H. Sato
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ON PERIODS OF MEROMORPHIC EICHLER INTEGRALS

HIROKI SATO

0. Introduction.

In this paper we treat cohomology groups $H^1(G, \boldsymbol{C}^{2q-1}, M)$ of meromorphic Eichler integrals for a finitely generated Fuchsian group G of the first kind. According to L. V. Ahlfors [2] and L. Bers [4], $H^1(G, \boldsymbol{C}^{2q-1}, M)$ is the space of periods of meromorphic Eichler integrals for G. In the previous paper [8], we had period relations and inequalities of holomorphic Eichler integrals for a certain Kleinian groups.

Let G be a Fuchsian group of the first kind which is generated by $\{A_1, B_1, \cdots, A_g, B_g\}$ with a relation $\prod_{j=1}^{g} B_j^{-1}A_j^{-1}B_jA_j = 1$. Set $S_j = B_j^{-1}A_j^{-1}B_jA_j$, $j = 1, \cdots, g$. We denote by $H_0^1(G, \boldsymbol{C}^{2q-1}, M)$ the space of cohomology classes Z with $Z_{S_j} = 0, j = 1, \cdots, g$. In general, $Z \in H^1 \cdot (G, \boldsymbol{C}^{2q-1}, M)$ is represented by direct sum of Eichler cohomology and Bers cohomology, that is, $Z = \alpha(\mathfrak{f}) + \beta^*(\mathfrak{g})$ (I. Kra [6], for notations see § 1). We denote by $H_1^1(G, \boldsymbol{C}^{2q-1}, M)$ the space of cohomology classes $Z = \alpha(\mathfrak{f}) + \beta^*(\mathfrak{g})$ with $\alpha_A(\mathfrak{f}(z)) = \overline{\beta_A^*(\mathfrak{g}(\bar{z}))}$ for $A \in G$ and $z \in U$, the upper half plane. We shall study some properties of the spaces $H_0^1(G, \boldsymbol{C}^{2q-1}, M)$ and $H_1^1(G, \boldsymbol{C}^{2q-1}, M)$. The main result is Theorem 3, that is, if E is a meromorphic Eichler integral whose S_j periods Z_j are all zero, $j = 1, \cdots, g$, then

$$ {}^t\tilde{Z}_{A_j^{-1}}I'_{n+1}Z_{B_j} - {}^t\tilde{Z}_{B_j^{-1}}I'_{n+1}Z_{A_j} = 0 \quad \text{and} \quad \sqrt{-1}({}^t\bar{\tilde{Z}}_{A_j^{-1}}I'_{n+1}Z_{B_j} - {}^t\bar{\tilde{Z}}_{B_j^{-1}}I'_{n+1}Z_{A_j}) $$

are real numbers and they may be positive, negative and zero (for notations see § 1).

In § 1 we state notations and preliminaries. In § 2 we enumerate theorems. In § 3 we state some lemmas which is necessary to prove the theorems. In § 4 we prove the theorems. In appendix, we state representations of period relation and inequalities by means of matrices.

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1. Notation.

Throughout this paper Γ denotes a non-elementary finitely generated Kleinian group and G denotes a Fuchsian group of the first kind which is generated by $\{A_1, B_1, \cdots, A_g, B_g\}$ with a relation $\prod_{j=1}^{g} B_j^{-1}A_j^{-1}B_jA_j = 1$. Let Ω be the region of discontinuity of Γ and let Δ be a component of Ω. We denote by Λ the limit set, $\lambda(z)|dz|$ the Poincaré metric on Ω. We denote by U and L the upper and lower half planes, respectively. Let $q \geqq 1$ be an integer.

We denote by $\boldsymbol{R}^n$ and $\boldsymbol{C}^n$ n dimensional vector spaces over $\boldsymbol{R}$ and $\boldsymbol{C}$, respectively, $n \geqq 0$ being an integer. We regard an element in $\boldsymbol{R}^n(\boldsymbol{C}^n)$ as a matrix with n rows and 1 column. We consider an element of Γ as a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. We denote by $GL(m, \boldsymbol{C})$ the group of $m \times m$ invertible matrices over $\boldsymbol{C}$. Let $\begin{pmatrix} u \\ v \end{pmatrix}$ be a vector in $\boldsymbol{C}^2$. For each $n = 2q - 2$, we denote by $\begin{pmatrix} u \\ v \end{pmatrix}^n$ the vector in $\boldsymbol{C}^{n+1}$ whose components are $u^n, u^{n-1}v, \cdots, uv^{n-1}, v^n$, where $\begin{pmatrix} u \\ v \end{pmatrix}^0 = 1$. For $A \in \Gamma$ we set $\begin{pmatrix} u_A \\ v_A \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}$ and define $M(A) \in GL(n+1, \boldsymbol{C})$ by

$$\begin{pmatrix} u_A \\ v_A \end{pmatrix}^n = M(A)\begin{pmatrix} u \\ v \end{pmatrix}^n .$$

For $m \times n$ matrix $N = (a_{ij}), (i = 1, \cdots, m; j = 1, \cdots, n)$, matrices $\bar{N}$ and $\tilde{N}$ are defined by $\bar{N} = (\bar{a}_{ij})$ and $\tilde{N} = (a_{m-i+1, n-j+1})$, respectively, where $\bar{a}_{ij}$ is the complex conjugate of a_{ij}. We denote by I_n the $n \times n$ identity matrix. We define $(n+1) \times (n+1)$ matrix I'_{n+1} and $n \times n$ matrix I''_n by

$$I'_{n+1} = \begin{bmatrix} {}_nC_0 & & & & & 0 \\ & -{}_nC_1 & & & & \\ & & \ddots & & & \\ & & & (-1)^q {}_nC_q & & \\ & & & & \ddots & \\ 0 & & & & & {}_nC_n \end{bmatrix}$$

and

$$I_n'' = \begin{bmatrix} {}_nC_0 & & & & & & \\ & -{}_nC_1 & & & 0 & & \\ & & \ddots & & & & \\ & & & (-1)^{q-2}{}_nC_{q-2} & & & \\ & 0 & & & (-1)^q{}_nC_q & & \\ & & & & & \ddots & \\ & & & & & & {}_nC_n \end{bmatrix},$$

respectively, where ${}_nC_j = n!/(n-j)!j!$. We define the product of matrices ${}^t(u_1, u_2, \cdots, u_m)$ and $(v_1, v_2, \cdots, v_m)$ by setting

$${}^t(u_1, u_2, \cdots, u_m)(v_1, v_2, \cdots, v_m) = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots u_1v_m \\ u_2v_1 & u_2v_2 & \cdots u_2v_m \\ \cdots & \cdots & \cdots \\ u_mv_1 & u_mv_2 & \cdots u_mv_m \end{bmatrix}.$$

A mapping $\chi: \Gamma \to \boldsymbol{C}^{2q-1}$ is called a cocycle if $\chi_{AB} = \chi_A + M(A)\chi_B$ for $A, B \in \Gamma$. A cocycle $\chi: \Gamma \to \boldsymbol{C}^{2q-1}$ is called a coboundary if there exists $V \in \boldsymbol{C}^{2q-1}$ such that $\chi_A = V - M(A)V$ for any $\chi_A \in \boldsymbol{C}^{2q-1}, A \in \Gamma$. Then the first cohomology group $H^1(\Gamma, \boldsymbol{C}^{2q-1}, M)$ is the space of cocycles factored by the space of coboundaries.

A holomorphic function ϕ on Δ is called an automorphic form of weight $(-2q)$ on Δ for $\Gamma, q \geqq 1$, if $\phi(Az)A'(z)^q = \phi(z)$ for all $A \in \Gamma$. For $q \geqq 2$, an automorphic form of weight $(-2q)$ on Δ is called integrable if

$$\iint_{\Delta/\Gamma} \lambda(z)^{2-q} |\phi(z)| \, dxdy < \infty .$$

We denote by $A_q(\Delta, \Gamma)$ the Banach space of integrable automorphic forms on Δ. The form ϕ is called bounded if

$$\sup \{\lambda(z)^{-q} |\phi(z)| \, | \, z \in \Delta\} < \infty .$$

The Banach space of bounded automorphic form on Δ is denoted by $\boldsymbol{B}_q(\Delta, \Gamma)$. For $\phi \in A_q(\Delta, \Gamma)$ and $\psi \in \boldsymbol{B}_q(\Delta, \Gamma)$, we define Petersson inner product by

$$(\phi, \psi) = \iint_{\Delta/\Gamma} \lambda(z)^{2-2q} \phi(z)\overline{\psi(z)} dxdy , \qquad q \geqq 2 .$$

For $q=1$, we shall interpret $A_1(\Delta,\Gamma)$ and $\boldsymbol{B}_1(\Delta,\Gamma)$ as the Hilbert space of square integrable automorphic forms of weight (-2) with inner product defined by

$$(\phi,\psi)=\iint_{\Delta/\Gamma}\phi(z)\overline{\psi(z)}dxdy\ .$$

A holomorphic function E on Δ_1 is called a holomorphic Eichler integral of order $(1-q)$ on Δ_1 if $E(Az)A'(z)^{1-q}-E(z)\in\Pi_{2q-2}$ on Δ_1, for all $A\in\Gamma$, where Π_{2q-2} is the vector space of polynomials of degree at most $2q-2$ and $\Delta_1=\bigcup_{A\in\Gamma}A(\Delta)$. We define a period of E for $A\in\Gamma$ by setting

$$\mathrm{pd}_A\,E(z)=E(Az)A'(z)^{1-q}-E(z)\ ,\qquad z\in\Delta_1\ .$$

We shall say that Eichler integral E of order $(1-q)$ is bounded if $\phi=D^{2q-1}E\in\boldsymbol{B}_q(\Delta_1,\Gamma)$, where D means differentiation with respect to z. $\boldsymbol{E}_{1-q}(\Delta_1,\Gamma)$ denotes the space of bounded Eichler integrals modulo Π_{2q-2}.

Let $f\in\boldsymbol{E}_{1-q}(\Delta_1,\Gamma)$ and E a representative of f and set $D^{2q-1}E=\phi$. We set

$$f_{n-j}(z)=\sum_{k=0}^{j}(-1)^k(j\,!/(j-k)\,!)z^{j-k}D^{2q-2-k}E(z)$$

and set

$$\mathfrak{f}(z)=\begin{bmatrix}f_0(z)\\ f_1(z)\\ \vdots\\ f_n(z)\end{bmatrix}.$$

We call $\mathfrak{f}(z)$ a column function vector of length $n+1$ associated with E (or f). Then we have

$$E(z)=(1/n\,!)^t\mathfrak{f}(z)I'_{n+1}\binom{1}{z}^n\ ,\qquad z\in\Delta_1\qquad(\text{Sato [8]})\ .$$

For each $A\in\Gamma$ we define X_A by

$$X_A=\mathfrak{f}(Az)-M(A)\mathfrak{f}(z)$$

and denote it by $\mathrm{pd}_A(\mathfrak{f})$. We call X_A period of $\mathfrak{f}$ for $A\in\Gamma$. The mapping $A\to X_A$ satisfies $X_{AB}=X_A+M(A)X_B$ for any $A,B\in\Gamma$, as is easily seen. Then a cohomology class is defined, which depends only on f and not E. We denote by $E_{1-q}(\Delta_1, \Gamma, M)$ the space of all $\mathfrak{f}(z)$ modulo C^{2q-1}. By the obvious way we may define a mapping $\alpha: E_{1-q}(\Delta_1, \Gamma, M) \to H^1(\Gamma, C^{2q-1}, M)$ as follows. Let $\mathfrak{f} \in E_{1-q}(\Delta_1, \Gamma, M)$. We define α by setting $\alpha_A(\mathfrak{f}(z)) = X_A$ for $A \in \Gamma$.

