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LARGE INCREMENTS OF BROWNIAN MOTION

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1. Let $X(t)$ denote Brownian motion on the line $0 \leq t < \infty$, let $g(h) = (2h \log h^{-1})^{1/2}$, and let $0 < \alpha < 1$. Orey and Taylor [5] have investigated the random set defined by the inequalities

$$E_\alpha : 0 \leq t \leq 1, \limsup X(t + h) - X(t)/g(h) \geq \alpha$$

and proved that $P\{\dim E_\alpha = 1 - \alpha^2\} = 1$. Here we prove two theorems on E_α that reflect more subtle properties of E_α than its Hausdorff dimension alone.

THEOREM 1. *With probability* 1, *a certain compact subset of* E_α *carries a probability measure* μ *such that* $\hat{\mu}(u) = o(u^{\frac{1}{2}(\alpha^2-1)}), 1 \leq u < \infty$.

THEOREM 2. *Let* F *be a closed set in* $(0,1)$ *of dimension* $d \geq \alpha^2$. *Then*

$$P\{\dim F \cap E_\alpha \geq d - \alpha^2\} = 1 .$$

For every pair d, α with $1 > d \geq \alpha^2$, there is almost-sure equality for a certain fixed set F_1 of dimension d. For every α there is a set F_2 of dimension $1 - \alpha^2$ such that $\dim F_2 \cap E_\alpha = 1 - \alpha^2$ almost surely.

The standard reference concerning relations between Fourier-Stieltjes transforms and Hausdorff measures is [3]: in particular, by a theorem of Beurling [3, Ch. III], the property of E_α claimed in Theorem 1 is stronger than the lower bound on $\dim E_\alpha$ found by Orey and Taylor. For example, by a theorem of Zygmund [1, p. 413; 6] the property of E_α is not even shared by certain sets of positive Lebesgue measure. Further examples concerning dimension and Fourier analysis are presented in [2], theorems on Brownian motion and dimension in [4 a $-$ d], while the indeterminacy of intersections of random sets and fixed sets (as in the second and third statements in Theorem 2) was observed in [4e].

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2. In the proofs we need estimates for sums $\sum (p - \xi_n)a_n$, where the ξ_n are independent random variables with common distribution

$$P\{\xi_n = 1\} = p = 1 - P\{\xi_n = 0\}; \sigma^2 = \sum |a_n|^2 ,$$

$B = \max |a_n|$. The basic inequality is

$$pe^{t(1-p)} + (1-p)e^{-pt} \leq 1 + p(1-p)t^2 \leq \exp t^2 p ,$$

valid for $0 \leq p \leq 1, -1 \leq t \leq 1$. In case the coefficients are real, we find by Chebyshev's inequality

$$P\{|\sum| \geq Y\} \leq 2 \exp t^2 p\sigma^2 \exp - tY , \qquad 0 \leq tB \leq 1 .$$

Choosing the best value of t we find

$$P\{|\sum| \geq Y\} \leq 2 \exp - 1/4 p^{-1}\sigma^{-2}Y^2 , \qquad \text{provided } YB \leq 2p\sigma^2 .$$

In the case of complex numbers a_n in the sum $\sum (p - \xi_n)a_n$, we have merely to replace Y by $\frac{1}{2}Y$ and double the bounds so obtained; this estimate is rough, but sufficient.

3. Let S be the functional $\max |X(b) - X(a)|$ $(0 \leq a < b \leq 1)$; we need only the 'tail' of the distribution of S, namely

$$P\{S \geq Y\} = \exp - \tfrac{1}{2}Y^2 \exp o(Y^2) , \qquad Y \to +\infty .$$

This estimate is obtained simply from the Gaussian law and the reflection principle, and is of course valid for $P\{X(1) - X(0) \geq Y\}$. We use it now to obtain an estimate from [5], involving parameters $0 < \beta \leq 1$, $0 < b < 1$. The event

$$X(h) - X(0) \geq \beta g(h), |X(t) - X(0)| \leq 2b^{\frac{1}{2}}g(h) \qquad \text{on } [0, bh]$$

