M. Nakai
Nagoya Math. J.
Vol. 56 (1974), 105–119

A TEST FOR PICARD PRINCIPLE

MITSURU NAKAI

A nonnegative locally Hölder continuous function $P(z)$ on $0 < |z| \leq 1$ will be referred to as a *density* on $0 < |z| \leq 1$. The *elliptic dimension* of a density $P(z)$ at $z=0$, $\dim P$ in notation, is defined to be the dimension of the half module of nonnegative solutions of the equation $\Delta u(z) = P(z)u(z)$ on the punctured unit disk $\Omega: 0 < |z| < 1$ with boundary values zero on $|z| = 1$. After Bouligand we say that the *Picard principle* is valid for a density P at $z = 0$ if $\dim P = 1$. The purpose of this paper is to establish the following practical test:

THEOREM. *The Picard principle is valid for a density $P(z)$ on $0 < |z| \leq 1$ at $z = 0$ if there exists a closed subset E of Ω such that $\Omega - E$ is connected and $z = 0$ is an irregular boundary point of the region $\Omega - E$ for the harmonic Dirichlet problem and*

$$(1) \qquad \int_{\Omega - E} P(z) \log \frac{1}{|z|} dxdy < \infty .$$

As a direct consequence of the theorem we see that if $P \in L^p(\Omega - E)$ $(1 < p \leq \infty)$ for an admissible exceptional set E as stated in the theorem, then the Picard principle is valid for P. Needless to say, here and also in the theorem the exceptional set E may be empty. We must also remark that (1) is *not* necessary for the validity of Picard principle as is seen by a simple example $P(z) = |z|^{-2}$ (cf. no. 12). The proof of the theorem will be given in nos. 9–8. In the last no. 12 we state four unsettled important problems related to elliptic dimensions.

1. Let $P(z)$ be a density on $0 < |z| \leq 1$, i.e. $P(z) \geq 0$ and $P(z)$ is locally Hölder continuous: $|P(z_1) - P(z_2)| \leq A_r |z_1 - z_2|^{\lambda_r}$ for every z_1 and z_2 in $0 < r \leq |z| \leq 1$ where $A_r \in (0, \infty)$ and $\lambda_r \in (0, 1]$ are constants which may depend on $r \in (0, 1)$. Such a density P can be considered to be a density on $\Omega: 0 < |z| < 1$ which is the restriction to Ω of a density on

Received March 22, 1974.

$\hat{\Omega}: 0 < |z| < \infty$. Let $\hat{P}(z)$ be the symmetric extension of a density $P(z)$ on $0 < |z| \leq 1$ to $\hat{\Omega}: \hat{P}(z) = P(z)$ on Ω and $\hat{P}(z) = P(1/\bar{z})$ for $1 \leq |z| < \infty$. Then $\hat{P}(z)$ is a density on $\hat{\Omega}$ and $\hat{P} \circ \tau = \hat{P}$, where τ is the involution of $\hat{\Omega}$ about $|z| = 1$, i.e. $\tau(z) = 1/\bar{z}$.

The basic tool of our proof is the unique solvability of the Dirichlet problem. Let R be a region in $\hat{\Omega}$ bounded by a finite number of disjoint analytic Jordan curves and Q be a density on $\hat{\Omega}$. For any $\varphi \in C(\partial R)$ there exists a unique function $Q_\varphi^R \in C(\bar{R})$ such that $Q_\varphi^R = \varphi$ on ∂R and Q_φ^R is a solution of $\Delta u = Qu$ on R. If $Q \equiv 0$, then we use the standard notation H_φ^R instead of Q_φ^R. The unique existence of H_φ^R can be seen e.g. by the Perron-Brelot method as can be found in any text book (cf. e.g. Tsuji [18]). By the same method we can see the unique existence of Q_φ^R but the following integral equation method is preferable for our purposes in the sense that it clarifies the relation between Q_φ^R and H_φ^R. Let $\mathrm{H}_R(z, \zeta)$ be the harmonic Green's function on R (cf. e.g. [18]) and consider the integral operator

$$(Tf)(z) = -\frac{1}{2\pi}\int_R H_R(z, \zeta)Q(\zeta)f(\zeta)d\xi d\eta \qquad (\zeta = \xi + i\eta) .$$

It is elementary to check that $f \in C^\alpha(D)$ implies $Tf \in C^{\alpha+1}(D)$ $(\alpha = 0, 1)$ and $\Delta Tf = Q \cdot f$ on D $(\alpha = 1)$ for an open set D in R and for an f on R for which Tf can be defined (cf. e. g. Miranda [9]). It is also easy to see that $f \in C(\bar{R})$ implies $Tf \in C(\bar{R})$ with $Tf = 0$ on ∂R, and that T is a compact operator from $C(\bar{R})$ into itself. By the maximum principle for subharmonic functions we see that 1 is not the proper value of T and therefore by the Riesz-Schauder theory, $I + T: C(\bar{R}) \to C(\bar{R})$ is bijective (cf. e.g. Yosida [19]). Hence Q_φ^R is obtained as $(I + T)^{-1}H_\varphi^R$:

$$Q_\varphi^R(z) = H_\varphi^R(z) - \frac{1}{2\pi}\int_R H_R(z, \zeta)Q(\zeta)Q_\varphi^R(\zeta)d\xi d\eta . \tag{2}$$

By the fact that $\varphi \to H_\varphi^R$ is a positive linear operator from $C(\partial R)$ into $C(\bar{R})$ with norm 1 and by the maximum principle of subharmonic functions, we see that $\varphi \to Q_\varphi^R$ is a positive linear operator from $C(\partial R)$ into $C(\bar{R})$ with norm 1.

