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ON A CLASSIFICATION OF THE FUNCTION FIELDS OF ALGEBRAIC TORI

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Let Π be a finite group and denote by $\boldsymbol{M}_\Pi$ the class of all (finitely generated Z-free) Π-modules. In the previous paper [3] we defined an equivalence relation in $\boldsymbol{M}_\Pi$ and constructed the abelian semigroup $T(\Pi)$ by giving an addition to the set of all equivalence classes in $\boldsymbol{M}_\Pi$. The investigation of the semigroup $T(\Pi)$ seems interesting and important, because this gives a classification of the function fields of algebraic tori defined over a field k which split over a Galois extension of k with group Π.

The purpose of this paper is to obtain information on the structure of the semigroup $T(\Pi)$.

We will recall the definitions given in [2] and [3]. A Π-module is called a permutation Π-module if it can be expressed as a direct sum of $\{Z\Pi/\Pi_i\}$ where each Π_i is a subgroup of Π. Further a Π-module M is called a quasi-permutation Π-module if there exists an exact sequence $0 \to M \to S \to S' \to 0$ where S and S' are permutation Π-modules. The dual module $\mathrm{Hom}_Z(M, Z)$ of a Π-module M is denoted by M^*. The augmentation ideal of $Z\Pi$ is denoted by I_Π and the dual module I_Π^* of I_Π is called the Chevalley's module of Π ([1], [2]).

Let k be a field. Let K be a Galois extension of k with group $\cong \Pi$ and let M be a Π-module with a Z-free basis $\{u_1, u_2, \cdots, u_n\}$. Define the action on the rational function field $K(X_1, X_2, \cdots, X_n)$ with n variables $X_1, X_2, \cdots, X_n$ over K by putting, for each $\sigma \in \Pi$ and $1 \leqq i \leqq n$, $\sigma(X_i) = \prod_{j=1}^n X_j^{m_{ij}}$ when $\sigma \cdot u_i = \sum_{j=1}^n m_{ij} u_j$, $m_{ij} \in Z$, and denote by $K(M)$ $K(X_1, X_2, \cdots, X_n)$ with this action of Π. It is well known ([7]) that there is a duality between the category of all algebraic tori defined over k which split over K and the category of all Π-modules. In fact, if T is an algebraic torus defined over k which splits over K, then the character

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group $X(T)$ of T can be regarded as a Π-module, and conversely, if M is a Π-module, then there is an algebraic torus T defined over k which splits over K such that $X(T) \cong M$ as Π-modules. It should be noted that the function field of an algebraic torus T defined over k which splits over K can be identified with the invariant subfield $K(X(T))^{\Pi}$ of $K(X(T))$.

For Π-modules M and N we define a relation $M \underset{(t)}{\text{——}} N$ if, for any Galois extension K of k with group $\cong \Pi$, there exist variables $X_1, X_2, \cdots, X_s$ and $Y_1, Y_2, \cdots, Y_t$ such that $K(M)^{\Pi}(X_1, X_2, \cdots, X_s)$ is k-isomorphic to to $K(N)^{\Pi}(Y_1, Y_2, \cdots, Y_t)$. Then this is evidently an equivalence relation in $\boldsymbol{M}_{\Pi}$. Let $T(\Pi)$ be the set of all equivalence classes in $\boldsymbol{M}_{\Pi}$ and denote by $[M]$ the equivalence class containing $M \in \boldsymbol{M}_{\Pi}$. Define an addition in $T(\Pi)$ by $[M] + [N] = [M \oplus N]$ for $M, N \in \boldsymbol{M}_{\Pi}$. This makes $T(\Pi)$ an abelian semigroup with unit element [0]. It is noted that, by the Swan's theorem ([12] and [3], (1.2)), a Π-module M is a quasi-permutation Π-module if and only if $[M] = [0]$ (i.e., $M \underset{(t)}{\text{——}} 0$). The subgroup of $T(\Pi)$ consisting of all invertible elements in $T(\Pi)$ is denoted by $T^{g}(\Pi)$.

As in [3] denote by $C(Z\Pi)$ the projective class group of $Z\Pi$ and define $C^{q}(Z\Pi) = \{[\mathfrak{A}] - [Z\Pi] \in C(Z\Pi) \mid \mathfrak{A}$ is a quasi-permutation, projective ideal of $Z\Pi\}$. Then $C^{q}(Z\Pi)$ is a subgroup of $C(Z\Pi)$ and the factor group $C(Z\Pi)/C^{q}(Z\Pi)$ can be regarded as a subgroup of $T^{g}(\Pi)$. Let $\Omega_{Z\Pi}$ be a maximal order in $Q\Pi$ containing $Z\Pi$ and denote by $C(\Omega_{Z\Pi})$ the projective class group of $\Omega_{Z\Pi}$. It has been shown in [3] and [4] that, for a fairly extensive class of finite groups Π, $C(Z\Pi)/C^{q}(Z\Pi) \cong C(\Omega_{Z\Pi})$. In such cases, the group $C(\Omega_{Z\Pi})$ can also be regarded as a subgroup of $T^{g}(\Pi)$. Further denote by $G(Z\Pi)$ the Grothendieck group of $Z\Pi$. Let $B_G(Z\Pi)$ be the subgroup of $G(Z\Pi)$ generated by all the images of permutation Π-modules in $G(Z\Pi)$ and define $Sw(\Pi) = G(Z\Pi)/B_G(Z\Pi)$. Then there exist natural homomorphisms $\theta_{\Pi} : T^{g}(\Pi) \to Sw(\Pi)$ and $\omega_{\Pi} : C(\Omega_{Z\Pi}) \to G(Z\Pi)$.

Our main results in this paper are the following:

[I] *The following statements on a finite group Π are equivalent*:

(1) *Every Sylow subgroup of Π is cyclic.*

(2) $T(\Pi) = T^{g}(\Pi)$, *i.e., $T(\Pi)$ is a group.*

(3) $[I_{\Pi}^{*}] \in T^{g}(\Pi)$.

[II] *The following statements on a finite group Π are equivalent*:

(1) *Π is a cyclic group or a direct product of a cyclic group of order n and a group with generators ρ, τ and relations $\rho^{k} = \tau^{2^d} = I$ and $\tau^{-1}\rho\tau$*

$=\rho^{-1}$ *where* $d \geqq 1$, $k \geqq 3$, *and both* n *and* k *are odd integers such that* $(n, k)=1$.

(2) I_{Π}^{*} *is a quasi-permutation* Π*-module, i.e.,* $I_{\Pi}^{*} \underset{(t)}{\text{——}} 0$.

(3) *The order of* $[I_{\Pi}^{*}]$ *in* $T(\Pi)$ *is finite.*

[III] *If* Π *is a finite p-group, then* $T^{q}(\Pi)=C(Z\Pi)/C^{q}(Z\Pi)$.

[IV] *The following statements on a finite group* Π *are equivalent:*

(1) Π *is* (i) *a cyclic group,* (ii) *a dihedral group of order* $2p^{c}$ *where* p *is an odd prime and* $c \geqq 1$, (iii) *a direct product of a cyclic group of order* q^{f} *and a dihedral group of order* $2p^{c}$ *where* $f, c \geqq 1$, p *and* q *are odd primes and* p *is a primitive* $q^{f-1}(q-1)$*-th root of unity modulo* q^{f}, *or* (iv) *a generalized quaternion group of order* $4p^{c}$ *where* p *is an odd prime congruent to* 3 *modulo* 4 *and* $c \geqq 1$.

(2) $T(\Pi)=C(Z\Pi)/C^{q}(Z\Pi)=C(\Omega_{Z\Pi})$.

(3) $T(\Pi)$ *is a finite group.*

[V] *The following statements on a finite group* Π *are equivalent:*

(1) $T(\Pi)=C(\Omega_{Z\Pi})$ *and* $\omega_{\Pi}: C(\Omega_{Z\Pi}) \rightarrow G(Z\Pi)$ *is a monomorphism.*

(2) $T(\Pi)=T^{q}(\Pi)$ *and* $\theta_{\Pi}: T(\Pi) \rightarrow Sw(\Pi)$ *is an isomorphism.*

(3) *The dual module of a quasi-permutation* Π*-module is always a quasi-permutation* Π*-module.*

(4) *If* $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ *is an exact sequence of* Π*-modules and any two of* M', M, M'' *are quasi-permutation* Π*-modules, then the third one is a quasi-permutation* Π*-module.*

[II] is a supplement to [2], (1.9) and [13], Cor. to Th. 7. [III], [IV] and [V] can be regarded as generalizations of the results in [3], § 5.

§ 1. Let Π be a finite group. Let $\boldsymbol{M}_{\Pi}$ be the class of all Π-modules, let $\boldsymbol{I}_{\Pi}$ be the class of all projective (left) ideals of $Z\Pi$ and let $\boldsymbol{S}_{\Pi}$ be the class of all permutation Π-modules. Further define:

$$\boldsymbol{H}_{\Pi}=\{M \in \boldsymbol{M}_{\Pi} \mid H^{1}(\Pi', M)=0 \text{ for every subgroup } \Pi' \text{ of } \Pi\};$$
$$\boldsymbol{D}_{\Pi}=\{M \in \boldsymbol{M}_{\Pi} \mid M \oplus M' \cong S \text{ for some } M' \in \boldsymbol{M}_{\Pi} \text{ and } S \in \boldsymbol{S}_{\Pi}\};$$
$$\boldsymbol{L}_{\Pi}=\{M \in \boldsymbol{M}_{\Pi} \mid M \oplus S \cong \mathfrak{A} \oplus S' \text{ for some } \mathfrak{A} \in \boldsymbol{I}_{\Pi} \text{ and } S, S' \in \boldsymbol{S}_{\Pi}\}.$$

Then it is easily seen that

$$\begin{matrix} \boldsymbol{S}_{\Pi} \subsetneqq \\ \boldsymbol{I}_{\Pi} \subsetneqq \end{matrix} \boldsymbol{L}_{\Pi} \subseteqq \boldsymbol{D}_{\Pi} \subseteqq \boldsymbol{H}_{\Pi} \subseteqq \boldsymbol{M}_{\Pi}\,.$$

LEMMA 1.1. *For every $M \in \boldsymbol{M}_\Pi$ there exists an exact sequence*

$$0 \longrightarrow N \longrightarrow S \longrightarrow M \longrightarrow 0$$

of Π-modules with $N \in \boldsymbol{H}_\Pi$ and $S \in \boldsymbol{S}_\Pi$.