has probability $h^{\beta^2}h^{o(1)} - o(h)$ as $h \to 0+$. Thus the event

$$X(h) - X(t) \geq (\beta - 2b^{\frac{1}{2}})g(h) \qquad \text{on } 0 \leq t \leq bh$$

has probability $> h^{\beta^2}h^{o(1)}$ for small $h > 0$. With the aid of this inequality we can begin to construct the measure μ. Let $0 \leq r < s \leq 1$ and let I_n $(1 \leq n \leq N)$ be the usual division of (r, s) into adjacent intervals of length $(s - r)N^{-1}$; supposing that b^{-1} is an integer (as in [5]) we have a further subdivision of each I_n into intervals I_n^q $(1 \leq q \leq \mathrm{b}^{-1})$ of length

$(s-r)bN^{-1}$. An interval I_n^q with left extremity x is *selected* if

$$X(x+h) - X(t) \geq (\beta - 2b^{\frac{1}{2}})g(h) \qquad \text{on } x \leq t \leq x + bh ,$$

with $h = N^{-1}(s-r)$. We put $\beta_1 = \beta - 2b^{\frac{1}{2}}$ and suppose that $0 < \beta_1 < \beta < \alpha^2$. The selections of the intervals I_n^q $(1 \leq n \leq N)$ are mutually independent for each q, with a probability $p = p_N \geq N^{-\beta^2}N^{o(1)}$, for large N. Let m_0 be Lebesgue measure in (r, s), let ξ be the characteristic function of the selected intervals, and let $m_1(dx) = p^{-1}\xi(x)m_0(dx)$.

LEMMA. *For any $\varepsilon > 0$ the inequality $|\hat{m}_1(u) - \hat{m}_0(u)| < \varepsilon(1+u)^{\frac{1}{2}(\alpha^2-1)}$ for all $u > 0$ holds, with probability approaching 1 as $N \to \infty$.*

Proof. The parameter $q = 1, \cdots, b^{-1}$ determines decompositions $m_1 = \sum m_1^q$ and $m_0 = \sum m_0^q$; because b is fixed it is sufficient to prove the inequality for each pair $\hat{m}_0^q$ and $\hat{m}_1^q$, as we now do, dropping the superscript q. Now $\hat{m}_0(u) - \hat{m}_1(u) = \sum (1 - p^{-1}\xi_n) f_n(u)$ where

$$|f_n| \leq bN^{-1} < N^{-1} \quad \text{and} \quad |f_n(u)| \leq 2|u|^{-1} .$$

The last inequality follows from $\left|\int_x^y e^{tut}dt\right| \leq 2|u|^{-1}$. Setting $C(u) = \max|f_n(u)|$, we cast the sum into the shape treated in paragraph 2, except for a factor p^{-1}. The inequality in question is thus $|\sum (p - \xi_n) f_n(u)| < \varepsilon p(1+u)^{\frac{1}{2}(\alpha^2-1)}$, where $B = C(u), \sigma^2 = NC^2(u)$.

On the interval $0 \leq u \leq N$ we replace σ^2 and B by their common upper bound N^{-1}, and choose $Y = \varepsilon p N^{\frac{1}{2}(\alpha^2-1)}$. Then $YB < p\sigma^2$ and we obtain an exponential bound $4 \exp - cp^{-1}\sigma^{-2}Y^2$. Here the exponent exceeds $cpNN^{\alpha^2-1} > N^\delta$ because $-\beta^2 + 1 + \alpha^2 - 1 > 0$. When $u > N$ we use $B = 2u^{-1}, \sigma^2 = 4Nu^{-2}, Y = \varepsilon p u^{\frac{1}{2}}(\alpha^2 - 1)$. To choose the best value of t in Chebyshev's inequality we must verify that $YB < 2p\sigma^2$, and this is true if $u^{\alpha^2+1} < N^2$. The exponent obtained exceeds $cp^{-1}\sigma^{-2}Y^2 > pN^{-1}u^{\alpha^2+1} \geq N^\delta$, as before. For the remaining numbers u, defined by the inequality $u^{\alpha^2+1} > N^2$, we choose $t = \eta B^{-1}$ with a small constant $\eta > 0$ and obtain from Chebyshev's inequality a bound $4 \exp - cB^{-1}Y$, wherein $B^{-1}Y > N^\delta$.