Fix a $\zeta_0 \in R$ and let R_n be the region obtained from R by deleting the closed disk about ζ_0 with radius $1/n$ for large integer n and T_n be the corresponding integral operator: $C(\bar{R}_n) \to C(\bar{R}_n)$. Then $u_n = (I_n$

$+ T_n)^{-1}H_R(\cdot, \zeta_0)$ forms a decreasing sequence dominated by $H_R(\cdot, \zeta_0)$, and if we denote by $G_R(\cdot, \zeta_0)$ the limit function, then

$$G_R(z, \zeta_0) = H_R(z, \zeta_0) - \frac{1}{2\pi}\int_R H_R(z, \zeta)Q(\zeta)G_R(\zeta, \zeta_0)d\xi d\eta\ . \tag{3}$$

The function $G_R(z, \zeta)$ is referred to as the *Green's function* of $\Delta u = Qu$ on R. By (3) we see that $G_R(\cdot, \zeta) \in C^1(\bar{R} - \{\zeta\})$ and $\partial G_R(z, \zeta)/\partial t = \partial H_R(z, \zeta)/\partial t + O(1)$ as $z \to \zeta$ where $t = x$ and y. By this and by the Green formula we have the symmetry $G_R(z, \zeta) = G_R(\zeta, z)$. We denote by $C^\omega(\partial R)$ the class of real analytic functions on ∂R. If $\varphi \in C^\omega(\partial R)$, then H_φ^R is easily seen to belong to $C^1(\bar{R})$, and by (2) we see that $Q_\varphi^R \in C^1(\bar{R})$. By the Green formula

$$Q_\varphi^R(z) = \frac{1}{2\pi}\int_{\partial R} \varphi(\zeta)\frac{\partial}{\partial \nu_\zeta}G_R(z, \zeta)ds_\zeta \tag{4}$$

where $\partial/\partial\nu$ denotes the inner normal derivative and ds is the line element. This is primarily derived for $\varphi \in C^\omega(\partial R)$ but the denseness of $C^\omega(\partial R)$ in $C(\partial R)$ assures the validity of (4) for every $\varphi \in C(\partial R)$. As a consequence of (4) we have the Harnack inequality and the Harnack principle for nonnegative solutions of $\Delta u = Qu$.

2. For a density $P(z)$ on $0 < |z| \leq 1$ we shall study the half module $\mathscr{P}$ of nonnegative solutions u of the equation $\Delta u(z) = P(z)u(z)$ on the punctured disk $\Omega: 0 < |z| < 1$ with boundary values zero on $\beta: |z| = 1$. For the study of $\mathscr{P}$ we need to consider the half module $\mathscr{B}$ of nonnegative bounded solutions of $\Delta u = Pu$ on Ω with continuous boundary values on β. Let Ω_t be $0 < |z| < t$ and β_t be $|z| = t$ for $t \in (0, 1]$. Thus $\Omega_1 = \Omega$ and $\beta_1 = \beta$. We also consider auxiliary classes $\mathscr{P}_t$ and $\mathscr{B}_t$ of nonnegative and nonnegative bounded solutions of $\Delta u = Pu$ on Ω_t with boundary values zero and continuous boundary values on β, respectively. In particular $\mathscr{P}_1 = \mathscr{P}$ and $\mathscr{B}_1 = \mathscr{B}$. The boundary point $z = 0$ is of *parabolic character* (cf. Brelot [1], Ozawa [14], Royden [17]) in the following sense:

$$\mathscr{P}_t \cap \mathscr{B}_t = \{0\}\ . \tag{5}$$

Let $u \in \mathscr{P}_t \cap \mathscr{B}_t$. Since $\Delta u = Pu \geq 0, u$ is subharmonic on Ω_t. For any $\varepsilon > 0$ $s_\varepsilon(z) = -\varepsilon \log|z| - u(z)$ is superharmonic on Ω_t with $\liminf_{z \to \partial\Omega_t} s_\varepsilon(z) \geq 0$. The minimum principle for superharmonic functions yields $s_\varepsilon(z) \geq 0$ for every $\varepsilon > 0$ and therefore $u = 0$ on Ω_t. For any $u \in \mathscr{B}_t$ let $u_{t,s}$

be the solution of $\Delta u = Pu$ on $\Omega_t - \bar{\Omega}_s$ $(0 < s < t \leq 1)$ with boundary values u on β_t and zero on β_s. As a consequence of (5) we have

$$(6) \qquad u(z) = \lim_{s\to 0} u_{t,s}(z)$$

on Ω_t. In fact, let $v = \lim_{s\to 0} u_{t,s} \in \mathscr{B}_t$. Then $0 \leq v \leq u$ on Ω_t with $v = u$ on β_t. Thus $u - v \in \mathscr{P}_t \cap \mathscr{B}_t$, i.e. (6) is valid. Therefore $u \in \mathscr{B}_t$ is determined uniquely by its boundary values φ on β_t. We shall denote this u by $P_\varphi^{\Omega_t}$. Then $\varphi \to P_\varphi^{\Omega_t}$ is a positive linear operator from $C(\beta_t)$ into $\mathscr{B}_t \ominus \mathscr{B}_t$ with norm 1.

Fix a t and an s with $0 < s < t \leq 1$. Consider the operator $S = S_{s,t}$ from $\mathscr{P}_t$ into $\mathscr{P}_s$ given by $Su = u - P_u^{\Omega_t}$. Then S is a bijective half linear operator between $\mathscr{P}_t$ and $\mathscr{P}_s$ (Heins [4], Ozawa (15, 16]). If $Su = 0$, then $u = P_u^{\Omega_s}$ and u is bounded, i.e. $u \in \mathscr{P}_t \cap \mathscr{B}_t$ and $u = 0$. Thus S is injective. Let $v \in \mathscr{P}_s$ and $u_r = P_{\bar v}^{\Omega_t - \Omega_r}$ where $\bar{v} = v$ on β_r and $\bar{v} = 0$ on β_t $(0 < r < s)$ and $w_r = u_r - v \geq 0$. Since $\{u_r\}$ is increasing as $r \to 0$, if $u = \lim_{r\to 0} u_r$ is convergent, then $u \in \mathscr{P}_t$ and $\lim_{r\to 0} w_r = P_u^{\Omega_s}$ and $Su = v$, i.e. S is surjective. Let $f_t u$ be the flux $\int_0^{2\pi} [\partial u(re^{i\theta})/\partial r]_{r=t} t d\theta$ and $D_A(\varphi) = \int_A (|\nabla\varphi(z)|^2 + P(z)\varphi(z)^2) dxdy$ where $\nabla\varphi = (\varphi_x, \varphi_y)$. Then by the Green formula

