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THE UNIQUENESS PROBLEM OF MEROMORPHIC MAPS INTO THE COMPLEX PROJECTIVE SPACE

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§ 1. Introduction.

In 1921, G. Pólya showed that non-constant meromorphic functions φ and ψ of finite genera on the complex plane $\boldsymbol{C}$ are necessarily equal if there are distinct five values a_i $(1 \leqq i \leqq 5)$ such that $\varphi(z) - a_i$ and $\psi(z) - a_i$ have the same zeros of the same multiplicities for each i ([8]). Afterwards, R. Nevanlinna obtained the same conclusion for arbitrary φ and ψ satisfying $\varphi^{-1}(a_i) = \psi^{-1}(a_i)$ $(1 \leqq i \leqq 5)$ regardless of multiplicities. And, some other results relating to this were given by H. Cartan ([2], [3]), E. M. Schmid ([9]) and others. The purpose of this paper is to give some types of generalizations of these results to the case of meromorphic maps into the N-dimensional complex projective space $P_N(\boldsymbol{C})$.

We consider q hyperplanes H_i in $P_N(\boldsymbol{C})$ located in general position and two non-constant meromorphic maps f and g of $\boldsymbol{C}^n$ into $P_N(\boldsymbol{C})$ with $f(\boldsymbol{C}^n) \not\subset H_i$, $g(\boldsymbol{C}^n) \not\subset H_i$ such that $\nu(f, H_i) = \nu(g, H_i)$ for any i, where $\nu(f, H_i)$ and $\nu(g, H_i)$ denote the pull-back of the divisors (H_i) on $P_N(\boldsymbol{C})$ by f and g respectively (c, f., Definition 3.1).

The first main result is the following

THEOREM I. *If $q = 3N + 1$, there is a projective linear transformation L of $P_N(\boldsymbol{C})$ such that $L \cdot f = g$.*

And, we shall prove also

THEOREM II. *If $q = 3N + 2$ and either f or g is non-degenerate, i.e., the image does not included in any hyperplane in $P_N(\boldsymbol{C})$, then $f = g$.*

Moreover, we shall give some other results on the uniqueness problem in the case $q = 3N + 1$ under suitable assumptions. From this we shall

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show that, if $N=2$, Theorem II remains valid under weaker assumption that $q=7$ $(=3N+1)$. For the case $N \geqq 3$, the author does not know if the number of given hyperplanes in Theorem II can be replaced by an integer smaller than $3N+2$. It is a very interesting problem to seek the smallest integer $q(N)$ for each N such that Theorem II holds for arbitrarily given $q(N)$ hyperplanes in general position.

These results will be proved by the use of the classical theorem of E. Borel ([1]) and some combinatorial lemmas given in § 2.

For a domain B and a thin analytic subset S of B we shall study also meromorphic maps defined on $B-S$ which have essential singularities of special type along S (c.f., Definition 5.5) and give some theorems similar to the above Theorems I and II. Moreover, meromorphic maps f and g into $P_2(C)$ will be studied more precisely in the last section.

§ 2. Combinatorial lemmas.

Let G be a torsion free abelian group and consider a q-tuple $A=(a_1, a_2, \cdots, a_q)$ of elements a_i in G. For the subgroup $\tilde{A}$ of G generated by $a_1, a_2, \cdots, a_q$, we can take a basis $\{b_1, b_2, \cdots, b_t\}$ of $\tilde{A}$, because $\tilde{A}$ is a free abelian group. Then, each a_i $(1 \leqq i \leqq q)$ can be uniquely represented as

$$a_i = b_1^{\ell_{i1}} b_2^{\ell_{i2}} \cdots b_t^{\ell_{it}} \tag{2.1}$$

with suitable integers $\ell_{i\tau}$.

(2.2) *For integers $\ell_{i\tau}$ $(1 \leqq i \leqq q,\ 1 \leqq \tau \leqq t)$, it is possible to choose integers $p_1, p_2, \cdots, p_t$ satisfying the condition that, for integers*

$$\ell_i := \ell_{i1}p_1 + \ell_{i2}p_2 + \cdots + \ell_{it}p_t \qquad (1 \leqq i \leqq q)\ ,$$

if $\ell_i = \pm \ell_j$, then

$$(\ell_{i1}, \ell_{i2}, \cdots, \ell_{it}) = \pm(\ell_{j1}, \ell_{j2}, \cdots, \ell_{jt})\ .$$

This is shown by induction on t. The case $t=1$ is trivial. Assume that there exist $p_1, \cdots, p_{t-1}$ with the property that

$$(\ell_{i1}, \ell_{i2}, \cdots, \ell_{it-1}) = \pm(\ell_{j1}, \ell_{j2}, \cdots, \ell_{jt-1})$$

if $\ell_i^* = \pm \ell_j^*$ for integers $\ell_i^* := \ell_{i1}p_1 + \cdots + \ell_{it-1}p_{t-1}$. Then, it is easy to show that there are only finitely many integers p_t such that $p_1, p_2, \cdots, p_t$ do not satisfy the desired condition.

DEFINITION 2.3. We shall call integers $p_1, p_2, \cdots, p_t$ with the property (2.2) to be *generic* with respect to $\ell_{i\tau}$ and the integers $\ell_i = \sum_{\tau=1}^{t} \ell_{i\tau} p_\tau$ to be *representations* of a_i $(1 \leqq i \leqq q)$.

We have

(2.4) *If* $a_{i_1}^{m_1} a_{i_2}^{m_2} \cdots a_{i_r}^{m_r} = a_{j_1}^{m'_1} a_{j_2}^{m'_2} \cdots a_{j_s}^{m'_s}$, *it holds that*

$$m_1 \ell_{i_1} + m_2 \ell_{i_2} + \cdots + m_r \ell_{i_r} = m'_1 \ell_{j_1} + m'_2 \ell_{j_2} + \cdots + m'_s \ell_{j_s} .$$

In fact, substituting the identity (2.1) into both sides, we see

$$b_1^{n_1} b_2^{n_2} \cdots b_t^{n_t} = b_1^{n'_1} b_2^{n'_2} \cdots b_t^{n'_t}$$

for integers $n_\tau := \sum_{\kappa=1}^{r} m_\kappa \ell_{i_\kappa \tau}$ and $n'_\tau := \sum_{\kappa=1}^{s} m'_\kappa \ell_{j_\kappa \tau}$. Since $b_1, b_2, \cdots, b_t$ are linearly independent in G, $n_\tau = n'_\tau$ for any τ $(1 \leqq \tau \leqq t)$. Therefore,

$$\begin{aligned} \sum_{\kappa=1}^{r} m_\kappa \ell_{i_\kappa} &= \sum_{\kappa=1}^{r} \sum_{\tau=1}^{t} m_\kappa \ell_{i_\kappa \tau} p_\tau \\ &= \sum_{\tau=1}^{t} n_\tau p_\tau \\ &= \sum_{\tau=1}^{t} n'_\tau p_\tau \\ &= \sum_{\kappa=1}^{s} m'_\kappa \ell_{j_\kappa} . \end{aligned}$$

Now, we give

DEFINITION 2.5. *Let* $q \geqq r > s \geqq 1$. We shall call a q-tuple $A = (a_1, a_2, \cdots, a_q)$ of elements a_i in G to have the *property* $(P_{r,s})$ if any chosen r elements $a_{\iota(1)}, a_{\iota(2)}, \cdots, a_{\iota(r)}$ in A satisfy the condition that, for any given $i_1, i_2, \cdots, i_s$ $(1 \leqq i_1 < \cdots < i_s \leqq r)$, there exist some other $j_1, j_2, \cdots, j_s$ $(1 \leqq j_1 < \cdots < j_s \leqq r, \{i_1, i_2, \cdots, i_s\} \neq \{j_1, j_2, \cdots, j_s\})$ such that

$$a_{\iota(i_1)} a_{\iota(i_2)} \cdots a_{\iota(i_s)} = a_{\iota(j_1)} a_{\iota(j_2)} \cdots a_{\iota(j_s)} .$$

Let us study relations among a_i for a q-tuple $A = (a_1, a_2, \cdots, a_q)$ with the property $(P_{r,s})$. To this end, we take representations $\ell_1, \ell_2, \cdots, \ell_q$ of $a_1, a_2, \cdots, a_q$ for suitably chosen basis and generic integers. Changing indices i of a_i if necessary, we assume

$$\ell_1 \leqq \ell_2 \leqq \cdots \leqq \ell_q .$$

LEMMA 2.6. *In the above situation, it holds that*

$$\ell_s = \ell_{s+1} = \cdots = \ell_{s+u}$$

and so

$$a_s = a_{s+1} = \cdots = a_{s+u}$$

for $u := q - r + 1$.

