

ON PRESERVING THE KOBAYASHI PSEUDODISTANCE

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§ 0.

If X is a complex space, the Kobayashi pseudo-distance d_X is an intrinsic pseudometric on X defined as follows. If p and q are points of X , a chain α from p to q consists of intermediate points p_0, \dots, p_r with $p_0 = p$ and $p_r = q$ together with maps f_i of the unit disc $D = \{z \in \mathbb{C}^1 \mid |z| < 1\}$ into X and points a_i and b_i in D such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $i = 1, \dots, r$. If $d_D(a, b)$ denotes the hyperbolic distance between the points a and b in the unit disc, then the length of the chain α is defined as $|\alpha| = d_D(a_1, b_1) + d_D(a_2, b_2) + \dots + d_D(a_r, b_r)$. The pseudo-distance between p and q is then defined as the infimum of the lengths of all chains from p to q : $d_X = \inf \{|\alpha| \mid \alpha \text{ a chain from } p \text{ to } q\}$. It is easy to establish that $d_X(p, q)$ is jointly continuous in p and q and that holomorphic maps are distance decreasing—i.e. if $f: X' \rightarrow X$ is holomorphic and $f(p') = p$, $f(q') = q$ then $d_X(p, q) \leq d_{X'}(p', q')$. If d_X is an actual distance—i.e. if $d_X(p, q) \neq 0$ for $p \neq q$ —then X is said to be hyperbolic and in that case the metric topology induced by d_X coincides with the original topology of X ([1]). A general reference for this subject is Kobayashi's book [4].

If A is a closed subset of X , then the inclusion map $X - A \rightarrow X$ is holomorphic, so that $d_X(p, q) \leq d_{X-A}(p, q)$ for p and q not in A . Removing an analytic set of codimension 1 often changes the pseudo-distance radically. For instance, the pseudo-distance on $\mathbb{C}^* = \{z \in \mathbb{C} \mid z \neq 0\}$ is identically zero, but, if we remove a single point from \mathbb{C}^* , what is left is a hyperbolic space. The same sort of phenomenon generally does not occur if A is an analytic set of codimension at least 2. For instance, Kobayashi proves ([4]) that if A is closed and nowhere dense in some hyperplane section of D^n (the unit polydisc in n -space), then removing

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A does not affect the distance between points not in A —i.e. $d_{D^n-A}(p, q) = d_{D^n}(p, q)$ for p and q not in A . The principal results of this paper (Proposition 2 and Theorems 1 and 2) are generalizations of that proposition. That is, theorems to the effect that the pseudo-distance is preserved if a “small” set (generally one of codimension 2) is removed. Such a result does not hold without some restriction on the space, as is shown by the following

EXAMPLE: Let Y be a hyperbolic projective algebraic manifold (for instance, a nonsingular curve of genus greater than 1) embedded in P^n (complex projective n -space). Let $\pi: C^{n+1} - \{0\} \rightarrow P^n$ be the map which takes a point to the line containing it, and let X be the cone over Y —i.e. $X = \pi^{-1}(Y) \cup \{0\}$. Then the pseudo-distance on X is identically zero, since X is a union of lines intersecting at the origin. Let $A = \{0\}$. The space $X - A$ has non-trivial pseudo-distance since $d_{X-A}(p, q) \geq d_Y(\pi(p), \pi(q)) > 0$ if p and q do not belong to the same line through the origin. Note that X is singular with singular locus A . By choosing Y to be of large dimension, we can make the codimension of A in X as large as we wish.

The methods we use to attack the problem are different from those used by Kobayashi, and essentially consist of showing that $\text{Hol}(D, X - A)$ is dense in $\text{Hol}(D, X)$ by using the flows of vector fields to push maps $D \rightarrow X$ away from A .