$$f_r u_r - f_t u_r = D_{\Omega_t - \bar{\Omega}_r}(u_r) , \qquad f_r v - f_s v = D_{\Omega_s - \bar{\Omega}_r}(v) = D_{\Omega_t - \bar{\Omega}_r}(v)$$

where we set $v = 0$ on $\Omega_t - \Omega_s$. Again by the Green formula we see the Dirichlet principle: $D_{\Omega_t - \bar{\Omega}_r}(u_r) \leq D_{\Omega_t - \bar{\Omega}_r}(v)$. Hence

$$0 \leq -f_t u_r \leq -f_s v - f_r(u_r - v) .$$

Since $u_r - v \geq 0$ and $u_r - v = 0$ on β_r, we have $f_r(u_r - v) \geq 0$. Therefore

$$0 \leq \limsup_{r\to 0} (-f_t u_r) \leq -f_s v < \infty$$

and $\lim_{r\to 0} u_r = \infty$ does not hold and thus $u = \lim_{r\to 0} u_r$ is convergent. This means that $\dim P$ depends only on the behavior of P at $z = 0$.

Another consequence of (5) and actually of (6) is

$$(7) \qquad \int_{\Omega_s} |\nabla u(z)|^2 \, dxdy < \infty$$

for every $u \in \mathscr{B}_t$ and $s \in (0, t)$. This will play the essential role in the next no. 3. For $0 < r' < r < s$, the Dirichlet principle which is a simple consequence of the Green formula yields $D_{\Omega_s}(u_{s,r}) = D_{\Omega_s - \bar{\Omega}_{r'}}(u_{s,r'}) \leq D_{\Omega_s - \bar{\Omega}_r}(u_{s,r}) = D_{\Omega_s}(u_{s,r})$ where we have set $u_{s,r} = 0$ on $\bar{\Omega}_r$ and $u_{s,r'} = 0$ on $\bar{\Omega}_{r'}$. By (2) and (6) we have

$$\lim_{r' \to 0} \frac{\partial}{\partial a} u_{s,r'}(z) = \frac{\partial}{\partial a} u(z)$$

where $a = x$ and y, and by the Fatou lemma

$$D_{\Omega_s}(u) \leq \liminf_{r' \to 0} D_{\Omega_s}(u_{s,r'}) \leq D_{\Omega_s}(u_{s,r}) < \infty$$

for any fixed $r \in (0, s)$ and in particular we have (7).

3. The mean operation $u \to u^*$ is useful for the study of subharmonic functions. Let u be defined on Ω_t such that

$$u^*(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta$$

can be defined for $r \in (0, t)$. This is the case e.g. when u is subharmonic on Ω_t. If u is bounded subharmonic on Ω_t, then u can be extended to $|z| < t$ so as to be subharmonic by giving the value $\limsup_{z \to 0} u(z)$ at $z = 0$, and hence we have (cf. e.g. Tsuji [18])

$$\ell(u) \equiv \lim_{r \to 0} \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta = \limsup_{z \to 0} u(z) \ . \tag{8}$$

If $u \in \mathscr{B}_t$, then u is bounded subharmonic on Ω_t and therefore the above relation (8) is applicable to every $u \in \mathscr{B}_t$.

We now maintain that for any $u \in \mathscr{B}_t$ there exists an exceptional closed subset $E = E_u$ of $(0, t)$ with finite logarithmic measure

$$\int_E d \log r < \infty \tag{9}$$

such that

$$\lim_{|z| \notin E, z \to 0} u(z) = \ell(u) \ . \tag{10}$$

For the proof consider Fourier coefficients $c_n(r)$ and $s_n(r)$ $(n = 1, 2, \cdots)$ of $u(re^{i\theta})$ for any $r \in (0, t)$ as a function of θ:

$$\begin{cases} c_n(r) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} u(re^{i\theta})\cos n\theta d\theta \\ s_n(r) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} u(re^{i\theta})\sin n\theta d\theta \end{cases}$$

for $n = 1, 2, \cdots$. On setting

$$\varphi(r) = \left(\sum_{n=1}^{\infty} n^2(c_n(r)^2 + s_n(r)^2)\right)^{1/2}$$

we assert that

$$\int_0^1 \varphi(r) d\log r \le \frac{1}{\pi}\int_{\Omega_t} |\nabla u(z)|^2 dxdy < \infty . \tag{11}$$

Observe that

$$u(re^{i\theta}) = u^*(r) + \sum_{n=1}^{\infty} (c_n(r)\cos n\theta + s_n(r)\sin n\theta)$$

for $(r, \theta) \in (0, t) \times T$ with $T = (-\infty, \infty)/\text{mod}\, 2\pi$ and thus

$$u_\theta(re^{i\theta}) = \sum_{n=1}^{\infty} (-nc_n(r)\sin n\theta + ns_n(r)\cos n\theta) .$$

Therefore, in view of $|\nabla u(re^{i\theta})|^2 = u_r(re^{i\theta})^2 + r^{-2}u_\theta(re^{i\theta})^2$, we have

$$r^{-2}\varphi(r)^2 = \frac{1}{\pi}\int_0^{2\pi} r^{-2}u_\theta(re^{i\theta})^2 d\theta \le \frac{1}{\pi}\int_0^{2\pi} |\nabla u(re^{i\theta})|^2 d\theta .$$