Proof. Assume that

$$\ell_1 \leqq \cdots \leqq \ell_s = \ell_{s+1} = \cdots = \ell_{s+v} < \ell_{s+v+1} \leqq \cdots \leqq \ell_q$$

for some v with $v < u$ $(= q - r + 1)$. Among a_i $(1 \leqq i \leqq q)$, we choose r elements

$$a_{\iota(1)} = a_1, \cdots, a_{\iota(s)} = a_s, a_{\iota(s+1)} = a_{s+u}, a_{\iota(s+2)} = a_{s+u+1}, \cdots, a_{\iota(r)} = a_q .$$

By the assumption, considering the case $i_1 = 1$, $i_2 = 2, \cdots, i_s = s$ in Definition 2.4, we can take some $j_1, j_2, \cdots, j_s$ $(1 \leqq j_1 < \cdots < j_s \leqq r, \{j_1, j_2, \cdots, j_s\} \neq \{1, 2, \cdots, s\})$ such that

$$a_{\iota(j_1)} a_{\iota(j_2)} \cdots a_{\iota(j_s)} = a_1 a_2 \cdots a_s .$$

Then, by (2.4), we have

$$\begin{aligned} &\ell_{\iota(j_1)} + \ell_{\iota(j_2)} + \cdots + \ell_{\iota(j_s)} - (\ell_1 + \ell_2 + \cdots + \ell_s) \\ &\quad = (\ell_{\iota(j_1)} - \ell_1) + (\ell_{\iota(j_2)} - \ell_2) + \cdots + (\ell_{\iota(j_s)} - \ell_s) \\ &\quad = 0 . \end{aligned}$$

On the other hand, we see easily $\kappa = i_\kappa \leqq \iota(j_\kappa)$ and so $\ell_{\iota(j_\kappa)} - \ell_\kappa \geqq 0$ for any κ $(1 \leqq \kappa \leqq s)$. This implies that

$$\ell_1 = \ell_{\iota(j_1)},\ \ell_2 = \ell_{\iota(j_2)}, \cdots, \ell_s = \ell_{\iota(j_s)} .$$

By the assumption, $\ell_i < \ell_{\iota(j)}$ for any i, j if $1 \leqq i \leqq s$ and $s + 1 \leqq j \leqq r$. We have necessarily $j_\kappa = \kappa$ $(1 \leqq \kappa \leqq s)$. This is a contradiction. We conclude thus $v \geqq u$. The proof of Lemma 2.6 is completed.

For the case $r = 2s$, we can give more precise conclusion.

LEMMA 2.7. *In the same situation as in Lemma* 2.6, *if* $r = 2s$ $(s > 2)$, $a_i = 1$ $(=$ *the unit element of* $G)$ *for any* i *with* $s \leqq i \leqq q - s + 1$, $a_{s-1} \neq 1$, $a_{q-s+2} \neq 1$ *and* $a_{q-s+2} \neq a_{q-s+3}$, *then* $a_{s-1} a_{q-s+2} = 1$.

Proof. By Lemma 2.6,

$$\begin{aligned} \ell_1 \leqq \cdots \leqq \ell_{s-2} \leqq \ell_{s-1} < \ell_s = \cdots \\ = \ell_{q-s+1} = 0 < \ell_{q-s+2} < \ell_{q-s+3} \leqq \cdots \leqq \ell_q . \end{aligned}$$

Considering the case $\iota(1) = 1, \cdots, \iota(s+1) = s + 1$, $\iota(s+2) = q - s + 2$, $\cdots, \iota(2s) = q$ and $i_1 = 1$, $i_2 = 2, \cdots, i_{s-1} = s - 1$ and $i_s = s + 2$ in Definition 2.5, we can take indices $j_1, j_2, \cdots j_s$ $(1 \leqq j_1 < \cdots < j_s < 2s,\ \{j_1, j_2, \cdots, j_s\} \neq \{1, 2, \cdots, s-1, s+2\})$ such that

$$a_{\iota(j_1)} a_{\iota(j_2)} \cdots a_{\iota(j_s)} = a_{\iota(1)} a_{\iota(2)} \cdots a_{\iota(s-1)} a_{\iota(s+2)} ,$$

whence

$$\ell_{\iota(j_1)} + \ell_{\iota(j_2)} + \cdots + \ell_{\iota(j_s)} = \ell_1 + \ell_2 + \cdots + \ell_{s-1} + \ell_{q-s+2} \tag{2.8}$$

by (2.4). We define the number k by the condition that

$$\iota(j_1) < \iota(j_2) < \cdots < \iota(j_{k-1}) < s \leqq \iota(j_k) < \cdots < \iota(j_s)$$

and put

$$\{m_1, m_2, \cdots, m_{s-k}\} = \{1, 2, \cdots, s-1\} - \{\iota(j_1), \iota(j_2), \cdots, \iota(j_{k-1})\} .$$

Here, $s > k$. In fact, if not, $\iota(j_1) = 1, \cdots, \iota(i_{s-1}) = s - 1$ and so $\ell_{q-s+2} = \ell_{\iota(j_s)}$, which contradicts the assumption. Canceling $\ell_{\iota(j_\kappa)}$ $(1 \leqq \kappa \leqq k-1)$ from the both sides of (2.8), we obtain

$$\ell_{\iota(j_k)} + \ell_{\iota(j_{k+1})} + \cdots + \ell_{\iota(j_s)} = \ell_{m_1} + \ell_{m_2} + \cdots + \ell_{m_{s-k}} + \ell_{q-s+2} .$$

If $\iota(j_s) \geqq q - s + 2$, then we get inequalities

$$\begin{aligned} 0 &\leqq \ell_{\iota(j_k)} + \ell_{\iota(j_{k+1})} + \cdots + \ell_{\iota(j_{s-1})} \\ &= \ell_{m_1} + \ell_{m_2} + \cdots + \ell_{m_{s-k}} + (\ell_{q-s+2} - \ell_{\iota(j_s)}) \\ &\leqq \ell_{m_1} + \ell_{m_2} + \cdots + \ell_{m_{s-k}} < 0 , \end{aligned}$$

which is a contradiction. Therefore, $j_s \leqq s + 1$. Then, we have necessarily $\iota(j_{s-1}) = s$ and $\iota(j_s) = s + 1$. By the relation (2.8), we conclude $\ell_{s-1} + \ell_{q-s+2} = 0$, whence $h_{s-1} h_{q-s+2} = 1$. This completes the proof.

§ 3. Two meromorphic maps with the same inverse images of hyperplanes.

Let f be a meromorphic map of a domain D in C^n into $P_N(C)$. For arbitrarily fixed homogeneous coordinates $w_1 : w_2 : \cdots : w_{N+1}$ on $P_N(C)$, we can write

$$f(z) = f_1(z) : f_2(z) : \cdots : f_{N+1}(z)$$

on a neighborhood U of every point a in D with holomorphic functions $f_i(z)$ $(1 \leqq i \leqq N+1)$ on U, where they can be chosen so as to satisfy the condition

$$\operatorname{codim} \{f_1(z) = f_2(z) = \cdots = f_{N+1}(z) = 0\} \geqq 2 .$$

In the following, such a representation of f is referred to as an admissible representation of f on U. If D is a Cousin-II domain, then f has an admissible representation on the totality of D.

Let us take a hyperplane

$$H: a^1 w_1 + a^2 w_2 + \cdots + a^{N+1} w_{N+1} = 0$$

in $P_N(C)$ with $f(D) \not\subset H$. For any $a = (a_1, a_2, \cdots a_n) \in D$, taking an admissible representation $f = f_1 : f_2 : \cdots : f_{N+1}$ on a neighborhood U of a, we define a holomorphic function

$$F = a^1 f_1 + a^2 f_2 + \cdots + a^{N+1} f_{N+1}$$

on U and expand it as a compactly convergent series

$$F(u_1 + a_1, \cdots, u_n + a_n) = \sum_{m=0}^{\infty} P_m(u_1, u_2, \cdots, u_n)$$

around a, where P_m is either identically zero or a homogeneous polynomial of degree m.

DEFINITION 3.1. We define

$$\nu(f, H)(a) = \min\{m : P_m(u) \not\equiv 0\} ,$$

which is obviously determined independently of any choice of homogeneous coordinates and admissible representations.