§ 1

For a complex space X we will denote by $\text{Hol}(D, X)$ the set of holomorphic maps of D into X . Note that $\text{Hol}(D, X)$ depends only on the reduction of X . We equip $\text{Hol}(D, X)$ with the compact-open topology. We will use the notation $U \subset\subset X$ to indicate that \bar{U} is a compact subset of X . We wish to consider the following three properties which a closed subset A of X may have:

- I. The Kobayashi pseudo-distance on X restricts to that on $X - A$ —i.e., $d_{X-A}(p, q) = d_X(p, q)$ for $p, q \in X - A$.
- II. $\text{Hol}(D, X - A)$ is dense in $\text{Hol}(D, X)$.
- III. Every $f \in \text{Hol}(D, X)$ with $f(D) \subset\subset X$ can be connected to a $g \in \text{Hol}(D, X - A)$ by a curve in $\text{Hol}(D, X)$ which lies entirely in $\text{Hol}(D, X - A)$ except for its initial point—i.e., there is a homotopy

$H(d, t): D \times I \rightarrow X$ such that H is holomorphic in d for each fixed t , $H(d, 0) = f(d)$ and $H(d, t) \notin A$ except for $t = 0$.

Remark. It is easy to verify that the two slightly different definitions of III are equivalent.

PROPOSITION 1. III \Rightarrow II \Rightarrow I.

Proof. III \Rightarrow II. Let $M = \{f \in \text{Hol}(D, X) \mid f(D) \subset \subset X\}$. We need only show that M is dense in $\text{Hol}(D, X)$. But if $f \in \text{Hol}(D, X)$ then f_t (defined by $f_t(x) = f(tx)$) belongs to M for every $0 \leq t < 1$ and $f_t \rightarrow f$ in the topology of $\text{Hol}(D, X)$ as $t \rightarrow 1$.

II \Rightarrow I. Let p and q be two points of X not in A . Let $r = d_X(p, q)$. Choose $\varepsilon > 0$, and let $f_i: D \rightarrow X, i = 1, \dots, m$ be holomorphic maps such that $f_1(0) = p, f_i(a_i) = f_{i+1}(0), f_m(a_m) = q$ for points $a_1, \dots, a_m \in D$ satisfying $\sum_{i=1}^m d_D(0, a_i) \leq r + \varepsilon$. (We are using a reformulation of the definition given in the introduction. The reformulation is obtained by using hyperbolic translations to map half the points involved to the origin.) Suppose that g_i is a map of D into $X - A, i = 1, \dots, m$ and we put $y_i = g_i(0), z_i = g_i(a_i), y_0 = p, y_{m+1} = q, x_i = f_i(a_i) = f_{i+1}(0), x_0 = p$. Then $d_{X-A}(p, q) \leq \sum_{i=0}^m d_{X-A}(y_i, y_{i+1}) \leq \sum_{i=0}^m [d_{X-A}(y_i, z_i) + d_{X-A}(z_i, y_{i+1})] \leq \sum_{i=0}^m d_D(0, a_i) + \sum_{i=0}^m d_{X-A}(z_i, y_{i+1}) \leq r + \varepsilon + \sum_{i=0}^m d_{X-A}(z_i, y_{i+1})$. Now, if the g_i are chosen close to the f_i in $\text{Hol}(D, X)$, then the points $z_i = g_i(a_i)$ and $y_{i+1} = g_{i+1}(0)$ will both be close to $x_i = f_i(a_i) = f_{i+1}(0)$ and hence close to each other. Since d_{X-A} is continuous, $d_{X-A}(z_i, y_{i+1})$ will be small. Choose the g_i so close to the f_i in $\text{Hol}(D, X)$ that $\sum_{i=0}^m d_{X-A}(z_i, y_{i+1}) < \varepsilon$. We obtain $d_{X-A}(p, q) \leq r + 2\varepsilon = d_X(p, q) + 2\varepsilon$. Finally, letting $\varepsilon \rightarrow 0$ we obtain $d_{X-A}(p, q) \leq d_X(p, q)$. Since the other inequality $d_{X-A}(p, q) \geq d_X(p, q)$ is always satisfied, we obtain $d_{X-A}(p, q) = d_X(p, q)$. End of proof.