A fortiori

$$\int_0^t \varphi(r)^2 d\log r \le \frac{1}{\pi}\int_0^{2\pi}\int_0^t |\nabla u(re^{i\theta})|^2 rdrd\theta ,$$

i.e. (11) is valid. Next set

$$a_n = \int_{t/(n+2)}^{t/(n+1)} \varphi(r)^2 d\log r$$

for $n = 1, 2, \cdots$. By (11) we have $\sum_{n=1}^{\infty} a_n < \infty$. We can find a decreasing sequence $\{\varepsilon_n\}$ converging to zero such that $\sum_{n=1}^{\infty} \varepsilon_n^{-2} a_n < \infty$. Let

$$E_n = \{r \in [t/(n+2), t/(n+1)] ; \varphi(r) \ge \varepsilon_n\}$$

and

$$E = E_u = \left(\bigcup_{n=1}^{\infty} E_n\right) \cap (0, t)$$

which is a closed subset of $(0, t)$. Observe that

$$\int_E d\log r = \sum_{n=1}^{\infty} \varepsilon_n^{-2} \int_{E_n} \varepsilon_n^2 d\log r \leq \sum_{n=1}^{\infty} \varepsilon_n^{-2} \int_{E_n} \varphi(r)^2 d\log r = \sum_{n=1}^{\infty} \varepsilon_n^{-2} a_n ,$$

i.e. we have (9). By the Schwarz inequality

$$(u(re^{i\theta}) - u^*(r))^2 = \left(\sum_{n=1}^{\infty} \left(nc_n(r)\frac{\cos n\theta}{n} + ns_n(r)\frac{\sin n\theta}{n}\right)\right)^2$$
$$\leq \left(\sum_{n=1}^{\infty} n^{-2}\right)\cdot\left(\sum_{n=1}^{\infty} n^2(c_n(r)^2 + s_n(r)^2)\right) .$$

Therefore we conclude that $|u(re^{i\theta}) - u^*(r)| \leq 6^{-1/2}\pi\varphi(r)$. For an arbitrary $\varepsilon > 0$ there exists by (8) an $r_1 \in (0, t)$ such that $|u^*(r) - \ell(u)| < \varepsilon/2$ for every $r \in (0, r_1)$. Let n be such that $6^{-1/2}\pi\varepsilon_n < \varepsilon/2$ and set $r_0 = \min(r_1, t/(n+1))$. Then

$$|u(re^{i\theta}) - \ell(u)| \leq 6^{-1/2}\pi\varphi(r) + |u^*(r) - \ell(u)| < \varepsilon$$

for every $r \in (0, r_0) - E$, i.e. we have (10).

4. The *P-unit* e_t on Ω_t is the function in $\mathscr{B}_t$ with $e_t | \beta_t = 1$. Using the P-unit $e = e_1$ on $\Omega = \Omega_1$ consider the equation

$$\Delta v(z) + 2\nabla \log e(z)\cdot\nabla v(z) = 0 \tag{12}$$

on Ω. Let v be a bounded nonnegative solution of (12) on Ω_s. Then the following maximum-minimum principle is valid for $t \in (0, s)$:

$$\sup_{z\in\Omega_t} v(z) = \max_{z\in\beta_t} v(z) , \qquad \inf_{z\in\Omega_t} v(z) = \min_{z\in\beta_t} v(z) . \tag{13}$$

By an easy computation one sees that $ev \in \mathscr{B}_s$. Let $c = \max_{\beta_t} v$ and $c' = \min_{\beta_t} v$. Then since $ev, ce, c'e \in \mathscr{B}_t$ and $c'e \leq ev \leq ce$ on β_t, we see, by (6) or by the remark after (6) in no. 2, that $c'e \leq ev \leq ce$ on Ω_t and thus $c' \leq v \leq c$ on Ω_t, i.e. (13) is valid.

5. Using results in nos. 3 and 4 we now maintain that under the assumption

$$\ell(e) > 0 \tag{14}$$

the following limit

$$\lim_{z\to 0} u(z)/e(z) = \ell(u)/\ell(e) \tag{15}$$

exists for every $u \in \mathscr{B}$.

For the proof, let E_u and E_e be exceptional sets in no. 3 for u and e on Ω, respectively. Then $E = E_u \cup E_e$ is also a closed subset of $(0,1)$ and (9) implies that

$$\int_E d\log r < \infty .$$

From this it follows that there exists a strictly decreasing sequence $\{r_n\}$ coverging to zero in $(0,1) - E$. Let ε be an arbitrary number in $(0, \ell(e))$. By (10) there exists an N such that

$$|u(z) - \ell(u)| < \varepsilon , \qquad |e(z) - \ell(e)| < \varepsilon$$

for every $z \in \beta_{r_n}$ with $n > N$. Let $u_e = u/e$. Then again by an easy computation u_e is a solution of (12). Since $u \leq ce$ on β with $c = \max_\beta u$, (6) or the remark after (6) in no. 2 implies that $u_e \leq c$ on Ω, i.e. u_e is bounded on Ω. Thus the maximum-minimum principle in no. 4 is applicable to u_e. Since

$$\frac{\ell(u) - \varepsilon}{\ell(e) + \varepsilon} \leq u_e(z) \leq \frac{\ell(u) + \varepsilon}{\ell(e) - \varepsilon}$$

on β_{r_n}, we have the same inequality on Ω_{r_n}. Therefore

$$\frac{\ell(u) - \varepsilon}{\ell(e) + \varepsilon} \leq \liminf_{z\to 0} \frac{u(z)}{e(z)} \leq \limsup_{z\to 0} \frac{u(z)}{e(z)} \leq \frac{\ell(u) + \varepsilon}{\ell(e) - \varepsilon}$$

is valid for every $\varepsilon \in (0, \ell(e))$ and (15) follows.