Now, let us consider two non-constant meromorphic maps f and g of D into $P_N(C)$ and q $(\geqq 2N + 2)$ hyperplanes H_i $(1 \leqq i \leqq q)$ in $P_N(C)$ located in general position. Suppose that $f(D^n) \not\subset H_i$, $g(D^n) \not\subset H_i$ and $\nu(f, H_i) = \nu(g, H_i)$ for any i. Let H_i be given as

$$H_i ; a_i^1 w_1 + a_i^2 w_2 + \cdots + a_i^{N+1} w_{N+1} = 0 . \tag{3.2}$$

For an arbitrarily given Cousin-II subdomain U of D, we take admissible representations $f = f_1 : f_2 : \cdots : f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$ on U. We define holomorphic functions

$$F_i^f = a_i^1 f_1 + a_i^2 f_2 + \cdots + a_i^{N+1} f_{N+1}$$

and

$$F_i^g = a_i^1 g_1 + a_i^2 g_2 + \cdots + a_i^{N+1} g_{N+1}$$

on U and put

$$h_i(z) = \frac{F_i^g(z)}{F_i^f(z)} \qquad (1 \leqq i \leqq q) . \tag{3.3}$$

By the assumption, each h_i is a nowhere zero holomorphic function on U. As is easily seen, the ratios $h_i : h_j$ are uniquely determined independently of any choices of homogeneous coordinates, representations (3.2) and admissible representations. Therefore, we can consider the well-defined holomorphic map

$$h = h_1 : h_2 : \cdots : h_q \tag{3.4}$$

of D into $P_{q-1}(\boldsymbol{C})$. If D itself is a Cousin-II domain, h has an admissible representation on the totality of D with functions $h_i(z)$ on D defined by (3.3).

We shall study the case $q = 2N + 2$. By $\mathscr{I}$ we denote the set of all combinations $I = (i_1, i_2, \cdots, i_{N+1})$ $(1 \leqq i_1 < \cdots < i_{N+1} \leqq 2N + 2)$ of indices $1, 2, \cdots, 2N + 2$. For a point $u = u_1 : u_2 : \cdots : u_{2N+2} \in P_{2N+1}(\boldsymbol{C})$ and $I = (i_1, i_2, \cdots, i_{N+1}) \in \mathscr{I}$, we put $u_I = u_{i_1} u_{i_2} \cdots u_{i_{N+1}}$ and consider the map Φ of $P_{2N+1}(\boldsymbol{C})$ into $P_{M-1}(\boldsymbol{C})$ defined as

$$\Phi(u) = (u_I : I \in \mathscr{I}) \in P_{M-1}(\boldsymbol{C}) ,$$

where $M = \binom{2N+2}{N+1}$.

PROPOSITION 3.5. *In the above situation, non-zero constants A_I $(I \in \mathscr{I})$ can be chosen independently of each f and g such that, for the maps h defined by* (3.4),

$$\Phi \cdot h(D) \subset H^* : = \{u \in P_{M-1}(\boldsymbol{C}) : \textstyle\sum_{I \in I} A_I u_I = 0\} .$$

Proof. Without loss of generality, we may assume that D is a Cousin-II domain. For, by the theorem of identity, Proposition 3.5 is true if it is shown that $\Phi \cdot h(U) \subset H^*$ for some non-empty open subset U of D. Let H_i $(1 \leqq i \leqq 2N + 2)$ be given by (3.2). By the assumption, any minor of degree $N + 1$ of the matrix $(a_j^i ; {}^{1 \leqq i \leqq N+1}_{1 \leqq j \leqq 2N+1})$ does not vanish. Taking admissible representations $f = f_1 : f_2 : \cdots f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$, we rewrite the definition (3.3) of h_i $(1 \leqq i \leqq 2N + 2)$ as

$$a_i^1 f_1 + a_i^2 f_2 + \cdots + a_i^{N+1} f_{N+1} = h_i (a_i^1 g_1 + a_i^2 g_2 + \cdots + a_i^{N+1} g_{N+1}) . \tag{3.6}$$

From these $2N + 2$ identities eliminating $2N + 2$ functions $f_1, f_2, \cdots, f_{N+1}, g_1, g_2, \cdots, g_{N+1}$, we get

$$\Psi : = \det (a_i^1, \cdots, a_i^{N+1}, a_i^1 h_i, \cdots, a_i^{N+1} h_i ; 1 \leqq i \leqq 2N + 2) = 0 . \tag{3.7}$$

For any combination $I = (i_1, i_2, \cdots, i_{N+1}) \in \mathscr{I}$, we take $J = (j_1, j_2, \cdots, j_{N+1}) \in \mathscr{I}$ such that

$$\{i_1, i_2, \cdots, i_{N+1}, j_1, j_2, \cdots, j_{N+1}\} = \{1, 2, \cdots, 2N + 2\} .$$

And, put

$$A_I = (-1)^{(N+1)(N+2)/2 + i_1 + \cdots + i_{N+1}} \det \left(a_{i_r}^j ; {}^{1 \leqq j \leqq N+1}_{1 \leqq r \leqq N+1}\right) \det \left(a_{j_s}^k ; {}^{1 \leqq k \leqq N+1}_{1 \leqq s \leqq N+1}\right) ,$$

Then, by the Laplace expansion formula,

$$\Psi = \sum_{(i_1 \cdots i_{N+1}) \in \mathscr{I}} A_{(i_1 \cdots i_{N+1})} h_{i_1} h_{i_2} \cdots h_{i_{N+1}} .$$

Since $A_I \neq 0$ for any $I \in \mathscr{I}$ by the assumption, this gives Proposition 3.5.

§ 4. Some consequences of E. Borel's theorem.

In the following, we shall study mainly functions and maps defined on $D := C^n$ or a domain D which is given as $D := B - S$ for a subdomain B of C^n and its irreducible analytic subset S. We denote by $\mathscr{H}^*$ the set of all nowhere zero holomorphic functions on D and by $\mathscr{C}$ the set of all constant functions for the case $D = C^n$ and of all holomorphic functions on D which can be meromorphically continuable to the totality of B for the case $D = B - S$. Moreover, we put $\mathscr{C}^* = \mathscr{C} \cap \mathscr{H}^*$. Then, as is easily seen, the multiplicative group $G = \mathscr{H}^* / \mathscr{C}^*$ is a torsion free abelian group. For two elements h and h^* in $\mathscr{H}^*$, we mean by the notation

$$h \sim h^*$$

that $h / h^* \in \mathscr{C}^*$.

Now, we recall the following theorem of E. Borel ([1]).

THEOREM 4.1. *If functions $h_1, h_2, \cdots, h_p$ in $\mathscr{H}^*$ satisfy the condition that $h_i \not\sim h_j$ for any i, j ($\neq$), then they are linearly independent over $\mathscr{C}$, i.e., a relation*

$$a^1 h_1 + a^2 h_2 + \cdots + a^p h_p = 0$$

($a^i \in \mathscr{C}$) implies always $a^1 = a^2 = \cdots = a^p = 0$.

For the proof, see [5], Theorem 3.5 and Theorem 4.1.

COROLLARY 4.2. *If* $a^1h_1 + a^2h_2 + \cdots + a^ph_p = 0$ *for functions* $h_i \in \mathscr{H}^*$ *and* $a^i \in \mathscr{C}$, *then there exists a partition of indices*

$$\{1, 2, \cdots, p\} = I_1 \cup I_2 \cup \cdots \cup I_k$$

$(I_\ell \cap I_m = \emptyset,\ I_\ell \neq \emptyset)$ *such that*

$$\textstyle\sum_{i \in I} a^i h_i = 0$$

for any ℓ *and* $h_i \sim h_j$ *for any* $i, j \in I_\ell$.

Remark. In Corollary 4.2, if $a^i \neq 0$ for any i, each I_ℓ contains obviously at least two indices. This shows that, for any h_i, there exists some h_j $(i \neq j)$ with $h_i \sim h_j$.

Proof of Corollary 4.2. Consider the partition $\{1, 2, \cdots, p\} = I_1 \cup \cdots \cup I_k$ such that i and j are in the same class if and only if $h_i \sim h_j$. Then, we can write

$$\textstyle\sum_{i=1}^{p} a^i h_i = \sum_{\ell=1}^{k} \sum_{i \in I_\ell} a^i h_i = \sum_{\ell=1}^{k} c^\ell h_{i_\ell} = 0$$

for some $c^\ell \in \mathscr{C}$ and any fixed $i_\ell \in I_\ell$. By Theorem 4.1, $c^\ell = 0$ for any ℓ, which yields Corollary 4.2.

After these preliminaries, we give

PROPOSITION 4.3. *Let* D *be a domain given as the above and assume that it is a Cousin-II domain. If meromorphic maps* f *and* g *of* D *into* $P_N(C)$ *satisfy the condition that* $f(D^n) \not\subset H_i$, $g(D^n) \not\subset H_i$ *and* $\nu(f, H_i) = \nu(g, H_i)$ *for* $q\,(\geqq 2N + 2)$ *hyperplanes* H_i $(1 \leqq i \leqq q)$ *in general position, then the* q*-tuple of the canonical images of the functions* h_i *defined by* (3.3) *into* $G = \mathscr{H}^*/\mathscr{C}^*$ *has the property* $(P_{2N+2,N+1})$ (*c.f.*, *Definition* 2.5).