Proof. We can construct an epimorphism $\varphi: S \to M$, $S \in \boldsymbol{S}_\Pi$ such that, for every subgroup Π' of Π, $\varphi/S^{\Pi'}: S^{\Pi'} \to M^{\Pi'}$ is an epimorphism. If we put $N = \operatorname{Ker}\varphi$, we have an exact sequence $0 \to N \to S \to M \to 0$. From this we get the exact sequence

$$0 \longrightarrow N^{\Pi'} \longrightarrow S^{\Pi'} \xrightarrow{\varphi/S^{\Pi'}} M^{\Pi'} \longrightarrow H^1(\Pi', N) \longrightarrow H^1(\Pi', S)\ .$$

Since $\varphi/S^{\Pi'}$ is an epimorphism and $H^1(\Pi', S) = 0$, it follows that $H^1(\Pi', N) = 0$, and therefore $N \in \boldsymbol{H}_\Pi$.

LEMMA 1.2 ([6]). *A Π-module M is contained in $\boldsymbol{D}_\Pi$ if and only if any exact sequence $0 \to N \to L \to M \to 0$ of Π-modules with $N \in \boldsymbol{H}_\Pi$ splits.*

Proof. The only if part is obvious and the if part follows directly from (1.1).

Let $M, N \in \boldsymbol{M}_\Pi$. Define a relation $M \underset{(t)}{=\!=} N$ if, for every Galois extension K of k with group $\cong \Pi$, $K(M)^\Pi$ is k-isomorphic to $K(N)^\Pi$. It is evident that if $M \underset{(t)}{=\!=} N$ then $M \underset{(t)}{-\!-} N$.

LEMMA 1.3 ([8], [6]). *Let $0 \to N \to L \to M \to 0$ be an exact sequence of Π-modules with $M \in \boldsymbol{D}_\Pi$. Then $L \underset{(t)}{=\!=} N \oplus M$.*

Proof. See [8], (1.2.2) or [6], (1.4).

LEMMA 1.4. *Let M be a Π-module. Then $M \in \boldsymbol{D}_\Pi$ if and only if $M \in \boldsymbol{D}_{\Pi'}$ for every Sylow subgroup Π' of Π.*

Proof. The only if part is obvious. Suppose that $M \in \boldsymbol{D}_{\Pi'}$ for every Sylow subgroup Π' of Π. By (1.1) there exists an exact sequence $0 \to N \to S \to M \to 0$ of Π-modules with $N \in \boldsymbol{H}_\Pi$ and $S \in \boldsymbol{S}_\Pi$. According to (1.2) this sequence is Π'-split for every Π'. Then this is also Π-split, as is well known, and therefore $M \oplus N \cong S$. Thus $M \in \boldsymbol{D}_\Pi$.

For $n \geqq 1$ we denote by $\Phi_n(X)$ the n-th cyclotomic polynomial and by ζ_n the primitive n-th root of unity.

THEOREM 1.5. *The following statements on a finite group Π are equivalent*:

(1) *Every Sylow subgroup of Π is cyclic.*

(2) $\boldsymbol{H}_\Pi = \boldsymbol{D}_\Pi$.

(3) $T(\Pi) = T^g(\Pi)$, *i.e.*, $T(\Pi)$ *is a group.*

(4) $[I_\Pi^*] \in T^g(\Pi)$.

Proof. To show (1) $\Rightarrow$ (2) we may assume by (1.4) that Π is a cyclic p-group. Let $\Pi = \langle \sigma \rangle$ and $|\Pi| = p^\ell$. By induction on ℓ we will prove that $\boldsymbol{H}_\Pi = \boldsymbol{D}_\Pi$. Let $M \in \boldsymbol{H}_\Pi$ and put $M' = \{u \in M \mid \Phi_{p^\ell}(\sigma)u = 0\}$ and $M'' = M/M'$. Then we have an exact sequence $0 \to M' \to M \to M'' \to 0$ of Π-modules. For every subgroup Π' of Π we have $H^2(\Pi', M') = \hat{H}^0(\Pi', M') = 0$, because $M'^{\Pi'} = 0$. Since $M \in \boldsymbol{H}_\Pi$ it follows from this that $M'' \in \boldsymbol{H}_\Pi$. Here M'' can be regarded as a $\Pi/\langle \sigma^{p^{\ell-1}} \rangle$-module. Therefore by induction we have $M'' \in \boldsymbol{D}_{\Pi/\langle \sigma^{p^{\ell-1}} \rangle} \subseteq \boldsymbol{D}_\Pi$, hence $M'' \oplus N \cong S$ for some $N \in \boldsymbol{M}_{\Pi/\langle \sigma^{p^{\ell-1}} \rangle}$ and $S \in \boldsymbol{S}_{\Pi/\langle \sigma^{p^{\ell-1}} \rangle}$. Thus we get an exact sequence $0 \to M' \to M \oplus N \to S \to 0$. On the other hand, M' can be considered as a $Z[\zeta_{p^\ell}]$-module. Since Π is a cyclic p-group, we can find an exact sequence $0 \to M' \to T \to T' \oplus \mathfrak{A} \to 0$ of Π-modules where $T, T' \in \boldsymbol{S}_\Pi$ and $\mathfrak{A} \in \boldsymbol{I}_\Pi$. Forming the push-out of $\begin{array}{l} M' \to M \oplus N \\ \downarrow \\ T \end{array}$ we get the exact sequence $0 \to M \oplus N \to T \oplus S \to T' \oplus \mathfrak{A} \to 0$. Since $M \oplus N \in \boldsymbol{H}_\Pi$, this sequence splits, hence $M \oplus N \oplus \mathfrak{A} \oplus T' \cong T \oplus S$. Consequently we have $M \in \boldsymbol{D}_\Pi$, which proves that $\boldsymbol{H}_\Pi = \boldsymbol{D}_\Pi$. Next suppose that $\boldsymbol{H}_\Pi = \boldsymbol{D}_\Pi$. Let $M \in \boldsymbol{M}_\Pi$. Then by (1.1) there exists an exact sequence $0 \to L \to S \to M^* \to 0$ of Π-modules with $L \in \boldsymbol{H}_\Pi$ and $S \in \boldsymbol{S}_\Pi$. Dualizing this we get an exact sequence $0 \to M \to S \to L^* \to 0$. By assumption we have $L \in \boldsymbol{D}_\Pi$ and so $L^* \in \boldsymbol{D}_\Pi$. Hence we have, by (1.3), $M \oplus L^* \underset{(t)}{-\!\!-\!\!-} 0$, which shows that $[M] \in T^g(\Pi)$. Thus $T(\Pi) = T^g(\Pi)$. This proves (2) $\Rightarrow$ (3). The implication (3) $\Rightarrow$ (4) is obvious. Finally suppose that $[I_\Pi^*] \in T^g(\Pi)$. Then $I_\Pi^* \oplus N \underset{(t)}{-\!\!-\!\!-} 0$ for some $N \in \boldsymbol{M}_\Pi$. Therefore there exists an exact sequence $0 \to S' \to S \to I_\Pi \oplus N^* \to 0$ with $S, S' \in \boldsymbol{S}_\Pi$. From this we get the exact sequence

$$0 \longrightarrow H^1(\Pi, I_\Pi) \oplus H^1(\Pi, N^*) \longrightarrow H^2(\Pi, S') .$$

However $H^1(\Pi, I_\Pi) \cong Z/|\Pi|Z$. Therefore $H^2(\Pi, S')$ contains an element of order $|\Pi|$. Then we easily see that every Sylow subgroup of Π is cyclic, which completes the proof of (4) $\Rightarrow$ (1).

We return to the general situation.

LEMMA 1.6. *Let M be a Π-module such that $[M] \in T^g(\Pi)$. Then there exists $L \in \boldsymbol{D}_\Pi$ such that $L \underset{(t)}{\text{——}} M$.*

Proof. There is a Π-module M' such that $M \oplus M' \underset{(t)}{\text{——}} 0$. By virtue of (1.1) there exist exact sequences

$$0 \longrightarrow M \longrightarrow S \longrightarrow N \longrightarrow 0$$
$$0 \longrightarrow M' \longrightarrow S' \longrightarrow N' \longrightarrow 0$$

of Π-modules where $S, S' \in \boldsymbol{S}_\Pi$ and $N^*, N'^* \in \boldsymbol{H}_\Pi$. Since $M \oplus M' \underset{(t)}{\text{——}} 0$, we also have an exact sequence $0 \to M \oplus M' \to T \to T' \to 0$ with $T, T' \in \boldsymbol{S}_\Pi$. Forming the pushout of $\begin{array}{c} M \oplus M' \to S \oplus S', \\ \downarrow \\ T \end{array}$ we get the exact sequence

$$0 \longrightarrow T \longrightarrow S \oplus S' \oplus T' \longrightarrow N \oplus N' \longrightarrow 0 .$$

Because $N^*, N'^* \in \boldsymbol{H}_\Pi$, we have $N \oplus N' \oplus T \cong S \oplus S' \oplus T$ and so $N, N' \in \boldsymbol{D}_\Pi$. From this we get the exact sequence

$$0 \longrightarrow M \longrightarrow S \oplus N' \oplus T \longrightarrow S \oplus S' \oplus T' \longrightarrow 0 .$$

Then from (1.3) it follows that $M \underset{(t)}{\text{——}} N'$. Thus $L = N'$ is as desired.