6. Let u be a continuous function on $\Omega \cup \beta$ such that u is a solution of $\Delta u = Pu$ on Ω. *Then the condition*

$$u(e^{i\theta}) = \left[\frac{\partial}{\partial r} u(re^{i\theta})\right]_{r=1} = 0$$

for every $\theta \in T = (-\infty, \infty)/\mathrm{mod}\, 2\pi$ *implies that* $u \equiv 0$ *on* Ω.

Let $\hat{P}$ be the symmetric extension to $\hat{\Omega}$ of P and fix a $t \in (0,1)$. Let R be the annulus $t < |z| < 1/t$. Consider the solution u_1 of $\Delta u = \hat{P}u$ on R with boundary values $u(z)$ on $|z| = t$ and $-u(1/\bar{z})$ on $|z| = 1/t$. By the symmetry of $\hat{P}$ about $|z| = 1, u_1(\tau(z))$ is also a solution of $\Delta u = Pu$ where $\tau(z) = 1/\bar{z}$. Since $u_1(\tau(z)) + u_1(z) = -u(1/(1/\bar{z})) + u(z) = 0$ on $|z| = t$ and similarly on $|z| = 1/t$, we have $u_1(z) + u_1(\tau(z)) = 0$ on R and in particular $u_1 = 0$ on $|z| = 1$. Thus $u_1(z) = u(z)$ on $t \leq |z| \leq 1$. This means that u has a C^2-extension to an open set containing $\Omega \cup \beta$. In particular $f(r) = r^{-2}u_{\theta\theta}(re^{i\theta})$ is continuous on $[t, 1]$ for any fixed $\theta \in T$ and the same is true of $g(r) = P(re^{i\theta})$. Consider the Cauchy problem for the linear ordinary equation

$$\varphi''(r) + r^{-1}\varphi'(r) + g(r)\varphi(r) = f(r)$$

whose coefficients are continuous on $[t, 1]$ with the initial condition

$$\varphi(1) = \varphi'(1) = 0$$

on $[t, 1]$. Then $u(re^{i\theta})$, as a function of r, is a solution of this problem besides the trivial solution $\varphi(r) \equiv 0$. By the uniqueness of the solution of the Cauchy problem we have $u(re^{i\theta}) \equiv 0$ on $[t, 1]$ for any fixed $\theta \in T$, and since t is arbitrary in $(0, 1)$, we conclude that $u(z) \equiv 0$ on Ω.

7. Let $G_{\Omega-\bar{\Omega}_t}(z, \zeta)$ be the Green's function of $\Delta u = Pu$ on $\Omega - \bar{\Omega}_t$ and $H_{\Omega-\bar{\Omega}_t}(z, \zeta)$ the harmonic Green's function. We simply denote by $H(z, \zeta)$ the harmonic Green's function of Ω and hence of $|z| < 1$, i.e.

$$H(z, \zeta) = \log\left|\frac{1 - \bar{\zeta}z}{z - \zeta}\right| .$$

by (3) we have

$$0 < G_{\Omega-\bar{\Omega}_t}(z, \zeta) \leq H_{\Omega-\bar{\Omega}_t}(z, \zeta) \leq H(z, \zeta) .$$

Since $G_{\Omega-\bar{\Omega}_t}(z, \zeta) \leq G_{\Omega-\bar{\Omega}_s}(z, \zeta)$ for $0 < s \leq t < 1$, the Harnack principle assures the existence of

$$G(z, \zeta) = \lim_{t \to 0} G_{\Omega-\bar{\Omega}_t}(z, \zeta)$$

which will be referred to as the Green's function of $\Delta u = Pu$ on Ω.

Under the assumption that the limit (15) exists for every $u \in \mathscr{B}$ we next prove the existence of

$$K(\zeta) \equiv \lim_{z \to 0} G(z, \zeta)/e(z) \in \mathscr{P} \tag{16}$$

for every fixed $\zeta \in \Omega$ (cf. Heins [4], Hayashi [3]).

Suppose $\zeta \in \Omega - \bar{\Omega}_t$ $(t \in (0, 1))$ and let $c(\zeta) = \max_{\beta_t} G(\cdot, \zeta)$ and $c'(\zeta) = \min_{\beta_t} e$. Since $G(\cdot, \zeta)$ and $(c(\zeta)/c'(\zeta))e$ are in $\mathscr{B}_t$ and the former is dominated by the latter on β_t, $\{G(z, \cdot)/e(z) ; z \in \Omega_t\}$ is a uniformly bounded family of positive solutions of $\Delta u = Pu$ on $\Omega - \bar{\Omega}_t$ for every $t \in (0, 1)$. Hence by the Harnack principle $\{G(z, \cdot)/e(z) ; z \to 0\}$ is a normal family on each compact set in Ω. Contrary to the assertion assume the limit

(16) does not exist. Then there exist two sequences $\{z_{j,n}\}$ $(j=1,2)$ in Ω coverging to zero such that

$$K_j(\zeta) = \lim_{n\to\infty} G(z_{j,n}, \zeta)/e(z_{j,n})$$

exist on Ω $(j=1,2)$ and $K_1(z) \not\equiv K_2(z)$ on Ω. Clearly $K_j \in \mathscr{P}$ $(j=1,2)$. For any $u \in \mathscr{B}$ let $u_t = u_{1,t}$ as in (6) and $G_t(z,\zeta) = G_{\Omega-\bar{\Omega}_t}(z,\zeta)$. By (4)

$$u_t(z) = -\frac{1}{2\pi}\int_0^{2\pi} u_t(e^{i\theta})\left[\frac{\partial}{\partial r}G_t(z, re^{i\theta})\right]_{r=1} d\theta$$

for $z \in \Omega - \bar{\Omega}_t$. On letting $t \to 0$ we have

$$u(z) = -\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\left[\frac{\partial}{\partial r}G(z, re^{i\theta})\right]_{r=1} d\theta$$

for every $z \in \Omega$ and a fortiori

$$\frac{u(z_{j,n})}{e(z_{j,n})} = -\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\left[\frac{\partial}{\partial r}\left(\frac{G(z_{j,n}, re^{i\theta})}{e(z_{j,n})}\right)\right]_{r=1} d\theta$$