Proof. We choose $2N + 2$ functions, say $h_1, h_2, \cdots, h_{2N+2}$, among h_i. With each combination $I = (i_1, i_2, \cdots, i_{N+1})$ of indices $1, 2, \cdots, 2N + 2$ associate the nowhere zero holomorphic functions $h_I = h_{i_1} h_{i_2} \cdots h_{i_{N+1}}$ on D. By Proposition 3.5, they satisfy the identity

$$\textstyle\sum_I A_I h_I = 0$$

for non-zero constants A_I. Then, by Remark to Corollary 4.2, we have easily Proposition 4.3.

Since any one of h_i may be assumed to be the constant 1 by a suitable change of admissible representations, Lemma 2.6 and Lemma 2.7 imply immediately

COROLLARY 4.4. *Under the same assumption of Proposition* 4.3, $q-2N$ *functions* $h_{k_1}, h_{k_2}, \cdots, h_{k_{q-2N}}$ *can be chosen such that* $h_{k_m} \sim 1$ $(1 \leqq m \leqq q-2N)$. *And, furthermare, if* $h_i \not\sim 1$ *for any other* i, *then there exist some* i, j *with* $i \neq j$ *and* $i, j \neq k_m$ $(1 \leqq m \leqq q-2N)$ *such that* $h_i \sim h_j$ *or* $h_i h_j \sim 1$.

Remark. Theorem 4.1 remains valid under the weaker assumption that each h_i can be written as $h_i = f_i^d$ with a not identically zero holomorphic function f_i on D such that, for any $i, j\,(\neq)$, $f_i/f_j \not\equiv$ const, in the case $D = \boldsymbol{C}^n$ and f_i/f_j has essential singularities along S in the case $D = B - S$ if $d > p(p-2)$ (c.f., [5], Remark 3.7, (ii)). By the same argument as the above, we can prove Proposition 4.3 under the assumption

$$\nu(f, H_i) \equiv \nu(g, H_i) \qquad (\mathrm{mod}\ d)$$

for a sufficiently large d depending only on N instead of the assumption $\nu(f, H_i) = \nu(g, H_i)$. And, many of the results in the following sections remain valid under these weaker conditions. We omit here the details in this direction.

We shall give now another application of E. Borel's theorem.

PROPOSITION 4.5. *Let* $P(X_1, X_2, \cdots, X_t)$ *be a polynomial of* t *variables with coefficients in* $\mathscr{C}$. *If*

$$P(h_1, h_2, \cdots, h_t) = 0$$

for some $h_1, h_2, \cdots, h_t$ *in* $\mathscr{H}^*$ *such that* $h_1^{\nu_1} h_2^{\nu_2} \cdots h_t^{\nu_t} \notin \mathscr{C}^*$ *for any integers* $(\nu_1, \nu_2, \cdots, \nu_t) \neq (0, 0, \cdots, 0)$,

$$P(X_1, X_2, \cdots, X_t) \equiv 0\ .$$

Namely, all coefficients of P *are equal to zero.*

Proof. We write

$$P(X_1, \cdots, X_t) = \textstyle\sum_{\nu_1, \nu_2, \cdots, \nu_t \geqq 0} a_{\nu_1 \nu_2 \cdots \nu_t} X_1^{\nu_1} X_2^{\nu_2} \cdots X_t^{\nu_t}$$

$(a_{\nu_1 \nu_2 \cdots \nu_t} \in \mathscr{C})$ and assume that $a_{\nu_1 \nu_2 \cdots \nu_t} \neq 0$ for some $\nu_1^0, \nu_2^0, \cdots, \nu_t^0$. Since

$$\textstyle\sum_{\nu_1, \nu_2, \cdots, \nu_t} a_{\nu_1 \nu_2 \cdots \nu_t} h_1^{\nu_1} h_2^{\nu_2} \cdots h_t^{\nu_t} = 0$$

and $h_1^{\nu_1}h_2^{\nu_2}\cdots h_t^{\nu_t}\in\mathscr{H}^*$, we can conclude by Remark to Corollary 4.2 that there exist some $\mu_1^0,\mu_2^0,\cdots,\mu_t^0$ with $(\nu_1^0,\nu_2^0,\cdots,\nu_t^0)\neq(\mu_1^0,\mu_2^0,\cdots,\mu_t^0)$ such that

$$h_1^{\nu_1^0}h_2^{\nu_2^0}\cdots h_t^{\nu_t^0}\sim h_1^{\mu_1^0}h_2^{\mu_2^0}\cdots h_t^{\mu_t^0}$$

and so $h_1^{\nu_1^0-\mu_1^0}h_2^{\nu_2^0-\mu_2^0}\cdots h_t^{\nu_t^0-\mu_t^0}\in\mathscr{C}^*$. This contradicts the assumption. We have Proposition 4.5.

§ 5. Uniqueness theorems of meromorphic maps.

As in the previous sections, we consider two meromorphic maps f and g of D into $P_N(\boldsymbol{C})$ and $q\,(\geqq 2N+2)$ hyperplanes H_i in $P_N(\boldsymbol{C})$ located in general position such that $f(D)\not\subset H_i$, $g(D)\not\subset H_i$ and $\nu(f,H_i)=\nu(g,H_i)$ $(1\leqq i\leqq q)$. We study first the case $D=\boldsymbol{C}^n$.

THEOREM 5.1. *If $q\geqq 3N+1$, then it is possible to choose homogeneous coordinates $w_1:w_2:\cdots:w_{N+1}$ on $P_N(\boldsymbol{C})$ such that*

$$g_1=c_1f_1,\qquad g_2=c_2f_2,\qquad g_{N+1}=c_{N+1}f_{N+1}\tag{5.2}$$

for suitable admissible representations $f=f_1:f_2:\cdots:f_{N+1}$ and $g=g_1:g_2:\cdots:g_{N+1}$, where c_i are some non-zero constants.

Proof. As in § 3, we define by (3.3) a nowhere zero holomorphic function h_i for each H_i. According to Corollary 4.4, we may assume that $N+1$ $(=(3N+1)-2N)$ functions among them, which we say $c_1:=h_1,\cdots,c_{N+1}:=h_{N+1}$, are of constants. Since the ratios $h_1:h_2:\cdots:h_{3N+1}$ are determined independently of a choice of homogeneous coordinates, each H_i $(1\leqq i\leqq N+1)$ may be assumed to be given as

$$H_i: w_i=0\ .$$

We have then Theorem 5.1 by the definition (3.3) of h_i.

Proof of Theorem I. Theorem I stated in § 1 is an immediate consequence of Theorem 5.1. In fact, it suffices to take a linear transformation

$$L: w_i'=c_iw_i\qquad 1\leqq i\leqq N+1$$

for constants c_i in Theorem 5.1.

Proof of Theorem II. In this case, $c_i:=h_i$ $(1\leqq i\leqq N+2)$ may be assumed to be of constants and each H_i $(1\leqq i\leqq N+2)$ may be given as

$$H_i: w_i = 0 \qquad 1 \leqq i \leqq N+1$$

and

$$H_{N+2}: w_1 + w_2 + \cdots + w_{N+1} = 0 .$$

For admissible representations $f = f_1 : f_2 : \cdots : f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$, we have the relation (5.2) and

$$g_1 + g_2 + \cdots + g_{N+1} = c_{N+2}(f_1 + f_2 + \cdots + f_{N+1}) .$$

Therefore,

$$(c_1 - c_{N+2})f_1 + (c_2 - c_{N+2})f_2 + \cdots + (c_{N+1} - c_{N+2})f_{N+1} = 0 .$$

Since f may be assumed to be non-degenerate, we conclude

$$c_1 = c_2 = \cdots = c_{N+1} = c_{N+2} .$$

This shows that $f = g$.