§ 2

In this section we obtain results for open subsets of C^n . Some of these could have been deduced as corollaries of later results but the proofs are easier to follow here and the results somewhat more detailed.

We recall that a subset A of a topological space B is said to be of first category in B if it is contained in a countable union of closed, nowhere dense subsets. We omit the (easy) proof of

LEMMA 1: Let $f: X \rightarrow Y$ be a holomorphic map between complex

spaces, and let A be of the first category in X . If X is a countable union of compact sets and $\dim_{f(x)} Y \geq \dim_x X$ (respectively, $\dim_{f(x)} Y > \dim_x X$) for every $x \in X$, then $f(A)$ (respectively, $f(X)$) is of the first category in Y .

PROPOSITION 2. *Let U be an open subset of \mathbf{C}^n . Let A be a closed subset of U which is of the first category in a nowhere dense closed analytic subset of U . Then $\text{Hol}(D, U - A)$ is dense in $\text{Hol}(D, U)$. Furthermore, if A is contained in a closed analytic subset of U of codimension ≥ 2 , then A has property III as a closed subset of U .*

Proof. Suppose A is of the first category in B , where B is a nowhere dense closed analytic subset of U . Let $M = \{f \in \text{Hol}(D, U) \mid f(D) \subset \subset U\}$. M is dense in $\text{Hol}(D, U)$ (see proof of Proposition 1). Let $g \in M$. Consider the map $G: D \times B \rightarrow \mathbf{C}^n$ defined by $G(d, b) = g(d) - b$. Since $\dim D \times B \leq n$, Lemma 1 shows that $G(D \times A)$ is of the first category in \mathbf{C}^n . In particular $G(D \times A)$ contains no neighborhood of the origin. Choose a sequence c_1, c_2, \dots , of points of \mathbf{C}^n such that $c_i \rightarrow 0$ as $i \rightarrow \infty$ and $c_i \notin G(D \times A)$. Define $g_i: D \rightarrow \mathbf{C}^n$ by $g_i(d) = g(d) - c_i$. Since $g(D) \subset \subset U$ there is a $N > 0$ such that for $i > N$, $g_i(D) \subset U$. The sequence $g_i, i > N$, has g as limit in $\text{Hol}(D, U)$ and $g_i(D) \subset U - A$ by construction. This completes the proof of the first assertion. Now suppose that A is contained in a closed analytic subset of U of codimension ≥ 2 . It suffices to consider the case where A is itself a closed analytic subset of U of codimension ≥ 2 . Let $f \in M$. Consider the map $F: D \times A \times \mathbf{C} \rightarrow \mathbf{C}^n$ defined by $F(d, a, t) = t(f(d) - a)$. Since $\dim D \times A \times \mathbf{C} \leq n$ and since $D \times A \times \mathbf{R}$ is of the first category in $D \times A \times \mathbf{C}$, Lemma 1 shows that $F(D \times A \times \mathbf{R})$ is of the first category in \mathbf{C}^n and, in particular, that it is a proper subset of \mathbf{C}^n . Let $c \in \mathbf{C}^n, c \notin F(D \times A \times \mathbf{R})$. Define $H(d, s): D \times \mathbf{C} \rightarrow \mathbf{C}^n$ by $H(d, s) = f(d) + sc$. Since $f(D)$ is relatively compact in U , $H(d, s) \in U$ for sufficiently small s (independently of d) and, by construction, $H(d, s) \notin A$ for any s and d except when $s = 0$. Obviously H provides the required homotopy. End of proof.