PROPOSITION 1.7. *Let Π be a finite group. Then the group $T^g(\Pi)$ is finitely generated.*

Proof. By (1.6) each element of $T^g(\Pi)$ has a representative in $\boldsymbol{D}_\Pi$. Then, according to [5], (5.8), $T^g(\Pi)$ is finitely generated.

The authors do not know whether, for any finite group Π, the semigroup $T(\Pi)$ is finitely generated or not.

§ **2.** In this section we will study the Chevalley's module I_Π^* more precisely. The torsion part of an abelian group A will be denoted by $t(A)$.

LEMMA 2.1. *Let Π be a finite group such that $(I_\Pi^*)^{(n)} \underset{(t)}{\text{——}} 0$ for some $n > 0$. Then:*

(1) *$(I_{\Pi'}^*)^{(n)} \underset{(t)}{\text{——}} 0$ for every subgroup Π' of Π;*

(2) *$(I_{\Pi/\Pi'}^*)^{(n)} \underset{(t)}{\text{——}} 0$ for every normal subgroup Π' of Π.*

Proof. Let Π' be a subgroup of Π with $[\Pi : \Pi'] = m$. Then it is easily seen that $I_\Pi \cong I_{\Pi'} \oplus Z\Pi'^{(m-1)}$ as Π'-modules and therefore we have $I_\Pi^* \cong I_{\Pi'}^* \oplus Z\Pi'^{(m-1)}$. This shows that $(I_{\Pi'}^*)^{(n)} \underset{(t)}{\text{———}} 0$. Next suppose that Π' is a normal subgroup of Π with $[\Pi:\Pi'] = m$. Then we have $I_{\Pi/\Pi'}^* \cong Z\Pi/\Pi'/(\sum_{\sigma\Pi'\in\Pi/\Pi'}\sigma\Pi')$ and $Z\Pi/\Pi' \otimes_{Z\Pi} I_\Pi^* \cong Z\Pi/\Pi'/(m\sum_{\sigma\Pi'\in\Pi/\Pi'}\sigma\Pi')$. Therefore $Z\Pi/\Pi' \otimes_{Z\Pi} I_\Pi^*/t(Z\Pi/\Pi' \otimes_{Z\Pi} I_\Pi^*) \cong I_{\Pi/\Pi'}^*$. Since $(I_\Pi^*)^{(n)} \underset{(t)}{\text{———}} 0$, there exists an exact sequence $0 \to (I_\Pi^*)^{(n)} \to S \to S' \to 0$ with $S, S' \in \boldsymbol{S}_\Pi$. Tensoring this with $Z\Pi/\Pi'$ over $Z\Pi$, we get the exact sequence

$$\Bigl(Z\Pi/\Pi' \underset{Z\Pi}{\otimes} I_\Pi^*\Bigr)^{(n)} \xrightarrow{\ \varphi\ } Z\Pi/\Pi' \underset{Z\Pi}{\otimes} S \longrightarrow Z\Pi/\Pi' \underset{Z\Pi}{\otimes} S' \longrightarrow 0\,.$$

Then $\operatorname{Im}\varphi \cong (Z\Pi/\Pi' \otimes_{Z\Pi} I_\Pi^*/t(Z\Pi/\Pi' \otimes_{Z\Pi} I_\Pi^*))^{(n)} \cong (I_{\Pi/\Pi'}^*)^{(n)}$. This shows that $(I_{\Pi/\Pi'}^*)^{(n)} \underset{(t)}{\text{———}} 0$.

LEMMA 2.2. *Let Π be a finite group whose Sylow subgroups are cyclic. Let $0 \to N \to L \to M \to 0$ be an exact sequence of Π-modules and suppose that $\hat{H}^0(\Pi', N) = 0$ for every subgroup Π' of Π. Then $L \underset{(t)}{\text{———}} M \oplus N$.*

Proof. We can construct an exact sequence $0 \to N \to F \to V \to 0$ of Π-modules where F is $Z\Pi$-free. Since every Sylow subgroup of Π is cyclic, we have $V \in \boldsymbol{H}_\Pi$ and therefore (1.5) shows that $V \in \boldsymbol{D}_\Pi$. Forming the pushout of $\begin{matrix} N \to L, \\ \downarrow \quad \\ F \quad \end{matrix}$ we get the exact sequence

$$0 \longrightarrow L \longrightarrow F \oplus M \longrightarrow V \longrightarrow 0\,.$$

By (1.3) we have $F \underset{(t)}{=\!=} N \oplus V$ and $F \oplus M \underset{(t)}{=\!=} L \oplus V$. Thus $L \underset{(t)}{\text{———}} M \oplus N$.

Let Π be a finite group whose Sylow subgroups are cyclic. Then Π is expressible as a semidirect product of a cyclic normal subgroup $\langle\sigma\rangle$ of order ℓ and a cyclic subgroup $\langle\tau\rangle$ of order m such that $(\ell, m) = 1$ and for every prime $q\,|\,m$ a q-Sylow subgroup of Π is not contained in the centralizer $C(\sigma)$ of σ. Let $\ell = \ell(\Pi)$ and $m = m(\Pi)$ and define $i(\Pi) = |\operatorname{Im}(\langle\tau\rangle \to \operatorname{Aut}\langle\sigma\rangle)|$.

THEOREM 2.3. *The following statements on a finite group Π are equivalent:*

(1) *Π is a cyclic group or a direct product of a cyclic group of order n and a group with generators ρ, τ and relations $\rho^k = \tau^{2^d} = I$ and $\tau^{-1}\rho\tau = \rho^{-1}$*

where $d \geqq 1$, $k \geqq 3$ *and both* n *and* k *are odd integers such that* $(n, k) = 1$.

(2) I_{Π}^{*} *is a quasi-permutation* Π*-module, i.e.,* $I_{\Pi}^{*} \underset{(\iota)}{\text{——}} 0$.

(3) *The order of* $[I_{\Pi}^{*}]$ *in* $T(\Pi)$ *is finite.*

Proof. By (1.5) we may assume that Π is a finite group whose Sylow subgroups are cyclic. It should be remarked that the statement (1) is equivalent to the following one:

(1′) $i(\Pi) = 1$ or 2.

The implication (2) $\Rightarrow$ (3) is evident. Hence we only need to prove (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2).

First suppose that $i(\Pi) = m(\Pi) = 2, 4$ or q and $\ell(\Pi) = p^c$, $c \geqq 1$ where p and q are distinct odd primes. Let $\Lambda = Z\Pi/(\Phi_{p^c}(\sigma)) \cong Z\langle\tau\rangle[\zeta_{p^c}]$. Then Λ is isomorphic to the trivial crossed product of $Z[\zeta_{p^c}]$ and $\langle\tau\rangle$ and so it is a hereditary order in $Q\Pi/(\Phi_{p^c}(\sigma))$. Further let $R = Z[\zeta_{p^c}]$ and $\mathfrak{P} = (\zeta_{p^c} - 1)$. Then $\mathfrak{P}$ is a prime ideal of R and both R and $\mathfrak{P}$ can be regarded as Λ-modules. According to a result in [9] we have $\Lambda \cong R \oplus \mathfrak{P} \oplus \cdots \oplus \mathfrak{P}^{i(\Pi)-1}$ as Λ-modules. It is easily seen that all of Λ, Λ^*, R and R^* are quasi-permutation Π-modules. Therefore, when $i(\Pi) = 2$, we have $\mathfrak{P} \underset{(\iota)}{\text{——}} 0$ and $\mathfrak{P}^* \underset{(\iota)}{\text{——}} 0$. However, when $i(\Pi) = 4$ or q, we have $\mathfrak{P}^{*(j)} \underset{(\iota)}{\not\text{——}} 0$ for any $j > 0$. In fact, if $\mathfrak{P}^{*(j)} \underset{(\iota)}{\text{——}} 0$, then there is an exact sequence $0 \to S' \to S \to \mathfrak{P}^{(j)} \to 0$ with $S', S \in \boldsymbol{S}_{\Pi}$. Tensoring this with Λ over $Z\Pi$ and eliminating the torsion parts, we get

$$\Lambda^{(u)} \oplus R^{(v)} \oplus \mathfrak{P}^{2(w)} \oplus \mathfrak{P}^{(j)} \cong \Lambda^{(u')} \oplus R^{(v')} \oplus \mathfrak{P}^{2(w')} \qquad \text{when } i(\Pi) = 4$$

and

$$\Lambda^{(u)} \oplus R^{(v)} \oplus \mathfrak{P}^{(j)} \cong \Lambda^{(u')} \oplus R^{(v')} \qquad \text{when } i(\Pi) = q\ ,$$

where u, v, w, u', v', w' are non negative integers. This contradicts the Rosen's result ([9]). Now the Π-module $\Phi_{p^c}(\sigma)I_{\Pi}$ has a free basis

$$\{\Phi_{p^c}(\sigma)(\sigma^{\alpha} - 1), \Phi_{p^c}(\sigma)(\sigma^{\alpha}\tau^{\beta} - 1), \Phi_{p^c}(\sigma)(\tau^{\beta} - 1)\}_{0<\alpha<p^{c-1}, 0<\beta<m(\Pi)}$$

and $\Phi_{p^c}(\sigma)I_{\Pi} \cong I_{\Pi/\langle\sigma^{p^{c-1}}\rangle}$ as Π-modules. Therefore $\Lambda \otimes_{Z\Pi} I_{\Pi}$ is torsion-free and we have an exact sequence

$$0 \longrightarrow I_{\Pi/\langle\sigma^{p^{c-1}}\rangle} \longrightarrow I_{\Pi} \longrightarrow \Lambda \underset{Z\Pi}{\otimes} I_{\Pi} \longrightarrow 0\ .$$