Here, we cannot conclude $f = g$ without the assumption of non-degeneracy of f or g in Theorem 5.1 even if any large number of hyperplanes H_i in general position with $\nu(f, H_i) = \nu(g, H_i)$ are given. We give an example. For an arbitrarily given $q\,(\geqq 6)$, take a matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_5 & a_6 & \cdots & a_q \\ 1 & b_5 & b_6 & \cdots & b_q \end{pmatrix}$$

such that any minor of M does not vanish and

$$\begin{vmatrix} a_5(a_i - 1) & a_i(a_5 - 1) \\ b_5(b_i - 1) & b_i(b_5 - 1) \end{vmatrix} = 0 \qquad (6 \leqq i \leqq q) \tag{5.3}$$

and consider hyperplanes

$$\begin{aligned} &H_i: w_i = 0 && 1 \leqq i \leqq 3 \\ &H_4: w_1 + w_2 + w_3 = 0 && \\ &H_j: w_1 + a_j w_2 + b_j w_3 = 0 && 5 \leqq j \leqq q . \end{aligned}$$

As is easily seen by (5.3), we can choose non-zero constants c_1, c_2, c_3, d_i $(5 \leqq i \leqq q)$ such that $c_1 \neq 1$ and

$$\frac{1 - c_3}{1 - d_i} = \frac{c_1 - c_3}{a_i(c_1 - d_i)} = \frac{c_2 - c_3}{b_i(c_2 - d_i)} \qquad (5 \leqq i \leqq q) .$$

If we take meromorphic maps $f = f_1 : f_2 : f_3$ on C^n into $P_2(C)$ with $f(C^n) \not\subset H_i$ $(1 \leqq i \leqq q)$ and

$$(1 - c_3)f_1 + (c_1 - c_3)f_2 + (c_2 - c_3)f_3 = 0$$

and $g = f_1 : c_1 f_2 : c_2 f_3$, then we see easily $f \neq g$ and $\nu(f, H_i) = \nu(g, H_i)$ for any i $(1 \leqq i \leqq q)$.

Consider next the case $D = B - S$, where B is a domain in C^n and S is an irreducible analytic subset of B. Let f be a meromorphic map of D into $P_N(C)$. Using inhomogeneous coordinates $u_i : = w_i / w_{N+1}$ $(1 \leqq i \leqq N)$ for homogeneous coordinates $w_1 : w_2 : \cdots : w_{N+1}$ with $f(D) \not\subset \{w_{N+1} = 0\}$, we can write

$$f = (\varphi_1^f, \varphi_2^f, \cdots, \varphi_N^f) ,$$

where φ_i^f are meromorphic functions on D.

THEOREM 5.4. *Let f, g be meromorphic maps of D into $P_N(C)$ such that $f(D) \not\subset H_i$, $g(D) \not\subset H_i$ and $\nu(f, H_i) = \nu(g, H_i)$ for $3N + 1$ hyperplanes H_i $(1 \leqq i \leqq 3N + 1)$ in general position. Then, it is possible to choose inhomogeneous coordinates $u_1, u_2, \cdots, u_N$ such that, for representations $f = (\varphi_1^f, \varphi_2^f, \cdots, \varphi_N^f)$ and $g = (\varphi_1^g, \varphi_2^g, \cdots, \varphi_N^g)$,*

$$\varphi_1^g = \alpha_1 \varphi_1^f, \varphi_2^f = \alpha_2 \varphi_2^f, \cdots, \varphi_N^g = \alpha_N \varphi_N^f ,$$

where α_i are meromorphic functions on the totality of B.

Proof. Take a regular point x in S arbitrarily. We can choose a neighborhood U of x such that

$$U = \{|z_1| < 1, |z_2| < 1, \cdots, |z_n| < 1\}$$

and

$$U^* = U \cap D = \{0 < |z_1| < 1, |z_2| < 1, \cdots, |z_n| < 1\} ,$$

for suitably chosen local coordinates $z_1, z_2, \cdots, z_n$ with $x = (0, 0, \cdots, 0)$. Since U^* is a Cousin-II domain, we can apply Corollary 4.4. By the same argument as in the proof of Theorem 5.1, for functions h_i on U^* defined by (3.3), we may assume that $h_1, h_2, \cdots, h_{N+1}$ have meromorphic continuations to U. And, we can find easily inhomogeneous coordinates on $P_N(C)$ such that $\alpha_i : = \varphi_i^g / \varphi_i^f$ $(1 \leqq i \leqq N)$ are meromorphically continuable to U for representations $f = (\varphi_1^f, \cdots, \varphi_N^f)$ and $g = (\varphi_1^g, \cdots, \varphi_N^g)$. Then, by the classical E. E. Levi's theorem, α_i are meromorphic on the totality of B. This completes the proof.

We want to get an analogy to Theorem II. To this end, we give

DEFINITION 5.5. We shall call a meromorphic map $f=(\varphi_1^f, \varphi_2^f, \cdots, \varphi_N^f)$ of $D(=B-S)$ into $P_N(C)$ to have *essential singularities of type* (*E*) *along* S if $\alpha^1\varphi_1^f+\alpha^2\varphi_2^f+\cdots+\alpha^N\varphi_N^f$ is not meromorphically continuable to S for any meromorphic functions α_i $(1 \leqq i \leqq N)$ on B except the case $\alpha^1 \equiv \alpha^2 \equiv \cdots \equiv \alpha^N \equiv 0$.

THEOREM 5.6. *Let f, g be meromorphic maps satisfying the same conditions as in Theorem* 5.3 *for* $3N+2$ *hyperplanes H_i in general position. If f or g has essential singularities of type* (*E*) *along S, then* $f=g$.

Proof. For a regular point x of S, as in the proof of Theorem 5.4, taking a neighborhood U of x, we may assume that h_i $(1 \leqq i \leqq N+2)$ are well-defined and meromorphic on U. Moreover, choosing suitable homogeneous coordinates and an admissible representation $f=f_1: f_2: \cdots : f_{N+1}$ on $U^*=U \cap D$, we have by the similar manner as in the proof of Theorem II

$$(h_1-h_{N+2})f_1+(h_2-h_{N+2})f_2+\cdots+(h_{N+1}-h_{N+2})f_{N+1}=0 .$$

Therefore,

$$(\alpha_1-1)\varphi_1^f+(\alpha_2-1)\varphi_2^f+\cdots+(\alpha_N-1)\varphi_N^f+(\alpha_{N+1}-1)=0$$

for well-defined meromorphic functions $\varphi_i^f := f_i/f_{N+1}$ $(1 \leqq i \leqq N)$ and $\alpha_j := h_j/h_{N+2}$ $(1 \leqq j \leqq N+1)$ which are also meromorphic on B by E. E. Levi's theorem. By the assumption,

$$\alpha_1=\alpha_2=\cdots=\alpha_{N+1}=1 .$$

This completes the proof.

§ 6. The case that $3N+1$ hyperplanes are given.

Let f, g be meromorphic maps of a domain D stated in § 4 into $P_N(C)$ and assume that, for $3N+1$ hyperplanes H_i $(1 \leqq i \leqq 3N+1)$, $f(D) \not\subset H_i$, $g(D) \not\subset H_i$ and $\nu(f, H_i)=\nu(g, H_i)$. Under these assumption, we shall give more precise informations in the previous section.

THEOREM 6.1. (i) *In the case $D=C^n$, if f or g is non-degenerate,* *then each c_i of Theorem 5.1 can be chosen to be $+1$ or -1, and, moreover, if $N \geqq 2$, it is impossible that exactly one c_i is equal to 1.*

(ii) *In the case $D = B - S$, if f or g has essential singularities of the type (E) along S, then each α_i in Theorem 5.4 can be chosen to be of constant $+1$ or -1 and, moreover, if $N \geqq 2$, it is impossible that $\alpha_i \equiv -1$ for any i $(1 \leqq i \leqq N+1)$.*

Proof. For the proof of the case $D = B - S$, it may be assumed that $B = \{|z_1| < 1, \cdots, |z_n| < 1\}$ and $S = \{z_1 = 0\} \cap B$ as in the proof of Theorem 5.4. In the following, we mean $D = C^n$ or $D = B - S$ for the above B and S and by $\mathscr{H}^*, \mathscr{C}, \mathscr{C}^*$ and h_i the ones defined as in §4 for such a domain D. By Corollary 4.4, we may assume that at least $N+1$ h_i's are in $\mathscr{C}^*$ and, moreover, $h_i \notin \mathscr{C}^*$ for the other h_i because, if $h_i \in \mathscr{C}^*$ for mutually distinct $N+2$ i's, $f = g$ by the same reason as in the proof of Theorem II. For convenience' sake, assume $h_i \notin \mathscr{C}^*$ $(1 \leqq i \leqq 2N)$ and $\alpha_j := h_j \in \mathscr{C}^*$ $(2N+1 \leqq j \leqq 3N+1)$. Let each H_i $(1 \leqq i \leqq 3N+1)$ be given as (3.2). We may assume here $a^i_{2N+j} = \delta^i_j$ $(1 \leqq i, j \leqq N+1)$ by a suitable change of homogeneous coordinates. Then, any minor of the matrix

$$(a^i_j\,;\, {\scriptstyle 1 \leqq i \leqq N+1 \atop 1 \leqq j \leqq 2N})$$

does not vanish. Take now functions $\eta_1, \eta_2, \cdots, \eta_t$ in $\mathscr{H}^*$ whose canonical images into $G = \mathscr{H}^*/\mathscr{C}^*$ constitute a basis of the subgroup $\tilde{A}$ of G generated by the canonical images of $h_1, h_2, \cdots, h_{3N+1}$ into G. Then, we can write uniquely as

$$h_i = \alpha_i \eta_1^{\ell_{i1}} \eta_2^{\ell_{i2}} \cdots \eta_t^{\ell_{it}} \qquad (1 \leqq i \leqq 3N+1) \tag{6.2}$$

for some $\alpha_i \in \mathscr{C}^*$ and integers $\ell_{i\tau}$. Choose here integers $p_1, p_2, \cdots, p_t$ which are generic with respect to $\ell_{i\tau}$ and put $\ell_i := \sum_{\tau=1}^{t} \ell_{i\tau} p_\tau$ $(1 \leqq i \leqq 3N+1)$.

