermore it is known (Vinberg [11]) that $V(A)$ is an indecomposable self-dual convex cone if and only if $m = 1$, or $m \ge 2$ and all n_{ik} $(i \ne k)$ are the same non-zero integer. For such T-algebra, we put $\nu = 1$ if $m = 1$, and $\nu = n_{ik}$ $(i \ne k)$ if $m \ge 2$. Then the pair (m, ν) serves as a complete invariant for the linear isomorphism classes of indecomposable self-dual homogeneous convex cones. It is not difficult to see that our invariant (m, ν) coincides with the one given in Introduction. We know that the possibilities of (m, ν) are as those in Introduction, seeing the classification of Vinberg [9] of such convex cones. For such pair (m, ν) we associate an algebra F_m^ν over $\boldsymbol{R}$ as in Introduction. Then the T-algebra A_m^ν with the invariant (m, ν) is constructed explicitly in the following way.

Let first $m = 2$, and

$$C_{\nu-1} = \sum_{0 \le s \le \nu-1} (C_{\nu-1})_s\,, \qquad (C_{\nu-1})_s = \sum_{i_1 < \cdots < i_s} \boldsymbol{R}c_{i_1}\cdots c_{i_s}$$

be the usual graduation of $C_{\nu-1} = F_m^\nu$. Identifying $\boldsymbol{R}$ with $(C_{\nu-1})_0 = \boldsymbol{R}1$, we define a linear map $\mathscr{R}e\colon F_m^\nu \to \boldsymbol{R}$ by the projection $C_{\nu-1} \to (C_{\nu-1})_0$. Putting $P_m^\nu = (C_{\nu-1})_0 + (C_{\nu-1})_1$, we also define a linear map $\mathscr{P}\colon F_m^\nu \to P_m^\nu$ by the projection $C_{\nu-1} \to (C_{\nu-1})_0 + (C_{\nu-1})_1$. Denoting by $\lambda \mapsto \bar{\lambda}$ the canonical positive involution of $C_{\nu-1}$ (defined by the correspondence $c_{i_1} c_{i_2} \cdots c_{i_s} \mapsto (-c_{i_s})\cdots(-c_{i_2})(-c_{i_1})$), we define the standard inner product $(\ ,\ )$ on F_m^ν by

$$(1.1) \qquad\qquad (\lambda, \mu) = \mathscr{R}e\,\lambda\bar{\mu} \qquad \text{for } \lambda, \mu \in F_m^\nu\,.$$

Note that P_m^ν is invariant under the involution $\lambda \mapsto \bar\lambda$ and

$$(1.2) \qquad \mathscr{R}e\, \lambda = \tfrac{1}{2}(\mathscr{P}\lambda + \overline{\mathscr{P}\lambda}) \qquad \text{for } \lambda \in F_m^\nu .$$

Let next $m \neq 2$. Then F_m^ν is the division algebra over R of dimension ν, and so we have the direct sum decomposition

$$F_m^\nu = R1 + \{\lambda \in F_m^\nu \,; \bar\lambda = -\lambda\} ,$$

where $\lambda \mapsto \bar\lambda$ denotes the canonical involution of F_m^ν. Identifying R with $R1$ and putting $P_m^\nu = F_m^\nu$, we define linear maps $\mathscr{R}e \colon F_m^\nu \to R$ and $\mathscr{P} \colon F_m^\nu \to P_m^\nu$ by the projection $F_m^\nu \to R1$ and the identity map of F_m^ν respectively. Then (1.2) holds also for these F_m^ν. The standard inner product $(\,,\,)$ on F_m^ν is also defined by the formula (1.1). Note that then for each m the involution $\lambda \mapsto \bar\lambda$ commutes with $\mathscr{R}e$ and $\mathscr{P}$.

Now, for a general pair (m,ν), we denote by $\hat A_m^\nu$ the algebra $M_m(F_m^\nu)$ of all matrices of degree m with coefficients in F_m^ν. The product in $\hat A_m^\nu$ will be denoted by $a \cdot b$. Put $a^* = {}^t\bar a$ for $a \in \hat A_m^\nu$. Then $\hat A_m^\nu$ becomes a matrix algebra with involution of rank m. A linear endomorphism p of $\hat A_m^\nu$ is defined by

$$(pa)_{ik} = \begin{cases} \mathscr{P}a_{ik} & \text{if } i \neq k \\ \mathscr{R}e\, a_{ik} & \text{if } i = k . \end{cases}$$

We put $A_m^\nu = p(\hat A_m^\nu)$ and define a product in A_m^ν by $ab = p(a \cdot b)$ for $a, b \in A_m^\nu$. A_m^ν has the natural bigrading induced from the one of $\hat A_m^\nu$. The involution $a \mapsto a^*$ of $\hat A_m^\nu$ leaves A_m^ν invariant and induces an involution of A_m^ν. Relative to these grading and involution, A_m^ν becomes a T-algebra of rank m, by taking $n_i = 1$ for each i. This T-algebra A_m^ν is the required one.

Put

$$X_m^\nu = X(A_m^\nu) , \qquad V_m^\nu = V(A_m^\nu) .$$

Then X_m^ν coincides with the space

$$H_m(P_m^\nu) = \{a \in M_m(P_m^\nu) \,; {}^t\bar a = a\}$$

of all hermitian matrices in the space $M_m(P_m^\nu)$ of matrices of degree m with coefficients in P_m^ν. It becomes a compact simple Jordan algebra by the product

$$x \,\square\, y = \tfrac{1}{2}(xy + yx) \qquad \text{for } x, y \in X_m^\nu .$$

The unit element e of the T-algebra A_m^ν is also the unit element of the Jordan algebra X_m^ν. The cone V_m^ν is also represented as the interior of the subset $\{x \square x\,;\, x \in X_m^\nu\}$ of X_m^ν. We define the *standard inner product* $(\ ,\ )$ on X_m^ν by

$$(x, y) = \operatorname{Tr} x \square y = \mathscr{R}e\,(\operatorname{Tr} x \cdot y) \qquad \text{for } x, y \in X_m^\nu\,,$$

where Tr denotes the usual trace of a matrix. This coincides with the restriction to X_m^ν of the inner product $\operatorname{tr} ab^*$ of the T-algebra A_m^ν. The cone V_m^ν is self-dual with respect to this inner product. We define an injective linear map $\psi\colon X_m^\nu \to \mathfrak{gl}(X_m^\nu)$ by

$$\psi(x)y = xy + yx = 2x \square y \qquad \text{for } x, y \in X_m^\nu\,.$$

Let $\mathfrak{p}_m^\nu = \psi(X_m^\nu)$ and $\mathfrak{k}_m^\nu$ be the subalgebra of $\mathfrak{gl}(X_m^\nu)$ consisting of all derivations of the Jordan algebra X_m^ν. Then the Lie algebra $\mathfrak{g}(V_m^\nu)$ of the group $G(V_m^\nu)$ is given by

$$\mathfrak{g}(V_m^\nu) = \mathfrak{k}_m^\nu + \mathfrak{p}_m^\nu\,,$$

and $\mathfrak{k}_m^\nu$ coincides with the stabilizer of the point $e \in V_m^\nu$ in $\mathfrak{g}(V_m^\nu)$. Furthermore the above is a Cartan decomposition of $\mathfrak{g}(V_m^\nu)$ and the associate Cartan involution is given by $A \mapsto -{}^t A$ with respect to the standard inner product of X_m^ν. If we put

$$T(x) = \tfrac{1}{2}\psi(x) \qquad \text{for } x \in X_m^\nu\,,$$

we have $T(x) \in \mathfrak{p}_m^\nu$ and $T(x)e = x$ for each $x \in X_m^\nu$.

Next we recall the notion of S-algebras. Let

$$C = \sum_{1 \le i,\ k \le m+1} C_{ik}$$

be a T-algebra of rank $m + 1$ and j complex structures on $C_{i,m+1}, C_{m+1,i}$ $(1 \le i \le m)$. C is called an *S-algebra* of rank m, if it satisfies the following additional conditions:

(VIII) $a_{ik}(jb_{k,m+1}) = j(a_{ik}b_{k,m+1})$ if $1 \le i < k \le m$.

(IX) $j(a_{i,m+1}^*) = (ja_{i,m+1})^*$ for each $1 \le i \le m$.

(X) $\theta(ja_{i,m+1}(ja_{i,m+1})^*) = \theta(a_{i,m+1}a_{i,m+1}^*)$ for each $1 \le i \le m$.

j will be called the *partial complex structure* of C. For an S-algebra C of rank m, we put

$$A = \sum_{1 \leq i,\, k \leq m} C_{ik} \, ,$$

$$B = \sum_{1 \leq i \leq m} (C_{i,m+1} + C_{m+1,i}) \, ,$$

$$X = \{a \in A \, ; \, a^* = a\} \, ,$$

$$Y = \{a \in B \, ; \, a^* = a\} \, .$$

Then A becomes a T-algebra of rank m and is called the *T-algebra associated to* C. The complex structure j on B leaves Y invariant and so induces a complex structure j on Y. This j makes Y into a complex vector space in the natural way, which will be denoted by (Y, j). The inner product $(\ ,\ )$ on C defined by the T-algebra structure on C satisfies

$$(ja, jb) = (a, b) \qquad \text{for } a, b \in B \, .$$

Remark. It is known (Takeuchi [7]) that the positive numbers

$$n_i^0 = 2\left(2 + \sum_{\substack{1 \leq k \leq m \\ k \neq i}} n_{ik} + \tfrac{1}{2} n_{i,m+1}\right) \quad (1 \leq i \leq m), \quad n_{m+1}^0 = 1$$

satisfy the axiom (III).

A permutation $i \mapsto \tilde{i}$ of the set $\{1, 2, \cdots, m\}$ is said to be *admissible for* C, if $i < k$, $\tilde{i} > \tilde{k}$ imply $n_{ik} = 0$. The renaming $C_{ik} \rightsquigarrow C_{\tilde{i}\tilde{k}}$ $(1 \leq i, \ k \leq m)$, $C_{i,m+1} \rightsquigarrow C_{\tilde{i},m+1}$, $C_{m+1,i} \rightsquigarrow C_{m+1,\tilde{i}}$ $(1 \leq i \leq m)$ of the grading of C by an admissible permutation $i \mapsto \tilde{i}$, is called an *inessential change of the grading* of C. S-algebras C of rank m and C' of rank m' are said to be *isomorphic*, if $m = m'$ and there exists an algebra isomorphism of C onto C' such that it becomes an isomorphism as a bigraded involutive algebra with partial complex structure (i.e., it preserves gradings and commutes with involutions and partial complex structures), after an inessential change of the grading of C.

Let C be an S-algebra of rank m. Put

$$\hat{u} = \sum_{1 \leq i \leq m} u_{i,m+1}, \ \underset{\vee}{u} = \sum_{1 \leq i \leq m} u_{m+1,i} \qquad \text{for } u = (u_{ik}) \in Y \, ,$$

and define a symmetric bilinear map $\Phi \colon Y \times Y \to X$ by

$$\Phi(u, v) = \tfrac{1}{2}(\hat{u}\underset{\vee}{v} + \hat{v}\underset{\vee}{u}) \qquad \text{for } u, v \in Y \, .$$

Then it satisfies $\Phi(ju, jv) = \Phi(u, v)$ for $u, v \in Y$, and hence the map $F \colon (Y, j) \times (Y, j) \to X^c$ defined by

$$F(u, v) = \tfrac{1}{2}\{\Phi(u, v) + \sqrt{-1}\Phi(u, jv)\} \qquad \text{for } u, v \in Y$$

is a hermitian map. The maps Φ and F are said to be *associated to C*. It is known (Takeuchi [7]) that F is a $V(A)$-positive hermitian map. The Siegel domain $D(V(A), F) \subset X^c \times (Y, j)$ associated to $V(A)$ and F will be denoted by $D(C)$. Then (Takeuchi [7]) the correspondence $C \rightsquigarrow D(C)$ induces a bijection from the set of isomorphism classes of S-algebras onto the set of affine isomorphism classes of affinely homogeneous Siegel domains.