If $a_1, a_2, \cdots, a_{2q-1}$ are distinct points in Λ, and $\psi \in B_q(\Delta, \Gamma)$, then we call

$$\frac{(z-a_1)\cdots(z-a_{2q-1})}{2\pi i}\iint_{\Omega}\frac{\lambda(\zeta)^{2-2q}\overline{\psi(\zeta)}d\zeta\wedge\overline{d\zeta}}{(\zeta-z)(\zeta-a_1)\cdots(\zeta-a_{2q-1})},$$

$z \in C$, $q \geqq 2$, a potential for ψ, and denote it by $\operatorname{Pot}(\psi)$. For $A \in \Gamma$, we define a period of $\operatorname{Pot}(\psi)$ by setting

$$\mathrm{pd}_A \operatorname{Pot}(\psi)(z) = \operatorname{Pot}(\psi)(Az)A'(z)^{1-q} - \operatorname{Pot}(\psi)(z), \qquad z \in C.$$

It is easily seen that $\operatorname{Pot}(\psi) \mid \Omega - \Delta_1 \in E_{1-q}(\Omega - \Delta_1, \Gamma)$ for $\psi \in B_q(\Delta_1, \Gamma)$. We set

$$g_{n-j}(z) = \sum_{k=0}^{j}(-1)^k(j!/(j-k)!)z^{j-k}D^{2q-2-k}\operatorname{Pot}(\psi)(z), \qquad z \in \Omega - \Delta_1.$$

We set

$$\mathfrak{g}(z) = \begin{pmatrix} g_0(z) \\ g_1(z) \\ \vdots \\ g_n(z) \end{pmatrix}.$$

We call $\mathfrak{g}(z)$ a column function vector of length $n+1$ associated with $\operatorname{Pot}(\psi)$ (or ψ). Then

$$\operatorname{Pot}(\psi)(z) = (1/n!)\mathfrak{g}(z)I'_{n+1}\binom{1}{z}^n, \qquad z \in \Omega - \Delta_1 \qquad \text{(Sato [8])}.$$

We denote by $L_\infty(\Delta_1, \Gamma, M)$ the space of all $\mathfrak{g}$ modulo C^{2q-1}. For each $A \in \Gamma$, we define Y_A by setting

$$Y_A = \mathfrak{g}(Az) - M(A)\mathfrak{g}(z), \qquad z \in \Omega - \Delta_1$$

and denote it by $\mathrm{pd}_A(\mathfrak{g})$. The mapping $A \to Y_A$ satisfies $Y_{AB} = Y_A + M(A)Y_B$, for any $A, B \in \Gamma$, as easily seen. Then a cohomology class is defined, which depends only on ψ. The definition Y_A applies to the case $\Omega - \Delta_1 \neq \phi$. Noting the Remark after Lemma 4 in [8], this function for the remaining case be defined. We define a mapping $\beta^*: L_\infty(\Delta_1, \Gamma, M) \to$

$H^1(\Gamma, \boldsymbol{C}^{2q-1}, M)$ as follows. Let $\mathfrak{g} \in \boldsymbol{L}_\infty(\Delta_1, \Gamma, M)$. We define β^* by setting $\beta_A^*(\mathfrak{g}) = Y_A$ for $A \in \Gamma$.***

Let G be a Fuchsian group of the first kind which is generated by $\{A_1, B_1, \cdots, A_g, B_g\}$ with a relation $\prod_{j=1}^g B_j^{-1}A_j^{-1}B_jA_j = 1$. Set $S_j = B_j^{-1}A_j^{-1}B_jA_j$, $j = 1, \cdots, g$. We denote by $H_0^1(G, \boldsymbol{C}^{2q-1}, M)$ the subspace of $H^1(G, \boldsymbol{C}^{2q-1}, M)$ whose elements are all cohomology classes Z such that $Z_{S_j} = 0$, $j = 1, \cdots, g$, that is Z_{S_j} are cohomologous to zero. For any $Z \in H^1(G, \boldsymbol{C}^{2q-1}, M)$,

$$Z = \alpha(\mathfrak{f}) + \beta^*(\mathfrak{g}) ,$$

for $\mathfrak{f} \in \boldsymbol{E}_{1-q}(U, G, M)$ and $\mathfrak{g} \in \boldsymbol{L}_\infty(U, G, M)$ by Kra [6]. We denote by $H_1^1(G, \boldsymbol{C}^{2q-1}, M)$ the subspace of $H^1(G, \boldsymbol{C}^{2q-1}, M)$ whose elements are all cohomology classes Z such that $\alpha_A(\mathfrak{f}(z)) = \overline{\beta_A^*(\mathfrak{g}(\bar{z}))}$, for every $A \in G$ and $z \in U$. We denote by $\boldsymbol{E}_{1-q}^0(U, G, M)$ and $\boldsymbol{L}_\infty^0(U, G, M)$ the subspaces of $\boldsymbol{E}_{1-q}(U, G, M)$ and $\boldsymbol{L}_\infty(U, G, M)$ formed by all $\mathfrak{f}$ and all $\mathfrak{g}$ which satisfy $\alpha_{S_j}(\mathfrak{f}) = 0$ and $\beta_{S_j}^*(\mathfrak{g}) = 0$, $j = 1, \cdots, g$, respectively. We define $\boldsymbol{E}_{1-q}^{01}(U, G, M)$ and $\boldsymbol{E}_{1-q}^{02}(U, G, M)$ by setting

$$\boldsymbol{E}_{1-q}^{01}(U, G, M) = \{\mathfrak{f} \in \boldsymbol{E}_{1-q}(U, G, M) \mid \operatorname{Re} \alpha_{S_j}(\mathfrak{f}) = 0, j = 1, \cdots, g\}$$

and

$$\boldsymbol{E}_{1-q}^{02}(U, G, M) = \{\mathfrak{f} \in \boldsymbol{E}_{1-q}(U, G, M) \mid \operatorname{Im} \alpha_{S_j}(\mathfrak{f}) = 0, j = 1, \cdots, g\} ,$$

respectively. Similarly we define $\boldsymbol{E}_{1-q}^{01}(U, G)$ and $\boldsymbol{E}_{1-q}^{02}(U, G)$ by setting

$$\boldsymbol{E}_{1-q}^{01}(U, G) = \{E \in \boldsymbol{E}_{1-q}(U, G) \mid \operatorname{Re} \alpha_{S_j}(\mathfrak{f}) = 0, j = 1, \cdots, g\}$$

and

$$\boldsymbol{E}_{1-q}^{02}(U, G) = \{E \in \boldsymbol{E}_{1-q}(U, G) \mid \operatorname{Im} \alpha_{S_j}(\mathfrak{f}) = 0, j = 1, \cdots, g\} ,$$

respectively, where $\mathfrak{f}$ is a column function vector associated with E. We define $\boldsymbol{B}_q^{01}(U, G)$ and $\boldsymbol{B}_q^{02}(U, G)$ as follows.

$$\boldsymbol{B}_q^{01}(U, G) = \{\phi \in \boldsymbol{B}_q(U, G) \mid \operatorname{Re} \beta_{S_j}^*(\mathfrak{g}) = 0, j = 1, \cdots, g\}$$

and

$$\boldsymbol{B}_q^{02}(U, G) = \{\phi \in \boldsymbol{B}_q(U, G) \mid \operatorname{Im} \beta_{S_j}^*(\mathfrak{g}) = 0, j = 1, \cdots, g\}$$

where $\mathfrak{g}$ is a column function vector associated with ϕ.

*** In the case where Γ contains parabolic elements, we may similarly define $\mathfrak{f}$, $\mathfrak{g}$, $\cdots$ as above (see Sato [8]).

By a similar method as above we define a meromorphic Eichler integral, $M_{1-q}(\Delta_1, \Gamma)$ the space of meromorphic Eichler integrals modulo Π_{2q-2}, the space $M_{1-q}(\Delta_1, \Gamma, M)$ and a mapping $\alpha: M_{1-q}(\Delta_1, \Gamma, M) \to H^1(\Gamma, \boldsymbol{C}^{2q-1}, M)$.

2. The main results.

In this section we state Theorems. Throughout this section let G be a Fuchsian group of the first kind which is generated by $\{A_1, B_1, \cdots, A_g, B_g\}$ with a relation $\prod_{j=1}^{g} B_j^{-1}A_j^{-1}B_jA_j = 1$. Set $S_j = B_j^{-1}A_j^{-1}B_jA_j$ and $T_j = S_j \cdots S_1$ $(j = 1, \cdots, g)$. We denote by ${}^tM(A)$ transposed matrix of $M(A), A \in G$. At first we write the main results in the previous paper [8] in the case of Fuchsian groups. Let $X, Y \in H^1(G, \boldsymbol{C}^{2q-1}, M)$. We define $\Phi_1(X, Y), \Phi_2(X, Y)$ and $\Phi_3(X, Y)$ by setting

$$\Phi_1(X, Y) = \sum_{j=1}^{g} ({}^t\tilde{X}_{A_j^{-1}} I'_{n+1} Y_{B_j} - {}^t\tilde{X}_{B_j^{-1}} I'_{n+1} Y_{A_j})$$

$$\Phi_2(X, Y) = \sum_{j=1}^{g} {}^t(\tilde{X}_{A_j} - \tilde{X}_{B_j^{-1}}) I'_{n+1} M(A_j) Y_{T_{j-1}}$$

and

$$\Phi_3(X, Y) = \sum_{j=1}^{g} {}^t(\tilde{X}_{A_j^{-1}} - \tilde{X}_{B_j}) I'_{n+1} M(B_j) Y_{T_j} ,$$

respectively. We define $\Phi_j(\overline{X}, Y), \Phi_j(X, \overline{Y})$ and $\Phi_j(\overline{X}, \overline{Y})$, $j = 1, 2, 3$, by the same way as above. We set $\Phi = \Phi_1 + \Phi_2 + \Phi_3$.

THEOREM A. (*Corollary* 1 *to Theorem* 2 *in* [8]). *Let* $f_1, f_2 \in \boldsymbol{E}_{1-q}(U, G)$, $p \geqq 1$ *and* E_1, E_2 *arbitrary representatives of* f_1 *and* f_2, *respectively. Set* $X_A^{(1)} = \mathrm{pd}_A\, \mathfrak{f}_1$ *and* $X_A^{(2)} = \mathrm{pd}_A\, \mathfrak{f}_2$ *for every* $A \in G$, *where* $\mathfrak{f}_j$ *are column function vectors associated with* E_j $(j = 1, 2)$. *Then*

$$\sum_{j=1}^{3} \Phi_j(X^{(1)}, X^{(2)}) = 0 .$$

THEOREM B. (*Corollary* 2 *to Theorem* 1 *in* [8]). *Let* $f \in \boldsymbol{E}_{1-q}(U, G)$, $q \geqq 1$ *and* E *a representative of* f *and let* $\mathfrak{f}$ *be a column function vector associated with* E. *Set* $\mathrm{pd}_A\, \mathfrak{f} = X_A$ *for* $A \in G$ *and set* $D^{2q-1}E = \phi$. *Then*

$$\sum_{j=1}^{3} \Phi_j(\overline{X}, X) = 2i(-1)^{q-1} \|\phi\|^2 .$$

THEOREM C. (*Kra* [6], *Sato* [8]). *Let* $X \in \alpha(\boldsymbol{E}_{1-q}(U, G, M))$. *If* X_A

is real for every $A \in G$, *then* $X_A = 0$.

Now we state our theorems. According to Kra [6]

$$\dim_C \alpha(E_{1-q}(U, G, M) = \dim_C \beta^*(L_\infty(U, G, M)) = (2q-1)(g-1)\ , \qquad q \geqq 2\ ,$$

where $\dim_C H$ denotes the dimension of H over C.