Now, let us take a combination $I = (i_1, i_2, \cdots, i_{2N+2})$ $(1 \leqq i_1 < \cdots < i_{2N+2} \leqq 3N+1)$ arbitrarily. As in the proof of Proposition 3.5, considering admissible representations $f = f_1 : f_2 : \cdots : f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$ related as (3.6), we obtain

$$\det(a^1_i, \cdots, a^{N+1}_i, a^1_i h_i, \cdots, a^{N+1}_i h_i\,;\, i = i_1, i_2, \cdots, i_{2N+2}) = 0\ . \tag{6.3}$$

Substitute the identities (6.2) into (6.3). Then, we can rewrite (6.3) as

$$P_I(\eta_1, \eta_2, \cdots, \eta_t) = 0\ ,$$

where $P_I(X_1, X_2, \cdots, X_t)$ is a polynomial of t variables with coefficients in $\mathscr{C}$. And, by Proposition 4.5, we have

$$(6.4) \qquad P_I(X_1, X_2, \cdots, X_t) \equiv 0 \ .$$

Consider a rational function

$$Q_I(\zeta) = P(\zeta^{p_1}, \zeta^{p_2}, \cdots, \zeta^{p_t})$$

of ζ, which is identically zero because of (6.4). On the other hand, $Q_I(\zeta)$ is also obtained by substituting $h_i = \alpha_i \zeta^{\ell_i}$ into (6.3). We have thus

$$(6.5) \quad Q_I(\zeta) = \det(a_i^1, \cdots, a_i^{N+1}, \alpha_i\zeta^{\ell_i}a_i^1, \cdots, \alpha_i\zeta^{\ell_i}a_i^{N+1}; i = i_1, \cdots, i_{2N+2}) = 0 \ .$$

Particularly, for a combination $I_0 = (1, 2, \cdots, 2N+2)$, we observe the coefficients of terms of $Q_{I_0}(\zeta)$ of the highest degree and of the lowest degree. To this end, we may assume by Lemma 2.6

$$\ell_1 \leqq \ell_2 \leqq \cdots \leqq \ell_N < \ell_{2N+1} = \cdots = \ell_{3N+1} = 0 < \ell_{N+1} \leqq \cdots \leqq \ell_{2N} \ .$$

Then, we have easily

$$\det\begin{pmatrix} 0 & A_1 \\ A_2 & 0 \\ A_3 & A_3^* \end{pmatrix} = \det\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \\ A_3 & A_3^* \end{pmatrix} = 0 \ ,$$

where

$$A_1 = (a_j^i \,;\, {}^{1 \leqq i \leqq N+1}_{1 \leqq j \leqq N}) \ , \qquad A_2 = (a_j^i \,;\, {}^{1 \leqq i \leqq N+1}_{N+1 \leqq j \leqq 2N})$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ 0 & 1 & 0 \cdots 0 \end{pmatrix}$$

and

$$A_3^* = \begin{pmatrix} \alpha_{2N+1}, 0, 0, \cdots, 0 \\ 0, \alpha_{2N+2}, 0, \cdots, 0 \end{pmatrix} .$$

By the Laplace expansion formula, we conclude

$$\alpha_{2N+1}D_1 - \alpha_{2N+2}D_2 = \alpha_{2N+1}D_2 - \alpha_{2N+2}D_1 = 0 \ ,$$

where

$$D_1 = \det\begin{pmatrix} A_1 \\ e_1 \end{pmatrix} \det\begin{pmatrix} A_2 \\ e_2 \end{pmatrix} \quad \text{and} \quad D_2 = \det\begin{pmatrix} A_1 \\ e_2 \end{pmatrix} \det\begin{pmatrix} A_2 \\ e_1 \end{pmatrix}$$

for $e_1 = (1, 0, 0, \cdots, 0)$ and $e_2 = (0, 1, 0, \cdots, 0)$. Since H_i are in general position, we know $D_1 \neq 0$ and $D_2 \neq 0$. Hence, $\alpha_{2N+1}^2 = \alpha_{2N+2}^2$. The same arguments are available for the other α_i's among $\alpha_{2N+1}, \cdots, \alpha_{3N+1}$. Thus, we can conclude $\alpha_i = \pm 1$ $(2N + 1 \leqq i \leqq 3N + 1)$, because we may assume $\alpha_{2N+1} = 1$.

To complete the proof, assume that exactly one among α_i $(2N + 1 \leqq i \leqq 3N + 1)$ is equal to -1, e.g., $\alpha_{2N+1} = \alpha_{2N+2} = \cdots = \alpha_{3N} = 1$ and $\alpha_{3N+1} = -1$. We shall prove first that there are at most $N - 1$ indices i $(1 \leqq i \leqq 2N)$ such that $\alpha_i \neq 1$. Suppose that $\alpha_j \neq 1$ for some mutually distinct $j_1, j_2, \cdots, j_N$ $(1 \leqq j_m \leqq 2N)$. Here, changing η_τ if necessary, we may assume that $\alpha_{j_{N+1}} = 1$ for some j_{N+1} with $j_{N+1} \neq j_m$ $(1 \leqq m \leqq N)$ and $1 \leqq j_{N+1} \leqq 2N$. Putting $j_{N+2} = 2N + 1, \cdots, j_{2N+2} = 3N + 1$, we consider the identity (6.5) for a combination $I_1 = (j_1, j_2, \cdots, j_{2N+2})$. Particularly, substituting $\zeta = 1$, we get

$$\begin{aligned}
&\det (a_{j_m}^1, \cdots, a_{j_m}^{N+1}, \alpha_{j_m} a_{j_m}^1, \cdots, \alpha_{j_m} a_{j_m}^{N+1} ; 1 \leqq m \leqq 2N + 2) \\
&\quad = \det \begin{pmatrix} B_1 & B_1^* \\ B_2 & 0 \\ e_{N+1} & -2e_{N+1} \end{pmatrix} \\
&\quad = \pm 2(\alpha_{j_1} - 1) \cdots (\alpha_{j_N} - 1) \det (B_2) \det \begin{pmatrix} B_1 \\ e_{N+1} \end{pmatrix} \\
&\quad = 0 ,
\end{aligned}$$

where

$$B_1 = (a_{j_m}^i ; {}^{1 \leqq i \leqq N+1}_{1 \leqq m \leqq N}) , \qquad B_1^* = ((\alpha_{j_m} - 1) a_{j_m}^i ; {}^{1 \leqq i \leqq N+1}_{1 \leqq m \leqq N})$$
$$B_2 = (a_{j_m}^i ; {}^{1 \leqq i \leqq N+1}_{N+1 \leqq m \leqq 2N+1})$$

and $e_{N+1} = (0, 0, \cdots, 0, 1)$. This is a contradiction, because

$$\det (B_2) \neq 0 \quad \text{and} \quad \det \begin{pmatrix} B_1 \\ e_{N+1} \end{pmatrix} \neq 0$$

by the assumption. Therefore, we can choose $N + 1 \, (= 2N - (N - 1))$ indices $i_1, i_2, \cdots, i_{N+1}$ $(1 \leqq i_m \leqq 2N)$ with $\alpha_{i_m} = 1$.