Thus, for each individual $u \geq 0$ the inequality sought holds except on a set of measure $\exp - N^\delta$; in particular, at $u = 0$, $\|m_1\| < 2$, except on such a set. Thus, with probability near 1, the result is valid for fractions $u = jN^{-2}, 0 \leq j \leq N^4$, and since $\hat{m}_1 - \hat{m}_0$ has derivative at most 2, this disposes of the interval $0 \leq u \leq N^2$ (since the error introduced by passing to real numbers u doesn't exceed N^{-2}). When $u > N^2$, we use the inequality

$$|\hat{m}_1(u)| < 2u^{-1}p^{-1}\sum \xi_n < 4u^{-1}N = o(u^{-\frac{1}{2}}) .$$

An approach more congenial in Fourier analysis is to prove the inequality for all integers k and then pass to real numbers u by expanding e^{iut} in terms of e^{ikt}, on the interval $0 \leq t \leq 1$, whose length is less than 2π.

Using the lemma carefully we can prove Theorem 1. Once η is specified in $(0, \frac{1}{2})$ we state once and for all that the n-th step, in the inductive process to be described, must be accomplished except on a set of $P < \eta + \cdots + \eta^n$, and the measures constructed in the n-th step must have mass m in the interval $|1 - m| < \eta + \cdots + \eta^n$. In an obvious way we make $\beta - 2b^{\frac{1}{2}}$ increase to α. The lemma is applied first with $(r, s) = (0, 1)$, and the random measures m_1 constructed are step-functions, with level intervals $jN^{-1}(0 \leq j < N)$. As N is fixed, we can apply the lemma to these N intervals (r, s), and then sum the N different measures constructed to obtain a random measure m_2 such that $|\hat{m}_2(t) - \hat{m}_1(t)| < 1/4(1 + t)^{\frac{1}{2}(\alpha^2 - 1)}$ for all $t > 0$. The closed support of m_2 is contained in that of $m_1, \cdots$. Clearly we can find a limit measure μ, of mass between $1 - 2\eta$ and $1 + 2\eta$ such that $|\hat{\mu}(t) - \hat{m}_n(t)| \leq 2^{-n}(1 + t)^{\frac{1}{2}(\alpha^2 - 1)}$, supported in a compact subset of E_α. Since μ is defined except on a set of $P < 2\eta$, Theorem 1 is completely proved.

4. In the proof of Theorem 2 we require a lemma somewhat analogous to the one already proved; Fourier transforms are of little use here, since the set F need not carry any measure whose transform $\hat{\mu}$ tends to zero. We therefore work directly on the metrical properties of measures, assigning to each measure μ_0 on $[0, 1]$ a random measure μ_1 by the same process as before. Let $0 \leq \beta < \alpha < 1$, $\alpha^2 < d \leq 1$.

LEMMA. *Suppose that $\mu_0(I) \leq C|I|^d$ for all intervals I of length $|I|$. Then the inequality $|\mu_0(I) - \mu_1(I)| \leq \varepsilon|I|^{d-\alpha^2}$ for all intervals I holds, with probability approaching* 1 *as $N \to \infty$.*

Proof. Because $\mu_1(S) \leq p^{-1}\mu_o(S)$ for all sets S, the inequality is valid for intervals I so small that $2p^{-1}C|I|^d < \varepsilon|I|^{d-\alpha^2}$, or $|I|^{\alpha^2} < \varepsilon' p$. Now $p = N^{-\beta^2}N^{o(1)}$, so that the upper bound on $|I|$ exceeds N^{-1} for large N. For larger intervals we have the partition of μ_o and μ_1 determined by q, and as before we omit the superscript. Let then $I \subseteq (0,1)$ and observe that

$$\mu_0(I) - \mu_1(I) = p^{-1} \sum (p - \xi_n)\mu_0(I \cap I_n) .$$

Now $\mu_0(I \cap I_n) \leq CN^{-d}$, while $\sigma^2 \leq CN^{-d}\mu_0(I) \leq C'N^{-d}|I|^d$. To estimate the probability of the event $|\sum| > \varepsilon p |I|^{d-\alpha^2}$ we use the exponential integrals with $t = \eta N^d$ ($\eta > 0$ small). Here tY majorizes $t^2 p\sigma^2$ because $|I| \leq 1$; moreover $tY > cN^d|I|^{d-\alpha^2}p > N^\delta$ because $|I| \geq N^{-1}$ and $\alpha^2 > \beta^2$. This estimate is strong enough to account for the N^2 intervals I composed of adjacent intervals I_n; because the μ_1-measure of an interval of length N^{-1} is at most $Cp^{-1}N^{-d} = o(N^{\alpha^2-d})$, this in turn accounts for all intervals of length $|I| \geq N^{-1}$; now the lemma is completely proved.