THEOREM 1. *Let* G *be a Fuchsian group of the first kind which is generated by* $\{A_1, B_1, \cdots, A_g, B_g\}$ *with a relation* $\prod_{j=1}^{g} B_j^{-1}A_j^{-1}B_jA_j = 1$. *Then*

(1) $\dim_C H_0^1(G, C^{2q-1}, M) = \begin{cases}(2q-1)(g-1), & q \geqq 2\\ 2g, & q = 1\end{cases}$

(2) $\dim_C H_1^1(G, C^{2q-1}, M) = (2q-1)(g-1),\ q \geqq 2$.

Remark. Let G be a Fuchsian group of the first kind which is generated by $\{A_1, B_1, \cdots, A_g, B_g, C_1, \cdots, C_\mu, D_1, \cdots, D_\nu\}$ with relations $D_\nu \cdots D_1C_\mu \cdots C_1 \prod_{j=1}^{g} B_j^{-1}A_j^{-1}B_jA_j = 1$ and $C_j^{e_j} = 1$ $(j = 1, \cdots, \mu)$. Then by the same method as in the proof of Theorem 1 (1) in § 4, we have that

$$\dim_C H_0^1(G, C^{2q-1}, M) = \begin{cases}(2q-1)(g-1) + \sum_{j=1}^{\mu} 2[q - (q/e_j)] + 2\nu(q-1)\ , & q \geqq 2\\ 2g\ , & q = 1\ ,\end{cases}$$

where the bracket [ ] denotes the Gaussian symbol.

THEOREM 2. *Let* G *be the same group as in Theorem* 1. *Then for any* $Z \in H^1(G, C^{2q-1}, M)$,

$$\Phi(Z, Z) = 0$$

and $\sqrt{-1}\Phi(\bar{Z}, Z)$ *is a real number. Especially if* $Z \in H_1^1(G, C^{2q-1}, M)$, *then*

$$\Phi(Z, Z) = \Phi(\bar{Z}, Z) = 0\ .$$

THEOREM 3. *Let* G *be the same group as in Theorem* 1 *and let* E *be a meromorphic Eichler integral such that* $\alpha_{S_j}(\mathfrak{f}) = 0$, $j = 1, \cdots, g$ *and set* $\alpha(\mathfrak{f}) = Z$, *where* $\mathfrak{f}$ *is a column function vector associated with* E. *Then*

(1) ${}^t\tilde{Z}_{A_j^{-1}}I'_{n+1}Z_{B_j} - {}^t\tilde{Z}_{B_j^{-1}}I'_{n+1}Z_{A_j} = 0,\ j = 1, \cdots, g.$

(2) *The quantity*

$$\sqrt{-1}({}^t\tilde{\bar{Z}}_{A_j^{-1}}I'_{n+1}Z_{B_j} - {}^t\tilde{\bar{Z}}_{B_j^{-1}}I'_{n+1}Z_{A_j})\ , \qquad j = 1, \cdots, g$$

are real numbers. Furthermore they may be positive, negative and zero.

We consider relations among the subspaces of $H^1(G, \boldsymbol{C}^{2q-1}, M)$ defined in § 1. We easily see that

$$\boldsymbol{E}^0_{1-q}(U, G, M) = \boldsymbol{E}^{01}_{1-q}(U, G, M) \cap \boldsymbol{E}^{02}_{1-q}(U, G, M)$$

and

$$\sqrt{-1}\boldsymbol{E}^{01}_{1-q}(U, G, M) = \boldsymbol{E}^{02}_{1-q}(U, G, M) .$$

According to Kra [6], $\alpha(\boldsymbol{E}_{1-q}(U, G, M)) \cap \beta^*(\boldsymbol{L}_\infty(U, G, M)) = \{0\}$. Furthermore it is easily seen by Theorem C that

$$H^1_1(G, \boldsymbol{C}^{2q-1}, M) \cap \alpha(\boldsymbol{E}_{1-q}(U, G, M)) = \{0\}$$

and

$$H^1_1(G, \boldsymbol{C}^{2q-1}, M) \cap \beta^*(\boldsymbol{L}_\infty(U, G, M)) = \{0\} .$$

THEOREM 4. *Let G be the same group as in Theorem* 1. *Then*

(1) $\dim_{\boldsymbol{R}} (H^1_0(G, \boldsymbol{C}^{2q-1}, M) \cap H^1_1(G, \boldsymbol{C}^{2q-1}, M)) = (2q-1)(g-1),\ q \geqq 2$

(2) $\dim_{\boldsymbol{R}} \boldsymbol{E}^{01}_{1-q}(U, G, M) = (2q-1)(g-1)$

(3) $\dim_{\boldsymbol{R}} \boldsymbol{E}^{01}_{1-q}(U, G, M) = (2q-1)(g-1)$,

where $\dim_{\boldsymbol{R}} H$ *means the dimension of* H *over* $\boldsymbol{R}$.

THEOREM 5. *Let G be the same group as in Theorem* 1. *Then*

(1) $D^{2q-1}\boldsymbol{E}^{01}_{1-q}(U, G) = \boldsymbol{B}^{01}_q(U, G)$

(2) $D^{2q-1}\boldsymbol{E}^{02}_{1-q}(U, G) = \boldsymbol{B}^{02}_q(U, G)$.

3. Lemmas.

In this section we state some lemmas which are necessary to prove the theorems in § 2. Especially Lemmas 1 and 3 play essential roles in the proof of Theorems 1, 2 and 3. For each $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we denote by $A(z) = (az + b)/(cz + d)$. We set $n = 2q - 2$, $q \geqq 1$ being an integer, once and for all.

LEMMA 1. (*see Sato* [8]). *For* $A \in G$,

$$M(A) = I'^{-1}_{n+1}({}^t\widetilde{M(A)})^{-1}I'_{n+1} .$$

LEMMA 2. *The determinant of matrix* $(M(A) - I_{n+1})$ *is zero, that is* $\det(M(A) - I_{n+1}) = 0$ *for any* A.

Proof. At first we remark the following. Let B be a Möbius transformation. Set $C = BAB^{-1}$. Then $\det(M(A) - I_{n+1}) = \det(M(C) - I_{n+1})$. Hence it suffices to show the lemma in the following special cases.

(1) Let A be hypabolic. We set $A = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$, $K > 1$. Then

$$M(A) = \begin{bmatrix} K^n & & & & & & 0 \\ & K^{n-2} & & & & & \\ & & \ddots & & & & \\ & & & K^2 & & & \\ & & & & 1 & & \\ & & & & & K^{-2} & \\ 0 & & & & & \ddots & \\ & & & & & & K^{-n} \end{bmatrix}.$$

Thus $\det(M(A) - I_{n+1}) = 0$.

(2) Let A be elliptic. We set $A = \begin{pmatrix} e^{\pi i/m} & 0 \\ 0 & e^{-(\pi i/m)} \end{pmatrix}$, $m \geqq 2$ being an integer. Then by the same way as above, we easily see that $\det(M(A) - I_{n+1}) = 0$.

(3) Let A be parabolic. We set $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then all elements in the first column of the matrix $(M(A) - I_{n+1})$ are zero. Hence $\det(M(A) - I_{n+1}) = 0$. Our proof is now complete.

LEMMA 3. *Let* $B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $ad - b^2 = 1$, $a, b \neq 0, d \in \boldsymbol{R}$. *Set* $M(B) = (b_{ij})_{i,j=1,\cdots,n+1}$. *Then* $b_{qq} \neq 1$.

Proof. Since $b_{qq} - 1$ is the coefficient of z^{q-1} in the expression $(az + b)^{q-1}(bz + d)^{q-1} - z^{q-1}$,

$$\begin{aligned} b_{qq} - 1 = {}& {}_mC_0{}_mC_m(ad)^m + {}_mC_1{}_mC_{m-1}(ad)^{m-1}b^2 + \cdots + {}_mC_{m-1}{}_mC_1(ad)(b^2)^{m-1} \\ & + {}_mC_m{}_mC_0(b^2)^m - 1\,, \end{aligned}$$

where $m = q - 1$. Since $ad = b^2 + 1$,

$$\begin{aligned} b_{qq} - 1 = {}& {}_mC_0^2(b^2 + 1)^m + {}_mC_1^2(b^2 + 1)^{m-1}b^2 + \cdots + {}_mC_{m-1}^2(b^2 + 1)b^{2(m-1)} \\ & + {}_mC_m^2 b^{2m} - 1\,. \end{aligned}$$

Now we set $b^2 = x > 0$. Then the quantity $b_{qq} - 1$ is positive whenever x is positive. Hence there is no $b \neq 0$ such that $b_{qq} - 1 = 0$. Our proof is now complete.

Let $A = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$, $K > 1$ and $B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $ad - b^2 = 1, a, b \neq 0, d \in \boldsymbol{R}$. We denote $M(B)$ by $M(B) = (b_{ij})_{i,j=1,\cdots,n+1}$. We set

$$\boldsymbol{B}_{1q} = (b_{1q} \cdots b_{q-1q} b_{q+1q} \cdots b_{n+1q}) \ ,$$
$$\boldsymbol{B}_{q1} = (b_{q1} \cdots b_{qq-1} b_{qq+1} \cdots b_{qn+1})$$

and

$$\boldsymbol{B}_1 = \begin{bmatrix} b_{11} & \cdots b_{1q-1} & b_{1q+1} & \cdots b_{1n+1} \\ & \cdots\cdots\cdots\cdots\cdots & & \\ b_{q-11} & \cdots b_{q-1q-1} & b_{q-1q+1} & \cdots b_{q-1n+1} \\ b_{q+11} & \cdots b_{q+1q-1} & b_{q+1q+1} & \cdots b_{q+1n+1} \\ & \cdots\cdots\cdots\cdots\cdots & & \\ b_{n+11} & \cdots b_{n+1q-1} & b_{n+1q+1} & \cdots b_{n+1n+1} \end{bmatrix} .$$

We define $n \times n$ matrices $M'(A)$ and $M'(B)$ by setting

$$M'(A) = \begin{bmatrix} K^n & & & & & & 0 \\ & K^{n-2} & & & & & \\ & & \ddots & & & & \\ & & & K^2 & & & \\ & & & & K^{-2} & & \\ & & & & & \ddots & \\ 0 & & & & & & K^{-n} \end{bmatrix} - I_n$$

and

$$M'(B) = \boldsymbol{B}_1 - I_{n'}$$

respectively. We set

$$\boldsymbol{B}_2 = (1/(b_{qq} - 1))^t\boldsymbol{B}_{1q}\boldsymbol{B}_{q1} \ ,$$
$$\boldsymbol{B}_3 = M'(B) - \boldsymbol{B}_2$$

and

$$\boldsymbol{B} = \boldsymbol{B}_3 + \boldsymbol{I}_n \qquad (= \boldsymbol{B}_1 - \boldsymbol{B}_2) \ .$$

Let $Z \in H^1(G, \boldsymbol{C}^{2q-1}, M)$. If we set $Z_A = {}^t(a_0, a_1, \cdots, a_n)$ and $Z_B = {}^t(b_0, b_1, \cdots, b_n)$, then we denote Z'_A and Z'_B as $Z'_A = {}^t(a_0, a_1, \cdots, a_{q-2}, a_q, \cdots, a_n)$ and $Z'_B = {}^t(b_0, b_1, \cdots, b_{q-2}, b_q, \cdots, b_n)$, respectively, where $A = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$

and $B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $K > 1$, $ad - b^2 = 1$, $a, b \neq 0$, $d \in \boldsymbol{R}$. We define $n \times n$ matrix $\boldsymbol{D}$ by setting

$$\boldsymbol{D} = {}^t\tilde{\boldsymbol{B}}_3 I''_n - {}^t\tilde{\boldsymbol{B}}_3 {}^t\widetilde{M'(A)}^{-1} I''_n \boldsymbol{B}_3 ,$$

LEMMA 4. *Let G be the same group as in Theorem 1 and let E be a meromorphic Eichler integral such that* $\mathrm{pd}_{S_j}(\mathfrak{f}) = 0, j = 1, \cdots, g$, *where* $\mathfrak{f}$ *is a column function vector associated with E. Set $\alpha(\mathfrak{f}) = Z$. Suppose that* $A_1 = A = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$, $K > 1$ *and* $B_1 = B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $ad - b^2 = 1$, $a, b \neq 0$, $d \in \boldsymbol{R}$. *Then*

(1) $(M(A_j) - I_{n+1})Z_{B_j} = (M(B_j) - I_{n+1})Z_{A_j}$, $j = 1, \cdots, g$.