Property III has a certain “staying power”. Thus if X has property III for closed analytic subsets of codimension ≥ 2 , then so does any smooth holomorphic retract of X (the requirement that the retraction $r: X \rightarrow A$, where $A \subset X$, be smooth is imposed so that the inverse image under r of any analytic subset of codimension ≥ 2 is again of codimen-

sion ≥ 2). Also, if every open subset of X has property III for its analytic subsets of codimension ≥ 2 , then so does any space which is spread over X (that is, admits a locally biholomorphic map onto X). Using the fact that any Stein manifold is a smooth holomorphic retract of a tubular neighborhood of any embedding of it into C^m , one easily proves the following proposition (which is also a consequence of Theorem 2 in the next section).

PROPOSITION 3. *Let X be spread over an open subset of a Stein manifold. Then any closed analytic subset of X of codimension $2 \geq$ has property III as a closed subset of X .*

Note that the analogous proposition for Stein spaces would be false, as is shown by the example in § 0.

§ 3

In this section we show how to use the flow along vector fields to accomplish the job done by translations in § 2.

For a complex space X and a point $x \in X$ we denote by $T_x(X)$ (respectively, $TC_x(X)$) the tangent space (respectively, cone) to X at x . If $\varphi: X \rightarrow Y$ is a holomorphic mapping of complex spaces, we denote by $d\varphi_x$ the differential of φ at x . If $\varphi(x) = y$, the differential is a linear mapping $T_x(X) \rightarrow T_y(Y)$ which sends $TC_x(X)$ to $TC_y(Y)$.

By a vector field on X we will mean an \mathcal{O}_X -module homomorphism of Ω_X —the sheaf of germs of holomorphic differential forms of degree one—to \mathcal{O}_X . The collection of all vector fields on X is thus the vector space $\text{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$, which we will denote by $\Theta(X)$. (See [3] for background). Given a vector field T on X there is, locally, an associated local one-parameter group of automorphisms of X , called the flow along T . The parameter in question is complex and the flow depends holomorphically on all the variables occurring (even T). More precisely, suppose we are given a finite dimensional vector subspace V of $\Theta(X)$. Then for any relatively compact open subsets X_0 of X and V_0 of V there exists an $\varepsilon > 0$ for which a holomorphic map $\varphi: X_0 \times V_0 \times D_\varepsilon \rightarrow X$ exists, with $D_\varepsilon = \{z \in D \mid |z| < \varepsilon\}$ and $\varphi(x, T, t) = \varphi_T(x, t)$ i.e. the flow along T , for the "time" t , starting at x . One proves this by showing local existence and uniqueness (where defined) for the flow. On sufficiently small coordinate neighborhoods $X_\alpha \subset U_\alpha \subset C^m$ the result is obtained

by lifting a basis for the finite dimensional space of vector fields involved to the ambient $U_\alpha \subset \mathbb{C}^m$ and by observing that the flow on U_α thus obtained preserves the ideal sheaf of X_α . Finally one uses the uniqueness to piece together these flows, obtaining a holomorphic map $\varphi: \Omega \rightarrow X$ where Ω is the largest connected open neighborhood of $X \times V \times \{0\}$ in $X \times V \times \mathbb{C}$ on which the flow can be defined. We denote by $T(x)$ the value of a vector field at a point $x \in X$ and by $V(x) = \{T(x) | T \in V\}$ the vector subspace of $T_x(X)$ obtained by evaluating vector fields in V at x .

In the following we will show that if A is a closed analytic subset of a reduced complex space X and there are “enough” global vector fields on X then $\text{Hol}(D, X - A)$ is dense in $\text{Hol}(D, X)$. The intuitive idea here is, given a map $f: D \rightarrow X$ and a point $d \in D$ such that $f(d) \in A$, to find a global vector field which is “parallel” neither to A nor to the image of D and to use the flow of that vectorfield to push the image of D away from A at d . If $f(D) \subset \subset X$ then it suffices to find a vectorfield that does the job over all of $f(D)$. To make what we have said more precise and to avoid lengthy repetitions we make the following temporary definition:

DEF: A vector subspace W of $\Theta(X)$ is said to be *sufficiently disjoint from the tangent cone to A* if given any $a \in A$ there exist T_1 and $T_2 \in W$ such that $T_1(a)$ and $T_2(a)$ are linearly independent and no nontrivial linear combination of $T_1(a)$ and $T_2(a)$ lies in $TC_a(A)$.