2. Linear automorphisms of a Siegel domain

We consider the pair $\mathscr{S} = (S; \{m_\alpha\}_{1 \leq \alpha \leq \mu+1})$ of a matrix algebra

$$S = \sum_{1 \leq \alpha, \beta \leq \mu+1} S_{\alpha\beta}$$

with involution $u \mapsto u^*$ of rank $\mu + 1$ and the set $\{m_\alpha\}_{1 \leq \alpha \leq \mu+1}$ of $\mu + 1$ positive integers with $m_{\mu+1} = 1$, such that (1) partial complex structures j on $S_{\alpha,\mu+1}, S_{\mu+1,\alpha}$ $(1 \leq \alpha \leq \mu)$ are defined; (2) linear endomorphisms $\mathscr{P}$ on $S_{\alpha\alpha}$ $(1 \leq \alpha \leq \mu + 1)$ are defined; (3) for each α $(1 \leq \alpha \leq \mu + 1)$, there exists an algebra isomorphism $\vartheta : S_{\alpha\alpha} \to F_{m_\alpha}^{\nu_\alpha}$ for some ν_α, satisfying $\overline{\vartheta\lambda} = \vartheta(\lambda^*), \mathscr{P}(\vartheta\lambda) = \vartheta(\mathscr{P}\lambda)$ for $\lambda \in S_{\alpha\alpha}$. Such pair $\mathscr{S}$ is called an *S-system* of rank μ. The general element of $S_{\alpha\beta}$ (resp. of S) will be denoted by $u_{\alpha\beta}, v_{\alpha\beta}, w_{\alpha\beta}, \cdots$ (resp. by $u, v, w, \cdots$).

Let $\mathscr{S} = (S; \{m_\alpha\}_{1 \leq \alpha \leq \mu+1})$ be an S-system of rank μ and ε_α the unit element of the algebra $S_{\alpha\alpha}$ $(1 \leq \alpha \leq \mu + 1)$. Identifying R with $R\varepsilon_\alpha$, we define a linear map $\mathscr{R}e : S_{\alpha\alpha} \to R$ by

$$\mathscr{R}e\,\lambda = \tfrac{1}{2}(\mathscr{P}\lambda + (\mathscr{P}\lambda)^*) \qquad \text{for } \lambda \in S_{\alpha\alpha}\,.$$

Then we have

$$\mathscr{R}e\,\lambda = \mathscr{R}e\,(\vartheta\lambda) \qquad \text{for } \lambda \in S_{\alpha\alpha}$$

and the involution $u \mapsto u^*$ of S commutes with $\mathscr{P}$ and $\mathscr{R}e$ on each $S_{\alpha\alpha}$. A permutation $\alpha \mapsto \tilde{\alpha}$ of the set $\{1, 2, \cdots, \mu\}$ of indices is said to be *admissible for $\mathscr{S}$*, if $\alpha < \beta$, $\tilde{\alpha} > \tilde{\beta}$ imply $S_{\alpha\beta} = \{0\}$. The renaming $S_{\alpha\beta} \rightsquigarrow S_{\tilde{\alpha}\tilde{\beta}}$ $(1 \leq \alpha, \beta \leq \mu)$, $S_{\alpha,\mu+1} \rightsquigarrow S_{\tilde{\alpha},\mu+1}, S_{\mu+1,\alpha} \rightsquigarrow S_{\mu+1,\tilde{\alpha}}$ $(1 \leq \alpha \leq \mu)$, $m_\alpha \rightsquigarrow m_{\tilde{\alpha}}$ $(1 \leq \alpha \leq \mu)$ by an admissible permutation $\alpha \mapsto \tilde{\alpha}$, is called an *inessential change of the grading* of $\mathscr{S}$. S-systems $\mathscr{S} = (S; \{m_\alpha\})$ of rank μ and $\mathscr{S}' = (S'; \{m'_\alpha\})$ of rank μ' are said to be *equivalent*, if (1) $\mu = \mu'$ and (2) after an inessential change of the grading of $\mathscr{S}, m_\alpha = m'_\alpha$ $(1 \leq \alpha \leq \mu)$ and there exists an algebra isomorphism of S onto S',

which preserves gradings and commutes with involutions, j and $\mathscr{P}$. The equivalence class of $\mathscr{S}$ will be denoted by $\{\mathscr{S}\}$.

Let C be an S-algebra of rank m. We define an equivalence relation $\sim$ in the set $I = \{1, 2, \cdots, m\}$ inductively as follows: Let $i \sim i$ for each i. Suppose that an equivalence relation $i \sim k$ is defined for each i, k with $|i - k| < p$. We define i, k with $|i - k| = p$ to be $i \sim k$ if (1) $n_{ik} \neq 0$ and $n_{i,m+1} = n_{k,m+1}$; (2) $n_{i\ell} = n_{k\ell}$ for each $\ell \neq i, k$; (3) for each s lying between i and k (except i and k), $n_{is} = n_{ks} = 0$ or $i \sim s$, $s \sim k$. A grading of C is said to be *adapted to the relation* $\sim$, if $i < k < \ell$, $i \sim \ell$ imply $i \sim k \sim \ell$. In this case we get the decomposition $I = \bigcup_{\alpha=1}^{\mu} I_\alpha$ of I into the $\sim$-equivalence classes I_α $(1 \leq \alpha \leq \mu)$ in such a way that

$$(2.1) \quad I_\alpha = [m_1 + \cdots + m_{\alpha-1} + 1,\ m_1 + \cdots + m_{\alpha-1} + m_\alpha] \qquad \text{for } 1 \leq \alpha \leq \mu \,,$$

and hence

$$(2.2) \qquad\qquad m = \sum_{\alpha=1}^{\mu} m_\alpha \,.$$

We put

$$(2.3) \qquad\qquad I_{\mu+1} = \{m + 1\} \,, \qquad m_{\mu+1} = 1 \,.$$

Then the cardinality $|I_\alpha|$ of I_α is equal to m_α for each $1 \leq \alpha \leq \mu + 1$. We define $\alpha(i)$ with $1 \leq \alpha(i) \leq \mu + 1$ by the relation

$$(2.4) \qquad\qquad i \in I_{\alpha(i)} \qquad (1 \leq i \leq m + 1) \,.$$

THEOREM 2.1. (A) *For each S-algebra C, we can make the grading of C adapted to the relation $\sim$, by an inessential change of the grading of C.*

(B) *Let C be an S-algebra of rank m with the grading adapted to the relation $\sim$. Then we can associate to C an S-system $\mathscr{S} = (S\,;\{m_\alpha\}_{1\leq\alpha\leq\mu+1})$ of rank μ with the following properties:*

(a) *μ is the number of the $\sim$-equivalence classes I_α in $I = \{1, 2, \cdots, m\}$, and $m_\alpha = |I_\alpha|$ $(1 \leq \alpha \leq \mu)$.*

(b) *If $\alpha \neq \beta$ or $\beta \neq \gamma$,*

$$u_{\alpha\alpha}(v_{\alpha\beta}w_{\beta\gamma}) = (u_{\alpha\alpha}v_{\alpha\beta})w_{\beta\gamma} \,,$$
$$u_{\alpha\beta}(v_{\beta\beta}w_{\beta\gamma}) = (u_{\alpha\beta}v_{\beta\beta})w_{\beta\gamma} \,.$$

(c) *There exist linear isomorphisms*

$$(2.5) \qquad \theta^{ik}: C_{ik} \longrightarrow \begin{cases} S_{\alpha(i)\alpha(k)} & \text{if } \alpha(i) \neq \alpha(k) \\ \mathscr{P}S_{\alpha(i)\alpha(i)} & \text{if } \alpha(i) = \alpha(k),\ i \neq k \\ \mathscr{R}e\, S_{\alpha(i)\alpha(i)} & \text{if } i = k \end{cases}$$

satisfying

$$(1) \qquad \theta^{i\ell}(a_{ik}b_{k\ell}) = \begin{cases} \theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell}) & \text{if } \alpha(i) \neq \alpha(\ell) \\ \mathscr{P}(\theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell})) & \text{if } \alpha(i) = \alpha(\ell),\ i \neq \ell \\ \mathscr{R}e\,(\theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell})) & \text{if } i = \ell\,, \end{cases}$$

and hence $\theta^{ii} = \theta$ *for each* i,

$$(2) \qquad \theta^{ik}(a_{ki}^*) = (\theta^{ki}(a_{ki}))^*\,,$$

$$(3) \qquad j\theta^{i,m+1}(a_{i,m+1}) = \theta^{i,m+1}(ja_{i,m+1}), \qquad j\theta^{m+1,i}(a_{m+1,i}) = \theta^{m+1,i}(ja_{m+1,i})$$
$$\text{for each } 1 \leqq i \leqq m\,.$$

Hence we have

$$(2.6) \qquad \dim C_{ik} = \begin{cases} \dim S_{\alpha(i)\alpha(k)} & \text{if } \alpha(i) \neq \alpha(k) \\ \nu_{\alpha(i)} & \text{if } \alpha(i) = \alpha(k),\ i \neq k \\ 1 & \text{if } i = k\,. \end{cases}$$

(C) *We associate to each S-algebra C an S-system $\mathscr{S}$ by the pro-cedures (A) and (B). Then the correspondence $C \rightsquigarrow \mathscr{S}$ induces an injective map from the set of isomorphism classes of S-algebras into the set of equivalence classes of S-systems.*

Proof. We give the proof of the injectivity of the last map, since the proof of other assertions was contained in Takeuchi [7].

Let $\mathscr{S} = (S\,;\{m_\alpha\}_{1\leqq\alpha\leqq\mu+1})$ be an S-system. We define m, I_α ($1 \leqq \alpha \leqq \mu + 1$) and $\alpha(i)$ ($1 \leqq i \leqq m + 1$) by (2.1)~(2.4). Take real vector spaces C_{ik} ($1 \leqq i,\ k \leqq m + 1$) with dimensions (2.6) and linear isomor-phisms θ^{ik} as in (2.5). Put

$$C = \sum_{1\leqq i,\ k\leqq m+1} C_{ik}\,.$$

We define a product ab, an involutive linear automorphism $a \mapsto a^*$ and a partial complex structure j on C by the relations (1), (2) and (3). It is easy to see that $a \mapsto a^*$ is an anti-automorphism of C, making use of the commutativity of $u \mapsto u^*$ with $\mathscr{P}$ and $\mathscr{R}e$, and hence C becomes a

matrix algebra with involution and partial complex structure. If furthermore $\mathscr{S}$ is associated to an S-algebra C', then C becomes an S-algebra, which is isomorphic to C'. This shows the injectivity of our map.   q.e.d.

For an S-algebra C of the grading adapted to the relation $\sim$, the subalgebra

$$C^c = \sum_{\alpha(i)=\alpha(k)} C_{ik}$$

of C is called the *kernel* of C. We have the following multiplicative properties related to the kernel C^c (Takeuchi [7]).

LEMMA. 1) *If* $i \neq p$, $k \neq q$ *and one of* a_{ik}, b_{kp}, c_{pq} *belongs to* C^c, *then*

$$a_{ik}(b_{kp}c_{pq}) = (a_{ik}b_{kp})c_{pq} .$$

2) *If* $i \neq \ell$ *and* $\alpha(k)$ *equals* $\alpha(i)$ *or* $\alpha(\ell)$, *then*

$$a_{ik}(b_{k\ell}b_{k\ell}^*) = (a_{ik}b_{kl})b_{k\ell}^* ,$$
$$a_{ik}^*(a_{ik}b_{k\ell}) = (a_{ik}^*a_{ik})b_{k\ell} ,$$

or equivalently,

$$a_{ik}(b_{k\ell}c_{k\ell}^*) + a_{ik}(c_{k\ell}b_{k\ell}^*) = (a_{ik}b_{k\ell})c_{k\ell}^* + (a_{ik}c_{k\ell})b_{k\ell}^* ,$$
$$a_{ik}^*(b_{ik}c_{k\ell}) + b_{ik}^*(a_{ik}c_{k\ell}) = (a_{ik}^*b_{ik})c_{k\ell} + (b_{ik}^*a_{ik})c_{k\ell} .$$

3) *If* $1 \leq i \leq m$ *and* $a_{ik} \in C^c$, *then*

$$a_{ik}(jb_{k,m+1}) = j(a_{ik}b_{k,m+1}) ,$$
$$(jb_{m+1,i})a_{ik} = j(b_{m+1,i}a_{ik}) .$$

Now we consider the group of linear automorphisms of Siegel domains. Let $D(V,F) \subset X^c \times Y$ be a Siegel domain. Put

$$GL(D(V,F)) = A(D(V,F)) \cap GL(X^c \times Y) .$$

Let $\mathfrak{gl}(D(V,F))$ and $\mathfrak{g}(V)$ denote the Lie algebras of $GL(D(V,F))$ and $G(V)$ respectively. Identifying as $\mathfrak{gl}(X) \oplus \mathfrak{gl}(Y) \subset \mathfrak{gl}(X^c \times Y)$, we have (Pjateckii-Šapiro [5])

$$\mathfrak{gl}(D(V,F)) = \{(\psi', \psi'') \in \mathfrak{g}(V) \oplus \mathfrak{gl}(Y) ; \psi'F(u,v)$$
$$= F(\psi''u, v) + F(u, \psi''v) \ (u, v \in Y)\} .$$