(2) $Z'_B = M'(A)^{-1}\boldsymbol{B}_3 Z'_A$.

(3) ${}^t\widetilde{M'(A)}^{-1} I''_n M'(A^{-1}) = I''_n$

(4) $\boldsymbol{D} + {}^t\boldsymbol{D} = {}^t\tilde{\boldsymbol{B}}_3 I''_n + I''_n \boldsymbol{B}_3 + {}^t\tilde{\boldsymbol{B}}_3 I''_n \boldsymbol{B}_3$

(5) $\tilde{\boldsymbol{B}}_{1q} I''_n \boldsymbol{B}_1 = (-1)^q {}_nC_{q-1} b_{qq} \boldsymbol{B}_{q1}$

(6) $\tilde{\boldsymbol{B}}_{1q} I''_n {}^t\boldsymbol{B}_{1q} = (-1)^{q-1}({}_nC_{q-1} - {}_nC_{q-1} b_{qq}^2)$.

Proof. (1) By the assumption, $Z_{S_j} = 0, j = 1, \cdots, g$. We have that

$$\begin{aligned} Z_{S_j} &= Z_{B_j^{-1}A_j^{-1}B_jA_j} = Z_{B_j^{-1}} + M(B_j^{-1})Z_{A_j^{-1}} + M(B_j^{-1}A_j^{-1})Z_{B_j} + M(B_j^{-1}A_j^{-1}B_j)Z_{A_j} \\ &= M(B_j^{-1})(M(A_j^{-1}) - I_{n+1})Z_{B_j} + M(B_j^{-1}A_j^{-1})(M(B_j) - I_{n+1})Z_{A_j} , \end{aligned}$$

so that

$$(M(B_j) - I_{n+1})Z_{A_j} = (M(A_j) - I_{n+1})Z_{B_j} .$$

(2) We will show that $M'(A)Z'_B = \boldsymbol{B}_3 Z'_A$. Since $(M(A) - I_{n+1})Z_B = (M(B) - I_{n+1})Z_A$ by the above (1), the $(q, 1)$-elements of the left and right hand sides are equal to zero and

$$\sum_{\substack{k=0 \\ k \neq q-1}}^{n} b_{q,k+1} a_k + (b_{qq} - 1) a_{q-1} ,$$

respectively. Hence by $b_{qq} \neq 1$ (Lemma 3),

(i) $a_{q-1} = -(1/(b_{qq} - 1)) \sum_{\substack{k=0 \\ k \neq q-1}}^{n} b_{q,k+1} a_k$.

The $(j, 1)$-element $(j \neq q)$ of $(M(B) - I_{n+1})Z_A$ is

(ii) $\sum_{\substack{k=0 \\ k \neq j-1}}^{n} b_{j,k+1}a_k + (b_{jj} - 1)a_{j-1}$.

Substituting (i) into (ii), the $(j, 1)$-element $(j \neq q)$ of $(M(B) - I_{n+1})Z_A$ is equal to

$$\sum_{\substack{k=0 \\ k \neq j-1, q-1}}^{n} b_{j,k+1}a_k + (b_{jj} - 1)a_{j-1} - (b_{jq}/(b_{qq} - 1)) \sum_{\substack{k=0 \\ k \neq q-1}}^{n} b_{q,k+1}a_k ,$$

that is

$$\begin{aligned} &\{(b_{j1}, \cdots, b_{j,j-1}, b_{jj} - 1, b_{j,j+1}, \cdots, b_{j,q-2}, b_{jq}, \cdots, b_{j,n+1}) \\ &\quad - (1/(b_{qq} - 1))b_{jq}\boldsymbol{B}_{q1}\}Z'_A . \end{aligned}$$

Hence $M'(A)Z'_B = (M'(B) - \boldsymbol{B}_2)Z'_A = \boldsymbol{B}_3 Z'_A$.

(3) We will show that $I''_n M'(A^{-1}) = {}^t\widetilde{M'(A)}I''_n$.

$$\begin{aligned} &I''_n M'(A^{-1}) - {}^t\widetilde{M'(A)}I''_n \\ &\quad = I''_n M'(A^{-1}) - \widetilde{M'(A)}I''_n \qquad \text{(Lemma 4(2))} \\ &\quad = M'(A^{-1})I''_n - \widetilde{M'(A)}I''_n = (M'(A^{-1}) - \widetilde{M'(A)})I''_n . \end{aligned}$$

Since

$$M'(A^{-1}) = \widetilde{M'(A)} = \begin{pmatrix} K^{-n} - 1 & & & & & & 0 \\ & K^{-n+2} - 1 & & & & & \\ & & \ddots & & & & \\ & & & K^{-2} - 1 & & & \\ & 0 & & & K^{2} - 1 & & \\ & & & & & \ddots & \\ & & & & & & K^{n} - 1 \end{pmatrix},$$

we have the desired result.

(4) $\boldsymbol{D} + {}^t\tilde{\boldsymbol{D}}$

$$\begin{aligned} &= {}^t\tilde{\boldsymbol{B}}_3 I''_n - {}^t\tilde{\boldsymbol{B}}_3 {}^t\widetilde{M'(A)}^{-1} I''_n \boldsymbol{B}_3 + I''_n \boldsymbol{B}_3 - {}^t\tilde{\boldsymbol{B}}_3 I''_n M'(A)^{-1}\boldsymbol{B}_3 \\ &= {}^t\tilde{\boldsymbol{B}}_3 I''_n + I''_n \boldsymbol{B}_3 - {}^t\tilde{\boldsymbol{B}}_3({}^t\widetilde{M'(A)}^{-1} I''_n - I''_n M'(A)^{-1})\boldsymbol{B}_3 \\ &= {}^t\tilde{\boldsymbol{B}}_3 I''_n + I''_n \boldsymbol{B}_3 - {}^t\tilde{\boldsymbol{B}}_3 I''_n(M'(A^{-1})^{-1} - M'(A)^{-1})\boldsymbol{B}_3 \qquad \text{(Lemma 4(3))}. \end{aligned}$$

After a slight computation we have

$$M'(A^{-1})^{-1} - M'(A)^{-1} = -I_n .$$

Hence $\boldsymbol{D} + {}^t\tilde{\boldsymbol{D}} = {}^t\tilde{\boldsymbol{B}}_3 I''_n + I''_n \boldsymbol{B}_3 + {}^t\tilde{\boldsymbol{B}}_3 I''_n \boldsymbol{B}_3$.

(5) By Lemma 1, ${}^t\widetilde{M(B)}I'_{n+1}M(B) = I'_{n+1}$. The (q, j)-elements $(j \neq q)$ of the left and right hand sides are equal to $\sum_{k=0}^{n}(-1)^k{}_nC_k b_{n+1-k,q}b_{k+1,j}$ and 0, respectively. Hence

$$\sum_{\substack{k=0\\k\neq q-1}}^{n}(-1)^k{}_nC_k b_{n+1-k,q}b_{k+1,j} = (-1)^q{}_nC_{q-1}b_{qq}b_{qj}\ ,$$

$$(j = 1, \cdots, q-1, q+1, \cdots, n+1)$$

The left hand side is the $(1, j)$-element of $\tilde{\boldsymbol{B}}_{1q}I''_n\boldsymbol{B}_1$ and the right hand side is the $(1, j)$-element of $(-1)^q{}_nC_{q-1}b_{qq}\boldsymbol{B}_{q1}$.

(6) By Lemma 1, ${}^t\widetilde{M(B)}I'_{n+1}M(B) = I'_{n+1}$. The (q, q)-elements of the right and left hand sides are equal to $(-1)^{q-1}{}_nC_{q-1}$ and

$$\sum_{k=0}^{n}(-1)^k{}_nC_k b_{n+1-k,q}b_{k+1,q} = \tilde{\boldsymbol{B}}_{1q}I''_n{}^t\boldsymbol{B}_{1q} + (-1)^{q-1}{}_nC_{q-1}b_{qq}^2\ ,$$

respectively. Thus we have the desired result. Our proof is now complete.

LEMMA 5. (1) *Let* $Z \in H_0^1(G, \boldsymbol{C}^{2q-1}, M)$. *Let* $\Gamma = \{CAC^{-1} | A \in G,\ C$: *Möbius transformation*$\}$. *If* $Z^*_{A_1} = M(C)Z_A, A_1 = CAC^{-1}$, *for all* $A \in G$, *then* $Z^* \in H_0^1(\Gamma, \boldsymbol{C}^{2q-1}, M)$.

(2) *For* $A_1 = CAC^{-1}$ *and* $B_1 = CAC^{-1}$,

$${}^t\tilde{Z}^*_{A_1^{-1}}I'_{n+1}Z^*_{B_1} - {}^t\tilde{Z}^*_{B_1^{-1}}I'_{n+1}Z^*_{A_1} = {}^t\tilde{Z}_{A^{-1}}I'_{n+1}Z_B - {}^t\tilde{Z}_{B^{-1}}I'_{n+1}Z_A$$

and

$${}^t\tilde{\tilde{Z}}^*_{A_1^{-1}}I'_{n+1}Z^*_{B_1} - {}^t\tilde{\tilde{Z}}^*_{B_1^{-1}}I'_{n+1}Z^*_{A_1} = {}^t\tilde{\tilde{Z}}_{A^{-1}}I'_{n+1}Z_B - {}^t\tilde{\tilde{Z}}_{B^{-1}}I'_{n+1}Z_A\ .$$

Proof. (1) is easily seen by the simple computation.

(2) We only show the first identity.

$$\begin{aligned}
&{}^t\tilde{Z}^*_{A_1^{-1}}I'_{n+1}Z^*_{B_1} - {}^t\tilde{Z}^*_{B_1^{-1}}I'_{n+1}Z^*_{A_1}\\
&\quad = {}^t\tilde{Z}^*_{CA^{-1}C^{-1}}I'_{n+1}Z^*_{CBC^{-1}} - {}^t\tilde{Z}^*_{CB^{-1}C^{-1}}I'_{n+1}Z^*_{CAC^{-1}}\\
&\quad = {}^t\tilde{Z}_{A^{-1}}{}^t\widetilde{M(C)}I'_{n+1}M(C)Z_B - {}^t\tilde{Z}_{B^{-1}}{}^t\widetilde{M(C)}I'_{n+1}M(C)Z_A\\
&\quad = {}^t\tilde{Z}_{A^{-1}}I'_{n+1}Z_B - {}^t\tilde{Z}_{B^{-1}}I'_{n+1}Z_A \qquad \text{(Lemma 1).}
\end{aligned}$$

Our proof is now complete.

Let $E \in \boldsymbol{E}_{1-q}(U, G)$. Set $D^{2q-1}E = \phi \in \boldsymbol{B}_q(U, G)$, Pot$(\phi)(z) = E_1(z) \in \boldsymbol{E}_{1-q}(L, G)$ and $E_2(z) = \overline{E_1(\bar{z})}$, $z \in U$. We set $D^{2q-1}E_1 = \phi_1$ and $D^{2q-1}E_2 = \phi_2$. Then we have

LEMMA 6. (*Bers* [3], *see Kra* [6]).

$$c_q\phi_2(z) = \phi(z) \quad \textit{and} \quad c_qE_2 - E \in \Pi_{2q-2}\ ,$$

where $c_q = (-1)^{q-1}(2q-2)\,!$.