It is easily seen that $\hat{H}^0(\Pi', \Lambda) = 0$ for every subgroup Π' of Π. Since

Λ is hereditary, we have $\hat{H}^0(\Pi', (\Lambda \otimes_{Z\Pi} I_\Pi)^*) = 0$ for every subgroup Π' of Π. Then, according to (2.2), we have $I_\Pi^* \underset{(t)}{\sim} I^*_{\Pi/\langle \sigma^{p^{c-1}} \rangle} \oplus (\Lambda \otimes_{Z\Pi} I_\Pi)^*$. Tensoring the exact sequence $0 \to I_\Pi \to Z\Pi \to Z \to 0$ with Λ over $Z\Pi$, we get the exact sequence

$$0 \longrightarrow \Lambda \underset{Z\Pi}{\otimes} I_\Pi \longrightarrow \Lambda \longrightarrow R/\mathfrak{P} \longrightarrow 0 .$$

Forming further the pullback of $\begin{array}{c}\Lambda \to R/\mathfrak{P} \\ \uparrow \\ R\end{array}$, we see that $(\Lambda \otimes_{Z\Pi} I_\Pi) \oplus R \cong \Lambda \oplus \mathfrak{P}$. Since $\Lambda^* \underset{(t)}{\sim} 0$ and $R^* \underset{(t)}{\sim} 0$, $(\Lambda \otimes_{Z\Pi} I_\Pi)^* \underset{(t)}{\sim} \mathfrak{P}^*$. Thus we have

$$I_\Pi^* \underset{(t)}{\sim} I^*_{\Pi/\langle \sigma^{p^{c-1}} \rangle} \oplus \mathfrak{P}^* \cdots\cdots\cdots \text{(a)} .$$

(3) $\Rightarrow$ (1): Suppose that $i(\Pi) > 2$. To show that Π does not satisfy (3) we may suppose by (2.1) that $i(\Pi) = m(\Pi) = 4$ or q and $\ell(\Pi) = p$ where p and q are distinct odd primes. In this case the group $\Pi/\langle\sigma\rangle$ is cyclic and so $I^*_{\Pi/\langle\sigma\rangle} \cong I_{\Pi/\langle\sigma\rangle} \underset{(t)}{\sim} 0$. By virtue of (a) we have $I_\Pi^* \underset{(t)}{\sim} \mathfrak{P}^*$. However $\mathfrak{P}^{*(j)} \underset{(t)}{\not\sim} 0$ for any $j > 0$. Therefore the order of $[I_\Pi^*]$ in $T(\Pi)$ is not finite. This completes the proof of (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Suppose that $i(\Pi) = 1$ or 2. If $i(\Pi) = 1$, then $I_\Pi^* \cong I_\Pi \underset{(t)}{\sim} 0$. Hence we suppose that $i(\Pi) = 2$. Let σ' be the generator of a cyclic group of order n and let $\mu = \sigma'\rho\tau^2$. Then $\langle\mu\rangle$ is the normal subgroup of Π of order $b = nk2^{d-1}$. Let p be a prime divisor of k. Let $k = p^c k'$, $(p, k') = 1$, and let $b' = b/p^c = nk'2^{d-1}$. Further let $F(X) = \prod_{r|b'} \Phi_{p^c r}(X)$ and $G(X) = F(X)/\Phi_{p^c}(X)$, and let $\Gamma = Z\Pi/(F(\mu))$, $\Gamma_0 = Z\Pi/(G(\mu))$ and $\Gamma_1 = Z\Pi/(\Phi_{p^c}(\mu))$. Since any of $F(1), G(1)$ and $\Phi_{p^c}(1)$ is not 0, we see that $t(\Gamma \otimes_{Z\Pi} I_\Pi) = t(\Gamma_0 \otimes_{Z\Pi} I_\Pi) = t(\Gamma_1 \otimes_{Z\Pi} I_\Pi) = 0$. Hence, tensoring the exact sequence $0 \to \Gamma_1 \to \Gamma \to \Gamma_0 \to 0$ with I_Π over $Z\Pi$, we get the exact sequence

$$0 \longrightarrow \Gamma_1 \underset{Z\Pi}{\otimes} I_\Pi \longrightarrow \Gamma \underset{Z\Pi}{\otimes} I_\Pi \longrightarrow \Gamma_0 \underset{Z\Pi}{\otimes} I_\Pi \longrightarrow 0 .$$

Because $G(1) = \pm 1$, we can show that $\Gamma_0 \otimes_{Z\Pi} Z = 0$. Therefore, tensoring the exact sequence $0 \to I_\Pi \to Z\Pi \to Z \to 0$ with Γ_0 over $Z\Pi$, we see that $\Gamma_0 \otimes_{Z\Pi} I_\Pi \cong \Gamma_0$. Let $H(X) = X^b - 1/G(X)$ and $\Gamma' = Z\Pi/(H(\mu))$. Then there is an exact sequence $0 \to \Gamma' \to Z\Pi \to \Gamma_0 \to 0$. Forming the pullback of $\begin{array}{c}\Gamma \otimes_{Z\Pi} I_\Pi \to \Gamma_0 \\ \uparrow \\ Z\Pi\end{array}$, we get the exact sequence

$$0 \longrightarrow \Gamma' \longrightarrow \left(\Gamma_1 \underset{Z\Pi}{\otimes} I_\Pi\right) \oplus Z\Pi \longrightarrow \Gamma \underset{Z\Pi}{\otimes} I_\Pi \longrightarrow 0 \cdots\cdots (\mathrm{b}) .$$

The group $\Pi/\langle\mu^{p^c}\rangle$ is dihedral and of order $2p^c$ and there is a natural epimorphism $Z\Pi/\langle\mu^{p^c}\rangle \to \Gamma_1$. It is seen that $\Gamma_1 \otimes_{Z\Pi} I_\Pi \cong \Gamma_1 \otimes_{Z\Pi/\langle\mu^{p^c}\rangle} I_{\Pi/\langle\mu^{p^c}\rangle}$. Therefore, as shown in the proof of (a), $(\Gamma_1 \otimes_{Z\Pi} I_\Pi)^* \underset{(t)}{-\!\!-} 0$. On the other hand, there exists the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & I_\Pi & \longrightarrow & \Gamma \otimes_{Z\Pi} I_\Pi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z\Pi/\langle\mu^{b/p}\rangle & \longrightarrow & Z\Pi & \longrightarrow & \Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & Z/pZ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then $K \cong I_{\Pi/\langle\mu^{b/p}\rangle}$. Thus we have the exact sequence

$$0 \longrightarrow I_{\Pi/\langle\mu^{b/p}\rangle} \longrightarrow I_\Pi \longrightarrow \Gamma \underset{Z\Pi}{\otimes} I_\Pi \longrightarrow 0 \cdots\cdots\cdots (\mathrm{c}) .$$

Now it is easily shown that $\hat{H}^0(\Pi', \Gamma) = 0$ for every subgroup Π' of Π. From the exact sequence $0 \to \Gamma \otimes I_\Pi \to \Gamma \to Z/pZ \to 0$ we get the exact sequence

$$\hat{H}^{-1}(\Pi', \Gamma) \xrightarrow{\varphi} \hat{H}^{-1}(\Pi', Z/pZ) \longrightarrow \hat{H}^0\left(\Pi', \Gamma \underset{Z\Pi}{\otimes} I_\Pi\right) \longrightarrow 0 .$$

By a direct computation we can show that φ is an epimorphism. Hence, for every subgroup Π' of $\Pi, \hat{H}^0(\Pi', \Gamma \otimes_{Z\Pi} I_\Pi) = 0$, and so $\hat{H}^0(\Pi', (\Gamma \otimes_{Z\Pi} I_\Pi)^*) = 0$. From (2.2) and (b) it follows that $(\Gamma \otimes_{Z\Pi} I_\Pi)^* \underset{(t)}{-\!\!-} 0$, because $\Gamma'^* \underset{(t)}{-\!\!-} 0$ and $(\Gamma_1 \otimes_{Z\Pi} I_\Pi)^* \underset{(t)}{-\!\!-} 0$. Furthermore, applying (2.2) to (c), we have $I_\Pi^* \underset{(t)}{-\!\!-} I_{\Pi/\langle\mu^{b/p}\rangle}^*$. Accordingly we can show by induction on k that $I_\Pi^* \underset{(t)}{-\!\!-} I_{\Pi/\langle\mu^{b/k}\rangle}^*$. Since $\Pi/\langle\mu^{b/k}\rangle$ is cyclic, this concludes that $I_\Pi^* \underset{(t)}{-\!\!-} 0$. Thus the proof of (1) $\Rightarrow$ (2) is completed.