The projection $\pi_X\colon \mathfrak{gl}(D(V,F)) \to \mathfrak{g}(V)$ is defined by the correspondence $(\psi',\psi'') \mapsto \psi'$. In particular, for the Siegel domain $D(C) \subset X^c \times (Y,j)$ associated to an S-algebra C, we have an identification

$$\mathfrak{gl}(D(C)) = \{(\psi',\psi'') \in \mathfrak{g}(V(A)) \oplus \mathfrak{gl}(Y) \, ; \, \psi''j = j\psi'',$$
$$\psi'\Phi(u,v) = \Phi(\psi''u,v) + \Phi(u,\psi''v) \; (u,v \in Y)\} \, ,$$

where $\Phi\colon Y \times Y \to X$ is the symmetric map associated to C. It is known (Vinberg-Gindikin-Pjateckii-Šapiro [12]) that $\mathfrak{gl}(D(C))$ (resp. $\mathfrak{g}(V(A))$) is an algebraic Lie algebra in $\mathfrak{gl}(X \times Y)$ (resp. in $\mathfrak{gl}(X)$), and $\pi_X\colon \mathfrak{gl}(D(C)) \to \mathfrak{g}(V(A))$ is a rational homomorphism. We shall describe the structure of $\mathfrak{gl}(D(C))$ in terms of the S-algebra C. Assume that C has the grading adapted to the relation $\sim$. Put

$$X^c = C^c \cap X \, , \qquad V^c = C^c \cap V(A) \, .$$

Then it can be shown that V^c is the direct product $V_1 \times \cdots \times V_\mu$ of the cones V_α corresponding to the T-algebras

$$A_\alpha = \sum_{\alpha(i)=\alpha(k)=\alpha} C_{ik} \qquad (1 \leq \alpha \leq \mu) \, ,$$

and hence V^c is a self-dual homogeneous convex cone in X^c, in virtue of the characterization of Vinberg cited in §1. The Lie algebra $\mathfrak{g}(V^c)$ of $G(V^c)$ has a Cartan decomposition

$$\mathfrak{g}(V^c) = \mathfrak{k}(V^c) + \mathfrak{p}(V^c) \, ,$$

where $\mathfrak{k}(V^c)$ is the stabilizer of $e_A = \sum_{1 \leq i \leq m} e_i \in V^c$ in $\mathfrak{g}(V^c)$, and $\mathfrak{p}(V^c)$ is the image $\psi^c(X^c)$ of the injective linear map $\psi^c\colon X^c \to \mathfrak{gl}(X^c)$ defined by

$$\psi^c(x)y = xy + yx \qquad \text{for } x,y \in X^c \, .$$

We define linear maps $\psi'\colon X^c \to \mathfrak{gl}(X)$ and $\psi''\colon X^c \to \mathfrak{gl}(Y)$ by

$$\psi'(x)y = xy + yx \qquad \text{for } x \in X^c, \; y \in X \, ,$$
$$\psi''(x)u = xu + ux \qquad \text{for } x \in X^c, \; u \in Y \, ,$$

and then define an injective linear map $\psi\colon X^c \to \mathfrak{gl}(X) \oplus \mathfrak{gl}(Y)$ by $\psi = \psi' \oplus \psi''$, i.e.,

$$\psi(x)(y,u) = (\psi'(x)y, \psi''(x)u) \qquad \text{for } x \in X^c, \; y \in X, \; u \in Y \, .$$

Put

$$T^u = \sum_{1 \leqq a(i) < a(k) \leqq \mu} C_{ik} \; .$$

Then T^u becomes a Lie algebra by the bracket product $[t, t'] = tt' - t't$. A faithful representation $\psi \colon T^u \to \mathfrak{gl}(X) \oplus \mathfrak{gl}(Y)$ is defined by

$$\psi(t)(x, u) = (tx + xt^*, \; tu + ut^*) \qquad \text{for } t \in T^u, x \in X, u \in Y \; .$$

We define subspaces $\mathfrak{p}^c$ and $\mathfrak{t}^u$ of $\mathfrak{gl}(X) \oplus \mathfrak{gl}(Y)$ by

$$\mathfrak{p}^c = \psi(X^c) \; , \qquad \mathfrak{t}^u = \psi(T^u) \; .$$

$\mathfrak{t}^u$ is nilpotent in the sense that it consists of nilpotent endomorphisms of $X \times Y$. Then we have

THEOREM 2.2. (Takeuchi [7]) *We have a direct sum decomposition*

$$\mathfrak{gl}(D(C)) = \mathfrak{g}_0 + \mathfrak{g}^c + \mathfrak{t}^u$$

as vector spaces with the following properties:

    1) *$\mathfrak{t}^u$ is the maximum nilpotent ideal of $\mathfrak{gl}(D(C))$.*

    2) *$\mathfrak{g}^c$ is a reductive algebraic subalgebra of $\mathfrak{gl}(D(C))$ without compact factors. $\mathfrak{g}^c X^c \subset X^c$ and the restriction to X^c induces an isomorphism of $\mathfrak{g}^c$ onto $\mathfrak{g}(V^c)$. We have a Cartan decomposition*

$$\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c \; ,$$

where $\mathfrak{k}^c$ is the stabilizer of e_A in $\mathfrak{g}^c$, and $\mathfrak{p}^c$ is the one defined in the above.

    3) *$\mathfrak{g}_0$ is a compact subalgebra of $\mathfrak{gl}(D(C))$ satisfying $\mathfrak{g}_0 X^c = \{0\}$ and $[\mathfrak{g}_0, \mathfrak{g}^c] = \{0\}$.*

    4) *The stabilizer $\mathfrak{k}$ of e_A in $\mathfrak{gl}(D(C))$ is a maximal compact subalgebra of $\mathfrak{gl}(D(C))$ with the decomposition*

$$\mathfrak{k} = \mathfrak{g}_0 \oplus \mathfrak{k}^c \; .$$

COROLLARY. *Let C be an S-algebra. Then the rank of an S-system associated to C is equal to 1 if and only if (1) the associated T-algebra A is isomorphic to one of the A_m^ν, and (2) the projection $\pi_X \colon \mathfrak{gl}(D(C)) \to \mathfrak{g}(V(A))$ is surjective.*

Proof. We may assume that C has the grading adapted to the relation $\sim$. Let $\mathscr{S} = (S ; \{m_a\}_{1 \leqq a \leqq \mu+1})$ be an S-system associated to C.

Assume $\mu = 1$. Then $T^u = \{0\}$, $X = X^c$, $V(A) = V^c$ and A is isomorphic to $A_{m_1}^{\nu_1}$. It follows from Theorem 2.2 that

$$\mathfrak{gl}(D(C)) = \mathfrak{g}_0 \oplus \mathfrak{g}^c \, ,$$

where $\mathfrak{g}_0 X = \{0\}$ and $\mathfrak{g}^c$ is isomorphic to $\mathfrak{g}(V(A))$ through the restriction to X, and hence π_X is surjective.

Conversely assume that (1) and (2) are satisfied. It follows from (2) that $\pi_X(\mathfrak{gl}(D(C)))$ is reductive and $\pi_X(\mathfrak{t}^u)$ is a nilpotent ideal in $\pi_X(\mathfrak{gl}(D(C)))$, and hence $\pi_X(\mathfrak{t}^u) = \{0\}$. Since π_X is injective on $\mathfrak{t}^u$, we have $T^u = \{0\}$. Now (1) implies $\mu = 1$.   q.e.d.

3. Siegel domains satisfying (i) and (ii)

LEMMA 1. *Let $D(V, F) \subset X^c \times Y$ be a Siegel domain associated to a self-dual homogeneous convex cone V. Then the Satake's second condition* (ii) *is equivalent to the following condition*

(ii)′ $\pi_X : \mathfrak{gl}(D(V, F)) \to \mathfrak{g}(V)$ *is surjective.*

If $D(V, F)$ satisfies (ii), *it is affinely homogeneous.*

Proof. If the condition (ii) is satisfied, we have $\mathfrak{p}(V) \subset \pi_X(\mathfrak{gl}(D(V, F)))$ in the notation in Introduction. On the other hand, we have $\mathfrak{g}(V) = \mathfrak{k}(V) + \mathfrak{p}(V)$ with $[\mathfrak{p}(V), \mathfrak{p}(V)] = \mathfrak{k}(V)$, and hence $\mathfrak{g}(V) \subset \pi_X(\mathfrak{gl}(D(V, F)))$. This shows (ii)′.

Conversely assume (ii)′. Then $D(V, F)$ is affinely homogeneous (cf. Pjateckii-Šapiro [5]). From the arguments in §1, we may assume that $D(V, F) = D(C)$ of an S-algebra C of rank m satisfying (1) the associated T-algebra $A = A_m^\nu$, and (2) $\pi_X : \mathfrak{gl}(D(C)) \to \mathfrak{g}(V_m^\nu)$ is surjective. Then as we have seen in Theorem 2.2 and Cor., $\mathfrak{gl}(D(C))$ contains the image $\mathfrak{p}^c = \psi(X_m^\nu)$ of the map $\psi = \psi' \oplus \psi''$ given by

$$\psi'(x)y = xy + yx \qquad \text{for } x, y \in X_m^\nu = X \, ,$$
$$\psi''(x)u = xu + ux \qquad \text{for } x \in X_m^\nu, \ u \in Y \, .$$

Take the base point $e_A \in V_m^\nu$ and the standard inner product $(\, ,)$ on X_m^ν. Recall that then

$$T(x) = \tfrac{1}{2}\psi'(x) \qquad \text{for } x \in X_m^\nu \, .$$

We shall show that if we put

$$R(x) = \tfrac{1}{2}\psi''(x) \qquad \text{for } x \in X_m^\nu \, ,$$

$R(x)$ is self-adjoint with respect to the hermitian inner product $(e_A, F(u, v))$ on (Y, j); this will imply (ii). We may assume $n_i = 1$ for $1 \leqq i \leqq m$,

in virtue of Remark of § 1. Thus the inner product $(\,,\,)$ on C coincides on X_m^ν with the standard inner product of X_m^ν. We define a hermitian inner product $\langle\,,\,\rangle$ on (Y, j) by

$$\langle u, v \rangle = (u, v) + \sqrt{-1}(u, jv) \qquad \text{for } u, v \in Y .$$

Then we have $4(e_A, F(u, v)) = \langle u, v \rangle$ for each $u, v \in Y$. In fact, for the symmetric map Φ associated to C we have

$$2(e_A, \Phi(u, v)) = (e_A, \hat{u}\hat{v} + \hat{v}\hat{u}) = (\hat{v}^*, \hat{u}) + (\hat{v}^*, \hat{u})$$
$$= (\hat{u}, \hat{v}) + (\hat{u}, \hat{u}) = (u, v) .$$

Furthermore we have

$$(\psi''(x)u, v) = (xu + ux, v) = (u, x^*v) + (u, vx^*)$$
$$= (u, xv + vx) = (u, \psi''(x)v) \qquad \text{for } x \in X_m^\nu, \ u, v \in Y ,$$

and hence $\psi''(x)$ is self-adjoint with respect to the hermitian inner product $\langle\,,\,\rangle$. This implies the required assertion.