4. Proof of Theorems.

Proof of Theorem 1. (1) At first let $q \geqq 2$. Let $Z \in H_0^1(G, \boldsymbol{C}^{2q-1}, M)$. By Lemma 4(1), $(M(B_j) - I_{n+1})Z_{A_j} = (M(A_j) - I_{n+1})Z_{B_j}$ $(j = 1, \cdots, g)$. By Lemma 5(1), we may normalize that $A_j = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$, $K > 1$ and $B_j = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $ad - b^2 = 1$, $a, b \neq 0$, $d \in \boldsymbol{R}$, for each $j = 1, \cdots, g$. Set $Z_{A_j} = {}^t(a_0, a_1, \cdots, a_n)$ and $Z_{B_j} = {}^t(b_0, b_1, \cdots, b_n)$, $n = 2q-2$. We show that if we give $(2q-1)$ complex numbers $a_0, a_1, \cdots, a_{q-2}, a_q, \cdots, a_n, b_{q-1}$, then we uniquely determine $b_0, b_1, \cdots, b_{q-2}, b_q, \cdots, b_n, a_{q-1}$. We see that

$$M(A_j) - I_{n+1} = \begin{bmatrix} K^n - 1 & & & & & & & 0 \\ & K^{n-2} - 1 & & & & & & \\ & & \ddots & & & & & \\ & & & K^2 - 1 & & & & \\ & & & & 0 & & & \\ & & & & & K^{-2} - 1 & & \\ & & & & & & \ddots & \\ 0 & & & & & & & K^{-n} - 1 \end{bmatrix}.$$

Set $M(B_j) - I_{n+1} = (b_{ij})_{i,j=1,\cdots,n+1} - I_{n+1}$. Since $(M(B_j) - I_{n+1})Z_{A_j} = (M(A_j) - I_{n+1})Z_{B_j}$ and $b_{qq} \neq 1$, we can uniquely determine a_{q-1} by $a_0, a_1, \cdots, a_{q-2}, a_q, \cdots, a_n$. Then $b_0, b_1, \cdots, b_{q-2}, b_q, \cdots, b_n$ are also uniquely determined.

Especially, we consider about Z_{A_g} and Z_{B_g}. Set $Z_{A_g} = {}^t(a_{g0}, \cdots, a_{gn})$ and $Z_{B_g} = {}^t(b_{g0}, \cdots, b_{gn})$. From coboundary property, we normalize that $a_{g0}, \cdots, a_{g,q-2}, a_{g,q}, \cdots, a_{g,n}$ and $b_{g,q-1}$ are all zero. Then by a similar way as above we conclude that $b_{g0}, \cdots, b_{g,q-2}, b_{g,q}, \cdots, b_{g,n}, a_{g,q-1}$ are all zero. Hence $Z_{A_g} = Z_{B_g} = 0$. Thus we conclude that

$$\dim_{\boldsymbol{C}} H_0^1(G, \boldsymbol{C}^{2q-1}, M) = (2q-1)(g-1)\ .$$

Next let $q = 1$. Then for any $Z \in H^1(G, \boldsymbol{C}, M)$, we easily see that $Z_{S_j} = 0$, $j = 1, \cdots, g$. Hence $H^1(G, \boldsymbol{C}, M) = H_0^1(G, \boldsymbol{C}, M)$. Thus $\dim_{\boldsymbol{C}} H_0^1 \cdot(G, \boldsymbol{C}, M) = 2g$.

(2) We will show that $H_1^1(G, \boldsymbol{C}^{2q-1}, M)$ is isomorphic to $\boldsymbol{B}_q(U, G)$.

Let $\phi \in \boldsymbol{B}_q(U, G)$. We will show that there uniquely exists $\mathfrak{f} \in \boldsymbol{E}_{1-q}(U, G, M)$ such that $\alpha_A(\mathfrak{f}(z)) = \overline{\beta_A^*(\mathfrak{g}(\bar{z}))}$, $z \in U$ and $D^{2q-1}E = \phi$, where $E(z) = (1/n!) \cdot {}^t\mathfrak{f}(z) I'_{n+1}\begin{pmatrix}1\\ z\end{pmatrix}^n$, $z \in U$, and $\mathfrak{g}$ is a column function vector associated with $c_q\phi$. Set $\mathrm{Pot}(\phi)(z) = E_1(z)$, $z \in L$. We set $E_2(z) = \overline{E_1(\bar{z})}$, $z \in U$ and set $E(z) = c_q E_2(z)$, $z \in U$. Then by Lemma 6, $D^{2q-1}E(z) = \phi(z)$. Furthermore we see that

$$\mathrm{pd}_A\, E(z) = \mathrm{pd}_A\, c_q E_2(z) = \mathrm{pd}_A\, c_q \overline{E_1(\bar{z})} = \mathrm{pd}_A\, c_q\, \overline{\mathrm{Pot}\,(\phi)(\bar{z})}\ , \qquad z \in U\ .$$

Let $\mathfrak{f}$ and $\mathfrak{g}$ be column function vectors associated with the above E and $c_q\phi$, respectively. Thus we obtain that $\alpha_A(\mathfrak{f}(z)) = \overline{\beta_A^*(\mathfrak{g}(\bar{z}))}$ for $A \in G$ and $z \in U$. If we set $Z = \alpha(\mathfrak{f}) + \beta^*(\mathfrak{g})$, then we have $Z \in H_1^1(G, \boldsymbol{C}^{2q-1}, M)$ by the above construction. Thus we have a mapping from $\phi \in \boldsymbol{B}_q(U, G)$ to $Z \in H_1^1(G, \boldsymbol{C}^{2q-1}, M)$ by the above way.

It is trivial that the mapping is injective and surjective. Our proof is now complete.

Proof of Theorem 2. By Kra's decomposition theorem (Kra [6]), $Z = \alpha(\mathfrak{f}) + \beta^*(\mathfrak{g})$ with $\mathfrak{f} \in \boldsymbol{E}_{1-q}(U, G, M)$ and $\mathfrak{g} \in \boldsymbol{L}_\infty(U, G, M)$. We set $\alpha(\mathfrak{f}) = X$ and $\beta^*(\mathfrak{g}) = Y$. Then $\mathfrak{g} \in \boldsymbol{E}_{1-q}(L, G, M)$. We set $\mathfrak{f}^*(z) = \overline{\mathfrak{g}(\bar{z})}$, $z \in U$. Then $\mathfrak{f}^* \in \boldsymbol{E}_{1-q}(U, G, M)$ and $X_A^* = \overline{Y}_A$, $A \in G$, where $X_A^* = \alpha_A(\mathfrak{f}^*(z))$. We define ϕ and ϕ^* by setting

$$D^{2q-1}(1/n!)\,{}^t\mathfrak{f}(z) I'_{n+1}\begin{pmatrix}1\\ z\end{pmatrix}^n \phi(z)$$

and

$$D^{2q-1}(1/n!)\,{}^t\mathfrak{f}^*(z) I'_{n+1}\begin{pmatrix}1\\ z\end{pmatrix}^n = \phi^*(z)\ ,$$

respectively. Then

$$\begin{aligned}\Phi(Z, Z) &= \Phi(X + Y, X + Y) = \Phi(X, X) + \Phi(X, Y) + \Phi(Y, X) + \Phi(Y, Y)\\ &= \Phi(X, X) + \Phi(X, \overline{X}^*) + \Phi(\overline{X}^*, X) + \Phi(\overline{X}^*, \overline{X}^*)\ .\end{aligned}$$

Since $\Phi(X, X) = \Phi(X^*, X^*) = 0$ (Theorem A) and $\Phi(\overline{X}^*, X) = -\Phi(X, \overline{X}^*) = 2\sqrt{-1}(-1)^{q-1}(\phi, \phi^*)$ (Corollary 2 to Theorem 1 in [8]), we have $\Phi(Z, Z) = 0$.

By Theorem B,

$$\begin{aligned}\Phi(\overline{Z}, Z) &= \Phi(\overline{X} + \overline{Y}, X + Y) = \Phi(\overline{X}, X) + \Phi(\overline{X}, Y) + \Phi(\overline{Y}, X) + \Phi(\overline{Y}, Y)\\ &= \Phi(\overline{X}, X) + \Phi(\overline{X}, \overline{X}^*) + \Phi(X^*, X) + \Phi(X^*, \overline{X}^*)\\ &= 2\sqrt{-1}(-1)^{q-1}\,\|\phi\|^2 - 2\sqrt{-1}(-1)^{q-1}\,\|\phi^*\|^2\ .\end{aligned}$$

Hence $\sqrt{-1}\Phi(\bar{Z}, Z)$ is a real number.

Next let $Z \in H_1^1(U, G, M)$. Then $Y_A = \bar{X}_A$, $A \in G$. Hence $X_A^* = X_A$, $A \in G$, so that $\phi = \phi^*$. Hence we have the desired result. Our proof is now complete.

Proof of Theorem 3. (1) In the case of $q = 1$, it is trivial, so that we only show the case of $q \geqq 2$. We may normalize that $A_j = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$, $K > 1$ and $B_j = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $ad - b^2 = 1$, a, $b \neq 0$, $d \in \boldsymbol{R}$. For the sake of brevity we consider A and B instead of A_j and B_j, respectively. Set $Z_A = {}^t(a_0, a_1, \cdots, a_n)$ and $Z_B = {}^t(b_0, b_1, \cdots, b_n)$. Let $M(A), M(B), M'(A)$, $M'(B)$ and I''_n be the same as defined in § 2. Set $Z'_A = {}^t(a_0, \cdots, a_{q-2}, a_q, \cdots, a_n)$ and $Z'_B = {}^t(b_0, \cdots, b_{q-2}, b_q, \cdots, b_n)$.

At first we show that if ${}^t\tilde{Z}'_B I''_n M'(A^{-1}) Z'_A = {}^t\tilde{Z}'_B I''_n M'(A) Z'_B$, then ${}^t\tilde{Z}_{A^{-1}} I'_{n+1} Z_B = {}^t\tilde{Z}_{B^{-1}} I'_{n+1} Z_A$. For, since

$$(M(A) - I_{n+1})Z_B = (M(B) - I_{n+1})Z_A ,$$

we have that

$$\begin{aligned}
&-{}^t\tilde{Z}_{A^{-1}} I'_{n+1} Z_B + {}^t\tilde{Z}_{B^{-1}} I'_{n+1} Z_A \\
&\quad = {}^t\tilde{Z}_A I'_{n+1} M(A) Z_B - {}^t\tilde{Z}_B I'_{n+1} M(B) Z_A \qquad \text{(by Lemma 1)} \\
&\quad = {}^t\tilde{Z}_B {}^t\widetilde{M(A)} I'_{n+1} Z_A - {}^t\tilde{Z}_B I'_{n+1} M(B) Z_A \\
&\quad = {}^t\tilde{Z}_B I'_{n+1} M(A^{-1}) Z_A - {}^t\tilde{Z}_B I'_{n+1} M(B) Z_A \qquad \text{(by Lemma 1)} \\
&\quad = {}^t\tilde{Z}_B I'_{n+1} M(A^{-1}) Z_A - {}^t\tilde{Z}_B I'_{n+1} Z_A - {}^t\tilde{Z}_B I'_{n+1} M(A) Z_B + {}^t\tilde{Z}_B I'_{n+1} Z_B \\
&\quad = {}^t\tilde{Z}_B I'_{n+1} (M(A^{-1}) - I_{n+1}) Z_A - {}^t\tilde{Z}_B I'_{n+1} (M(A) - I_{n+1}) Z_B .
\end{aligned}$$