§ **3.** We first give

PROPOSITION 3.1. *The following statements on a finite group Π are equivalent*:

(1) $\boldsymbol{D}_\Pi = \boldsymbol{L}_\Pi$.

(2) $T^q(\Pi) = C(Z\Pi)/C^q(Z\Pi)$.

Proof. This is an immediate consequence of (1.6).

THEOREM 3.2. *Let Π be a finite p-group. Then $T^q(\Pi) = C(Z\Pi)/C^q(Z\Pi)$.*

Proof. By (3.1) it suffices to show that $\boldsymbol{D}_\Pi = \boldsymbol{L}_\Pi$. If $M \in \boldsymbol{D}_\Pi$, there are $M' \in \boldsymbol{D}_\Pi$ and $S \in \boldsymbol{S}_\Pi$ such that $M \oplus M' \cong S$. Since Π is a p-group, for every subgroup Π' of $\Pi, Z_p\Pi/\Pi'$ is an indecomposable $Z_p\Pi$-module. It is well known that the Krull-Schmidt theorem holds for finitely generated $Z_p\Pi$-modules. Hence there exists $T \in \boldsymbol{S}_\Pi$ such that $T_p \cong M_p$ as $Z_p\Pi$-modules. This implies that T and M have the same genus. Then there is $\mathfrak{A} \in \boldsymbol{I}_\Pi$ such that $M \oplus Z\Pi \cong T \oplus \mathfrak{A}$. This proves that $M \in \boldsymbol{L}_\Pi$.

We did not succeed in determining all finite groups Π such that $T^q(\Pi) = C(Z\Pi)/C^q(Z\Pi)$. It is shown that a finite nilpotent group Π is a p-group if and only if, for any $M \in \boldsymbol{M}_\Pi$ such that $M \oplus S' \cong S$ for some $S', S \in \boldsymbol{S}_\Pi$, there is $T \in \boldsymbol{S}_\Pi$ whose genus is the same as that of M. Hence our method used in the proof of (3.2) can not be applied to nilpotent groups.

However we can determine all finite groups Π such that $T(\Pi) = C(Z\Pi)/C^q(Z\Pi)$.

THEOREM 3.3. *The following statements on a finite group Π are equivalent*:

(1) *Π is* (i) *a cyclic group,* (ii) *a dihedral group of order $2p^c$ where p is an odd prime and $c \geqq 1$,* (iii) *a direct product of a cyclic group of order q^f and a dihedral group of order $2p^c$ where $f, c \geqq 1$, p and q are odd primes and p is a primitive $q^{f-1}(q-1)$-th root of unity modulo q^f, or* (iv) *a generalized quaternion group of order $4p^c$ where $c \geqq 1$ and p is an odd prime congruent to* 3 *modulo* 4.

(2) $T(\Pi) = C(Z\Pi)/C^q(Z\Pi) = C(\Omega_{Z\Pi})$.

(3) *$T(\Pi)$ is a finite group.*

Proof. It can easily be seen that the statement (1) is equivalent to the following one:

(1′) *Π is* (i′) *a cyclic group or* (ii′) *a direct product of a cyclic group of order n and a group with generators ρ, τ and relations $\rho^{p^c} = \tau^{2^d} = I$ and $\tau^{-1}\rho\tau = \rho^{-1}$ where $c, d \geqq 1$, p is an odd prime which is a prime in $Z[\zeta_{n2^d}]$ and $(2p, n) = 1$.*

The implication (2) $\Rightarrow$ (3) is obvious. Hence it suffices to prove (1′) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1′).

(1′) $\Rightarrow$ (2): It follows directly from [4], (3.6) that $C(Z\Pi)/C^q(Z\Pi) = C(\Omega_{Z\Pi})$. Therefore, by (1.5) and (3.1), it suffices to show that $\boldsymbol{D}_\Pi = \boldsymbol{L}_\Pi$. In case (i′) $\ell = \ell(\Pi) = |\Pi|$ and $m = m(\Pi) = 1$, while, in case (ii′), $\ell = \ell(\Pi) = np^c$, $m = m(\Pi) = 2^d$ and $i(\Pi) = 2$. Let σ be an element of Π of order ℓ. In case (i′) we define $\mu = \sigma$ and $b = \ell$. On the other hand, in case (ii′), we define $\mu = \sigma\tau^2$ and $b = \ell 2^{d-1}$. Then $\langle\mu\rangle$ is the normal subgroup of Π of order b. Now, by induction on b, we will prove that $\boldsymbol{D}_\Pi = \boldsymbol{L}_\Pi$. For every $b' | b$, let $\Psi_{b'}(X) = X^b - 1/X^{b'} - 1$. Then we can construct the following exact sequences:

$$0 \longrightarrow Z\Pi/(\Psi^{(1)}(\mu)) \longrightarrow Z\Pi/(\chi^{(0)}(\mu)) \longrightarrow Z\Pi/(\mu^{b_1} - 1) \longrightarrow 0\,,$$
$$0 \longrightarrow Z\Pi/(\Psi^{(1)}(\mu)) \longrightarrow Z\Pi/(\chi^{(1)}(\mu)) \longrightarrow Z\Pi/(\mu^{b_2} - 1) \longrightarrow 0\,,$$
$$0 \longrightarrow Z\Pi/(\Psi^{(2)}(\mu)) \longrightarrow Z\Pi/(\chi^{(1)}(\mu)) \longrightarrow Z\Pi/(\mu^{b_3} - 1) \longrightarrow 0\,,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$0 \longrightarrow Z\Pi/(\Psi^{(t-1)}(\mu)) \longrightarrow Z\Pi/(\chi^{(t-1)}(\mu)) \longrightarrow Z\Pi/(\mu^{b_{2t-2}} - 1) \longrightarrow 0\,,$$
$$0 \longrightarrow Z\Pi/(\Psi^{(t)}(\mu)) \longrightarrow Z\Pi/(\chi^{(t-1)}(\mu)) \longrightarrow Z\Pi/(\mu^{b_{2t-1}} - 1) \longrightarrow 0\,,$$

where, for $1 \leqq i \leqq 2t-1$, $b_i \| b$ and, for $1 \leqq j \leqq t-1$, $\Psi^{(j)}(X) | \Psi_{b_{2j-1}}(X)$, $\Psi^{(j)}(X) | \Psi_{b_{2j}}(X)$, $\chi^{(j)}(X) = \Psi^{(j)}(X)(X^{b_{2j}} - 1) = \Psi^{(j+1)}(X)(X^{b_{2j+1}} - 1)$, $\chi^{(0)}(X) = X^b - 1$ and $\Psi^{(t)}(X) = \Phi_b(X)$. Let $M \in \boldsymbol{D}_\Pi$. Then, for any $b' | b$, $M/(\mu^{b'} - 1)M \in \boldsymbol{D}_{\Pi/\langle\mu^{b'}\rangle}$. For any $\Psi(X) | X^b - 1$ we define $\overline{M}_\Psi = M/\Psi(\mu)M/t(M/\Psi(\mu)M)$. Tensoring the above exact sequences with M over $Z\Pi$, we get the exact sequences:

$$0 \longrightarrow \overline{M}_{\Psi^{(1)}} \longrightarrow M \longrightarrow M/(\mu^{b_1} - 1)M \longrightarrow 0\,,$$
$$0 \longrightarrow \overline{M}_{\Psi^{(1)}} \longrightarrow \overline{M}_{\chi^{(1)}} \longrightarrow M/(\mu^{b_2} - 1)M \longrightarrow 0\,,$$
$$0 \longrightarrow \overline{M}_{\Psi^{(2)}} \longrightarrow \overline{M}_{\chi^{(1)}} \longrightarrow M/(\mu^{b_3} - 1)M \longrightarrow 0\,,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$0 \longrightarrow \overline{M}_{\Psi^{(t-1)}} \longrightarrow \overline{M}_{\chi^{(t-1)}} \longrightarrow M/(\mu^{b_{2t-2}} - 1)M \longrightarrow 0\,,$$

$$0 \longrightarrow \bar{M}_{\Phi_b} \longrightarrow \bar{M}_{\chi^{(t-1)}} \longrightarrow M/(\mu^{b_{2t-1}} - 1)M \longrightarrow 0 .$$

Since $M/(\mu^{b_i} - 1)M \in \boldsymbol{D}_{\Pi/\langle\mu^{b_i}\rangle}$, applying (1.3) repeatedly, we get

$$M \oplus \bigoplus_{j=1}^{t-1} (M/(\mu^{b_{2j}} - 1)M) \underset{(*)}{=\!=} \bar{M}_{\Phi_b} \oplus \bigoplus_{j=1}^{t} (M/(\mu^{b_{2j-1}} - 1)M) .$$