To prove the first statement in Theorem 2, let $\dim F > e \geq \alpha^2$, so that F carries a measure μ_0 subject to a Lipschitz condition with exponent e [3, Ch. III]. The lemma can then be applied to construct a sequence of measures μ_N, concentrated in F, whose limit measure is concentrated in $F \cap E_\alpha$ and has mass $>\frac{1}{2}$; each μ_N fulfills a Lipschitz condition with exponent e, while the entire sequence fulfills a *uniform* (with respect to N) Lipschitz condition with exponent $e - \alpha^2$, ensuring that $F \cap E_\alpha$ has dimension at least $e - \alpha^2$. As before, this can be accomplished on a set of probability arbitrarily close to 1, so $P\{\dim F \cap E_\alpha \geq e - \alpha^2\} = 1$.

To prove the remaining statements in Theorem 2 we choose for F_1 and F_2 certain dyadic sets, defined as follows. To each strictly increasing sequence $M = (m_k)$ of positive integers we associate the set of all infinite sums $\sum \varepsilon_k 2^{-m_k}$ ($\varepsilon_k = 0, 1$). The Hausdorff dimension of F is then $\liminf k/m_k$, the lower density of M [2, Ch. II]. For F_1 we choose $m_k = [d^{-1}m_k]$, so that $m_{k+1} > m_k \geq 1$ and M has density $d \geq 1 - \alpha^2$. Each integer $k \geq 1$ determines a covering of F_1 by 2^k intervals J of length 2^{-m_k}; let us estimate the number of intervals J that contain a number t, for which $|X(t+h) - X(t)| > \beta g(h)$, for some number h in the range $2^{-m_k} \leq h \leq k2^{-m_k}$. The expected number is at most $2^k 2^{-\beta^2 m_k} 2^{o(k)}$, and it is almost sure that for large k a bound of this type is valid. Clearly this implies that $\dim F_1 \cap E_\alpha \leq d - \beta^2$ (whenever $\beta < \alpha^2$) hence $P\{\dim F_1 \cap E_\alpha \leq d - \alpha^2\} = 1$. Moreover, when $d < \alpha^2$ $F_1 \cap E_\alpha = \phi$ almost surely.

We now sketch briefly a curious result about the critical case $d = \alpha^2$, choosing a sequence M with $m_k = d^{-1}k + o(k), m_k - d^{-1}k \to \infty$. As will be explained below, F_1 carries a measure μ satisfying the Lipschitz condition in each exponent $d_1 < d$. Adapting the second lemma we can prove that $F_1 \cap E_\alpha$ almost surely supports a continuous measure and must then be uncountable; a proper choice of M, taking account of the distribution of S, yields a set F_1 of dimension α^2 such that

$$|X(t + h) - X(t)| \leq \alpha g(h) \qquad \text{for } h < h_0 \text{ and all } t \text{ in } F_1$$

almost surely. (The argument in this paragraph is adapted from [5].)

The sequence M defining F_2 is described in terms of its counting function v: $v(s) = k$ if $m_k \leq s < m_{k+1}$. We require that $d = 1 - \alpha^2$ and that

(1) $v(s) \geq ds + s^{1/2}$ for $s \geq s_0$,

(2) $\liminf s^{-1}v(s) = d$,

(3) $v(t^6) \geq t^6 - t$ for all integers t in an infinite set T.

Then F_2 carries a product measure μ_0 derived from its representation as a Cantor set; its modulus of continuity $w(h) = \sup \mu_0(a, a + h)$ is governed by the inequalities $w(2^{-s}) \leq 2 \cdot 2^{-v(s)}$.