Take now an index μ such that

$$|\ell_{i_\mu}| = \max (\ell_{i_1}, \ell_{i_2}, \cdots, \ell_{i_{N+1}}) .$$

Then, $|\ell_{i_\mu}| = |\ell_{i_{\mu'}}|$ for some μ' ($\neq \mu$). In fact, if not, substitute an ℓ_{i_μ}-th primitive root of unity into the identity (6.5) for a combination $I_2 := (i_1, i_2, \cdots, i_{N+1}, 2N+1, \cdots, 3N+1)$. We have then a contradiction by the same argument as the above. The fact $|\ell_{i_\mu}| = |\ell_{i_{\mu'}}|$ means that $h_{i_{\mu'}} = h_{i_\mu}^m$ for $m = \pm 1$. For admissible representations $f = f_1 : f_2 : \cdots : f_{N+1}$ and $g = g_1 : g_2 : \cdots : g_{N+1}$, we know $g_i = f_i$ $(1 \leqq i \leqq N)$ and $g_{N+1} = -f_{N+1}$. We may assume here $a^i_{j_\mu} = 1$ $(1 \leqq i \leqq N+1)$ by a change of homogeneous coordinates and put $b^i := a^i_{j_{\mu'}}$. Then,

$$(f_1 + f_2 + \cdots + f_{N+1})^{-m}(b^1 f_1 + b^2 f_2 + \cdots + b^{N+1} f_{N+1})$$
$$= (f_1 + f_2 + \cdots + f_N - f_{N+1})^{-m}(b^1 f_1 + b^2 f_2 + \cdots + b^N f_N - b^{N+1} f_{N+1}),$$

whence

$$(b^{N+1} - mb^1)f_1 + (b^{N+1} - mb^2)f_2 + \cdots + (b^{N+1} - mb^N)f_N = 0 .$$

Since f may be assumed to be non-degenerate,

$$b^{N+1} - mb^1 = b^{N+1} - mb^2 = \cdots = b^{N+1} - mb^N = 0 .$$

Then, $b^i = b^j$ for some i, j ($\neq$) in the case $N \geqq 2$, which is a contradiction. This completes the proof of Theorem 6.1.

COROLLARY 6.6. *Under the same assumption of Theorem* 6.1, *if* $N = 2$, *then,* $f = g$.

Proof. For the case $D = \boldsymbol{C}^n$, Theorem 6.1 implies that $c_1 = c_2 = c_3 = 1$ or $c_1 = c_2 = c_3 = -1$. In any case, we have $f = g$. Similarly, for the case $D = B - S$ too, we conclude also $f = g$.

THEOREM 6.7. *Let* f, g *be meromorphic maps of* $\boldsymbol{C}^n$ *into* $P_N(\boldsymbol{C})$ *such that* $f(\boldsymbol{C}^n) \not\subset H_i$, $g(\boldsymbol{C}^n) \not\subset H_i$ *and* $\nu(f, H_i) = \nu(g, H_i)$ *for* $3N+1$ *hyperplanes* H_i $(1 \leqq i \leqq 3N+1)$ *in general position. If the image of* f *is not included in any subvariety of* $P_N(\boldsymbol{C})$ *which is defined as the zero set of a homogeneous polynomial of degree* $\leqq 2$, *then* $f = g$.

Proof. Let H_i be given as (3.2). By Theorem 6.1, we may put $g_i = c_i f_i$ $(1 \leqq i \leqq N+1)$ for admissible representations $f = f_1 : f_2 : \cdots : f_{N+1}$, where $c_i := h_i = \pm 1$. Moreover, by Corollary 4.4, if $f \neq g$, we may assume that $h_{N+2} h_{N+3} \sim 1$ or $h_{N+2} \sim h_{N+3}$, i.e., $h_{N+3} = d h^m_{N+2}$ for $m = \pm 1$ and $d \in \mathscr{C}^*$. As in the proof of Theorem 6.1,

$$
\begin{aligned}
&(a_{2N+1}^1 f_1 + \cdots + a_{2N+1}^{N+1} f_{N+1})^{-m}(a_{2N+2}^1 f_1 + \cdots + a_{2N+2}^{N+1} f_{N+1}) \\
&\quad - d(a_{2N+1}^1 c_1 f_1 + \cdots + a_{2N+1}^{N+1} c_{N+1} f_{N+1})^{-m}(a_{2N+2}^2 c_1 f_1 + \cdots + a_{2N+2}^{N+1} c_1 f_{N+1}) = 0 .
\end{aligned}
$$

By the assumption, the left hand side vanishes identically as a polynomial of $N+1$ indeterminates $f_1, f_2, \cdots, f_{N+1}$. By simple calculations, we can conclude $f = g$.

§ 7. Meromorphic maps into $P_2(C)$.

Let us consider in this section two meromorphic maps f and g of C^n into $P_2(C)$ such that $f(C^n) \not\subset H_i$, $g(C^n) \not\subset H_i$ and $\nu(f, H_i) = \nu(g, H_i)$ for six hyperplanes H_i $(1 \leqq i \leqq 6)$ in general position. We shall study relations between the functions h_i defined by (3.3). By the equivalence relation $h_i \sim h_j$, i.e., $h_i/h_j \equiv$ const., we classify the set $\{h_1, h_2, \cdots, h_6\}$ into the subclasses $J_1, J_2, \cdots, J_k$. By M we denote the maximum of the numbers of elements in J_ℓ $(1 \leqq \ell \leqq k)$.

We study first the case $M = 2$. To this end, take functions $\eta_1, \eta_2, \cdots, \eta_t$ in $\mathscr{H}^*$ whose canonical images to $G = \mathscr{H}^*/\mathscr{C}^*$ constitute a basis of the subgroup of G generated by the canonical images of h_i $(1 \leqq i \leqq 6)$ into G. Writing each h_i as

$$
h_i \sim \eta_1^{\ell_{i1}} \eta_2^{\ell_{i2}} \cdots \eta_t^{\ell_{it}} ,
$$

we choose integers $p_1, p_2, \cdots, p_t$ which are generic with respect to $\ell_{i\tau}$ and put $\ell_i := \sum_{\tau=1}^{t} \ell_{i\tau} p_\tau$. By Lemma 2.6 and Proposition 4.3, it may be assumed that

$$
\ell_1 \leqq \ell_2 < \ell_3 = \ell_4 = 0 < \ell_5 \leqq \ell_6
$$

after a suitable change of indices. Let us assume $\ell_1 < \ell_2$ and $\ell_5 < \ell_6$. Then, by Corollary 4.4, we see $-\ell_2 = \ell_5$. Moreover, exchanging each η_τ by η_τ^{-1} if necessary, we may assume $\ell_1 + \ell_6 \geqq 0$. By Proposition 4.3, we can take indices i, j, k $(1 \leqq i < j < k \leqq 6$, $\{i, j, k\} \neq \{1, 5, 6\})$ such that $h_i h_j h_k \sim h_1 h_5 h_6$. Then, $\ell_i + \ell_j + \ell_k = \ell_1 + \ell_5 + \ell_6$ by (2.4). Let $l_1 + l_6 > 0$. If $k \leqq 5$, we have a contradiction that

$$
\ell_i + \ell_j + \ell_k \leqq \ell_3 + \ell_4 + \ell_5 = \ell_5 < \ell_1 + \ell_5 + \ell_6 .
$$

Therefore, $k = 6$, $i \geqq 2$, $j \leqq 4$, and so $h_1 h_5 = h_i h_j$ for some i, j $(2 \leqq i < j \leqq 4)$. In conclusion, there are two possible cases (i) $h_1 h_6 \sim 1$ and (ii) $h_1 h_5 \sim h_2$. For the case (i), changing notations, we have the type

(I) $$(h_1, h_2, \cdots, h_6) = (c_1 h^{*-1}, c_2 h^{-1}, 1, c_3, h, h^*) ,$$

where $h, h^* \in \mathscr{H}^*$ with $h \not\sim 1$, $h^* \not\sim 1$, $h \not\sim h^*$ and $hh^* \not\sim 1$ and $c_1 \in \boldsymbol{C}^*$.

For the case (ii), if we put $h := h_5$, then $h_1 \sim h^{-2}$. Observe the types of functions $h_r h_s h_t$ $(r < s < t, \{r, s, t\} \neq \{1, 2, 6\})$ such that $h_r h_s h_t \sim h_1 h_2 h_6$. Using the assumption $\ell_5 < \ell_6$, we can easily conclude $h_6 \sim h^\ell$ for $\ell = 2$, 3 or 4. The case $\ell = 2$ can be reduced to the type (I). For the case $\ell = 3$, we have the type

(II) $$(h_1, h_2, \cdots, h_6) = (c_1 h^{-2}, c_2 h^{-1}, 1, c_3, h, c_4 h^3) ,$$

where $h \not\sim 1$.

On the other hand, we can prove that the case $\ell = 4$ is impossible. In fact, suppose that

(7.1) $$(h_1, h_2, \cdots, h_6) = (c_1 h^{-2}, c_2 h^{-1}, 1, c_3, h, c_4 h^4)$$

for some $h \in \mathscr{H}^*$ with $h \sim 1$ and $c_i \in \mathscr{C}^*$. For fixed admissible representations of f and g we consider the identity (3.7) as in the proof of Proposition 3.5. Substituting (7.1) into them, we have a relation

$$P(h) = 0 ,$$

where $P(X)$ is polynomial of degree $\leqq 8$ with constant coefficients. According to Proposition 4.4, the coefficients of $P(h)$ are all zeros. Thus, we get nine relations among unknowns c_i and a_j^i $(1 \leqq i \leqq 3,\ 1 \leqq j \leqq 6)$. By elementary computations, it is possible to conclude that all solutions contradicts the assumption that H_i are in general position.