The last assertion of the Lemma is clear from the above arguments.

$$\text{q.e.d.}$$

LEMMA 2. *Let $\mathscr{S} = (S\,;\,\{m_\alpha\}_{1 \leq \alpha \leq 2})$ be an S-system of rank $\mu = 1$. Then $\mathscr{S}$ is associated to an S-algebra if and only if it satisfies the following conditions* 1)~8):

1) $\varepsilon_\alpha u_{\alpha\beta} = u_{\alpha\beta}$.
2) *There exist* $\ell_\alpha > 0$ $(1 \leq \alpha \leq \mu + 1)$ *such that*

$$\ell_\alpha \, \mathscr{R}e\,(u_{\alpha\beta}v_{\beta\alpha}) = \ell_\beta \, \mathscr{R}e\,(v_{\beta\alpha}u_{\alpha\beta}) .$$

3) $\mathscr{R}e\,(u_{\alpha\beta}u_{\alpha\beta}^*) > 0$ *if* $u_{\alpha\beta} \neq 0$.
4) $\mathscr{R}e\,(u_{\alpha\beta}(v_{\beta\gamma}w_{\gamma\alpha})) = \mathscr{R}e\,((u_{\alpha\beta}v_{\beta\gamma})w_{\gamma\alpha})$.
5) $u_{\alpha\beta}(jv_{\beta,\mu+1}) = j(u_{\alpha\beta}v_{\beta,\mu+1})$ *if* $1 \leq \alpha \leq \beta \leq \mu$.
6) $j(u_{\alpha,\mu+1}^*) = (ju_{\alpha,\mu+1})^*$ *if* $1 \leq \alpha \leq \mu$.
7) $\mathscr{R}e\,(ju_{\alpha,\mu+1}(ju_{\alpha,\mu+1})^*) = \mathscr{R}e\,(u_{\alpha,\mu+1}u_{\alpha,\mu+1}^*)$ *if* $1 \leq \alpha \leq \mu$.
8) 8–1) $\lambda(\mu u) = (\lambda\mu)u$ *for* $\lambda, \mu \in S_{11}$, $u \in S_{12}$,
    8–2) $\lambda(uv) = (\lambda u)v$ *for* $\lambda \in S_{11}$, $u \in S_{12}$, $v \in S_{21}$.

Remark 1. For a general S-system $\mathscr{S} = (S\,;\,\{m_\alpha\}_{1 \leq \alpha \leq \mu+1})$ of rank μ, the conditions in order that $\mathscr{S}$ is associated to an S-algebra, are given by the above conditions 1)~7) for general μ, together with the following associativity conditions:

8) If $\alpha \neq \beta$ or $\beta \neq \gamma$,

$$u_{\alpha\alpha}(v_{\alpha\beta}w_{\beta\gamma}) = (u_{\alpha\alpha}v_{\alpha\beta})w_{\beta\gamma}\,,$$

$$u_{\alpha\beta}(v_{\beta\beta}w_{\beta\gamma}) = (u_{\alpha\beta}v_{\beta\beta})w_{\beta\gamma}\,.$$

9) $u_{\alpha\beta}(v_{\beta\gamma}w_{\gamma\delta}) = (u_{\alpha\beta}v_{\beta\gamma})w_{\gamma\delta}$ if $\alpha < \beta < \gamma$ and $\beta < \delta$.

10) $u_{\alpha\beta}\mathscr{P}(v_{\beta\gamma}w_{\gamma\beta}) = (u_{\alpha\beta}v_{\beta\gamma})w_{\gamma\beta}$ if $\alpha < \beta < \gamma$ and $m_\beta \geqq 2$,

$u_{\alpha\beta}(v_{\beta\gamma}v_{\beta\gamma}^*) = (u_{\alpha\beta}v_{\beta\gamma})v_{\beta\gamma}^*$ if $\alpha < \beta < \gamma$ and $m_\beta = 1$.

Proof of Lemma 2. Let $\mathscr{S}$ be associated as in Theorem 2.1, (B) to an S-algebra of the grading adapted to the relation $\sim$. Then 1) follows from the axiom (II) of S-algebras. In virtue of Remark of §1, we may assume that $n_i = n_k$ if $\alpha(i) = \alpha(k)$. Put $\ell_{\alpha(i)} = n_i$. Then 2) follows from the axiom (III) and the equality $(\lambda, \mu) = (\bar\lambda, \bar\mu)$ for the standard inner product $(\ ,\ )$ of F_m^ν, seeing that F_m^ν is generated by $\mathscr{P}F_m^\nu$. We get 3) making use of the axiom (IV). Theorem 2.1, (B), (b) implies 8–1), 8–2) and

8–3) $u(\lambda v) = (u\lambda)v$ for $u \in S_{21}$, $v \in S_{12}$, $\lambda \in S_{11}$.

Applying the involution $u \mapsto u^*$ on 8–2), we get

8–2)' $(uv)\lambda = u(v\lambda)$ for $u \in S_{12}$, $v \in S_{21}$, $\lambda \in S_{11}$.

Now 4) follows from 8–3), 8–2)' and the equality $(\lambda\mu, \nu) = (\mu, \bar\lambda\nu)$ for the standard inner product of F_m^ν. 5) follows from the axiom (VIII), seeing also that F_m^ν is generated by $\mathscr{P}F_m^\nu$. 6) and 7) follow from the axioms (IX) and (X) respectively.

Conversely assume that $\mathscr{S}$ satisfies the conditions 1)$\sim$8). Let

$$C = \sum_{1 \leqq i,\, k \leqq m+1} C_{ik}\,, \qquad m = m_1$$

be the matrix algebra with involution $a \mapsto a^*$ and partial complex structure j, constructed from $\mathscr{S}$ as in (C) of Theorem 2.1. We shall prove axioms (I)$\sim$(X) for C.

(I) Put $\theta = \theta^{ii}: C_{ii} \to \boldsymbol{R}\varepsilon_{\alpha(i)} = \boldsymbol{R}$. Then

$$\theta^{ii}(a_{ii}b_{ii}) = \mathscr{R}e\,(\theta^{ii}(a_{ii})\theta^{ii}(b_{ii})) = \theta^{ii}(a_{ii})\theta^{ii}(b_{ii})$$

implies that θ is an algebra isomorphism.

(II) The unit element $e_i \in C_{ii}$ satisfies $\theta^{ii}(e_i) = \varepsilon_{\alpha(i)}$. It follows from 1)

$$\theta^{ik}(e_i a_{ik}) = \begin{cases} \theta^{ii}(e_i)\theta^{ik}(a_{ik}) = \theta^{ik}(a_{ik}) & \text{if } \alpha(i) \neq \alpha(k) \\ \mathscr{P}(\theta^{ii}(e_i)\theta^{ik}(a_{ik})) = \mathscr{P}\theta^{ik}(a_{ik}) & \text{if } \alpha(i) = \alpha(k), \ i \neq k \\ \mathscr{R}e\,(\theta^{ii}(e_i)\theta^{ik}(a_{ik})) = \mathscr{R}e\,\theta^{ik}(a_{ik}) & \text{if } i = k \end{cases}$$
$$= \theta^{ik}(a_{ik}) \ .$$

This implies $e_i a_{ik} = a_{ik}$.

(III)   Put $n_i = \ell_{\alpha(i)}$.   Then from 2) it follows

$$n_i \theta(a_{ik}b_{ki}) = \ell_{\alpha(i)}\theta^{ii}(a_{ik}b_{ki}) = \ell_{\alpha(i)}\,\mathscr{R}e\,(\theta^{ik}(a_{ik})\theta^{ki}(b_{ki}))$$
$$= \ell_{\alpha(k)}\,\mathscr{R}e\,(\theta^{ki}(b_{ki})\theta^{ik}(a_{ik})) = \ell_{\alpha(k)}\theta^{kk}(b_{ki}a_{ik})$$
$$= n_k \theta(b_{ki}a_{ik}) \ .$$

(IV)   If $a_{ik} \neq 0$,

$$\theta(a_{ik}a_{ik}^*) = \theta^{ii}(a_{ik}a_{ik}^*) = \mathscr{R}e\,(\theta^{ik}(a_{ik})\theta^{ki}(a_{ik}^*))$$
$$= \mathscr{R}e\,(\theta^{ik}(a_{ik})\theta^{ik}(a_{ik})^*) > 0$$

in virtue of 3).

(V): $a_{ik}(b_{k\ell}c_{\ell i}) = (a_{ik}b_{k\ell})c_{\ell i}$.   We may assume that i, k, ℓ are mutually distinct, in virtue of (II).   Then

$$\theta^{ii}(a_{ik}(b_{k\ell}c_{\ell i})) = \mathscr{R}e\,(\theta^{ik}(a_{ik})\theta^{ki}(b_{k\ell}c_{\ell i}))$$
$$= \begin{cases} \mathscr{R}e\,[\theta^{ik}(a_{ik})(\theta^{k\ell}(b_{k\ell})\theta^{\ell i}(c_{\ell i}))] & \text{if } \alpha(k) \neq \alpha(i) \\ \mathscr{R}e\,[\theta^{ik}(a_{ik})\mathscr{P}(\theta^{k\ell}(b_{k\ell})\theta^{\ell i}(c_{\ell i}))] & \text{if } \alpha(k) = \alpha(i) \end{cases}$$
$$= \mathscr{R}e\,[\theta^{ik}(a_{ik})(\theta^{k\ell}(b_{k\ell})\theta^{\ell i}(c_{\ell i}))] \ ,$$

since $\mathscr{R}e\,(\lambda\mu) = \mathscr{R}e\,(\lambda\mathscr{P}u)$ for $\lambda \in \mathscr{P}F_m^\nu, \mu \in F_m^\nu$.   In the same way we get

$$\theta^{ii}((a_{ik}b_{k\ell})c_{\ell i}) = \mathscr{R}e\,[(\theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell}))\theta^{\ell i}(c_{\ell i})] \ .$$

Now (V) follows from 4).

(VI): $a_{ik}(b_{k\ell}c_{\ell p}) = (a_{ik}b_{k\ell})c_{\ell p}$ if $i < k < \ell$, $k < p$.   In virtue of (II), we may consider the following three cases:

(a) $\alpha(i) = \alpha(k) = \alpha(\ell) = 1 < \alpha(p) = 2$, $m \geqq 3$,

(b) $\alpha(i) = \alpha(k) = \alpha(p) = 1 < \alpha(\ell) = 2$, $m \geqq 3$,

(c) $\alpha(i) = \alpha(k) = \alpha(\ell) = \alpha(p) = 1$, $\ell \neq p$, $m \geqq 4$.

In case (a), we have

$$\theta^{ip}(a_{ik}(b_{k\ell}c_{\ell p})) = \theta^{ik}(a_{ik})\theta^{kp}(b_{k\ell}c_{\ell p}) = \theta^{ik}(a_{ik})(\theta^{k\ell}(b_{k\ell})\theta^{\ell p}(c_{\ell p}))$$
$$= (\theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell}))\theta^{\ell p}(c_{\ell p}) \quad \text{by 8-1)}$$
$$= \theta^{i\ell}(a_{ik}b_{k\ell})\theta^{\ell p}(c_{\ell p}) \quad \text{since } m \neq 2$$
$$= \theta^{ip}((a_{ik}b_{k\ell})c_{\ell p}) \ .$$

This implies (VI). In case (b), making use of 8–2) and $m \neq 2$, we have

$$\theta^{ip}(a_{ik}(b_{k\ell}c_{\ell p})) = \theta^{ip}((a_{ik}b_{k\ell})c_{\ell p}) ,$$

which implies (VI). In case (c), making use of $m \neq 2$, we have

$$\theta^{ip}(a_{ik}(b_{k\ell}c_{\ell p})) = \theta^{ik}(a_{ik})(\theta^{k\ell}(b_{k\ell})\theta^{\ell p}(c_{\ell p})) ,$$
$$\theta^{ip}(a_{ik}b_{k\ell})c_{\ell p} = (\theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell}))\theta^{\ell p}(c_{\ell p}) .$$

Since F_m^ν is an associative algebra for $m \geq 4$, we get (VI).

(VII): $a_{ik}(b_{k\ell}b_{k\ell}^*) = (a_{ik}b_{k\ell})b_{k\ell}^*$ if $i < k < \ell$. We may consider following two cases:

(a) $\alpha(i) = \alpha(k) = 1 < \alpha(\ell) = 2, \ m \geq 2$,

(b) $\alpha(i) = \alpha(k) = \alpha(\ell) = 1, \ m \geq 3$.