Now we follow the proof of the first statement, setting $N = 2^{t^3}$ for some t in T. The inequality necessary for one step in the construction is $|\mu_0(I) - \mu_1(I)| \leq \varepsilon |I|^d$ for all intervals I. First of all $w(h) = o(h^d)$; thus $w(h) \leq \varepsilon p h^d$ for $h < N^{-1}$. In fact, for $h \leq 2^{-t^6}$, say $h = 2^{-s}$, $w(h)h^{-d} \leq 2^{-s^{1/2}}$, while $p > N^{-\beta^2} N^{o(1)}$, so the inequality $s \geq t^6$ yields $w(h)h^{-d} \leq N^{-1} = o(p)$. For $h > 2^{-t^6}$ we use the elementary inequality $w(h) \leq 2h \cdot 2^{t^6} w(2^{-t^6}) < 4h2^t$. Also, $4h2^t < \varepsilon p h^d$ when $h \leq N^{-1}$, because $d + \alpha^2 = 1$. Thus we have disposed of intervals I of length $< N^{-1}$.

For remaining numbers $h \geq N^{-1}$ we study sums $p^{-1} \sum \equiv p^{-1} \sum (p - \xi_n)\mu(I \cap I_n)$; in $\sum$ we have $B \leq w(N^{-1})$ and $\sigma^2 \leq w(h)w(N^{-1})$. In the exponential integrals we take $t = \eta w^{-1}(N^{-1})$ with a small $\eta > 0$ and obtain a bound

$$P\{|\textstyle\sum| > \varepsilon p h^d\} \leq 2 \exp \eta^2 p w(h) w^{-1}(N^{-1}) \exp - \varepsilon p h^d w^{-1}(N^{-1}) \,.$$

Now $w(h) = 0(h^d)$ so the exponent is negative for small η and has modulus $> cph^d w^{-1}(N^{-1}) \geq cpN^{-d} w^{-1}(N^{-1}) > N^\delta$ for a certain $\delta > 0$; these inequalities are sufficient to construct a measure on $F_2 \cap E_\alpha$ with modulus of continuity $0(h^d)$; so $F_2 \cap E_\alpha$ has dimension $1 - \alpha^2$.

By the same method we can prove an even stronger property for a set F_3 of dimension $1 - \alpha^2$. Let S be a sequence of positive numbers tending to 0, and let $E_\alpha(S)$ be defined by the functional $\limsup X(t+h) - X(t)/g(h), h \in S$. Then F_3 will be a compact set of dimension $1-\alpha^2$, and $\dim F_3 \cap E_\alpha(S) = 1-\alpha^2$ almost-surely, for each fixed null sequence S.

F_3 is a "compound" dyadic set, slightly more complicated than F_1 and F_2 in structure. Using the dyadic representation as before, we have sets D_q defined as follows: $x = \sum \varepsilon_k 2^{-k}$ is in D_k if either $\varepsilon_k = 0$ on $q \leq k \leq d^{-1}q$, or $\varepsilon_k = 0$ on $q^2 \leq k \leq d^{-1}q^2$; F_3 is the intersection $\cap D(q_j)$, where $q_{j+1} > (j+1)q_j^2$. Since each $D(q)$ has an efficient covering by dyadic intervals, $\dim F_3 \leq 1-\alpha^2 = d$. Each sequence of symbols $a_j = I$ or II determines a dyadic set contained in F_3: when $a_j = I$ we take the first alternative allowed in $D(q_j)$, and the second when $\alpha_j = II$. When the sequence $s = h_1 > h_2 > \cdots > h_m > \cdots$ is specified, there is a subsequence of S, say (h_m^*) and a choice of the symbols $\alpha_j = I, II$ with this property: the numbers $-\log h_m^*$ and the digits ε_k *omitted* from the dyadic set become far apart in the sense that for large m all integers k in $[-\varepsilon \log h_m^*, -\varepsilon^{-1} \log h_m^*]$ are unrestricted. Now, choosing the product measure on this special subset of F_3 and using $N \cong h_m^{-1}$ in the construction leading to F_2, we can construct a subset of $F_3 \cap E_\alpha(S)$ of dimension $1-\alpha^2$.

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