Since the elements of the q-th rows and the q-th column of the matrices $(M(A^{-1}) - I_{n+1})$ and $(M(A) - I_{n+1})$ are all zero, we obtain that if ${}^t\tilde{Z}'_B I''_n M' \cdot (A^{-1}) Z'_A = {}^t\tilde{Z}'_B I''_n M'(A) Z'_B$, then

$${}^t\tilde{Z}_B I'_{n+1} (M(A^{-1}) - I_{n+1}) Z_A = {}^t\tilde{Z}_B I'_{n+1} (M(A) - I_{n+1}) Z_B .$$

Let $\boldsymbol{B}_{1q}, \boldsymbol{B}_{q1}, \boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3$ and $\boldsymbol{B}$ be the same as defined in § 2. Then since $(M(A) - I_{n+1})Z_B = (M(B) - I_{n+1})Z_A$, by Lemma 4(2)

$$Z'_B = M'(A)^{-1} \boldsymbol{B}_3 Z'_A .$$

If we substitute $Z'_B = M'(A)^{-1} \boldsymbol{B}_3 Z'_A$ in ${}^t\tilde{Z}'_B I''_n M'(A^{-1}) Z'_A - {}^t\tilde{Z}'_B I''_n M'(A) Z'_B$, then we have by using Lemma 4(3) that

$$
\begin{aligned}
&{}^t\tilde{Z}'_A {}^t\tilde{\boldsymbol{B}}_3\{{}^t\widetilde{M'(A)}^{-1}I''_n M'(A^{-1}) - {}^t\widetilde{M'(A)}^{-1}I''_n M'(A)M'(A)^{-1}\boldsymbol{B}_3\}Z'_A \\
&\quad = {}^t\tilde{Z}'_A {}^t\tilde{\boldsymbol{B}}_3\{I''_n - {}^t\widetilde{M'(A)}^{-1}I''_n\boldsymbol{B}_3\}Z'_A .
\end{aligned}
$$

If we set

$$\boldsymbol{D} = {}^t\tilde{\boldsymbol{B}}_3 I''_n - {}^t\tilde{\boldsymbol{B}}_3 {}^t\widetilde{M'(A)}^{-1}I''_n\boldsymbol{B}_3 ,$$

then

$${}^t\tilde{\boldsymbol{D}} = I''_n\boldsymbol{B}_3 - {}^t\tilde{\boldsymbol{B}}_3 I''_n {}^t M'(A)^{-1}\boldsymbol{B}_3 .$$

By Lemma 4(4),

$$\boldsymbol{D} + {}^t\tilde{\boldsymbol{D}} = {}^t\tilde{\boldsymbol{B}}_3 I''_n + I''_n\boldsymbol{B}_3 + {}^t\tilde{\boldsymbol{B}}_3 I''_n\boldsymbol{B}_3 .$$

If $\boldsymbol{D} + {}^t\tilde{\boldsymbol{D}} = 0_{n,n}$, then ${}^t\tilde{Z}'_B I''_n M'(A^{-1})Z'_A = {}^t\tilde{Z}'_B I''_n M'(A)Z'_B$, where $0_{n,n}$ is a $n \times n$ matrix whose elements are all zero. For

$$
\begin{aligned}
&{}^t\tilde{Z}'_B I''_n M'(A^{-1})Z'_A - {}^t\tilde{Z}'_B I''_n M'(A)Z'_B \\
&\quad = {}^t\tilde{Z}'_A \boldsymbol{D} Z'_A = {}^t\tilde{Z}'_A {}^t\tilde{\boldsymbol{D}} Z'_A = (1/2){}^t\tilde{Z}'_A(\boldsymbol{D} + {}^t\tilde{\boldsymbol{D}})Z'_A = 0 .
\end{aligned}
$$

Now we will show that $\boldsymbol{D} + {}^t\tilde{\boldsymbol{D}} = 0_{n,n}$. Since

$$
\begin{aligned}
{}^t\tilde{\boldsymbol{B}} I''_n \boldsymbol{B} - I''_n &= ({}^t\tilde{\boldsymbol{B}}_3 + I_n)I''_n(\boldsymbol{B}_3 + I_n) - I''_n \\
&= {}^t\tilde{\boldsymbol{B}}_3 I''_n + I''_n\boldsymbol{B}_3 + {}^t\tilde{\boldsymbol{B}}_3 I''_n\boldsymbol{B}_3 = \boldsymbol{D} + {}^t\tilde{\boldsymbol{D}} ,
\end{aligned}
$$

it suffices to show that ${}^t\tilde{\boldsymbol{B}} I''_n \boldsymbol{B} = I''_n$. Since ${}^t\widetilde{M(B)}I'_{n+1}M(B) = I'_{n+1}$ (Lemma 1),

$${}^t\tilde{\boldsymbol{B}}_1 I''_n \boldsymbol{B}_1 + (-1)^{q-1}{}_nC_{q-1}{}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1} = I''_n .$$

On the other hand, ${}^t\tilde{\boldsymbol{B}} I''_n \boldsymbol{B} = I''_n$ is equivalent to

$${}^t\tilde{\boldsymbol{B}}_1 I''_n \boldsymbol{B}_1 - {}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_1 - {}^t\tilde{\boldsymbol{B}}_1 I''_n \boldsymbol{B}_2 + {}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_2 = I''_n .$$

Hence if we show that

$${}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_1 + {}^t\tilde{\boldsymbol{B}}_1 I''_n \boldsymbol{B}_2 = {}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_2 - (-1)^{q-1}{}_nC_{q-1}{}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1} ,$$

we have ${}^t\tilde{\boldsymbol{B}} I''_n \boldsymbol{B} = I''_n$.

By Lemma 4(5),

$$\tilde{\boldsymbol{B}}_{1q} I''_n \boldsymbol{B}_1 = (-1)^q{}_nC_{q-1}b_{qq}\boldsymbol{B}_{q1} ,$$

so that

$${}^t\tilde{\boldsymbol{B}}_1 I''_n {}^t\boldsymbol{B}_{1q} = (-1)^q{}_nC_{q-1}b_{qq}{}^t\tilde{\boldsymbol{B}}_{q1} .$$

Thus

$$\begin{aligned}{}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_1 + {}^t\tilde{\boldsymbol{B}}_1 I''_n \boldsymbol{B}_2 &= (1/(b_{qq}-1))({}^t\tilde{\boldsymbol{B}}_{q1}\tilde{\boldsymbol{B}}_{1q} I''_n \boldsymbol{B}_1 + {}^t\tilde{\boldsymbol{B}}_1 I''_n {}^t\boldsymbol{B}_{1q}\boldsymbol{B}_{q1}) \\ &= 2(-1)^q {}_nC_{q-1} b_{qq} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1}/(b_{qq}-1)\ .\end{aligned}$$

On the other hand

$$\begin{aligned}&{}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_2 - (-1)^{q-1} {}_nC_{q-1} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1} \\ &\quad= (1/(b_{qq}-1)^2){}^t\tilde{\boldsymbol{B}}_{q1}\tilde{\boldsymbol{B}}_{1q} I''_n {}^t\boldsymbol{B}_{1q}\boldsymbol{B}_{q1} - (-1)^{q-1} {}_nC_{q-1} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1} \\ &\quad= (-1)^{q-1}(-{}_nC_{q-1}b_{qq}^2 + {}_nC_{q-1}){}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1}/(b_{qq}-1)^2 \\ &\qquad - (-1)^{q-1} {}_nC_{q-1} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1} \qquad \text{(Lemma 4(6))}\ .\end{aligned}$$

Hence

$$\begin{aligned}&{}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_2 - (-1)^{q-1} {}_nC_{q-1} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1} \\ &\quad= \frac{(-1)^{q-1}(-{}_nC_{q-1}b_{qq}^2 + {}_nC_{q-1}) - (-1)^{q-1}{}_nC_{q-1}(b_{qq}-1)^2}{(b_{qq}-1)^2} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1} \\ &\quad= 2(-1)^q {}_nC_{q-1} b_{qq} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1}/(b_{qq}-1)\ .\end{aligned}$$

Hence we obtain that

$${}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_1 + {}^t\tilde{\boldsymbol{B}}_1 I''_n \boldsymbol{B}_2 = {}^t\tilde{\boldsymbol{B}}_2 I''_n \boldsymbol{B}_2 - {}_nC_{q-1} {}^t\tilde{\boldsymbol{B}}_{q1}\boldsymbol{B}_{q1}\ .$$

(2) Let $q \geqq 2$. By the same method as in the above proof, we have $\boldsymbol{D} + {}^t\tilde{\boldsymbol{D}} = 0_{n,n}$. From this we will show that $\sqrt{-1}({}^t\bar{\tilde{Z}}_{A_j^{-1}} I'_{n+1} Z_{B_j} - {}^t\bar{\tilde{Z}}_{B_j^{-1}} I'_{n+1} Z_{A_j})$ are real numbers, that is, that $\sqrt{-1}({}^t\bar{\tilde{Z}}'_{B_j} I''_n M'(A_j^{-1}) Z'_{A_j} - {}^t\bar{\tilde{Z}}_{B_j} I''_n M'(A_j) Z'_{B_j})$ are real numbers. We consider A and B instead of A_j and B_j, respectively. Set

$$\boldsymbol{D} = (d_{ij})_{i,j=1,\cdots,n}\ .$$

By the same method as in the above proof, we have that

$$\begin{aligned}&{}^t\bar{\tilde{Z}}'_B I''_n M'(A^{-1}) Z'_A - {}^t\bar{\tilde{Z}}_B I''_n M'(A) Z'_B \\ &\quad= {}^t\bar{\tilde{Z}}'_A \boldsymbol{D} Z'_A = \sum_{k=0}^{q-2} \bar{a}_{n-k}\left(\sum_{j=0}^{q-2} d_{k+1,j+1}a_j + \sum_{j=q}^{n} d_{k+1,j}a_j\right) \\ &\qquad + \sum_{k=q}^{n} \bar{a}_{n-k}\left(\sum_{k=0}^{q-2} d_{k,j+1}a_j + \sum_{j=q}^{n} d_{kj}a_j\right) \\ &\quad= \sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{k+1,j+1}a_j\bar{a}_{n-k} + \sum_{k=q}^{q-2}\sum_{j=q}^{n} d_{k+1,j}a_j\bar{a}_{n-k} \\ &\qquad + \sum_{k=q}^{n}\sum_{j=0}^{q-2} d_{k,j+1}a_j\bar{a}_{n-k} + \sum_{k=q}^{n}\sum_{j=q}^{n} d_{kj}a_j\bar{a}_{n-k}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{k+1,j+1} a_j \bar{a}_{n-k} + \sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{n-k,n-j} a_{n-j} \bar{a}_k \\
&\quad + (1/2)\Big(\sum_{k=0}^{q-2}\sum_{j=q}^{n} d_{k+1,j} a_j \bar{a}_{n-k} + \sum_{k=q}^{n}\sum_{j=0}^{q-2} d_{n-k+1,n-j} a_{n-j} \bar{a}_k\Big) \\
&\quad + (1/2)\Big(\sum_{k=q}^{n}\sum_{j=0}^{q-2} d_{k,j+1} a_j \bar{a}_{n-k} + \sum_{k=0}^{q-2}\sum_{j=q}^{n} d_{n-k,n-j+1} a_{n-j} \bar{a}_k\Big) \\
&= \sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{k+1,j+1} a_j \bar{a}_{n-k} + \sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{n-j,n-k} a_{n-k} \bar{a}_j \\
&\quad + (1/2)\Big(\sum_{k=0}^{q-2}\sum_{j=q}^{n} d_{k+1,j} a_j \bar{a}_{n-k} + \sum_{k=0}^{q-2}\sum_{j=q}^{n} d_{n-j+1,n-k} a_{n-k} \bar{a}_j\Big) \\
&\quad + (1/2)\Big(\sum_{k=q}^{n}\sum_{j=0}^{q-2} d_{k,j+1} a_j \bar{a}_{n-k} + \sum_{j=0}^{q-2}\sum_{k=q}^{n} d_{n-j,n-k+1} a_{n-k} \bar{a}_j\Big) \\
&= \sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{k+1,j+1} a_j \bar{a}_{n-k} - \sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{k+1,j+1} a_{n-k} \bar{a}_j \\
&\quad + (1/2)\Big(\sum_{k=0}^{q-2}\sum_{j=q}^{n} d_{k+1,j} a_j \bar{a}_{n-k} - \sum_{k=0}^{q-2}\sum_{j=q}^{n} d_{k+1,j} a_{n-k} \bar{a}_j\Big) \\
&\quad + (1/2)\Big(\sum_{k=q}^{n}\sum_{j=0}^{q-2} d_{k,j+1} a_j \bar{a}_{n-k} - \sum_{k=q}^{n}\sum_{j=0}^{q-2} d_{k,j+1} \bar{a}_j a_{n-k}\Big) \\
&= 2\sqrt{-1}\Big\{\sum_{k=0}^{q-2}\sum_{j=0}^{q-2} d_{k+1,j+1} \operatorname{Im}(a_j \bar{a}_{n-k}) + \sum_{k=0}^{q-2}\sum_{j=q}^{n} d_{k+1,j} \operatorname{Im}(a_j \bar{a}_{n-k}) \\
&\qquad + \sum_{k=q}^{n}\sum_{j=0}^{q-2} d_{k,j+1} \operatorname{Im}(a_j \bar{a}_{n-k})\Big\}\,.
\end{aligned}
$$

Hence we have the desired result for the case of $q \geqq 2$.