In case (a), we have

$$\theta^{ik}(a_{ik}(b_{k\ell}b_{k\ell}^*)) = \mathscr{P}(\theta^{ik}(a_{ik})\theta^{kk}(b_{k\ell}b_{k\ell}^*))$$
$$= \mathscr{P}[\theta^{ik}(a_{ik}) \, \mathscr{R}e \, (\theta^{k\ell}(b_{k\ell})\theta^{k\ell}(b_{k\ell})^*)]$$
$$= \mathscr{P}[\theta^{ik}(a_{ik})(\theta^{k\ell}(b_{k\ell})\theta^{k\ell}(b_{k\ell})^*)] ,$$

since $\mathscr{P}(\lambda \, \mathscr{R}e \, \mu) = \mathscr{P}(\lambda\mu)$ for $\lambda \in \mathscr{P}F_m^\nu$ and $\mu \in F_m^\nu$ with $\bar\mu = \mu$. On the other hand

$$\theta^{ik}((a_{ik}b_{k\ell})b_{k\ell}^*) = \mathscr{P}(\theta^{i\ell}(a_{ik}b_{k\ell})\theta^{\ell k}(b_{k\ell}^*))$$
$$= \mathscr{P}[(\theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell}))\theta^{k\ell}(b_{k\ell})^*]$$
$$= \mathscr{P}[(\theta^{ik}(a_{ik})(\theta^{k\ell}(b_{k\ell})\theta^{k\ell}(b_{k\ell})^*)] \qquad \text{by 8–2) .}$$

These imply (VII). In case (b), we have

$$\theta^{ik}(a_{ik}(b_{k\ell}b_{k\ell}^*)) = \theta^{ik}(a_{ik})(\theta^{k\ell}(b_{k\ell})\theta^{k\ell}(b_{k\ell})^*) ,$$
$$\theta^{ik}((a_{ik}b_{k\ell})b_{k\ell}^*) = (\theta^{ik}(a_{ik})\theta^{k\ell}(b_{k\ell}))\theta^{k\ell}(b_{k\ell})^* .$$

Together with the equality $\lambda(\mu\bar\mu) = (\lambda\mu)\bar\mu$ in F_m^ν, we get (VII).

(VIII) follows from 5) and 8–1).

(IX) and (X) are immediate consequences of 6) and 7) respectively.

<div align="right">q.e.d.</div>

Remark 2. For an S-system $(S; \{m_\alpha\})$ of rank μ satisfying 2) and 3), we put

$$\mathrm{tr}\, u = \sum_\alpha \ell_\alpha \, \mathscr{R}e \, u_{\alpha\alpha} \qquad \text{for } u = (u_{\alpha\beta}) \in S ,$$
$$(u, v) = \mathrm{tr}\, uv^* \qquad \text{for } u, v \in S .$$

Then $(\ ,\ )$ is an inner product on S satisfying

$$(u^*, v^*) = (u, v) \qquad \text{for } u, v \in S ,$$

and the condition 4) is equivalent to

$$(uv, w) = (v, u^*w) \qquad \text{for } u, v, w \in S ,$$

or

$$(uv, w) = (u, wv^*) \qquad \text{for } u, v, w \in S .$$

Let $\mathfrak{M}(F_m^\nu)$ be the set of all equivalence classes of (finite dimensional and unital) F_m^ν-modules over C, and $\mathfrak{M}^*(F_m^\nu)$, $\mathrm{Aut}^*\,(F_m^\nu, \mathscr{P})$ and $\mathfrak{M}(m, \nu)$ be as in Introduction. Then the natural map from $\mathfrak{M}^*(F_m^\nu)$ into $\mathfrak{M}(F_m^\nu)$ is a bijection. This can be proved in the same way as Lemma 21 in Ozeki-Takeuchi [4]. Since the natural map is equivariant with respect to the action of $\mathrm{Aut}^*\,(F_m^\nu, \mathscr{P})$, we have the identification

$$\mathfrak{M}(m, \nu) = \mathrm{Aut}^*\,(F_m^\nu, \mathscr{P}) \backslash \mathfrak{M}^*(F_m^\nu) = \mathrm{Aut}^*\,(F_m^\nu, \mathscr{P}) \backslash \mathfrak{M}(F_m^\nu)$$

of orbit spaces. The point of $\mathfrak{M}(m, \nu)$ represented by an F_m^ν-module W will be denoted by $[W]$.

In virtue of Lemma 1 and Cor. of Theorem 2.2, the classification of affine isomorphism classes of those Siegel domains $D(V, F)$ such that V is an indecomposable self-dual homogeneous convex cone and it satisfies the condition (ii), is equivalent to the classification of isomorphism classes of those S-algebras such that the rank of an associated S-system is 1. Thus, in virtue of Theorem 2.1 and Lemma 2, our Theorem A in Introduction is equivalent to the following

THEOREM 3.1. *The set $\mathfrak{S}(m, \nu)$ of equivalence classes of those S-systems $(S; \{m_\alpha\})$ of rank 1 satisfying $m_1 = m$, 1)~8) and $\vartheta: S_{11} \cong F_m^\nu$, is in a bijective correspondence with the orbit space $\mathfrak{M}(m, \nu)$.*

Proof. Let $\mathscr{S} = (S; \{m_\alpha\}_{1 \le \alpha \le 2})$ be an S-system of rank 1 satisfying $m_1 = m$, 1)~8) and $S_{11} \cong F_m^\nu$. We define

$$(u, v) = \mathscr{R}e\ uv^* \qquad \text{for } u, v \in S_{12} .$$

Then 3) and 7) show that $(\ ,\ )$ is an inner product on S_{12} satisfying $(ju, jv) = (u, v)$ for $u, v \in S_{12}$. Put $W = (S_{12}, j)$ and

$$\langle u, v \rangle = (u, v) + \sqrt{-1}(u, jv) \qquad \text{for } u, v \in S_{12} .$$

Then $\langle\,,\,\rangle$ is a hermitian inner product on W. 1), 5) and 8–1) show that W is an F_m^ν-module over C. Furthermore Remark 2 implies $(\lambda u, v) = (u, \bar\lambda v)$ for $\lambda \in F_m^\nu, u, v \in S_{12}$, and hence

$$(*) \quad \langle\lambda u, v\rangle = \langle u, \bar\lambda v\rangle \qquad \text{for } \lambda \in F_m^\nu, \; u, v \in W \; .$$

Now the correspondence $\mathscr{S} \rightsquigarrow W$ induces the map $\mathfrak{S}(m, \nu) \ni \{\mathscr{S}\} \mapsto [W] \in \mathfrak{M}(m, \nu)$.

Conversely let W be an F_m^ν-module with a hermitian inner product $\langle\,,\,\rangle$ satisfying the condition $(*)$, and W_R the scalar restriction to R of W. We define an inner product $(\,,\,)$ on W_R by

$$(u, v) = \mathscr{R}e \langle u, v\rangle \qquad \text{for } u, v \in W \; .$$

It satisfies

$$(3.1) \qquad (\lambda u, v) = (u, \bar\lambda v) \qquad \text{for } \lambda \in F_m^\nu, \; u, v \in W \; .$$

The natural complex structure on W_R (defined by the multiplication $\sqrt{-1}$ on W) will be denoted by j. Making use of the standard inner product $(\lambda, \mu) = \mathscr{R}e \,\lambda\bar\mu$ of F_m^ν, we define a bilinear map $W_R \times W_R \ni (u, v) \mapsto uv \in F_m^\nu$ by

$$(uv, \lambda) = (u, \lambda v) \qquad \text{for each } \lambda \in F_m^\nu \; .$$

It satisfies

$$(3.2) \qquad \lambda(uv) = (\lambda u)v \; , \qquad (uv)\lambda = u(\bar\lambda v) \; ,$$

$$(3.3) \qquad \mathscr{R}e \, uv = (u, v) \; ,$$

$$(3.4) \qquad \overline{uv} = vu \; ,$$

$$(3.5) \qquad (ju)(jv) = uv \; ,$$

for $u, v \in W$ and $\lambda \in F_m^\nu$. Now we put

$$S = \left\{ \begin{pmatrix} \lambda & u \\ v & \xi \end{pmatrix}; \begin{array}{l} \lambda \in F_m^\nu, \; \xi \in R \\ u, v \in W_R \end{array} \right\} \; , \qquad m_1 = m, \; m_2 = 1$$

and define on S a product ss', an involution $s \mapsto s^*$ and a partial complex structure j by

$$\begin{pmatrix} \lambda & u \\ v & \xi \end{pmatrix} \begin{pmatrix} \lambda' & u' \\ v' & \xi' \end{pmatrix} = \begin{pmatrix} \lambda\lambda' + uv' & \lambda u' + \xi'u \\ \bar\lambda'v + \xi v' & (v, u') + \xi\xi' \end{pmatrix} \; ,$$

$$\begin{pmatrix} \lambda & u \\ v & \xi \end{pmatrix}^* = \begin{pmatrix} \bar\lambda & v \\ u & \xi \end{pmatrix},$$

$$j\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} = \begin{pmatrix} 0 & ju \\ jv & 0 \end{pmatrix}.$$

Then $\mathscr{S} = (S; \{m_\alpha\}_{1\leq\alpha\leq2})$ becomes an S-system of rank 1 with $S_{11} = F_m^\nu$. We define

$$\mathrm{tr}\, s = \mathscr{R}e\,\lambda + \xi \qquad \text{for } s = \begin{pmatrix} \lambda & u \\ v & \xi \end{pmatrix} \in S,$$

$$(s, s') = \mathrm{tr}\, ss'^* \qquad \text{for } s, s' \in S,$$

and hence

$$(s, s) = (\lambda, \lambda) + (u, u) + (v, v) + \xi^2 \qquad \text{for } s = \begin{pmatrix} \lambda & u \\ v & \xi \end{pmatrix} \in S.$$

We can show

$$(s, s') = (s^*, s'^*) \qquad \text{for } s, s' \in S,$$
$$(ss', s'') = (s, s''s'^*) \qquad \text{for } s, s', s'' \in S,$$

making use of (3.1), (3.2) and (3.3). It follows from Remark 2 that $\mathscr{S}$ satisfies 2), 3) and 4). 1), 5), 6) and 8–1) follow from the definition. 7) and 8–2) follow from (3.5) and (3.2) respectively. Thus we get an S-system $\mathscr{S}$ of rank 1 satisfying $m_1 = m$, 1)~8) and $S_{11} = F_m^\nu$. It is easy to see that the correspondence $W \rightsquigarrow \mathscr{S}$ induces the map $\mathfrak{M}(m,\nu) \ni [W] \mapsto \{\mathscr{S}\} \in \mathfrak{S}(m,\nu)$, and that it is the inverse of the former map $\mathfrak{S}(m,\nu) \to \mathfrak{M}(m,\nu)$. This proves Theorem 3.1. q.e.d.

Next we shall describe explicitly the hermitian map of the Siegel domain corresponding to $[W] \in \mathfrak{M}(m,\nu)$. The C-linear extension to $(P_m^\nu)^C$ of the involutive linear automorphism $\lambda \mapsto \bar\lambda$ of P_m^ν, will be also denoted by $\lambda \mapsto \bar\lambda$. Then $(X_m^\nu)^C$ is identified with the space

$$H_m((P_m^\nu)^C) = \{z \in M_m((P_m^\nu)^C); {}^t\bar z = z\}$$

of all hermitian matrices in $M_m((P_m^\nu)^C)$. The (i,k)-th component of $z \in H_m((P_m^\nu)^C)$ will be denoted by z_{ik}. Denoting the C-linear extension to $(P_m^\nu)^C$ of the standard inner product $(\,,\,)$ on P_m^ν by the same notation $(\,,\,)$, and the complex conjugation of $(P_m^\nu)^C$ with respect to P_m^ν by $\lambda \mapsto \tilde\lambda$, we define a hermitian inner product $\langle\,,\,\rangle$ on $(P_m^\nu)^C$ by

$$\langle \lambda, \mu \rangle = (\lambda, \tilde{\mu}) \qquad \text{for } \lambda, \mu \in (P^\nu_m)^C .$$

We define a map $H \colon W \times W \to (P^\nu_m)^C$ by

$$\langle \lambda, H(u, v) \rangle = \langle \lambda v, u \rangle \qquad \text{for each } \lambda \in P^\nu_m .$$

It is sesqui-linear (i.e., C-linear with respect to u and conjugate linear with respect to v) and satisfies

$$\widetilde{H(u, v)} = \overline{H(v, u)} \qquad \text{for } u, v \in W .$$

In terms of the standard units $c_0, c_1, \cdots, c_{\nu-1}$ of P^ν_m, H is given by

$$H(u, v) = \sum_{0 \leqq i \leqq \nu-1} \langle u, c_i v \rangle c_i \qquad \text{for } u, v \in W .$$

THEOREM 3.2. *The V^ν_m-positive hermitian map $F \colon Y \times Y \to H_m((P^\nu_m)^C)$ of the Siegel domain $D(V^\nu_m, F)$ corresponding to $[W] \in \mathfrak{M}(m, \nu)$ is described as follows: Put $Y = \overbrace{W + \cdots + W}^{m}$. The i-th component of $u \in Y$ will be denoted by $u_i \in W$ $(1 \leqq i \leqq m)$. Then F is given by*

$$F(u, v)_{ik} = \tfrac{1}{2}\{H(u_i, v_k) + \overline{H(u_k, v_i)}\} \qquad \textit{for } 1 \leqq i, \ k \leqq m, \ i \neq k ,$$
$$F(u, v)_{ii} = \langle u_i, v_i \rangle \qquad \textit{for } 1 \leqq i \leqq m .$$