for $j=1,2$. By (15), on letting $n\to\infty$, we have

$$\ell(u)/\ell(e) = -\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\left[\frac{\partial}{\partial r}K_j(re^{i\theta})\right]_{r=1} d\theta$$

for $j=1,2$. On setting $L(z) = K_1(z) - K_2(z)$, we conclude that

$$\int_0^{2\pi} u(e^{i\theta})\left[\frac{\partial}{\partial r}L(re^{i\theta})\right]_{r=1} d\theta = 0$$

for every $u \in \mathscr{B}$ and hence for every $u \in C(\beta)$. Thus

$$L(e^{i\theta}) = \left[\frac{\partial}{\partial r}L(re^{i\theta})\right]_{r=1} = 0$$

on T and therefore, by no. 6, $L(z) \equiv 0$, i.e. $K_1(z) \equiv K_2(z)$, a contradiction.

8. Under the assumption (16) we finally conclude that any function $u(z)$ in $\mathscr{P}$ is a constant multiple of $K(z)$, i.e. $\dim P = 1$ and thus the Picard principle is valid (cf. Martin [8], Nakai [11], S. Itô [5], etc.).

For any u in $\mathscr{P}$, let $\hat{u}_{t,s}$ be the solution of $\Delta u = Pu$ on $\Omega_t - \bar{\Omega}_s$ $(0 < s < t < 1)$ with boundary values u on β_t and zero on β_s. Let $G_s(z,\zeta) = G_{\Omega-\bar{\Omega}_s}(z,\zeta)$. Fix a $z \in \Omega - \bar{\Omega}_t$. The Green formula applied to u and $G(z,\cdot)$ for the region $\Omega - \bar{\Omega}_t$ yields

$$2\pi u(z) = -\int_{\beta_t} G(z,\zeta)\frac{\partial}{\partial\nu_\zeta}u(\zeta)ds_\zeta + \int_{\beta_t} u(\zeta)\frac{\partial}{\partial\nu_\zeta}G(z,\zeta)ds_\zeta$$

and also to $G_s(z,\cdot)$ and $\hat{u}_{t,s}$ for the region $\Omega_t - \bar{\Omega}_s$ with making $s \to 0$ yields

$$0 = -\int_{\beta_t} G(z,\zeta)\frac{\partial}{\partial\nu_\zeta}\hat{u}_t(\zeta)ds_\zeta + \int_{\beta_t} u(\zeta)\frac{\partial}{\partial\nu_\zeta}G(z,\zeta)ds_\zeta$$

where $\hat{u}_t = \lim_{s\to 0}\hat{u}_{t,s} \in \mathscr{B}_t$ with $\hat{u}_t = u$ on β_t. Subtraction of the latter from the former in the above two identities gives

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} G(z, te^{i\theta})\left[\frac{\partial}{\partial r}(\hat{u}_t(re^{i\theta}) - u(re^{i\theta}))\right]_{r=t-0} td\theta .$$

Since $u(re^{i\theta}) - \hat{u}_t(re^{i\theta}) \geq 0$ on Ω_t and zero on β_t,

$$d\mu_t(\theta) = \frac{1}{2\pi}e(te^{i\theta})\left[\frac{\partial}{\partial r}(u_t(re^{i\theta}) - u(re^{i\theta}))\right]_{r=t-0} td\theta \geq 0$$

on U. Let

$$K(z,\zeta) = G(z,\zeta)/e(\zeta) .$$

By (16), $K(z,\zeta)$, as a function of ζ, is continuous on $(|\zeta| < 1) - \{z\}$ and $K(z,0) = K(z)$, and we have

$$u(z) = \int_0^{2\pi} K(z, te^{i\theta})d\mu_t(\theta)$$

for $z \in \Omega - \bar{\Omega}_t$. Fix a $t_0 \in (0,1)$ and a $z_0 \in \Omega - \bar{\Omega}_{t_0}$. Since $K(z_0,\zeta)$, as a function of ζ, is a bounded solution of (12) on Ω_{t_0}, (13) implies that

$$a = \inf_{\Omega_{t_0}} K(z_0,\zeta) > 0 .$$

Set $c_t = \int_0^{2\pi} d\mu_t(\theta)$. Then $0 \leq c_t \leq u(z_0)/a$ for $t \in (0, t_0)$, and thus we can find a decreasing sequence $\{t_n\} \subset (0, t_0)$ coverging to zero such that $c = \lim_{n\to\infty} c_{t_n}$ exists. Hence by

$$u(z) = \int_0^{2\pi} (K(z, te^{i\theta}) - K(z))d\mu_{t_n}(\theta) + c_{t_n}K(z) ,$$

we deduce

$$|u(z) - cK(z)| \leq c_{t_n} \sup_{|\zeta|<t_n} |K(z,\zeta) - K(z,0)| + |c_{t_n} - c|\cdot K(z)$$

for $z \in \Omega - \bar{\Omega}_{t_n}$. On letting $n \to \infty$, we conclude that $u(z) = cK(z)$ on Ω.

9. We are ready to proceed to the proof of our theorem. All we have to prove is that the condition (1) implies (14), i.e. $\ell(e) > 0$. Then, by no. 5, (15) is valid and a fortiori (16) follows by no. 7, which in turn implies $\dim P = 1$ by no. 8.