By induction on b we have $M/(\mu^{b_i} - 1)M \in \boldsymbol{L}_{\Pi/\langle\mu^{b_i}\rangle} \subseteq \boldsymbol{L}_\Pi$ for each $1 \leqq i \leqq 2t - 1$. Accordingly we only need to show that $\bar{M}_{\Phi_b} \underset{(*)}{-\!-} \mathfrak{A}$ for some $\mathfrak{A} \in \boldsymbol{I}_\Pi$. In case (i′) $Z\Pi/(\Phi_b(\mu)) \cong Z[\zeta_b]$, and therefore there exist $\mathfrak{A} \in \boldsymbol{I}_\Pi$ and $k \geqq 0$ such that $\bar{M}_{\Phi_b} \oplus Z\Pi \cong Z[\zeta_b]^{(k)} \oplus \mathfrak{A}$. Hence $\bar{M}_{\Phi_b} \underset{(*)}{-\!-} \mathfrak{A}$. In case (ii′) we put $\Lambda = Z\Pi/(\Phi_b(\mu))$. Then $\Lambda \underset{(*)}{-\!-} 0$. If $d = 1$, Λ is isomorphic to the trivial crossed product of $Z[\zeta_b]$ and $\langle\tau\rangle$. Let $R = Z[\zeta_b]$ and $\mathfrak{P} = (\zeta_{p^c} - 1)$. Then both R and $\mathfrak{P}$ can regarded as Λ-modules. By assumption $\mathfrak{P}$ is the unique prime ideal of R ramified over $Z[\zeta_n] \subseteq Z[\zeta_b]^{\langle\tau\rangle}$. Therefore $\Lambda \cong R \oplus \mathfrak{P}$ as Λ-modules and any ambiguous ideal of R has the same genus as that of R or $\mathfrak{P}$. Hence there exist $\mathfrak{A} \in \boldsymbol{I}_\Pi$ and $u, v \geqq 0$ such that $\bar{M}_{\Phi_b} \oplus Z\Pi \cong R^{(u)} \oplus \mathfrak{P}^{(v)} \oplus \mathfrak{A}$. Since $R \underset{(*)}{-\!-} 0$ and $\mathfrak{P} \underset{(*)}{-\!-} 0$, this implies that $\bar{M}_{\Phi_b} \underset{(*)}{-\!-} \mathfrak{A}$. If $d \geqq 2$, $\Lambda/(\zeta_{p^c} - 1)\Lambda \cong Z/pZ[X]/(\Phi_{n2^d}(X))$ and so $\Lambda/(\zeta_{p^c} - 1)\Lambda$ is a field because p is a prime in $Z[\zeta_{n2^d}]$. Therefore Λ is a maximal order in $Q\Lambda$ and $Q\Lambda$ is a division ring. Then we have $\bar{M}_{\Phi_b} \oplus Z\Pi \cong \Lambda^{(u)} \oplus \mathfrak{A}$ for some $\mathfrak{A} \in \boldsymbol{I}_\Pi$ and $u \geqq 0$ and so $\bar{M}_{\Phi_b} \underset{(*)}{-\!-} \mathfrak{A}$. Thus the proof of (1′) $\Rightarrow$ (2) is completed.

(3) $\Rightarrow$ (1′): Assume that Π does not satisfy the condition (1′). We will prove that $T(\Pi)$ is not a finite group. If Π has a noncyclic Sylow subgroup, this follows from (1.5). Hence we may assume that every Sylow subgroup of Π is cyclic. If $\mathrm{i}(\Pi) > 2$, it follows from (2.3) that $T(\Pi)$ is not finite. Therefore we may further assume that $i(\Pi) \leqq 2$. If $T(\Pi)$ is a finite group, then, for any normal subgroup Π' of Π, $T(\Pi/\Pi')$ is a finite group. Hence we only need to show that $T(\Pi)$ is not finite, in each of the following cases:

(a) Π is a group with generators σ, τ and relations $\sigma^{p_1p_2} = \tau^2 = I$ and $\tau^{-1}\sigma\tau = \sigma^{-1}$ where p_1 and p_2 are distinct odd primes.

(b) Π is a group with generators σ, τ and relations $\sigma^{np} = \tau^{2^d} = I$, $d \geqq 1$, $\tau^{-1}\sigma^n\tau = \sigma^{-n}$ and $\sigma^p\tau = \tau\sigma^p$ where p is an odd prime which is not a prime in $Z[\zeta_{n2^d}]$ and $(2p, n) = 1$.

We define $\mu = \sigma$ and $b = p_1p_2$ in case (a) and $\mu = \sigma\tau^2$ and $b = np2^{d-1}$ in case (b). Let $\Lambda = Z\Pi/(\Phi_b(\mu))$. If Π has the type (a) or the type (b), $d = 1$, then Λ is isomorphic to the trivial crossed product of $Z[\zeta_b]$ and $\langle\tau\rangle$.

Further let $R = Z[\zeta_b]$. Then $\Lambda \underset{(t)}{\text{——}} 0$ and $R \underset{(t)}{\text{——}} 0$. There is an ambiguous ideal $\mathfrak{Q}$ of R such that $\Lambda \cong R \oplus \mathfrak{Q}$. Then we also have $\mathfrak{Q} \underset{(t)}{\text{——}} 0$. If $T \in \boldsymbol{S}_\Pi$, $\Lambda T \cong R^{(u)} \oplus \mathfrak{Q}^{(v)}$ for some $u, v \geqq 0$. In each case we can find an ambiguous ideal $\mathfrak{S}$ of R whose genus is different from those of R and $\mathfrak{Q}$. Now we can see that $(\mathfrak{S}^*)^{(j)} \underset{(t)}{\text{——}} 0$ for every $j > 0$. In fact, if $(\mathfrak{S}^*)^{(j)} \underset{(t)}{\text{——}} 0$, there is an exact sequence $0 \to S' \to S \to \mathfrak{S}^{(j)} \to 0$ of Π-modules with $S', S \in \boldsymbol{S}_\Pi$. Tensoring this with Λ over $Z\Pi$ and eliminating the torsion parts, we get $\mathfrak{S}^{(j)} \oplus \Lambda S' \cong \Lambda S$ and so $\mathfrak{S}^{(j)} \oplus R^{(u)} \oplus \mathfrak{Q}^{(v)} \cong R^{(u')} \oplus \mathfrak{Q}^{(v')}$ for some $u, v, u', v' \geqq 0$, which is a contradiction. Thus $T(\Pi)$ is not a finite group. Finally suppose that Π has the type (b) and $d \geqq 2$. Since p is not a prime in $Z[\zeta_{n2^d}]$, Λ is a non-maximal hereditary order in $Q\Lambda$. There is an ideal $\mathfrak{Q}$ of Λ whose genus is different from that of Λ such that $Q\mathfrak{Q} \cong Q\Lambda$. It is easily seen that, if Π' is a proper subgroup of Π, then $\Lambda \cdot Z\Pi/\Pi' = 0$. Hence we can show that $(\mathfrak{Q}^*)^{(j)} \underset{(t)}{\not\text{——}} 0$ for every $j > 0$, and so $T(\Pi)$ is not a finite group. This completes the proof of (3)$\Rightarrow$(1′).

§4. Let Γ be a ring with unit element. We denote by $G(\Gamma)$ the Grothendieck group of the category of finitely generated Γ-modules. Let Π be a finite group. Then there is a natural epimorphism $\nu_\Pi : G(Z\Pi) \to G(Q\Pi)$. There is also a natural homomorphism $\omega_\Pi : C(\Omega_{Z\Pi}) \to G(Z\Pi)$. It was shown in [11] that the sequence $C(\Omega_{Z\Pi}) \xrightarrow{\omega_\Pi} G(Z\Pi) \xrightarrow{\nu_\Pi} G(Q\Pi) \to 0$ is exact.

Let K be an algebraic number field and let R be the ring of all algebraic integers in K. Let Σ be a central simple K-algebra. We denote the completion of K at a valuation v of K by $\hat{K}_v$. It is said that an archimedean valuation v of K is ramified in Σ if $\hat{K}_v \otimes_K \Sigma$ is isomorphic to a full matrix algebra over the quaternion field. Denote by I_K the group of all fractional ideals in K and define $P'_\Sigma = \{\alpha R \in I_K \mid v(\alpha) > 0$ at every archimedean valuation v ramified in $\Sigma\}$. Let Ω be a maximal R-order in Σ. It was proved in [10] that $C(\Omega) \cong I_K/P'_\Sigma$.

Suppose that Π is a finite group whose Sylow subgroups are cyclic. Let Σ be a simple component of $Q\Pi$. Let K_Σ be the center of Σ and let R_Σ be the ring of all algebraic integers in K_Σ. Define $\Pi_\Sigma = \mathrm{Ker}\,(\Pi \to \Sigma)$ and let $\tilde{\Pi}_\Sigma = \langle \sigma_\Sigma \rangle$ be the maximal normal cyclic subgroup of Π/Π_Σ. Further put $a_\Sigma = |\Pi_\Sigma||\tilde{\Pi}_\Sigma|$ and $b_\Sigma = |\tilde{\Pi}_\Sigma|$. We have $\Sigma \cong Q\Pi/\Pi_\Sigma/(\Phi_{b_\Sigma}(\sigma_\Sigma))$ and so Σ can be expressed as a crossed product of $Q(\zeta_{b_\Sigma})$ and $\Pi/\Pi_\Sigma/\langle \sigma_\Sigma \rangle$.