Let $q = 1$. Then

$$
{}^t\bar{\bar{Z}}_{A-1} I'_{n+1} Z_B - {}^t\bar{\bar{Z}}_{B-1} I'_{n+1} Z_A = -\bar{Z}_A Z_B + Z_A \bar{Z}_B = 2\sqrt{-1}\operatorname{Im}(Z_A \bar{Z}_B)\,.
$$

Next we show some examples. Let $q = 2$. Let

$A = \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$, $K > 1$ and $B = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, $ad - b^2 = 1$, $a, b \neq 0$, $d \in \boldsymbol{R}$. Set

$$
Z_A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \quad \text{and} \quad Z_B = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}.
$$

Then since $Z \in H_0^1(G, \boldsymbol{C}^3, M)$, we have that $a_1 = -(1/2b)(aa_0 + da_2)$, $b_0 = -(1/(K^2 - 1))(a_0 + a_2)$ and $b_2 = (K^2/(K^2 - 1))(a_0 + a_2)$. By these identities,

$$
\begin{aligned}
&-{}^t\bar{\bar{Z}}_{A-1} I'_3 Z_B + {}^t\bar{\bar{Z}}_{B-1} I'_3 Z_A \\
&\quad = ((K^2 + 1)/(K^2 - 1))(\bar{a}_0 a_2 - \bar{a}_2 a_0) + (a/b)(\bar{a}_0 b_1 - \bar{b}_1 a_0) \\
&\qquad + (d/b)(\bar{a}_2 b_1 - \bar{b}_1 a_2)\,.
\end{aligned}
$$

Set $a_0 = r_0 > 0$, $b_1 = r_1 e^{i\theta}$ and $a_2 = r_2 > 0$, $r_0, r_1 > 0$, $r_2, \theta \in \boldsymbol{R}$. Then

$$\begin{aligned}\sqrt{-1}({}^t\bar{\bar{Z}}_{A-1}I_3' Z_B - {}^t\bar{\bar{Z}}_B I_3' Z_A) \\ = 2r_1\{(a/b)r_0 + (d/b)r_2\}\sin\theta \;.\end{aligned}$$

If $ab > 0$, then

$$\sqrt{-1}({}^t\bar{\bar{Z}}_{A-1}I_3' Z_B - {}^t\bar{\bar{Z}}_{B-1}I_3' Z_A) \begin{array}{ll} > 0 & (\theta = \pi/2) \\ < 0 & (\theta = -\pi/2) \end{array}$$

If $ab < 0$, then

$$\sqrt{-1}({}^t\bar{\bar{Z}}_{A-1}I_3' Z_B - {}^t\bar{\bar{Z}}_B I_3' Z_A) \begin{array}{ll} > 0 & (\theta = -\pi/2) \\ < 0 & (\theta = \pi/2) \;. \end{array}$$

Let $\theta = 0$. Then

$$\sqrt{-1}({}^t\bar{\bar{Z}}_{A-1}I_3' Z_B - {}^t\bar{\bar{Z}}_{B-1}I_{n+1}' Z_A) = 0 \;.$$

We remark that by the proof of Theorem 1(1), we may choose r_0, r_1, r_2 and θ arbitrary real numbers. Our proof is now complete.

Remark. By the above theorem, we see that even if Z_A are real for all $A \in G$, we cannot conclude that $Z_A = 0$. In this case Theorem C does not hold.

Proof of Theorem 4. (1) We give $(2q-1)(g-1)$ real numbers $a_{j0}, \cdots, a_{j,q-2}, a_{jq}, \cdots, a_{jn}, b_{j,q-1}$ $(j = 1, \cdots, g-1)$. Then we will show that there uniquely exists $\mathfrak{f} \in \boldsymbol{E}^{01}_{1-q}(U, G, M)$ such that

$$x_{A_j} = {}^t(a_{j0}, \cdots, a_{j,q-2}, *, a_{jq}, \cdots, a_{jn})$$

and

$$x_{B_j} = {}^t(*, \cdots, *, b_{j,q-1}, *, \cdots, *) \;,$$

where $\alpha_A(\mathfrak{f}) = X_A = x_A + \sqrt{-1}y_A$ for $A \in G$. Since $(M(A_j) - I_{n+1})x_{B_j} = (M(B_j) - I_{n+1})x_{A_j}$, we uniquely determine $x_{A_1}, x_{B_1}, \cdots, x_{A_{g-1}}, x_{B_{g-1}}$ by the same method as in the proof of Theorem 1(1). By coboundary property and $x_{S_g} = 0$, we may set $x_{A_g} = x_{B_g} = 0_{n+1}$. By Theorem C, there uniquely exists $\mathfrak{f} \in \boldsymbol{E}^{01}_{1-q}(U, G, M)$ such that $\operatorname{Re}\alpha_A(\mathfrak{f}) = x_A$ for $A \in G$.

We set $E(z) = (1/n!)^t\mathfrak{f}(z)I_{n+1}'\left(\begin{matrix}1 \\ z\end{matrix}\right)^n$, $z \in U$. Then $E \in \boldsymbol{E}^{01}_{1-q}(U, G)$. Set $D^{2q-1}E(z) = \phi(z) \in \boldsymbol{B}_q(U, G)$ and $\operatorname{Pot}(c_q\phi)(z) = E_1(z)$, $z \in \boldsymbol{L}$. Set $E_2(z) = \overline{E_1(\bar{z})}$, $z \in U$. Then by Lemma 6, we have that $E - E_2 \in \Pi_{2q-2}$. Noting that

$$\mathrm{pd}_A\, E(z) = (1/n\,!)^t\widetilde{\alpha_A(\mathfrak{f}(z))}^t\widetilde{M(A)}^{-1}I'_{n+1}\binom{z}{1}^n, \qquad z \in U$$

and

$$\mathrm{pd}_A\, \mathrm{Pot}\,(c_q\phi)(z) = (1/n\,!)^t\widetilde{\beta_A^*(\mathfrak{g}(z))}^t\widetilde{M(A)}^{-1}I'_{n+1}\binom{z}{1}^n, \qquad z \in L\,,$$

we have that $\alpha_A(\mathfrak{f}(z)) = \overline{\beta_A^*(\mathfrak{g}(\bar{z}))}$ for $A \in G$ and $z \in U$, where $\mathfrak{g}(z)$ is a column function vector associated with $c_q\phi$, We set $Z = \alpha(\mathfrak{f}) + \beta^*(\mathfrak{g})$. Then by the above thing, $Z \in H_1^1(G, \boldsymbol{C}^{2q-1}, M)$. Noting that $Z_A = 2x_A$ for $A \in G$, by the above construction, we easily see that $Z \in H_0^1(G, \boldsymbol{C}^{2q-1}, M)$. Hence $Z \in H_0^1(G, \boldsymbol{C}^{2q-1}, M) \cap H_1^1(G, \boldsymbol{C}^{2q-1}, M)$.

(2) and (3) are proved by a similar method as in the first half of the above proof. Our proof is now complete.

Proof of Theorem 5. (1) At first we remark the following. Let $E \in \boldsymbol{E}_{1-q}(U, G)$ and $\phi \in \boldsymbol{B}_q(U, G)$. Let $\mathfrak{f}$ and $\mathfrak{g}$ be column function vectors associated with E and ϕ, respectively. Then $\mathrm{pd}_{S_i}\mathfrak{f} = 0$ if and only if $\mathrm{pd}_{S_i} E = 0$ and $\mathrm{pd}_{S_j}\mathfrak{g} = 0$ if and only if $\mathrm{pd}_{S_j}\mathrm{Pot}\,(\phi) = 0$.

Let $E \in \boldsymbol{E}_{1-q}^{01}(U, G)$ and let $\mathfrak{f}$ be a column function vector associated with E such that $\mathrm{Re}\,\alpha_{S_j}(\mathfrak{f}) = 0$, $j = 1, \cdots, g$. Set $\phi = D^{2q-1}E \in \boldsymbol{B}_q(U, G)$. We will show that $\phi \in \boldsymbol{B}_q^{01}(U, G)$. Set $E_1(z) = \mathrm{Pot}\,(\phi)(z)$, $z \in L$. Then $E_1 \in \boldsymbol{E}_{1-q}(L, G)$. Set $E_2(z) = \overline{E_1(\bar{z})}$, $z \in U$. Then $E_2 \in \boldsymbol{E}_{1-q}(U, G)$. Then by Lemma 6, $c_qE_2(z) - E(z) \in \prod_{2q-2}$, $z \in U$. Since $\mathrm{Re}\,\alpha_{S_j}(\mathfrak{f}(z)) = 0$, $z \in U$, $\mathrm{Re}\,\beta_{S_j}^*(\mathfrak{g}(z)) = \mathrm{Re}\,c_q^{-1}\alpha_{S_j}\overline{(\mathfrak{f}(\bar{z}))} = \mathrm{Re}\,c_q^{-1}\alpha_{S_j}(\mathfrak{f}(\bar{z})) = 0$, $z \in L$, where $\mathfrak{g}$ is a column function vector associated with ϕ. Hence $\phi \in \boldsymbol{B}_q^{01}(U, G)$. Thus $D^{2q-1}\boldsymbol{E}_{1-q}^{01}$ $\cdot(U, G) \subset \boldsymbol{B}_q^{01}(U, G)$.

Conversely, we assume that $\phi \in \boldsymbol{B}_q^{01}(U, G)$ and $\mathfrak{g}$ be a column function vector associated with ϕ. Then there exists $f \in \boldsymbol{E}_{1-q}(U, G)$ such that $D^{2q-1}f = \phi$. We will show that $E \in \boldsymbol{E}_{1-q}^{01}(U, G)$, where E is a representative of f. We construct E_1 and E_2 from ϕ as above, and define $E(z)$ by setting $E(z) = c_qE_2(z)$, $z \in U$. Then by Lemma 6, $D^{2q-1}E(z) = \phi(z)$. Since $\mathrm{Re}\beta_{S_j}^*(\mathfrak{g}(z)) = 0$, $z \in L$, $\mathrm{Re}\,\alpha_{S_j}(\mathfrak{f}(z)) = \mathrm{Re}\,c_q\beta_{S_j}^*\overline{(\mathfrak{g}(\bar{z}))} = \mathrm{Re}\,c_q\beta_{S_j}^*(\mathfrak{g}(\bar{z})) = 0$, $z \in U$, where $\mathfrak{f}$ is a column function vector associated with E. Hence $D^{2q-1}\boldsymbol{E}_{1-q}^{01}(U, G) \supset \boldsymbol{B}_q^{01}(U, G)$. Thus $D^{2q-1}\boldsymbol{E}_{1-q}^{01}(U, G) = \boldsymbol{B}_q^{01}(U, G)$.