Proof. Let $\mathscr{S}$ be an S-system constructed from W as in Theorem 3.1, and C an S-algebra constructed from $\mathscr{S}$ as in Theorem 2.1, (C). We may assume that the T-algebra A associated to C is A^ν_m and $Y = \{a \in B \,; a^* = a\}$ is identified with the totality of "matrices" of the form

$$u = \left[\begin{array}{c|c} 0 & \begin{matrix} u_1 \\ \vdots \\ u_m \end{matrix} \\ \hline u_1 \cdots u_m & 0 \end{array} \right] ; \quad u_i \in W_R \ (1 \leqq i \leqq m) ,$$

with the natural complex structure j. The associated symmetric map Φ is given by

$$(3.6) \quad \begin{cases} \Phi(u, v)_{ik} = \tfrac{1}{2}\mathscr{P}(u_i v_k + v_i u_k) & \text{for } 1 \leqq i, \ k \leqq m, \ i \neq k , \\ \Phi(u, v)_{ii} = \mathscr{R}e \, (u_i v_i) = (u_i, v_i) & \text{for } 1 \leqq i \leqq m , \end{cases}$$

in the notation in Theorem 3.1, and hence

$$(3.6)' \quad \begin{cases} \Phi(u, jv)_{ik} = \tfrac{1}{2}\mathscr{P}(u_i(jv_k) - v_i(ju_k)) & \text{for } 1 \leqq i, \ k \leqq m, \ i \neq k \\ \Phi(u, jv)_{ii} = (u_i, jv_i) & \text{for } 1 \leqq i \leqq m . \end{cases}$$

We define an R-bilinear map $H \colon W_R \times W_R \to (P_m^\nu)^C$ by

$$H(u, v) = \mathscr{P}(uv + \sqrt{-1}u(jv)) \qquad \text{for } u, v \in W ,$$

denoting the C-linear extension of $\mathscr{P} \colon F_m^\nu \to P_m^\nu$ by the same $\mathscr{P} \colon (F_m^\nu)^C \to (P_m^\nu)^C$. Then it is characterized by

$$\langle \lambda, H(u, v) \rangle = \langle \lambda v, u \rangle \qquad \text{for each } \lambda \in P_m^\nu ,$$

and hence it coincides with the formerly defined sesqui-linear map H. In fact,

$$\begin{aligned}
\langle \lambda, H(u, v) \rangle &= \langle \lambda, uv + \sqrt{-1}u(jv) \rangle = (\lambda, uv) - \sqrt{-1}(\lambda, u(jv)) \\
&= (\lambda, uv) + \sqrt{-1}(\lambda, (ju)v) \\
&= (\lambda v, u) + \sqrt{-1}(\lambda v, ju) = \langle \lambda v, u \rangle .
\end{aligned}$$

Now, by (3.6), (3.6)', the associated hermitian map $F(u, v) = \frac{1}{2}\{\Phi(u, v) + \sqrt{-1}\Phi(u, jv)\}$ is described as

$$\begin{aligned}
2F(u, v)_{ik} &= \tfrac{1}{2}\{H(u_i, v_k) + \overline{H(u_k, v_i)}\} && \text{for } 1 \le i,\ k \le m,\ i \ne k , \\
2F(u, v)_{ii} &= \langle u_i, v_i \rangle && \text{for } 1 \le i \le m .
\end{aligned}$$

This implies the Theorem.   q.e.d.

Remark 3. Let $D(V_m^\nu, F) \subset X^c \times Y$ be the Siegel domain constructed from $[W] \in \mathfrak{M}(m, \nu)$ in the Theorem. As we have seen in the proof of Theorem 2.2, Cor., we have the direct sum decomposition

$$\mathfrak{gl}(D(V_m^\nu, F)) = \mathfrak{g}_0 \oplus \mathfrak{g}^c ,$$

where $\mathfrak{g}_0$ is the kernel of the projection π_X. The ideal $\mathfrak{g}_0$ is described as follows: Let $\mathfrak{D}_0$ denote the real subalgebra of $\mathfrak{gl}(W)$ consisting of those linear endomorphisms of W, leaving invariant the hermitian inner product $\langle\, ,\, \rangle$ of W, and commuting with the action of F_m^ν on W. We define $\psi_D \in \mathfrak{gl}(X^c \times Y)$ for $D \in \mathfrak{D}_0$ by

$$\psi_D\Big(z, \sum_i u_i\Big) = \Big(0, \sum_i Du_i\Big) \qquad \text{for } z \in X^c,\ u_i \in W\ (1 \le i \le m) .$$

Then the correspondence $D \mapsto \psi_D$ gives an isomorphism of $\mathfrak{D}_0$ onto $\mathfrak{g}_0$. This follows from the Theorem or from the detailed description of $\mathfrak{g}_0$ in terms of derivations of C, given in Takeuchi [7]. The subalgebra $\mathfrak{g}^c$ is also described explicitly by means of the F_m^ν-module structure of W [7].

It is easy to determine $\mathfrak{M}(m, \nu)$, since the set $\mathfrak{M}(F_m^\nu)$ is well known.

In the following we will give $\mathfrak{M}(m, \nu)$ for each (m, ν).

Case $m = 2$, $\nu \geqq 1$.

$F_m^\nu = C_{\nu-1}$. Note that the action of each $\sigma \in O(\nu - 1)$ on $R^{\nu-1} = (C_{\nu-1})_1$ is uniquely extended to an automorphism $\tilde{\sigma}$ of $C_{\nu-1}$, and that $\mathrm{Aut}^*(F_m^\nu, \mathscr{P})$ coincides with the totality of $\tilde{\sigma}$ for $\sigma \in O(\nu - 1)$.

(a) $\nu = 2\ell + 1$ $(\ell \geqq 0)$.

$$\mathfrak{M}(m, \nu) = \{[W_r]\,;\, r \in Z,\ r \geqq 0\}\,, \qquad \dim W_r = 2^\ell r\,.$$

(b) $\nu = 2\ell + 2$ $(\ell \geqq 0)$.

$$\mathfrak{M}(m, \nu) = \{[W_{r,s}]\,;\, r, s \in Z,\ r \geqq s \geqq 0\}\,, \qquad \dim W_{r,s} = 2^\ell (r + s)\,.$$

$W_{r,s} = \underbrace{W^{(1)} \oplus \cdots \oplus W^{(1)}}_{r} \oplus \underbrace{W^{(2)} \oplus \cdots \oplus W^{(2)}}_{s}$, where $W^{(1)}$ and $W^{(2)}$ are inequivalent irreducible $C_{2\ell+1}$-modules over C.

Case $m \neq 2$, $\nu = 1, 2, 4$ or 8.

(a) $\nu = 1$ $(m \neq 2)$. $F_m^\nu = R$ is isomorphic to C_0 as an algebra with involution and projection $\mathscr{P}$.

$$\mathfrak{M}(m, \nu) = \{[W_r]\,;\, r \in Z,\ r \geqq 0\}\,, \qquad \dim W_r = r\,.$$

(b) $\nu = 2$ $(m \geqq 3)$. $F_m^\nu = C$ (as a real algebra) is isomorphic to C_1 as an algebra with involution and $\mathscr{P}$.

$$\mathfrak{M}(m, \nu) = \{[W_{r,s}]\,;\, r, s \in Z,\ r \geqq s \geqq 0\}\,, \qquad \dim W_{r,s} = r + s\,.$$

$W_{r,s} = \underbrace{W^{(1)} \oplus \cdots \oplus W^{(1)}}_{r} \oplus \underbrace{W^{(2)} \oplus \cdots \oplus W^{(2)}}_{s}$, where $W^{(1)}$ and $W^{(2)}$ are inequivalent irreducible C_1-modules over C.

(c) $\nu = 4$ $(m \geqq 3)$. $F_m^\nu = H$ (real quaternion algebra) is isomorphic to C_2 as an involutive algebra.

$$\mathfrak{M}(m, \nu) = \{[W_r]\,;\, r \in Z,\ r \geqq 0\}\,, \qquad \dim W_r = 2r\,.$$

(d) $\nu = 8$ $(m = 3)$. $F_m^\nu = K$ (real Cayley algebra).

$$\mathfrak{M}(m, \nu) = \{[W_0]\}\,, \qquad \dim W_0 = 0\,.$$

4. Symmetric Siegel domains

Let C be an S-algebra corresponding to $[W] \in \mathfrak{M}(m, \nu)$ as in the previous section, and $F: (Y, j) \times (Y, j) \to X^c$, $X = X_m^\nu$, be the V_m^ν-positive

hermitian map associated to C. We want to describe the Satake's third condition (iii) for $D(V_m^v, F)$ in terms of the S-algebra C. The relations cited in Lemma of §2 will be used consistently throughout the following computations. Recall

$$F(u, v) = \tfrac{1}{4}\{(\hat{u}\hat{v} + \hat{v}\hat{u}) + \sqrt{-1}(\hat{u}(j\hat{v}) + (j\hat{v})\hat{u})\} \quad \text{for } u, v \in Y ,$$
$$R(x)u = \tfrac{1}{2}\psi''(x)u = \tfrac{1}{2}(xu + ux) = \tfrac{1}{2}(x\hat{u} + \hat{u}x) \quad \text{for } x \in X,\ u \in Y ,$$

and hence

$$\widehat{R(x)u} = \tfrac{1}{2}x\hat{u} , \qquad \underset{\smile}{R(x)u} = \tfrac{1}{2}\hat{u}x \quad \text{for } x \in X,\ u \in Y ,$$

$$R(x + \sqrt{-1}y)u = \tfrac{1}{2}\{(x\hat{u} + \hat{u}x) + j(y\hat{u} + \hat{u}y)\} \quad \text{for } x, y \in X,\ u \in Y .$$

We have

$$16R(F(u, v))R(x)u = (\hat{u}\hat{v} + \hat{v}\hat{u})(x\hat{u}) + (\hat{u}x)(\hat{u}\hat{v} + \hat{v}\hat{u})$$
$$+ j\{(\hat{v}(j\hat{v}) + (j\hat{v})\hat{u})(x\hat{u}) + (\hat{u}x)(\hat{u}(j\hat{v}) + (j\hat{v})\hat{u})\} ,$$

$$8F(u, R(x)v) = \hat{u}(\hat{v}x) + (x\hat{v})\hat{u} + \sqrt{-1}\{\hat{u}(j(\hat{v}x)) + (j(x\hat{v}))\hat{u}\} ,$$

$$16R(F(u, R(x)v))u = (\hat{u}(\hat{v}x) + (x\hat{v})\hat{u})\hat{u} + \hat{u}(\hat{u}(\hat{v}x) + (x\hat{v})\hat{u})$$
$$+ j\{[\hat{u}(j(\hat{v}x)) + (j(x\hat{v}))\hat{u}]\hat{u} + u[\hat{u}(j(\hat{v}x)) + (j(x\hat{v}))\hat{u}]\} ,$$

and hence the condition (iii) is equivalent to

$$(\hat{u}\hat{v} + \hat{v}\hat{u})(x\hat{u}) + j\{(\hat{u}(j\hat{v}) + (j\hat{v})\hat{u})(x\hat{u})\}$$
$$= (\hat{u}(\hat{v}x) + (x\hat{v})\hat{u})\hat{u} + j\{[\hat{u}(j(\hat{v}x)) + (j(x\hat{v}))\hat{u}]\hat{u}\}$$
$$\text{for each } u, v \in Y,\ x \in X .$$

Comparing the $C_{i, m+1}$-components, we get the equivalent conditions

$$\sum_{1 \leq k,\, \ell \leq m} \{(u_{i, m+1} v_{m+1, k})(x_{k\ell} u_{\ell, m+1}) + (v_{i, m+1} u_{m+1, k})(x_{k\ell} u_{\ell, m+1})$$
$$+ (u_{i, m+1}(j v_{m+1, k}))(x_{k\ell}(j u_{\ell, m+1})) + ((j v_{i, m+1}) u_{m+1, k})(x_{k\ell}(j u_{\ell, m+1}))\}$$
$$= \sum_{1 \leq k,\, \ell \leq m} \{(u_{i, m+1}(v_{m+1, k} x_{k\ell}))u_{\ell, m+1} + ((x_{ik} v_{k, m+1})u_{m+1, \ell})u_{\ell, m+1}$$
$$+ [u_{i, m+1}((j v_{m+1, k}) x_{k\ell})]j u_{\ell, m+1} + [(x_{ik}(j v_{k, m+1}))u_{m+1, \ell}]j u_{\ell, m+1}\}$$
$$\text{for each } 1 \leq i \leq m,\ u = (u_{k\ell}),\ v = (v_{k\ell}) \in Y,\ x = (x_{k\ell}) \in X .$$