LEMMA 4.1. *Let Π be a finite group whose Sylow subgroups are* *cyclic.*

(1) *If, for each simple component Σ of $Q\Pi$, every prime divisor of a_Σ in R_Σ is contained in P'_Σ, then $\omega_\Pi : C(\Omega_{Z\Pi}) \to G(Z\Pi)$ is a monomorphism.*

(2) *If $\omega_\Pi : C(\Omega_{Z\Pi}) \to G(Z\Pi)$ is a monomorphism, then, for each simple component Σ of $\Omega\Pi$, every prime divisor of b_Σ in R_Σ is contained in P'_Σ.*

Proof. (1) It suffices to prove that, for any exact sequence $0 \to M' \to M \to M'' \to 0$ of Π-modules, $\Omega_{Z\Pi}M \oplus \Omega_{Z\Pi} \cong \Omega_{Z\Pi}M' \oplus \Omega_{Z\Pi}M'' \oplus \Omega_{Z\Pi}$. Let Σ be any simple component of $Q\Pi$, and let Λ and Ω be the images of $Z\Pi$ and $\Omega_{Z\Pi}$ under the projection $Q\Pi \to \Sigma$, respectively. Now we will show that $\Omega M \oplus \Omega \cong \Omega M' \oplus \Omega M'' \oplus \Omega$. Taking the cohomology as Π_Σ-modules, we have the exact sequence:

$$0 \longrightarrow M'^{\Pi_\Sigma} \longrightarrow M^{\Pi_\Sigma} \xrightarrow{\ f\ } M''^{\Pi_\Sigma} \xrightarrow{\ g\ } H^1(\Pi_\Sigma, M')\ .$$

This yields the exact sequences.

$$\begin{array}{l} 0 \longrightarrow M'^{\Pi_\Sigma} \longrightarrow M^{\Pi_\Sigma} \longrightarrow L \longrightarrow 0 \\ 0 \longrightarrow L \longrightarrow M''^{\Pi_\Sigma} \longrightarrow X \longrightarrow 0 \end{array} \quad \cdots\cdots\cdots (*)$$

where $L = \operatorname{Im} f$, $X = \operatorname{Im} g$ and $|\Pi_\Sigma| X = 0$. Let $0 \to N' \to N \to Y \to 0$ be an exact sequence where N', N are Π/Π_Σ-modules and $a_\Sigma^s Y = 0$ for some $s > 0$. Then $a_\Sigma^s \Omega N \subseteq \Omega N'$. Since $C(\Omega) \cong I_k/P'_\Sigma$ and every prime divisor of a_Σ in K is contained in P'_Σ, we have $\Omega N \oplus \Omega \cong \Omega N' \oplus \Omega$. Let $\Psi_\Sigma = \sum_{\tau \in \Pi_\Sigma} \tau$. For any Π-module N $\Psi_\Sigma N \subseteq N^{\Pi_\Sigma}$ and $|\Pi_\Sigma| N^{\Pi_\Sigma} \subseteq \Psi_\Sigma N$, and so $\Omega N^{\Pi_\Sigma} \oplus \Omega \cong \Omega \Psi_\Sigma N \oplus \Omega$. As is easily seen, $\Omega \Psi_\Sigma N \cong \Omega N$. Thus $\Omega N \oplus \Omega \cong \Omega N^{\Pi_\Sigma} \oplus \Omega$. Applying these to (*), we get $\Omega M' \oplus \Omega \cong \Omega M'^{\Pi_\Sigma} \oplus \Omega$, $\Omega M \oplus \Omega \cong \Omega M^{\Pi_\Sigma} \oplus \Omega$ and $\Omega M'' \oplus \Omega \cong \Omega M''^{\Pi_\Sigma} \oplus \Omega \cong \Omega L \oplus \Omega$. Hence we only need to prove that $\Omega M^{\Pi_\Sigma} \oplus \Omega \cong \Omega M'^{\Pi_\Sigma} \oplus \Omega L \oplus \Omega$. Therefore it suffices to prove, under the assumption that $\Pi_\Sigma = \{I\}$, that $\Omega M \oplus \Omega \cong \Omega M' \oplus \Omega M'' \oplus \Omega$. To simplify our notation we write $\sigma = \sigma_\Sigma$, $b = b_\Sigma$, $K = K_\Sigma$ and $R = R_\Sigma$. If Π is cyclic, then $\Lambda = \Omega$. On the other hand, if Π is not cyclic, then Λ is a crossed product of $Z[\zeta_b]$ and $\Pi/\langle\sigma\rangle$ and therefore, for every prime ideal $\mathfrak{p}$ of R which does not contain b, $\Lambda_\mathfrak{p} = \Omega_\mathfrak{p}$. Since $|\Pi|\,\Omega_{Z\Pi} \subseteq Z\Pi$, we can find $t > 0$ such that $b^t \Omega \subseteq \Lambda$. Let N be a Π-module and define $N^{\Phi_b} = \{u \in N \mid \Phi_b(\sigma)u = 0\}$. Then $N^{\Phi_b} \subseteq \Lambda N$ and $b\Lambda N \subseteq N^{\Phi_b}$, and therefore $\Omega N \oplus \Omega \cong \Omega N^{\Phi_b} \oplus \Omega$. Returning to the sequence $0 \to M' \to M \to M'' \to 0$, we get the exact sequence:

$$0 \longrightarrow M'^{\Phi_b} \longrightarrow M^{\Phi_b} \longrightarrow M''^{\Phi_b} \longrightarrow U \longrightarrow 0$$

of Λ-modules where $bU = 0$. This can be separated into two short exact sequences:

$$0 \longrightarrow M'^{\Phi_b} \longrightarrow M^{\Phi_b} \longrightarrow L \longrightarrow 0\,,$$
$$0 \longrightarrow L \longrightarrow M''^{\Phi_b} \longrightarrow U \longrightarrow 0\,.$$

Since $b^t\Omega \subseteq \Lambda$ and $bU = 0$, $\Omega M^{\Phi_b} \oplus \Omega \cong \Omega M'^{\Phi_b} \oplus \Omega L \oplus \Omega \cong \Omega M'^{\Phi_b} \oplus \Omega M''^{\Phi_b} \oplus \Omega$. However $\Omega M \oplus \Omega \cong \Omega M^{\Phi_b} \oplus \Omega$, $\Omega M' \oplus \Omega \cong \Omega M'^{\Phi_b} \oplus \Omega$ and $\Omega M'' \oplus \Omega \cong \Omega M''^{\Phi_b} \oplus \Omega$. Thus $\Omega M \oplus \Omega \cong \Omega M' \oplus \Omega M'' \oplus \Omega$.

(2) Let Σ be a simple component of $Q\Pi$ and let $\mathfrak{p}$ be a prime divisor of b_Σ in R_Σ. In order to show that $\mathfrak{p} \in P'_\Sigma$, we may assume that $\Pi_\Sigma = \{I\}$, and hence we can use the notation as in the proof of (1). There is an exact sequence

$$0 \longrightarrow N \longrightarrow \Omega \longrightarrow X \longrightarrow 0$$

of Ω-modules where X is a simple Ω-module such that $\mathfrak{p}X = 0$. Since $C(\Omega) = I_K/P'_\Sigma$, we only need to prove that $N \oplus \Omega \cong \Omega \oplus \Omega$. Let p be a rational prime such that $pZ = \mathfrak{p} \cap Z$ and let $b = p^c b'$, $p \nmid b'$. Further let $\Sigma' = Q\Pi/(\Phi_{b'}(\sigma))$ and let Λ' and Ω' be the images of $Z\Pi$ and $\Omega_{Z\Pi}$ under the projection $Q\Pi \to \Sigma'$, respectively. Then $\Lambda' = Z\Pi/(\Phi_b(\sigma))$. Since $\Phi_{b'}(\zeta_b) \in \mathfrak{p}$, X can be regarded as a Λ'-module. We see that $\Lambda'_p = \Omega'_p$ and so X can be regarded as a Ω'-module. Therefore there is an exact sequence

$$0 \longrightarrow N' \longrightarrow \Omega'^{(t)} \longrightarrow X \longrightarrow 0,\ t > 0$$

of Ω'-modules. Forming the pullback $\begin{array}{c}\Omega \to X \\ \uparrow \\ \Omega'^{(t)}\end{array}$ we get $\Omega \oplus N' \cong N \oplus \Omega'^{(t)}$.

Since $\omega_\Pi : C(\Omega_{Z\Pi}) \to G(Z\Pi)$ is a monomorphism, the natural epimorphism $G(\Omega_{Z\Pi}) \to G(Z\Pi)$ is an isomorphism ([11], (5.4)). Then $G(\Omega) \oplus G(\Omega')$ is the direct summand of $G(\Omega_{Z\Pi})$ and so $G(\Omega) \cap G(\Omega') = 0$ in $G(\Omega_{Z\Pi})$. Hence the fact that $\Omega \oplus N' \cong N \oplus \Omega'^{(t)}$ implies that $N \oplus \Omega \cong \Omega \oplus \Omega$ and $N' \oplus \Omega' \cong \Omega'^{(t+1)}$. Thus the proof of (2) is completed.

We denote by $B_G(Z\Pi)$ the subgroup of $G(Z\Pi)$ generated by all the images of permutation Π-modules in $G(Z\Pi)$ and define $Sw(\Pi) = G(Z\Pi)/B_G(Z\Pi)$. Then there exists a natural homomorphism $\theta_\Pi : \dot{T}^g(\Pi) \to Sw(\Pi)$. We will denote the image of a Π-module L in $G(Z\Pi)$ (resp. $Sw(\Pi)$) by $\langle L \rangle$ (resp. $\langle\langle L \rangle\rangle$).