Let $\{r_n\}$ be a decreasing sequence in $(0, 1)$ converging to zero and ε_n be such that $(r_{n+1} + r_n)/2 < r_n - \varepsilon_n < r_n + \varepsilon_n < (r_n + r_{n-1})/2$ for $n = 1, 2, \cdots$ with $r_0 = 1$ and that

$$\int_{A_n} P(z) \log \frac{1}{|z|} dxdy < 2^{-n}, \qquad A_n = \{r_n - \varepsilon_n < |z| < r_n + \varepsilon_n\} .$$

This can be achieved by taking $\varepsilon_n > 0$ sufficiently small. Replacing E in (1) by $E - \bigcup_{n=1}^{\infty} A_n$ we can thus assume

$$E \cap \beta_{r_n} = \phi \qquad (n = 1, 2, \cdots) .$$

We denote by e_n the P-unit on Ω_{r_n}. Let $\{S_{n,m}\}$ be an increasing sequence $(m = 1, 2, \cdots)$ of subregions $S_{n,m}$ of $\Omega_{r_n} - E$ such that $\partial S_{n,m}$ consists of a finite number of disjoint Jordan curves with β_{r_n} a component of $\partial S_{n,m}$ and $\bigcup_{m=1}^{\infty} S_{n,m} = \Omega_{r_n} - E$. We denote by $u_{n,m}$ ($h_{m,n}$, resp.) the solution of $\Delta u = Pu$ (the harmonic function, resp.) on $S_{n,m}$ with boundary values 1 on β_{r_n} and zero on $\partial S_{n,m} - \beta_{r_n}$. Let $H_{n,m}(z, \zeta)$ be the harmonic Green's function of $S_{n,m}$. Then $H_n(z, \zeta) = \lim_{m\to\infty} H_{n,m}(z, \zeta)$ is the harmonic Green's function of $\Omega_{r_n} - E$. By (2)

$$h_{n,m}(z) = u_{n,m}(z) + \frac{1}{2\pi} \int_{S_{n,m}} H_{n,m}(z, \zeta) P(\zeta) u_{n,m}(\zeta) d\xi d\eta .$$

Since $\{h_{n,m}\}$ ($\{u_{n,m}\}$, resp.) is increasing $(m = 1, 2, \cdots)$, $h_m = \lim_{m\to\infty} h_{n,m}$ ($u_n = \lim_{m\to\infty} u_{n,m}$, resp.) is a bounded harmonic function (a bounded solution of $\Delta u = Pu$, resp.) on $\Omega_{r_n} - E$ with boundary values 1 on β_{r_n}. Moreover, since $H_{n,m}$ is increasing, the Lebesgue-Fatou theorem yields

$$h_n(z) = u_n(z) + \frac{1}{2\pi} \int_{\Omega_{r_n} - E} H_n(z, \zeta) P(\zeta) u_n(\zeta) d\xi d\eta .$$

Let $H(z, \zeta) = \log(|1 - \bar{\zeta} z| / |z - \zeta|)$ be the harmonic Green's function on Ω and hence on $|z| < 1$. Observe that $u_n \le e_n \le 1$ and $H_n(z, \zeta) \le H(z, \zeta)$. Therefore

$$h_n(z) \le e_n(z) + \frac{1}{2\pi} \int_{\Omega_{r_n} - E} H(z, \zeta) P(\zeta) d\xi d\eta \tag{17}$$

for $z \in \Omega_{r_n}$ where we set $h_n = 0$ on E. On integrating both sides of (17) on the circle $|z| = r \in (0, r_n)$ and using the Fubini theorem and the circle mean formula of Green's function:

$$\frac{1}{2\pi}\int_0^{2\pi} H(re^{i\theta}, \zeta)d\theta = \min\left(\log\frac{1}{r}, \log\frac{1}{|\zeta|}\right) \leq \log\frac{1}{|\zeta|},$$

we deduce (cf. no. 3)

$$h_n^*(r) \leq e_n^*(r) + \frac{1}{2\pi}\int_{\Omega_{r_n}-E} P(\zeta)\log\frac{1}{|\zeta|}d\xi d\eta . \tag{18}$$

10. By comparing the boundary values we see that $h_{n+1,m} \geq h_{n,k}$ on $S_{n+1,m} \cap S_{n,k}$ if m is sufficiently large for any fixed k. Therefore $h_{n+1} \geq h_n$ on $\Omega_{r_{n+1}}$ and a fortiori $h_{n+1}^* \geq h_n^*$ on $(0, r_{n+1}]$ for $n = 1, 2, \cdots$. It is also clear that $e_{n+1}^* \geq e_n^*$ on $(0, r_{n+1}]$ for $n = 1, 2, \cdots$. Since we have set $h_n = 0$ on E, h_n is subharmonic on Ω_{r_n}, and clearly e_n is subharmonic on Ω_{r_n}. Therefore

$$a_n = \lim_{r\to 0} h_n^*(r) \geq 0 , \qquad b_n = \lim_{r\to 0} e_n^*(r) \geq 0$$

exist (cf. no. 3) and $a_n \leq a_{n+1} \leq 1$ and $b_n \leq b_{n+1} \leq 1$ for $n = 1, 2, \cdots$, and thus

$$a = \lim_{n\to\infty} a_n \in [0, 1] , \qquad b = \lim_{n\to\infty} b_n \in [0, 1]$$

exist. By (18) we have

$$a_n \leq b_n + \frac{1}{2\pi}\int_{\Omega_{r_n}-E} P(\zeta)\log\frac{1}{|\zeta|}d\xi d\eta .$$

In view of (1) we have

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_{\Omega_{r_n}-E} P(\zeta)\log\frac{1}{|\zeta|}d\xi d\eta = 0$$

and finally we conclude that $a \leq b$, i.e.

$$\lim_{n\to\infty}\left(\lim_{r\to 0} h_n^*(r)\right) \leq \lim_{n\to\infty}\left(\lim_{r\to 0} e_n^*(r)\right) . \tag{19}$$