In the following, the general element of C_{ik} $(1 \leq i, k \leq m)$ and $C_{i, m+1}$ $(1 \leq i \leq m)$ will be denoted by x_{ik} and $u_i, v_i, w_i, \cdots$ respectively. If we write the above conditions for

$$x = x_{k\ell} + x_{k\ell}^* , \qquad k \neq i , \ \ell \neq i ,$$
$$x = x_{ik} + x_{ik}^* , \qquad k \neq i ,$$
$$x = x_{ii} , \qquad x_{ii}^* = x_{ii} ,$$

we have the following three conditions:

(iii–1) $(u_i v_k^*)(x_{k\ell} u_\ell) + (u_i v_\ell^*)(x_{k\ell}^* u_k) + (v_i u_k^*)(x_{k\ell} u_\ell) + (v_i u_\ell^*)(x_{k\ell}^* u_k)$

$\qquad\qquad + (u_i j v_k^*)(x_{k\ell} j u_\ell) + (u_i j v_\ell^*)(x_{k\ell}^* j u_k) + ((j v_i) u_k^*)(x_{k\ell} j u_\ell)$

$\qquad\qquad + ((j v_i) u_\ell^*)(x_{k\ell}^* j u_k)$

$\qquad = (u_i(v_k^* x_{k\ell}))u_\ell + (u_i(v_\ell^* x_{k\ell}^*))u_k + [u_i((j v_k^*) x_{k\ell})] j u_\ell$

$\qquad\qquad + [u_i((j v_\ell^*) x_{k\ell}^*)] j u_k \qquad\qquad\qquad$ if $k \neq i$, $\ell \neq i$.

(iii–2) $(u_i v_i^*)(x_{ik} u_k) + (u_i v_k^*)(x_{ik}^* u_i) + (v_i u_i^*)(x_{ik} u_k) + (v_i u_k^*)(x_{ik}^* u_i)$

$\qquad\qquad + (u_i j v_i^*)(x_{ik} j u_k) + (u_i j v_k^*)(x_{ik}^* u_i) + ((j v_i) u_i^*)(x_{ik} j u_k)$

$\qquad\qquad + ((j v_i) u_k^*)(x_{ik}^* j u_i)$

$\qquad = (u_i(v_i^* x_{ik}))u_k + (u_i(v_k^* x_{ik}^*))u_i + \displaystyle\sum_{\substack{1 \leq \ell \leq m \\ \ell \neq i}} ((x_{ik} v_k) u_\ell^*)u_\ell$

$\qquad\qquad + ((x_{ik} v_k) u_i^*)u_i + [u_i((j v_i^*) x_{ik})] j u_k + [u_i((j v_k^*) x_{ik}^*)] j u_i$

$\qquad\qquad + \displaystyle\sum_{\substack{1 \leq \ell \leq m \\ \ell \neq i}} ((x_{ik} j v_k) u_\ell^*) j u_\ell + [(x_{ik}(j v_k)) u_i^*] j u_i \qquad$ if $k \neq i$.

(iii–3) $(u_i v_i^*)(x_{ii} u_i) + (v_i u_i^*)(x_{ii} u_i) + (u_i j v_i^*)(x_{ii} j u_i) + ((j v_i) u_i^*)(x_{ii} u_i)$

$\qquad = (u_i(v_i^* x_{ii}))u_i + \displaystyle\sum_{1 \leq k \leq m} ((x_{ii} v_i) u_k^*)u_k + (u_i(j v_i^* x_{ii})) j u_i$

$\qquad\qquad + \displaystyle\sum_{1 \leq k \leq m} ((x_{ii} j v_i) u_k^*) j u_k \qquad\qquad$ for each $1 \leq i \leq m$.

Thus the condition (iii) is equivalent to the conditions (iii–1), (iii–2), (iii–3) as a whole. In the equality (iii–1), $(u_i v_k^*)(x_{k\ell} u_\ell) = ((u_i v_k^*) x_{k\ell})u_\ell = (u_i(v_k^* x_{k\ell}))u_\ell$. In the same way, $(u_i v_k^*)(x_{k\ell}^* u_k) = (u_i(v_\ell^* x_{k\ell}^*))u_k$, $(u_i j v_k^*)(x_{k\ell} j u_\ell) = [u_i((j v_k^*) x_{k\ell})] j u_\ell$ and $(u_i j v_\ell^*)(x_{k\ell}^* j u_k) = [u_i((j v_\ell^*) x_{k\ell}^*)] j u_\ell$. Furthermore $(v_i u_k^*)(x_{k\ell}^* u_k) = ((v_i u_k^*) x_{k\ell}^*)u_k = (v_i(u_k^* x_{k\ell}^*))u_k = (v_i(x_{k\ell} u_k)^*)u_k$. In the same way, $((j v_i) u_\ell^*)(x_{k\ell}^* j u_k) = (j v_i(x_{k\ell} u_\ell)^*) j u_k$. Thus (iii–1) is equivalent to

(iii–1)′ $(v_i u_k^*)(x_{k\ell} u_\ell) + (v_i(x_{k\ell} u_\ell)^*)u_k + ((j v_i) u_k^*) j(x_{k\ell} u_\ell)$

$\qquad\qquad + ((j v_i)(x_{k\ell} u_\ell)^*) j u_k = 0 \qquad$ if $k \neq i$, $\ell \neq i$.

In the lefthand side of the equality (iii–2), $(u_i v_i^*)(x_{ik} u_k) + (v_i u_i^*)(x_{ik} u_k) = ((u_i v_i^*) x_{ik})u_k + ((v_i u_i^*) x_{ik})u_k = (u_i(v_i^* x_{ik}))u_k + (v_i(u_i^* x_{ik}))u_k$. In the same way, $(u_i j v_i^*)(x_{ik} j u_k) + ((j v_i) u_i^*)(x_{ik} j u_k) = [u_i((j v_i^*) x_{ik})] j u_k + [(j v_i)(u_i^* x_{ik})] j u_k$. In the righthand side of (iii–2), $(u_i(v_k^* x_{ik}^*))u_i + ((x_{ik} v_k) u_i^*)u_i = ((u_i v_k^*) x_{ik}^*)u_i$

$+ (x_{ik}(v_k u_i^*))u_i = (u_i v_k^*)(x_{ik}^* u_i) + x_{ik}((v_k u_i^*)u_i)$.   In the same way, $[u_i((jv_k^*)$
$\times x_{ik}^*)]ju_i + ((x_{ik}jv_k)u_i^*)ju_i = (u_i jv_k^*)(x_{ik}^* ju_i) + x_{ik}[((jv_k)u_i^*)ju_i]$.   Thus (iii–2)
is equivalent to

(iii–2)′ $(v_i(x_{ik}^* u_i)^*)u_k + (v_i u_k^*)(x_{ik}^* u_i) + ((jv_i)(x_{ik}^* u_i)^*)ju_k + ((jv_i)u_k^*)j(x_{ik}^* u_i)$

$$= x_{ik}\{(v_k u_i^*)u_i + ((jv_k)u_i^*)ju_i\}$$
$$+ \sum_{\substack{1 \leq \ell \leq m \\ \ell \neq i}} \{((x_{ik}v_k)u_\ell^*)u_\ell + [(j(x_{ik}v_k))u_\ell^*]ju_\ell\} \qquad \text{if } i \neq k \, .$$

Furthermore (iii–3) is equivalent to

(iii–3)′ $\displaystyle\sum_{\substack{1 \leq k \leq m \\ k \neq i}} \{(v_i u_k^*)u_k + ((jv_i)u_k^*)ju_k\} = 0$ for each $1 \leq i \leq m$.

Now, if $m = 1$, the conditions (iii–1)′, (iii–2)′, (iii–3)′ are always satisfied.
If $m \geq 2$, the condition (iii–1)′ for $k = \ell \neq i$ implies

(4.1) $(v_i u_k^*)u_k + ((jv_i)u_k^*)ju_k = 0$ if $i \neq k$,

or equivalently

(4.1)′ $(v_i u_k^*)w_k + (v_i w_k^*)u_k + ((jv_i)u_k^*)jw_k + ((jv_i)w_k^*)ju_k = 0$ if $i \neq k$.

Actually the condition (4.1) is equivalent to three conditions (iii–1)′,
(iii–2)′, (iii–3)′.   In fact; assume (4.1), then replacing $w_k = x_{k\ell}u_\ell$ in (4.1)′,
we get (iii–1)′.   Replacing $u_k = x_{ik}^* u_i$ and $w_k = u_k$ in (4.1)′, we know
that the lefthand side of (iii–2)′ is 0.   The righthand side of (iii–2)′ is
also 0 by (4.1).   (iii–3)′ follows from (4.1).   These prove the required
assertion.

Now we are in a position to prove Theorem B in Introduction.

Proof of Theorem B.   Passing to the F_m^ν-module W, the condition
(4.1) is equivalent to

(4.2) $(\mathscr{P}(vu))u + [\mathscr{P}((jv)u)]ju = 0$ for $u, v \in W_R$.

Recalling that $(\mathscr{P}(uv), \lambda) = (uv, \lambda) = (u, \lambda v)$ for $\lambda \in P_m^\nu$, we get

$$\mathscr{P}(uv) = \sum_{0 \leq i \leq \nu-1} (u, c_i v)c_i \qquad \text{for } u, v \in W_R \, .$$

Thus (4.2) is equivalent to

$$\sum_{0 \leq i \leq \nu-1} \{(v, c_i u)(c_i u, w) - (jv, c_i u)(c_i u, jw)\} = 0 \qquad \text{for } u, v, w \in W_R \, ,$$

or equivalently

$$\sum_{0 \leq i \leq \nu - 1} \{(v, c_i u)(c_i u, jw) + (jv, c_i u)(c_i u, w)\} = 0 \qquad \text{for } u, v, w \in W_R \ .$$

Thus the condition (4.2) is equivalent to

$$(\sharp) \qquad \sum_{0 \leq i \leq \nu - 1} \langle c_i u, v \rangle \langle c_i u, w \rangle = 0 \qquad \text{for } u, v, w \in W \ .$$

This proves Theorem B.   q.e.d.

Let $\mathfrak{M}_s(m, \nu)$ denote the set of those $[W] \in \mathfrak{M}(m, \nu)$ defining symmetric Siegel domains.

LEMMA. *Let* $[W] \in \mathfrak{M}_s(m, \nu)$, $N = \dim W$ *and* $\rho : F_m^\nu \to \operatorname{End} W$ *be the representation defined by the* F_m^ν-*module structure on* W.

1) *If* $m \geq 2$, *then*

$$N(N - \nu + 2) = \sum_{k=1}^{\nu - 1} |\operatorname{Tr} \rho(c_k)|^2 \ .$$

2) *If* $\nu \geq 3$, *then* $\operatorname{Tr} \rho(c_k) = 0$ *for each* $1 \leq k \leq \nu - 1$.

3) *If* $m \geq 2$ *and* $\nu \neq 2$, *then* $N = 0$ *or* $N = \nu - 2$.

4) *If* $m \geq 2$ *and* $\nu = 2$, *then* $[W] = [W_{N,0}]$, *i.e.,* W *is the direct sum of* N-*copies of an irreducible* C_1-*module over* C.