THEOREM 4.2. *The following statements on a finite group Π are equivalent:*

(1) *Π is one of the following groups:* (i) *a cyclic group of order n where for every $m|n$ any prime ideal of $Z[\zeta_m]$ containing n is principal.* (ii) *a dihedral group of order $2p^c$ where $c \geqq 1$ and p is an odd prime.* (iii) *a direct product of a cyclic group of order q^f and a dihedral group of order $2p^c$ where $f, c \geqq 1$, p and q are odd primes, p is a primitive $q^{f-1}(q-1)$-th root of unity modulo q^f, for every $1 \leqq f' \leqq f$ any prime ideal of $Z[\zeta_{q^{f'}}]$ containing 2 is principal and for every $1 \leqq f' \leqq f$ and every $1 \leqq c' \leqq c$ any prime ideal of $Z[\zeta_{q^{f'}}, \zeta_{p^{c'}} + \zeta_{p^{c'}}^{-1}]$ containing q is principal.* (iv) *a generalized quaternion group of order $4p^c$ where $c \geqq 1$, p is an odd prime congruent to 3 modulo 4 and for every $1 \leqq c' \leqq c$ any prime ideal of $Z[\zeta_{p^{c'}} + \zeta_{p^{c'}}^{-1}]$ containing 2 is generated by a totally positive element.*

(2) *$T(\Pi) = C(\Omega_{Z\Pi})$ and $\omega_\Pi : C(\Omega_{Z\Pi}) \to G(Z\Pi)$ is a monomorphism.*

(3) *$T(\Pi) = T^q(\Pi)$ and $\theta_\Pi : T(\Pi) \to Sw(\Pi)$ is an isomorphism.*

(4) *The dual module of a quasi-permutation Π-module is always a quasi-permutation Π-module.*

(5) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of Π-modules and any two of M', M, M'' are quasi-permutation Π-modules, then the third one is a quasi-permutation Π-module.*

Proof. From (3.3) and (4.1) it follows immediately that (1) and (2) are equivalent. The implications (3) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious. Hence we only need to prove (1) $\Rightarrow$ (3), (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3).

(1) $\Rightarrow$ (3): We suppose that Π satisfies (1). Then by (1.5) and (3.3) we have $T(\Pi) = T^q(\Pi) = C(\Omega_{Z\Pi})$. Let $[\mathfrak{A}] - [\Omega_{Z\Pi}] \in C(\Omega_{Z\Pi})$ such that $\theta_\Pi([\mathfrak{A}] - [\Omega_{Z\Pi}]) = 0$, i.e., $\langle\!\langle \mathfrak{A} \rangle\!\rangle = \langle\!\langle \Omega_{Z\Pi} \rangle\!\rangle$. We will show that $\mathfrak{A} \oplus \Omega_{Z\Pi} \cong \Omega_{Z\Pi} \oplus \Omega_{Z\Pi}$. There exist $S_1, S_2 \in \boldsymbol{S}_\Pi$ such that $\langle \mathfrak{A} \oplus S_1 \rangle = \langle \Omega_{Z\Pi} \oplus S_2 \rangle$. Hence we have exact sequences

$$0 \longrightarrow L' \longrightarrow \mathfrak{A} \oplus S_1 \oplus L \longrightarrow L'' \longrightarrow 0 ,$$
$$0 \longrightarrow L' \longrightarrow \Omega_{Z\Pi} \oplus S_2 \oplus L \longrightarrow L'' \longrightarrow 0$$

with $L', L, L'' \in \boldsymbol{M}_\Pi$. Therefore, by virtue of (4.1), we have

$$\mathfrak{A} \oplus \Omega_{Z\Pi} S_1 \cong \Omega_{Z\Pi} \oplus \Omega_{Z\Pi} S_2 .$$

Let Σ be a simple component of $Q\Pi$ and let Λ and Ω be the images of $Z\Pi$ and $\Omega_{Z\Pi}$ under the projection $Q\Pi \to \Sigma$, respectively. Now it suffices to show that $\Omega\mathfrak{A} \oplus \Omega \cong \Omega \oplus \Omega$. If Σ is a division ring, then $\Omega S_1 \cong \Omega S_2 \cong \Omega^{(t)}$ for some $t \geqq 0$ so that $\Omega\mathfrak{A} \oplus \Omega \cong \Omega \oplus \Omega$. On the other hand, if Σ is not a division ring, then Λ is the trivial crossed product of $Z[\zeta_{qf'p^{c'}}]$ and a cyclic group $\langle\tau\rangle$ of order 2 where $0 \leqq f' \leqq f$ and $1 \leqq c'$, because Π is one of the groups as in (1). Let $R = Z[\zeta_{qf'p^{c'}}]$ and $\mathfrak{P} = (\zeta_{p^{c'}} - 1)$. Then $\mathfrak{P}$ is a prime ideal of R. Both R and $\mathfrak{P}$ can be regarded as Λ-modules and $\Lambda \cong R \oplus \mathfrak{P}$ as Λ-modules. Further we see that $\Lambda S_1 \cong \Lambda^{(u_1)} \oplus R^{(v_1)}$ and $\Lambda S_2 \cong \Lambda^{(u_2)} \oplus R^{(v_2)}$ for some $u_1, v_1, u_2, v_2 \geqq 0$. Consider the exact sequence $0 \to \mathfrak{P} \to R \to R/\mathfrak{P} \to 0$. Since $p\cdot(R/\mathfrak{P}) = 0$, we get $\Omega R \cong \Omega\mathfrak{P}$ and so $\Omega \cong \Omega R \oplus \Omega\mathfrak{P} \cong \Omega R \oplus \Omega R$. Hence we have $\Omega S_1 \cong \Omega R^{(u_1+v_1)}$ and $\Omega S_2 \cong \Omega R^{(u_2+v_2)}$. Therefore $\Omega\mathfrak{A} \oplus \Omega R^{(u_1+v_1)} \cong \Omega \oplus \Omega R^{(u_2+v_2)}$. Thus we can conclude that $\Omega\mathfrak{A} \cong \Omega$.

(3) $\Rightarrow$ (2): Suppose that $T(\Pi) = T^g(\Pi)$ and $\theta_\Pi : T(\Pi) \to Sw(\Pi)$ is an isomorphism. Since $Sw(\Pi)$ is a finite group, $T(\Pi)$ is so. Then by (3.3) we have $T(\Pi) = C(\Omega_{Z\Pi})$. Considering the commutative diagram

$$\begin{array}{ccc} C(\Omega_{Z\Pi}) & \xrightarrow{\omega_\Pi} & G(Z\Pi) \\ \| & & \downarrow \\ T(\Pi) & \underset{\sim}{\xrightarrow{\theta_\Pi}} & Sw(\Pi) \end{array},$$

we see that $\omega_\Pi : C(\Omega_{Z\Pi}) \to G(Z\Pi)$ is a monomorphism.

(4) $\Rightarrow$ (3): Suppose that Π satisfies (4). Then we have by (1.5) $T(\Pi) = T^g(\Pi)$. Let $[M] \in T(\Pi)$ such that $\theta_\Pi([M]) = \langle\!\langle M \rangle\!\rangle = 0$. Then there exist $S', S \in \boldsymbol{S}_\Pi$ such that $\langle M \oplus S' \rangle = \langle S \rangle$. Hence we have exact sequences

$$0 \longrightarrow L' \longrightarrow M \oplus S' \oplus L \longrightarrow L'' \longrightarrow 0 ,$$
$$0 \longrightarrow L' \longrightarrow S \oplus L \longrightarrow L'' \longrightarrow 0$$

with $L', L, L'' \in \boldsymbol{M}_\Pi$. Here we may assume by (1.5) that $L' \underset{(\mathfrak{t})}{\text{———}} 0$ and $L'' \underset{(\mathfrak{t})}{\text{———}} 0$. By assumption $L'^* \underset{(\mathfrak{t})}{\text{———}} 0$ and therefore there exists an exact sequence $0 \to L' \to T \to T' \to 0$ with $T, T' \in \boldsymbol{S}_\Pi$. Forming the pushout of $\begin{array}{l} L' \to S \oplus L \\ \downarrow \\ T \end{array}$, we get the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L' & \longrightarrow & S\oplus L & \longrightarrow & L'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & T & \longrightarrow & N & \longrightarrow & L'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & T' & = & T' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By assumption we have $L''^* \underset{(t)}{—} 0$. Then by (1.3) $N^* \underset{(t)}{—} 0$, and again by assumption $N \underset{(t)}{—} 0$. Therefore we see by (1.3) that $L \underset{(t)}{—} 0$. Similarly, forming the pushout of $\begin{array}{l} L' \to M \oplus S' \oplus L, \\ \downarrow \\ T \end{array}$ we can show that $M \oplus L \underset{(t)}{—} 0$. Consequently we have $M \underset{(t)}{—} M \oplus L \underset{(t)}{—} 0$. This proves that $\theta_\Pi : T(\Pi) \to Sw(\Pi)$ is an isomorphism.

Let $\Pi = \langle\sigma\rangle$ be a cyclic group of order n and let L be a Π-module. For every $m|n$ we define $L^{\Phi_m} = \{u \in L \,|\, \Phi_m(\sigma)u = 0\}$.

COROLLARY 4.3. *Let Π be a cyclic group of order n. If Π satisfies the conditions in* (4.2), *then it satisfies the following condition*:

(6) *A Π-module M is a quasi-permutation Π-module if and only if, for every $m|n$, M^{Φ_m} is a free $Z[\zeta_m]$-module.*

Proof. It has been shown in the proof of (4.2) that if L is a Π-module then for every $m|n$ $L^{\Phi_m} \cong Z[\zeta_m]L$. Hence the only if part follows immediately from (4.1). Let M be a Π-module such that for every $m|n$ M^{Φ_m} is $Z[\zeta_m]$-free. By (1.5) and (3.3) there is a projective ideal $\mathfrak{A}$ of $Z\Pi$ such that $M \oplus \mathfrak{A} \underset{(t)}{—} 0$. Then $M^{\Phi_m} \oplus \mathfrak{A}^{\Phi_m}$ is $Z[\zeta_m]$-free, hence $\mathfrak{A}^{\Phi_m}$ is so. Therefore by [3], (2.5) $\mathfrak{A} \underset{(t)}{—} 0$. Thus $M \underset{(t)}{—} 0$.

(4.2) and (4.3) show that the conjecture given in [3], p. 416 is true. It should be noted that the converse to (4.3) is not true ([3]).

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