With this definition the precise meaning of having “enough” global vector fields on X will be that $\Theta(X)$ is sufficiently disjoint from the tangent cone to A . First we prove a

TRANSVERSALITY LEMMA. *Let $\varphi: X \rightarrow Y$ be a holomorphic map from a nonsingular complex space X to a complex space Y , and let A be a closed analytic subspace of Y . If for every $x \in \varphi^{-1}(A)$ the image $d\varphi_x(T_x(X))$ of the tangent space contains two linearly independent vectors whose nontrivial linear combinations are never in $TC_{\varphi(x)}(A)$, then $\varphi^{-1}(A)$ is of codimension at least two.*

Proof. Let B be $\varphi^{-1}(A)$ with its reduced complex structure and let $i: B \rightarrow X$ be the inclusion. Let b be a nonsingular point of B . We have the following commutative diagram

$$\begin{array}{ccc}
 T_b(X) & \longrightarrow & T_{\varphi(b)}(Y) \\
 di_b \uparrow & & \uparrow \\
 T_b(B) & \longrightarrow & TC_{\varphi(b)}(A)
 \end{array}$$

since φ induces a map $B \rightarrow A$ and since $TC_b(B) = T_b(B)$. Since b is a nonsingular point of both B and X , the vector space codimension of $T_b(B)$ in $T_b(X)$ is the same as the codimension of B in X at b . If that codimension is ≤ 1 , then some nontrivial linear combination of *any* two vectors in $T_b(X)$ would get mapped to $TC_{\varphi(b)}(A)$, contradicting our assumption. Since the codimension of B is the infimum of its codimensions at regular points, we have the desired result. End of proof.

Given a finite dimensional subspace V of $\Theta(X)$ and the associated flow $\varphi: \Omega \rightarrow X$ where Ω is an open subset of $X \times V \times \mathbb{C}$, the middle partial derivative of φ at a point $\zeta = (x, T, t)$ of Ω is a linear map $D_2\varphi|_{\zeta}: V \rightarrow T_{\varphi(\zeta)}(X)$. We need to know that $D_2\varphi|_{\zeta}$ is, to first order, just $t \cdot$ (evaluation at $\varphi(\zeta)$).

LEMMA: For any $x \in X$ and $T \in V$, letting $\zeta = (x, T, t)$, we have

$$D_2\varphi|_{\zeta}(S) = tS(\varphi(\zeta)) + 0(t^2) \quad \text{as } t \text{ tends to } 0.$$

The bounds implied by “ $0(t^2)$ ” are locally uniform in x and T .

Proof. The desired result is local and φ can be calculated locally, so we may suppose that X is a closed analytic subspace of an open subset U of \mathbb{C}^m and even that there is a basis T_1, \dots, T_r of V that gives rise to vectorfields on X that can be lifted to vectorfields on U . Specifically, suppose that $\sum_j f_{ij}(\partial/\partial z_j)$, $i = 1, \dots, r$ are vectorfields on U that restrict to T_i on X . Let $F(x, a_1, \dots, a_r, t) = (F_j(x, a, t))$ denote the flow starting at $x \in U$ for time t along the vectorfield $\sum_{i,j} a_i f_{ij}(\partial/\partial z_j)$. By definition F satisfies

$$\text{i) } F(x, a, 0) = x$$

and

$$\text{ii) } \frac{\partial F_j}{\partial t}(x, a, t) = \sum_i a_i f_{ij}(F(x, a, t)).$$