(2) is similarly proved as above. Our proof is now complete.

Appendix. We will represent by means of matrices the period relation and inequalities obtained by Sato [8]. At first we introduce some notations. Let Γ be a finitely generated Kleinian group and $\varDelta$ be a simply connected component of the region of discontinuity of Γ. Let $e = \dim_C \boldsymbol{E}_{1-q}(\Delta_1, \Gamma)$ and let $E_1, \cdots, E_e$ a basis of $\boldsymbol{E}_{1-q}(\Delta_1, \Gamma)$, where $\Delta_1 = \bigcup_{A \in \Gamma} A(\Delta)$. Set $\mathrm{pd}_{A_j} \mathfrak{f}_i = X^i_{A_j}$ and $\mathrm{pd}_{B_j} \mathfrak{f}_i = X^i_{B_j}$, where $\mathfrak{f}_i$ are column function vectors associated with E_i $(i = 1, \cdots, e)$. We define $\Omega[A_1, \cdots, A_g, B_1, \cdots, B_g]$, χ_e and $M(A_1, \cdots, A_g, B_1, \cdots, B_g)$ as follows.

$$\Omega[A_1, \cdots, A_g, B_1, \cdots, B_g] = \begin{bmatrix} X^1_{A_1} X^2_{A_1} \cdots X^e_{A_1} \\ \cdots\cdots\cdots\cdots \\ X^1_{A_g} X^2_{A_g} \cdots X^e_{A_g} \\ X^1_{B_1} X^2_{B_1} \cdots X^e_{B_1} \\ \cdots\cdots\cdots\cdots \\ X^1_{B_g} X^2_{B_g} \cdots X^e_{B_g} \end{bmatrix},$$

$$\chi_e = \begin{array}{c} \left[\begin{array}{c|c} \begin{matrix} 0 & & & -I'_{n+1} \\ & & -I'_{n+1} & \\ & \iddots & & \\ -I'_{n+1} & & & 0 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0 & & & I'_{n+1} \\ & & I'_{n+1} & \\ & \iddots & & \\ I'_{n+1} & & & 0 \end{matrix} \end{array}\right] \begin{matrix} \}e \\ \\ \}e \end{matrix} \\ \quad\underbrace{\hphantom{xxxxxxxxxx}}_{e} \quad \underbrace{\hphantom{xxxxxxxxxx}}_{e} \end{array}$$

and

$$M(A_1, \cdots, A_g, B_1, \cdots, B_g) = \begin{bmatrix} M(A_1) & & & & & 0 \\ & \ddots & & & & \\ & & M(A_g) & & & \\ & & & M(B_1) & & \\ & & & & \ddots & \\ 0 & & & & & M(B_g) \end{bmatrix}.$$

Let G be a Fuchsian group of the first kind generated by $\{A_1, B_1, \cdots, A_g, B_g\}$ with a relation $\prod_{j=1}^{g} B_j^{-1} A_j^{-1} B_j A_j = 1$, Let $f_1, \cdots, f_e$ be a basis of $\boldsymbol{E}_{1-q}(U, G)$ and $E_1, \cdots, E_e$ representatives of $f_1, \cdots, f_e$, respectively. Let $\mathfrak{f}_j$ be column function vectors associated with E_j $(j = 1, \cdots, e)$. Set $D^{2q-1} E_j = \phi_j \in \boldsymbol{B}_q(U, G)$ and $\mathrm{pd}_A \mathfrak{f}_j = X^j_A$, $A \in G$ $(j = 1, \cdots, e)$. Then we have the following.

THEOREM A′. *Let G be as in Theorem* A. *Then*

$$\begin{aligned}&{}^t\tilde{\Omega}[A_1^{-1},\cdots,A_g^{-1},B_1^{-1},\cdots,B_g^{-1}]\chi_e\Omega[A_1,\cdots,A_g,B_1,\cdots,B_g]\\&\qquad+{}^t\tilde{\Omega}[A_1,\cdots,A_g,B_1^{-1},\cdots,B_g^{-1}]\chi_e M(A_1,\cdots,A_g,A_1,\cdots,A_g)\\&\qquad\quad\Omega[T_0,\cdots,T_{g-1},T_0,\cdots,T_{g-1}]\\&\qquad+{}^t\tilde{\Omega}[A_1^{-1},\cdots,A_g^{-1},B_1,\cdots,B_g]\chi_e M(B_1,\cdots,B_g,B_1,\cdots,B_g)\\&\qquad\quad\Omega[T_1,\cdots,T_g,T_1,\cdots,T_g]\\&\quad=0_{n+1,n+1}\,.\end{aligned}$$

THEOREM B′. *Let G be as in Theorem* B. *Then*

$$\begin{aligned}\boldsymbol{P}=\{(-1)^{q-1}/2i\}\{&{}^t\bar{\tilde{\Omega}}[A_1^{-1},\cdots,A_g^{-1},B_1^{-1},\cdots,B_g^{-1}]\chi_e\Omega[A_1,\cdots,A_g,B_1,\cdots,B_g]\\&+{}^t\bar{\tilde{\Omega}}[A_1,\cdots,A_g,B_1^{-1},\cdots,B_g^{-1}]\chi_e M(A_1,\cdots,A_g,A_1,\cdots,A_g)\\&\quad\Omega[T_0,\cdots,T_{g-1},T_0,\cdots,T_{g-1}]\\&+{}^t\bar{\tilde{\Omega}}[A_1^{-1},\cdots,A_g^{-1},B_1,\cdots,B_g]\chi_e M(B_1,\cdots,B_g,B_1,\cdots,B_g)\\&\quad\Omega[T_1,\cdots,T_g,T_1,\cdots,T_g]\}\end{aligned}$$

is positive definite, that is, this means if we set $P_{ij}=((-1)^{q-1}/2\sqrt{-1})\cdot\Phi(X^i,X^j)$, then $\sum_{i,j}c_iP_{ij}c_j\geqq 0$.

Let Γ_1 be a subgroup of Γ which leaves Δ invariant and which is generated by $\{A_1,\cdots,A_g,B_1,\cdots,B_g\}$ with a relation $\prod_{j=1}^g B_j^{-1}A_j^{-1}B_jA_j=1$.

THEOREM D (*Theorem* 1 *in* [8]). *Let Γ and Γ_1 be as defined above. Let $f\in\boldsymbol{E}_{1-q}(\Delta_1,\Gamma)$, and E a representative of f and let $\mathfrak{f}$ be a column function vector associated with E. Set $D^{2q-1}E=\phi$, $q\geqq 2$ and $\mathrm{pd}_A\,\mathfrak{f}=X_A$, $A\in\Gamma_1$. Let $\psi\in\boldsymbol{B}_q(\Delta,\Gamma_1)$. Let $\mathfrak{g}$ be a column function vector associated with ψ and set $\mathfrak{G}(z)=I'_{n+1}\widetilde{\mathfrak{G}(z)}$. Set $\mathrm{pd}_A\,\mathfrak{G}=Q_A$, $A\in\Gamma_1$. Then*

$$\begin{aligned}&\sum_{j=1}^g{}^tQ_{A_j}[X_{A_j^{-1}B_j^{-1}A_jT_{j-1}}-X_{T_{j-1}}]+\sum_{j=1}^g{}^tQ_{B_j^{-1}}[X_{B_jA_jT_{j-1}}-X_{A_j^{-1}B_jA_jT_{j-1}}]\\&\quad=2\,in\,!\,(\phi,\psi)\,.\end{aligned}$$

By using Lemma 1, we can rewrite the above identity as follows.

$$\begin{aligned}&\sum_{j=1}^g[{}^tQ_{A_j}X_{B_j}-{}^tQ_{B_j}X_{A_j}]+\sum_{j=1}^g{}^t(Q_{A_j^{-1}}-Q_{B_j})M(A_j)X_{T_{j-1}}\\&\qquad+\sum_{j=1}^g{}^t(Q_{A_j}-Q_{B_j^{-1}})M(B_j)X_{T_j}=2\,in\,!\,(\phi,\psi)\,.\end{aligned}$$

Now $f_1,\cdots,f_e$ be a basis of $\boldsymbol{E}_{1-q}(\Delta_1,\Gamma)$ and $E_1,\cdots,E_e$ representatives of $f_1,\cdots,f_e$, respectively. Let $\mathfrak{f}_j$ be column function vectors associated with E_j $(j = 1, \cdots, e)$ and set $\mathrm{pd}_A \mathfrak{f}_j = X_A^j$, $A \in \Gamma_1$ and $D^{2q-1}E_j = \phi_j$. Let $\psi_1, \cdots, \psi_b$ be a basis of $\boldsymbol{B}_q(\Delta, \Gamma_1)$, where $b = \dim_C \boldsymbol{B}_q(\Delta, \Gamma_1)$. Let $\mathfrak{g}_j$ be column function vectors associated with ψ_j $(j = 1, \cdots, b)$, and set $\mathfrak{G}_j(z) = I'_{n+1}\widetilde{\mathfrak{g}_j(z)}$. Set $\mathrm{pd}_A \mathfrak{G}_j = Q_A^j$, $A \in \Gamma_1$. Then we have the following

THEOREM D′.

$$
\begin{aligned}
&{}^t\Omega^*[A_1, \cdots, A_g, B_1, \cdots, B_g]\chi^*\Omega[A_1, \cdots, A_g, B_1, \cdots, B_g] \\
&\quad + {}^t\Omega^*[A_1^{-1}, \cdots, A_g^{-1}, B_1, \cdots, B_g]\chi^* M(A_1, \cdots, A_g, A_1, \cdots, A_g) \\
&\qquad\qquad \Omega[T_0, \cdots, T_{g-1}, T_0, \cdots, T_{g-1}] \\
&\quad + {}^t\Omega^*[A_1, \cdots, A_g, B_1^{-1}, \cdots, B_g^{-1}]\chi^* M(B_1, \cdots, B_g, B_1, \cdots, B_g) \\
&\qquad\qquad \Omega[T_1, \cdots, T_g, T_1, \cdots, T_g] \\
&\quad = 2\,in! \begin{bmatrix} (\phi_1, \psi_1)(\phi_2, \psi_1) & \cdots & (\phi_e, \psi_1) \\ (\phi_1, \psi_2)(\phi_2, \psi_2) & \cdots & (\phi_e, \psi_2) \\ \cdots\cdots & \cdots & \cdots\cdots \\ (\phi_1, \psi_b)(\phi_2, \psi_b) & \cdots & (\phi_e, \psi_b) \end{bmatrix},
\end{aligned}
$$

where

$$
\chi^* = \left[\begin{array}{ccc|ccc} & & & I_{n+1} & & 0 \\ & 0 & & & \ddots & \\ & & & 0 & & I_{n+1} \\ \hline -I_{n+1} & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & -I_{n+1} & & & \end{array}\right]
$$

and

$$
\Omega^*[A_1, \cdots, A_g, B_1, \cdots, B_g] = \begin{bmatrix} Q_{A_1}^1 Q_{A_1}^2 & \cdots & Q_{A_1}^b \\ & \cdots\cdots & \\ Q_{A_g}^1 Q_{A_g}^2 & \cdots & Q_{A_g}^b \\ Q_{B_1}^1 Q_{B_1}^2 & \cdots & Q_{B_1}^b \\ & \cdots\cdots & \\ Q_{B_g}^1 Q_{B_g}^2 & \cdots & Q_{B_g}^b \end{bmatrix}.
$$

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Department of Mathematics
Shizuoka University