Proof. 1) Let $m \geq 2$. Take an orthonormal basis $\{u_1, \cdots, u_N\}$ of W, and put

$$a_{\alpha\beta k} = \langle c_k u_\beta, u_\alpha \rangle \qquad \text{for } 1 \leq k \leq \nu - 1, \ 1 \leq \alpha, \beta \leq N \ .$$

Then $\bar{a}_{\alpha\beta k} = -a_{\beta\alpha k}$ and $\rho(c_k)$ $(1 \leq k \leq \nu - 1)$ has the matricial representation $(a_{\alpha\beta k})_{1 \leq \alpha, \beta \leq N}$. From the polarized form

$$\sum_{i=0}^{\nu - 1} \{\langle c_i u, v \rangle \langle c_i u', w \rangle + \langle c_i u', v \rangle \langle c_i u, w \rangle\} = 0 \qquad \text{for } u, u', v, w \in W$$

of the condition $(\sharp)$, we have

$$\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\gamma\beta} \delta_{\alpha\delta} + \sum_{k=1}^{\nu - 1} (a_{\beta\alpha k} a_{\delta\gamma k} + a_{\beta\gamma k} a_{\delta\alpha k}) = 0$$

for each $1 \leq \alpha, \beta, \gamma, \delta \leq N$. Putting $\beta = \alpha$ and $\delta = \gamma$, we get

$$1 + \delta_{\alpha\gamma} + \sum_k (a_{\alpha\alpha k} a_{\gamma\gamma k} + a_{\alpha\gamma k} a_{\gamma\alpha k}) = 0 \ .$$

Taking the sum over α and γ, we get

$$N^2 + N + \sum_k (\mathrm{Tr}\,\rho(c_k))^2 + \sum_k \mathrm{Tr}\,(\rho(c_k)^2) = 0 \;.$$

Since $\rho(c_k)^2 = \rho(c_k^2) = -1_W$ and hence $\mathrm{Tr}\,(\rho(c_k)^2) = -N$, we have

$$N(N+1) + \sum_k (\mathrm{Tr}\,\rho(c_k))^2 - (\nu - 1)N = 0 \;.$$

Hence

$$N(N - \nu + 2) = -\sum_k (\mathrm{Tr}\,\rho(c_k))^2 = \sum_k |\mathrm{Tr}\,\rho(c_k)|^2 \;.$$

2) We may asume $F_m^\nu \neq K$, since K has no modules other than the trivial one. Let $(F_m^\nu)^0$ denote the orthogonal complement in F_m^ν of the center of F_m^ν, with respect to the standard inner product on F_m^ν. Then $(F_m^\nu)^0$ is closed under the bracket product $[\lambda, \mu] = \lambda\mu - \mu\lambda$ and coincides with the commutator subalgebra $[(F_m^\nu)^0, (F_m^\nu)^0]$. The assertion follows from the fact that each c_k $(1 \leq k \leq \nu - 1)$ belongs to $(F_m^\nu)^0$ for $\nu \geq 3$.

3) is an immediate consequence of 1) and 2).

4) As we have seen in §3, if $m \geq 2$ and $\nu = 2$, then $\mathfrak{M}(m, \nu) = \{[W_{r,s}]; r, s \in \mathbf{Z}, r \geq s \geq 0\}$, $\dim W_{r,s} = r + s$, $W_{r,s} = \underbrace{W^{(1)} \oplus \cdots \oplus W^{(1)}}_{r} \oplus$ $\underbrace{W^{(2)} \oplus \cdots \oplus W^{(2)}}_{s}$, where $W^{(1)}$ and $W^{(2)}$ are inequivalent irreducible C_1-modules over C. Let $W = W_{r,s}$ and $[W] \in \mathfrak{M}_s(m, \nu)$. We may assume that $\rho(c_1)$ has the matricial representation

$$\begin{pmatrix} \sqrt{-1}\,1_r & \\ & -\sqrt{-1}\,1_s \end{pmatrix},$$

and hence $|\mathrm{Tr}\,\rho(c_1)|^2 = (r - s)^2$. Now the assertion 1) implies

$$(r + s)^2 = (r - s)^2 \;,$$

and hence $r \geq s = 0$.   q.e.d.

The set $\mathfrak{M}_s(m, \nu)$ can be determined by the above Lemma. For example, let $m = 2$, $\nu = 2\ell + 2$ $(\ell \geq 1)$. Recall that $F_m^\nu = C_{2\ell+1}$ and

$$\mathfrak{M}(m, \nu) = \{[W_{r,s}]; r, s \in \mathbf{Z}, r \geq s \geq 0\}, \quad \dim W_{r,s} = 2^\ell(r + s) \;.$$

If $[W_{r,s}] \in \mathfrak{M}_s(m, \nu)$, from Lemma 3) it follows

$$r + s = 0 \quad \text{or} \quad 2^\ell(r + s) = 2\ell ,$$

and hence the possibilities are one of

1) $\ell \geqq 1$, $r = s = 0$,
2) $\ell = 1$, $r = 1$, $s = 0$ ($\nu = 4$, dim $W_{1,0} = 2$),
3) $\ell = 2$, $r = 1$, $s = 0$ ($\nu = 6$, dim $W_{1,0} = 4$).

We shall show that for each of the above (r, s), $W = W_{r,s}$ satisfies the condition ($\sharp$).

Case 1). The condition ($\sharp$) is clearly satisfied, since $W = \{0\}$.

Case 2). Note first that in terms of the sesqui-linear map H, the condition ($\sharp$) is written equivalently as

$$\langle H(u, v)u, w \rangle = 2\langle u, v \rangle \langle u, w \rangle \quad \text{for } u, v, w \in W .$$

Let $W = C^2$ with the hermitian inner product $\langle u, v \rangle = {}^t u \bar{v}$. The R-linear map $\rho : P_m^\nu \to M_2(C)$ defined by

$$\rho(c_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho(c_1) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},$$

$$\rho(c_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(c_3) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$

gives rise to an irreducible F_m^ν-module structure on W satisfying $\langle \lambda u, v \rangle = \langle u, \bar{\lambda} v \rangle$. Through the injective map ρ, we identify P_m^ν with a real subspace of $M_2(C)$, and thus $(P_m^\nu)^c$ with $M_2(C)$. Then the hermitian inner product $\langle \, , \, \rangle$ on $(P_m^\nu)^c$ is given by

$$\langle \lambda, \mu \rangle = \tfrac{1}{2} \operatorname{Tr} (\lambda {}^t \bar{\mu}) \quad \text{for } \lambda, \mu \in M_2(C) .$$

Now the sesqui-linear map $H : C^2 \times C^2 \to M_2(C)$ is given by

$$H(u, v) = 2u {}^t \bar{v} \quad \text{for } u, v \in C^2 .$$

In fact, $\langle \lambda, 2u {}^t \bar{v} \rangle = \operatorname{Tr} \lambda (v {}^t \bar{u}) = \operatorname{Tr} (\lambda v) {}^t \bar{u} = \langle \lambda v, u \rangle$ for each $\lambda \in M_2(C)$. Hence we have

$$\langle H(u, v)u, w \rangle = \langle 2(u {}^t \bar{v})u, w \rangle = 2\langle u({}^t \bar{v} u), w \rangle$$
$$= 2\langle u, v \rangle \langle u, w \rangle \quad \text{for } u, v, w \in C^2 .$$

This shows the condition ($\sharp$) for W.

Case 3). Let $W = C^4$ with the hermitian inner product $\langle u, v \rangle = {}^t u \bar{v}$. The homomorphism $\rho : F_m^\nu \to M_4(C)$ defined by

$$\rho(c_0) = \begin{bmatrix} \begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & & 0 \\ \hline & & 1 & 0 \\ 0 & & 0 & 1 \end{array} \end{bmatrix}, \qquad \rho(c_1) = \begin{bmatrix} \begin{array}{cc|cc} \sqrt{-1} & 0 & & \\ 0 & \sqrt{-1} & & 0 \\ \hline & & -\sqrt{-1} & 0 \\ 0 & & 0 & -\sqrt{-1} \end{array} \end{bmatrix},$$

$$\rho(c_2) = \begin{bmatrix} \begin{array}{cc|cc} & & -1 & 0 \\ 0 & & 0 & -1 \\ \hline 1 & 0 & & \\ 0 & 1 & & 0 \end{array} \end{bmatrix}, \qquad \rho(c_3) = \begin{bmatrix} \begin{array}{cc|cc} & & \sqrt{-1} & 0 \\ 0 & & 0 & -\sqrt{-1} \\ \hline \sqrt{-1} & 0 & & \\ 0 & -\sqrt{-1} & & 0 \end{array} \end{bmatrix},$$

$$\rho(c_4) = \begin{bmatrix} \begin{array}{cc|cc} & & 0 & -1 \\ 0 & & 1 & 0 \\ \hline 0 & -1 & & \\ 1 & 0 & & 0 \end{array} \end{bmatrix}, \qquad \rho(c_5) = \begin{bmatrix} \begin{array}{cc|cc} & & 0 & -\sqrt{-1} \\ 0 & & -\sqrt{-1} & 0 \\ \hline 0 & -\sqrt{-1} & & \\ -\sqrt{-1} & 0 & & 0 \end{array} \end{bmatrix},$$

gives rise to an irreducible F_m^ν-module structure on W satisfying $\langle \lambda u, v \rangle = \langle u, \bar{\lambda} v \rangle$. It can be checked that the F_m^ν-module W satisfies the condition ($\sharp$).

It is verified that the above $[W_{r,s}]$ defines actually the symmetric domain of type $(IV)_{\nu+2}$, $(II)_5$ and $(EIII)$ respectively.

The set $\mathfrak{M}_s(m, \nu)$ for each (m, ν) is given as follows.

Case $m = 1$. $\nu = 1$, $F_m^\nu = \boldsymbol{R}$.

$$\mathfrak{M}_s(m, \nu) = \{[W_r \, ; \, r \in \boldsymbol{Z}, \, r \geqq 0\}, \qquad \dim W_r = r \, .$$

$[W_r]$ defines the symmetric domain of type $(I)_{1,1+r}$.

Case $m = 2$.

(a) $\nu = 1$. $F_m^\nu = C_0 = \boldsymbol{R}$.

$$\mathfrak{M}_s(m, \nu) = \{[W_0]\}, \qquad \dim W_0 = 0 \, .$$

$[W_0]$ defines the symmetric domain of type $(III)_2$.

(b) $\nu = 2$. $F_m^\nu = C_1 = \boldsymbol{C}$.

$$\mathfrak{M}_s(m, \nu) = \{[W_{r,0}] \, ; \, r \in \boldsymbol{Z}, \, r \geqq 0\}, \qquad \dim W_{r,0} = r \, .$$

$[W_{r,0}]$ defines the symmetric domain of type $(I)_{2,2+r}$.

(c) $\nu = 2\ell + 1$ $(\ell \geqq 1)$. $F_m^\nu = C_{2\ell}$.

$$\mathfrak{M}_s(m, \nu) = \{[W_0]\}, \qquad \dim W_0 = 0 \, .$$

$[W_0]$ defines the symmetric domain of type $(IV)_{\nu+2}$.

(d) $\nu = 2\ell + 2$ $(\ell \geqq 1)$. $F_m^\nu = C_{2\ell+1}$.

$$\nu \neq 4, 6 \quad \mathfrak{M}_s(m, \nu) = \{[W_{0,0}]\} \ , \quad \dim W_{0,0} = 0 \ .$$

$[W_{0,0}]$ defines the symmetric domain of type $(IV)_{\nu+2}$.

$$\nu = 4 \quad \mathfrak{M}_s(m, \nu) = \{[W_{0,0}], [W_{1,0}]\} \ , \quad \dim W_{0,0} = 0 \ , \quad \dim W_{1,0} = 2 \ .$$

$[W_{0,0}], [W_{1,0}]$ define symmetric domains of type $(IV)_6$, $(II)_5$ respectively.

$$\nu = 6 \quad \mathfrak{M}_s(m, \nu) = \{[W_{0,0}], [W_{1,0}]\} \ , \quad \dim W_{0,0} = 0 \ , \quad \dim W_{1,0} = 4 \ .$$

$[W_{0,0}], [W_{1,0}]$ define symmetric domains of type $(IV)_8$, $(EIII)$ respectively.

Case $m \geq 3$.

(a) $\nu = 1$. $F_m^\nu = R$.

$$\mathfrak{M}_s(m, \nu) = \{[W_0]\} \ , \qquad \dim W_0 = 0 \ .$$

$[W_0]$ defines the symmetric domain of type $(III)_m$.

(b) $\nu = 2$. $F_m^\nu = C$.

$$\mathfrak{M}_s(m, \nu) = \{[W_{r,0}] \, ; \, r \in Z, \ r \geq 0\} \ , \qquad \dim W_{r,0} = r \ .$$

$[W_{r,0}]$ defines the symmetric domain of type $(I)_{m,m+r}$.

(c) $\nu = 4$. $F_m^\nu = H$.

$$\mathfrak{M}_s(m, \nu) = \{[W_0], [W_1]\} \ , \quad \dim W_0 = 0 \ , \quad \dim W_1 = 2 \ .$$

$[W_0], [W_1]$ define symmetric domains of type $(II)_{2m}$, $(II)_{2m+1}$ respectively.

(d) $\nu = 8$. $F_m^\nu = K$.

$$\mathfrak{M}_s(m, \nu) = \{[W_0]\} \ , \qquad \dim W_0 = 0 \ .$$

$[W_0]$ defines the symmetric domain of type $(EVII)$.

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