Expanding F in powers of t we obtain

$F_j(x, a, t) = x + t \sum_i a_i f_{ij}(x) + \text{terms involving } t^2 \text{ and higher powers of } t.$

Differentiating with respect to a_i we obtain

$$\frac{\partial F_j}{\partial a_i}(x, a, t) = t f_{ij}(x) + 0(t^2)$$

and hence also

$$\frac{\partial F_j}{\partial a_i}(x, a, t) = t f_{ij}(F(x, a, t)) + 0(t^2)$$

But this is the desired result, for, if $S = b_1 T_1 + \cdots + b_r T_r$, then $D_{z\varphi}|_{\zeta}(S) = \sum_i b_i \left(\frac{\partial F_j}{\partial a_i} \Big|_{(x, a, t)} \right) = t \sum_i b_i f_{ij}(F(x, a, t)) + 0(t^2) = tS(\varphi(\zeta)) + 0(t^2)$. End of proof.

We are now in a position to prove

THEOREM 1. *Let X be a complex space and let A be a closed analytic subspace of X . If $\Theta(X)$ is sufficiently disjoint from the tangent cone to A , then A has property III as a closed subset of X (see §1 for the definition of property III).*

Proof. Let $f: D \rightarrow X$ be a map of the disc into X such that $f(D) \subset \subset X$. As the first step in the proof we show that there is a finite dimensional subspace of Θ which is "sufficiently disjoint from the relevant portion of the tangent cone to A ". Let $a \in A$. By hypothesis we can choose vectorfields T_1, T_2 in $\Theta(X)$ such that $\alpha T_1(a) + \beta T_2(a) \notin TC_a(A)$ for $|\alpha| + |\beta| = 1$. Since $\{(\alpha, \beta) \mid |\alpha| + |\beta| = 1\}$ is compact and $TC(A) = \bigcup_{a \in A} TC_a(A)$ is a closed analytic subset of $T(X) = \bigcup_{x \in X} T_x(X)$ (with its natural analytic structure) and since the map $x \mapsto \alpha T_1(x) + \beta T_2(x)$ is continuous, there is a neighborhood N_a of a in X such that $\alpha T_1(x) + \beta T_2(x) \notin TC(A)$ for $|\alpha| + |\beta| = 1$ and $x \in N_a$. Since $A \cap \overline{f(D)}$ is compact we can choose a finite number of points a_1, \dots, a_i of A such that the corresponding neighborhoods N_{a_i} cover $A \cap \overline{f(D)}$. Let V be the finite dimensional vector subspace of $\Theta(X)$ spanned by $T_1^{(1)}, T_2^{(1)}, T_1^{(2)}, T_2^{(2)}, \dots, T_2^{(i)}$ where $T_1^{(j)}$ and $T_2^{(j)}$ are the vectorfields that were chosen at the point a_j to define the neighborhood N_{a_j} . V is, in an obvious sense, sufficiently disjoint from the tangent cone to A over $A \cap \overline{f(D)}$.

Let $\varphi: \Omega \rightarrow X$ be the flow associated to V , where Ω is an open neighborhood of $X \times V \times \{0\}$ in $X \times V \times \mathbb{C}$. Let $\Omega' = (f \times 1 \times 1)^{-1}(\Omega) =$

$\{(d, T, t) \in D \times V \times \mathbf{C} \mid (f(d), T, t) \in \Omega\}$, and define $\varphi': \Omega' \rightarrow X$ by $\varphi'(d, T, t) = \varphi(f(d), T, t)$. Choose some norm $||$ on V . For $\varepsilon > 0$ let $V_\varepsilon = \{T \in V \mid |T| < \varepsilon\}$. We wish to show that there is an $\varepsilon > 0$ such that $D \times V_\varepsilon \times D_\varepsilon \subset \Omega'$ and such that the restriction of φ' to $D \times V_\varepsilon \times (D_\varepsilon - \{0\})$ satisfies the hypotheses of the transversality lemma (with $X = D \times V_\varepsilon \times (D_\varepsilon - \{0\})$, $Y = X$, $A = A$). Observe first that (by the lemma preceding this proof) the map $\psi: (\Omega' - (D \times V \times \{0\})) \times V \rightarrow T(X)$ given by $\psi((d, T, t), S) = (1/t)D_z\varphi'|_{(d, T, t)}(S)$ can be extended to a map $\tilde{\psi}: \Omega' \times V \rightarrow T(X)$ by putting $\tilde{\psi}((d, T, 0), S)$ equal to $S(f(d))$. Next let us construct for each $x \in X$ a neighborhood N_x of x and an $\varepsilon_x > 0$ as follows:

Case I. $x \notin A$. Then $\varphi(x, 0, 0) \notin A$. By continuity there exist a neighborhood N_x of x and an $\varepsilon_x > 0$ such that $(y, T, t) \in \Omega$ and $\varphi(y, T, t) \notin A$ for $y \in N_x$ and $|T|, |t| < \varepsilon$.

Case II. $x \in A \cap \overline{f(D)}$. By the construction of V , there exist $T_1, T_2 \in V$ such that no nontrivial linear combination of $T_1(x)$ and $T_2(x)$ belongs to $TC_x(A)$. We can restate this as $\tilde{\psi}((x, 0, 0), \alpha T_1 + \beta T_2) \notin TC(A)$ for $|\alpha| + |\beta| = 1$ (since ψ is linear in its last variable). By continuity there exist a neighborhood N_x of x and an $\varepsilon_x > 0$ such that $(y, T, t) \in \Omega$ and $\tilde{\psi}((y, T, t), \alpha T_1 + \beta T_2) \notin TC(A)$ for $|\alpha| + |\beta| = 1$, $y \in N_x$, $|T| < \varepsilon_x$ and $|t| < \varepsilon_x$. Note that, by the definition of $\tilde{\psi}$, this says that if $t \neq 0$ and $\zeta = (y, T, t)$ then $D_z\varphi'|_\zeta(T_1)$ and $D_z\varphi'|_\zeta(T_2)$ are linearly independent and that no nontrivial linear combination of them lies in $TC(A)$.

Case III. $x \in A - \overline{f(D)}$. Let $N_x = X - \overline{f(D)}$ and $\varepsilon_x = 1$.

Since $\overline{f(D)}$ is compact we can choose a finite number of points $x_1, \dots, x_n \in X$ such that the corresponding neighborhoods N_{x_i} cover $f(D)$. We can discard any points in $A - \overline{f(D)}$ and still have the same property. Let $\varepsilon = \min(\varepsilon_{x_1}, \dots, \varepsilon_{x_n})$. It is clear that ε has the desired property by construction.

Let $\tilde{\varphi}$ be the restriction of φ' to $D \times V_\varepsilon \times (D_\varepsilon - \{0\})$. Applying the transversality lemma we conclude that $\tilde{\varphi}^{-1}(A)$ is of codimension at least two, i.e., $\dim \tilde{\varphi}^{-1}(A) \leq \dim V_\varepsilon = \text{vector space dimension of } V$. Let $\pi: \tilde{\varphi}^{-1}(A) \rightarrow V_\varepsilon$ be the projection map and let Z be the "real section of $\tilde{\varphi}^{-1}(A)$ ", i.e., $Z = \tilde{\varphi}^{-1}(A) \cap (D \times V \times \mathbf{R})$. We claim that $\pi(Z)$ is of the first category in V_ε . Before proving this let us see how we can use it to complete the proof of the Theorem. If $\pi(Z)$ is of the first category in V_ε , then certainly $\pi(Z) \neq V_\varepsilon$. Choose $T \in V_\varepsilon$ which is not in the image of π . $T \notin \pi(Z)$ says precisely that $\varphi'(d, T, t) \notin A$ for $d \in D$ and $t \in \mathbf{R}$, $0 < |t| < \varepsilon$. Thus