11. Since $h_n > 0$ on $\Omega_{r_n} - E$ and $z = 0$ is an irregular boundary point of $\Omega - E$ and hence of $\Omega_{r_n} - E$, the Bouligand criteriond assures that

$$\limsup_{z\in\Omega_{r_n}-E,z\to 0} h_n(z) > 0 .$$

On the other hand h_n is subharmonic on Ω_{r_n} by the fact that we have defined $h_n = 0$ on E, and therefore (cf. no. 3)

$$a_n = \lim_{r\to 0} h_n^*(r) = \limsup_{z\to 0} h_n(z) = \limsup_{z\in\Omega_{r_n}-E,z\to 0} h_n(z) > 0$$

for every $n = 1, 2, \cdots$. Since $a_n \leq a_{n+1}$, we conclude that $a = \lim_{n\to\infty} a_n > 0$ and by (19) $\lim_{n\to\infty}(\lim_{r\to 0} e_n^*(r)) > 0$. Thus there exists an n such that

$$\lim_{r\to 0} e_n^*(r) > 0 .$$

Let $c = \inf_{\beta_{r_n}} 1/e > 0$. Then $e_n \leq ce$ on β_{r_n} implies that $e_n \leq ce$ on Ω_{r_n} and thus $e_n^* \leq ce^*$. Therefore

$$c\ell(e) = \ell(ce) = \lim_{r\to 0}(ce)^*(r) = \lim_{r\to 0} ce^*(r) \geq \lim_{r\to 0} e_n^*(r) > 0 ,$$

i.e. we have shown that $\ell(e) > 0$, i.e. (14) is valid.

The proof of the theorem is herewith complete.

12. At the end we state several important open problems related to elliptic dimensions. Let P and Q be densities on $0 < |z| \leq 1$ and $c \geq 1$ a real number. We ask:

PROBLEM 1. Is the relation $\dim cP = \dim P$ valid;

PROBLEM 2. Does the inequality $P \leq Q$ imply $\dim P \leq \dim Q$?

In the affirmative case we can deduce the important *order comparison theorem*: If $c^{-1}P \leq Q \leq cP$ then $\dim P = \dim Q$, which is in question at present. These problems should also be asked for densities on Riemann surfaces (cf. Royden [17], Nakai [12], Lathtinen [7], etc.).

If we restrict ourselves to rotation free densities P on $0 < |z| \leq 1$, i.e. densities satisfying $P(z) = P(|z|)$ on Ω, then we know that $\dim P$ is either 1 or the cardinal number $\mathfrak{c}$ of continuum and we have a complete criterion for $\dim P = 1$ (cf. Nakai [13]). It is also instructive for the study of elliptic dimensions to observe the following example: For densities $P_\lambda(z) = |z|^{-\lambda}$, $\dim P_\lambda = 1$ if $\lambda \in [-\infty, 2]$ and $\dim P_\lambda = \mathfrak{c}$ if $\lambda \in (2, \infty)$ where $P_{-\infty} \equiv 0$ (see [13]). Related to these we ask for general densities P on $0 < |z| \leq 1$ the following

PROBLEM 3. How widely the range of $P \to \dim P$ can cover cardinals;

PROBLEM 4. What is the comprehensive complete condition for dim $P = 1$?

These can also be discussed in the frame of Riemann surface setting, e.g. for densities on ends in the sense of Heins [4] (cf. Ozawa [15, 16], Myrberg [10], Kuramochi [6], Constantinescu-Cornea [2], Hayashi [3], etc.).

REFERENCES

[1] M. Brelot: Étude de l'équation de la chaleur $\Delta u = c(M)u(M)$, $c(M)$, au voisinage d'un point singulier du coefficient, Ann. Ec. N. Sup., **48** (1931), 153–246.
[2] C. Constantinescu-A. Cornea: Über einige Problem von M. Heins, Rev. math. pures appl., **4** (1959), 277–281.
[3] K. Hayashi: Les solutions positives de l'équation $\Delta u = Pu$ sur une surface de Riemann, Kōdai Math. Sem. Rep., **13** (1961), 20–24.
[4] M. Heins: Riemann surfaces of infinite genus, Ann. Math., **55** (1952), 296–317.
[5] S. Itô: Martin boundary for linear elliptic differential operators of second order in a manifold, J. Math. Soc. Japan, **16** (1964), 307–334.
[6] Z. Kuramochi: An example of a null-boundary Riemann surface, Osaka Math. J., **6** (1954), 83–91.
[7] A. Lahtinen: On the equation $\Delta u = Pu$ and the classification of acceptable densities on Riemann surfaces, Ann. Acad. Sci. Fenn., **533** (1973), 1–26.
[8] R. Martin: Minimal positive harmonic functions, Trans. Amer. Math. Soc. **49** (1941).
[9] C. Miranda: Partial Differential Equations of Elliptic Type, Springer, 1970.
[10] L. Myrberg: Über die Existenz der Greenschen Funktion der Gleichung $\Delta u = c(P)u$ auf Riemannschen Flächen, Ann. Acad. Sci. Fenn., **170** (1954).
[11] M. Nakai: The space of nonnegative solutions of the equation $\Delta u = Pu$ on a Riemann surface, Kōdai Math. Sem. Rep., **12** (1960), 151–178.
[12] M. Nakai: Order comparisons on canonical isomorphisms, Nagoya Math. J., **50** (1973), 67–87.
[13] M. Nakai: Martin boundary over an isolated singularity of rotation free density, J. Math. Soc. Japan, **26** (1974), 483–507.
[14] M. Ozawa: Classification of Riemann surfaces, Kōdai Math. Sem. Rep., **4** (1952), 63–76.
[15] M. Ozawa: Some classes of positive solutions of $\Delta u = Pu$ on Riemann surfaces, I, Kōdai Math. Sem. Rep., **6** (1954), 121–126.
[16] M. Ozawa: Some classes of positive solutions of $\Delta u = Pu$ on Riemann surfaces, II. Kōdai Math. Sem. Rep., **7** (1955), 15–20.
[17] H. Royden: The equation $\Delta u = Pu$, and the classification of open Riemann surfaces, Ann. Acad. Sci. Fenn., **271** (1959).
[18] M. Tsuji: Potential Theory in Mordern Function Theory, Maruzen, 1959.
[19] K. Yosida: Functional Analysis, Springer, 1965.

Nagoya University