Consider next the case $M = 2$ and $\ell_1 = \ell_2$. If $\ell_5 = \ell_6$, then we get the type

(III) $$(h_1, h_2, \cdots, h_6) = (h, c_1 h, 1, c_2, h^*, c_3 h^*) ,$$

where $h \not\sim 1$, $h^* \not\sim 1$ and $h \not\sim h^*$.

Suppose that $\ell_5 < \ell_6$. In this case, by Lemma 2.7, we see $h_1 h_5 \sim h_2 h_5 \sim 1$. Observe the possible types of functions $h_i h_j h_k$ $(i < j < k, \{i, j, k\} \neq \{1, 2, 6\})$ such that $h_i h_j h_k \sim h_1 h_2 h_6$. Putting $h := h_5$, we have easily $h_6 \sim h^m$ for $m = 2$ or 3. Therefore, we have one of the types

(IV) $$(h_1, h_2, \cdots, h_6) = (c_1 h^{-1}, c_2 h^{-1}, 1, c_3, h, c_4 h^2)$$

$$(\mathrm{V}) \qquad (h_1, h_2, \cdots, h_6) = (c_1h^{-1}, c_2h^{-1}, 1, c_3, h, c_4h^3)\ ,$$

where $h \not\sim 1$.

For the case $M = 2$ and $\ell_5 = \ell_6$, we have also one of the types (III), (IV) and (V), because this case can be reduced to the above by exchanging each η_τ by η_τ^{-1} $(1 \leqq \tau \leqq t)$.

Now, let us study the case $M = 3$. Without loss of generality, we may assume $h_1 \sim h_2 \sim h_3 \sim 1$. Observe all possible types of functions $h_ih_jh_k$ such that $h_ih_jh_k \sim h_1h_2h_3$, where $i < j < k$ and $\{i, j, k\} \neq \{1, 2, 3\}$. There are two possible subcases (a) $h_ih_j \sim 1$ $(4 \leqq i < j \leqq 6)$ and (b) $h_4h_5h_6 \sim 1$. We consider first the subcase (a). Changing indices if necessary, we may write

$$(7.2) \qquad (h_1, h_2, \cdots, h_6) = (1, c_1, c_2, h, c_3h^{-1}, h^*)\ ,$$

where $h, h^* \in \mathscr{H}^*$ with $h \not\sim 1$, $h^* \not\sim 1$ and $c_i \in \mathscr{C}^*$. If we substitute (7.2) into the identity (3.7), we have a relation

$$(7.3) \qquad A_1h^2h^* + A_2h^2 + A_3hh^* + A_4h + A_5h^* + A_6 = 0\ ,$$

where A_i $(1 \leqq i \leqq 6)$ are some constants. If $h \not\sim h^*$, $h^2 \not\sim h^*$, $h^2h^* \not\sim 1$ and $h \sim h^*$, then $A_s = 0$ for any s because of Proposition 4.5. This means that (7.3) vanishes identically as a polynomial of h and h^*. By substituting $h = h^* = 1$, we have easily $c_1 = 1$, $c_2 = 1$ or $c_3 = 1$. In any case, it is not difficult to conclude that (7.3) has no solution. On the other hand, if $h^2 \sim h^*$, $h^2h^* \sim 1$ or $h \sim h^*$, by exchanging h by h^{-1} and indices if necessary, we have one of the types

$$(\mathrm{VI}) \qquad (h_1, h_2, \cdots, h_6) = (1, c_2, c_3, h, c_4h^{-1}, c_5h)$$

and

$$(\mathrm{VII}) \qquad (h_1, h_2, \cdots, h_6) = (1, c_2, c_3, h, c_4h^{-1}, c_5h^2)\ ,$$

where $h \not\sim 1$.

We study next the subcase (b). Put

$$(h_1, h_2, \cdots, h_6) = (1, c_2, c_3h, h^*, c_4(hh^*)^{-1})$$

for $h, h^* \in \mathscr{H}^*$ with $h \not\sim 1$, $h^* \not\sim 1$ and $hh^* \not\sim 1$. As the above, by the use of (3.7), we have a relation

$$(7.4) \qquad B_1h^2h^{*2} + B_2h^2h^* + B_3hh^{*2} + B_4hh^* + B_5h + B_6h^* + B_7 = 0\ ,$$

where B_i are some constants. If $h^2h^* \not\sim 1$, $hh^{*2} \not\sim 1$ and $h \not\sim h^*$, then $B_i = 0$ for any i by Proposition 4.5. In this case too, we can show easily that (7.4) has no solution. On the other hand, if $h^2h^* \sim 1$, $hh^{*2} \sim 1$ or $h \sim h^*$, we can reduce all possible cases to the type

$$\text{(VIII)} \qquad (h_1, h_2, \cdots, h_6) = (1, c_2, c_3, h, c_4h, c_5h^{-2}) ,$$

where $h \not\sim 1$.

For the case $M \geqq 4$, we may assume $h_1 \sim h_2 \sim h_3 \sim h_4$. By the similar way as above, we have the only cases (α) $h_5 \sim h_6 \sim 1$, (β) $h_5h_6 \sim 1$ ($h_5 \not\sim 1$), (γ) $h_5 \not\sim 1$ and $h_6 \not\sim 1$ and (δ) $h_5 \sim h_6 \sim 1$. But, for the case (β), we see always $f(C^n) \subset H_6$, which contradicts the assumption. Thus, we obtain one of the following types;

$$\text{(IX)} \qquad (h_1, h_2, \cdots, h_6) = (1, c_2, c_3, c_4, h, c_5h)$$

$$\text{(X)} \qquad (h_1, h_2, \cdots, h_6) = (1, c_2, c_3, c_4, h, c_5h^{-1})$$

$$\text{(XI)} \qquad (h_1, h_2, \cdots, h_6) = (1, c_2, c_3, c_4, c_5, c_6) ,$$

where $h \not\sim 1$.

As is easily seen, one of these eleven types cannot be constructed from the others by changing indices $1, 2, \cdots, 6$, by multiplying all h_i by a common function in $\mathscr{H}^*$ or by choosing other generators h, h^*. And, it is not difficult to find concrete examples of meromorphic maps f and g and hyperplanes of these types.

Summalizing them, we give

THEOREM 7.5. *Let f, g be meromorphic maps of C^n into $P_2(C)$ such that $f(C^n) \not\subset H_i$, $g(C^n) \not\subset H_i$ and $\nu(f, H_i) = \nu(g, H_i)$ for six hyperplanes H_i in general position. Then, after a suitable change of indices, the functions h_i defined by* (3.3) *as in* § 3 *are related with one of the above types* (I) $\sim$ (XI).

As a consequences of this, we can prove

COROLLARY 7.6. *Uuder the same assumption of Theorem* 7.5, *it is possible to choose homogeneous coordinates $w_1 : w_2 : w_3$ such that, for suitable admissible representations $f = f_1 : f_2 : f_3$ and $g = g_1 : g_2 : g_3$, f and g are related with*

$$
(7.7) \qquad \begin{aligned} g_1 &= f_1 \\ g_2 &= cf_2 \\ g_3 &= P(f_1, f_2, f_3)/Q(f_1, f_2, f_3) \end{aligned}
$$

where $P(w_1, w_2, w_3)$ *and* $Q(w_1, w_2, w_3)$ *are homogeneous polynominals of degree* $\leqq 3$ *and* $\leqq 2$ *respectively and* c *is a non-zero constant.*

Proof. Let each H_i be given as (3.2). Assume that $\{h_i\}$ is of type (I). Without loss of generality, we may assume $a_3^1 = a_4^2 = a_5^3 = 1$, $a_3^2 = a_3^3 = a_4^1 = a_4^3 = a_5^1 = a_5^2 = 0$. We have then

$$
g_3(a_2^1 f_1 + a_2^2 c_3 f_2 + a_2^3 g_3) = f_3(a_2^1 f_1 + a_2^2 f_2 + a_2^3 f_3)
$$

by the identities (3.6) for $i = 2, 3, 4, 5$, and

$$
\begin{aligned} &(a_1^1 f_1 + a_1^2 c_3 f_2 + a_1^3 g_3)(a_6^1 f_1 + a_6^2 c_3 f_2 + a_6^3 g_3) \\ &\qquad = c_1(a_1^1 f_1 + a_1^2 f_2 + a_1^3 f_3)(a_6^1 f_1 + a_6^2 f_2 + a_6^3 f_3) \end{aligned}
$$

by (3.6) for $i = 1, 3, 4, 6$. From these two relations we can conclude easily the relations of the type (7.7). In the same manner, it is easy to obtain the desired relations for the other types of $\{h_i\}$. We have thus Corollary 7.6.

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