$H(d, t) = \varphi'(d, T, t)$, $0 \leq t \leq \varepsilon/2$, provides the desired homotopy. Finally, to show that $\pi(Z)$ is of the first category, observe that since $\tilde{\varphi}^{-1}(A)$ has at most countably many irreducible components, it suffices to show that $\pi(W \cap Z)$ is of the first category for every irreducible component W of Z . Let $t: W \subset D \times V \times C \rightarrow C$ be the third projection. Since W is irreducible and t is holomorphic, t is either an open mapping or constant. If t is an open mapping then, since R is nowhere dense in C , $W \cap Z = t^{-1}(R)$ is nowhere dense in W , and we may apply lemma 1 of §2 to conclude that $\pi(W \cap Z)$ is nowhere dense in V_* . If t is constant, let t_0 be its unique value. Then W is contained in the inverse image of A under the map $D \times V_* \times \{t_0\} \rightarrow X$ induced by φ' . The argument given above to show that $\tilde{\varphi}$ satisfies the hypotheses of the transversality lemma also shows that this map does as well (since the "same" partial derivative produces the required tangent vectors in either case). Hence $\dim W \leq 1 + \dim V - 2 < \dim V_*$ and we again conclude by applying lemma 1 of §2. End of proof.

Remark. One can see why $\pi(Z)$ is of the first category by observing that $(D \times V \times R) \cap \Omega'$ is a *real* analytic manifold and that an application of transversality and a count of *real* dimensions show that $\pi(Z)$ should be lower dimensional than V_* .

As a corollary we obtain

THEOREM 2. *Let X be a complex manifold whose tangent bundle is spanned by its global sections. If A is any analytic subset of X of codimension ≥ 2 then A has property III as a closed subset of X . In particular $\text{Hol}(D, X - A)$ is dense in $\text{Hol}(D, X)$ and the restriction to $X - A$ of the Kobayashi pseudo-distance on X is the Kobayashi pseudo-distance on $X - A$.*

Proof. We will show that $\Theta(X)$ is sufficiently disjoint from the tangent cone to A . Let $a \in A$. If a is a nonsingular point of A , choose T_1 and T_2 so that $T_1(a)$ and $T_2(a)$ span a two dimensional subspace of $T_a(X)$ complementary to $T_a(A)$. If a is a singular point, $TC_a(A)$ is an algebraic cone in $T_a(X)$ of dimension equal to $\dim_a(A)$. By one well known definition of dimension there is a linear subspace, of codimension equal to $\dim_a(A)$, lying $T_a(X)$ and whose intersection with $TC_a(A)$ has $\{0\}$ as an isolated point. But then, since $TC_a(A)$ is a cone, the intersection of the linear subspace and $TC_a(A)$ reduces to $\{0\}$. Simply choose

T_1 and T_2 so that $T_1(a)$ and $T_2(a)$ are linearly independent and lie in that linear subspace. End of proof.

Remarks. 1) The reason for requiring X to be nonsingular does not lie in the proof given above. The reason is the fact, due to Rossi ([5]), that if X is reduced and its tangent spaces are spanned by the values of vectorfields, then X must be nonsingular.

2) Any compact manifold whose tangent bundle is spanned by its global sections necessarily has trivial Kobayashi pseudo-distance. For, if X is compact, $\theta(X)$ is finite dimensional and vectorfields generate complete one parameter groups and the flow becomes a map $\varphi: X \times \theta(X) \times \mathbf{C} \rightarrow X$. If $x_0 \in X$ then $(T, t) \rightarrow \varphi(x_0, T, t)$ is holomorphic and its image contains a neighborhood of x_0 . Since $x_0 \times \theta(X) \times \mathbf{C}$ has trivial pseudo-distance, so does that neighborhood. Since x_0 was arbitrary, it follows that X has trivial pseudo-distance.

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