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CANONICAL ISOMORPHISMS OF ENERGY FINITE SOLUTIONS OF $\Delta u = Pu$ ON OPEN RIEMANN SURFACES

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We call a second order differential $P(z)dxdy$ on a Riemann surface R a *density* if it is not identically zero and $P(z)$ is a nonnegative Hölder continuous function of the local parameter $z = x + iy$ in each parametric disk. To each density P on R we associate the linear space $P(R)$ of C^2 solutions of the equation $\Delta u(z) = P(z)u(z)$ invariantly defined on R. We also consider subspaces $PX(R)$ of $P(R)$ consisting of solutions with certain boundedness properties X. As for X we consider B meaning the boundedness, D the finiteness of the Dirichlet integral $D_R(u) = \int_R du \wedge^* du$, E the finiteness of the *energy integral*

$$E_R(u) = D_R(u) + \int_R u^2(z)P(z)dxdy \tag{1}$$

which is the variation whose Euler-Lagrange equation is $\Delta u = Pu$, and their combinations BD and BE. If the base surface R is parabolic, then $PX(R) = \{0\}$ for every X under considerations (cf. Ozawa [5]), and to avoid such trivial cases we always assume that R is *hyperbolic*, i.e. there exists the harmonic Green kernel $G_R(z, \zeta)$ on R.

As usual we denote by $H(R)$ the space of harmonic functions on R and similarly by $HX(R)$ the subspace of harmonic functions with the property $X = B, D$, and BD. The operator

$$Tu = u + \frac{1}{2\pi}\int_R G_R(\cdot\,, \zeta)u(\zeta)P(\zeta)d\xi d\eta \qquad (\zeta = \xi + i\eta) \tag{2}$$

is an injective linear order-preserving mapping from $PB(R)$ ($PD(R)$, $PE(R)$, resp.) into $HB(R)$ ($HD(R)$, $HD(R)$, resp.) (cf. [4]). The mapping T is referred to as the *canonical isomorphism* since u and Tu have the same "ideal boundary values". Since $PBD(R)$ ($PBE(R)$, resp.) is dense

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in $PD(R)$ ($PE(R)$, resp.) with respect to the simultaneous uniform convergence on compact and the convergence in $D_R(\cdot)$ ($E_R(\cdot)$, resp.) (cf. [3] (Glasner-Katz [2], resp.)) and similarly $HBD(R)$ is dense in $HD(R)$ (cf. Royden [6], [8]), one might suspect that $T(PBD(R)) = HBD(R)$ ($T(PBE(R)) = HBD(R)$, resp.) implies $T(PD(R)) = HD(R)$ ($T(PE(R)) = HD(R)$, resp.) (cf. Royden [7, p. 23]). In this context the following recent result of Singer [9] is remarkable: *There exists a density P on the simply connected Riemann surface R such that $T(PBD(R)) = HBD(R)$ but $T(PD(R)) < HD(R)$ (strict inclusion).* The main idea of Singer's work mentioned above can be used to show the corresponding phenomenon for the space of energy finite solutions, which is the purpose of the present paper:

THEOREM. *There exists a density P on the simply connected hyperbolic Riemann surface R with $\int_R P(z)dxdy < \infty$ such that $T(PBE(R)) = HBD(R)$ but $T(PE(R)) < HD(R)$.*

This negates an assertion of Royden [7, p. 23] in its original form. After preparations in nos. 1–3, the proof of Theorem will be given in no. 4.

1. Let Ω be a regular subregion of an open Riemann surface R. We denote by e_Ω the function in $P(\Omega) \cap C(\bar{\Omega})$ with $e_\Omega | \partial\Omega = 1$. The limit $e = \lim_{\Omega \to R} e_\Omega$ exists and is referred to as the *P-unit*. Singer [9] showed that under the assumption $T(PBD(R)) = HBD(R)$ a necessary and sufficient condition for an $h \in HD(R)$ to belong to $T(PD(R))$ is $D_R(eh) < \infty$. We first show its counter part:

LEMMA. *Under the assumption $T(PBE(R)) = HBD(R)$, a necessary and sufficient condition for an $h \in HD(R)$ to belong to $T(PE(R))$ is $E_R(eh) < \infty$.*

Proof. Suppose $E_R(eh) < \infty$. Then there exists a $u \in PE(R)$ such that $u = eh$ on the Royden harmonic boundary $\Delta(R)$ of R (cf. Glasner-Katz [2]). On the other hand, $Tu = u$ on $\Delta(R)$ and therefore $Tu = h$ on $\Delta(R)$, which implies $Tu = h$ on R (cf. [8]).

Conversely suppose that $h \in T(PE(R))$, i.e. there exists a $u \in PE(R)$ with $h = Tu$. Let Ω be a regular subregion of R. Take h_Ω in $H(\Omega) \cap C(\bar{\Omega})$ such that $h_\Omega | \partial\Omega = u$ and set $g_\Omega = u - h_\Omega$. Observe that $D_\Omega(u) = D_\Omega(h) + D_\Omega(g_\Omega)$, and thus $D_\Omega(g_\Omega) \leq D_\Omega(u) \leq D_R(u)$. Since $h = \lim_{\Omega \to R} h_\Omega$ (cf. [8]), the Fatou lemma yields

$$(3)\quad E_R(eh)^{1/2} \leq \liminf_{\Omega \to R} E_\Omega(e_\Omega h)^{1/2} = \liminf_{\Omega \to R} (E_\Omega(e_\Omega u)^{1/2} + E_\Omega(e_\Omega g_\Omega)^{1/2}) .$$

By the repeated applications of the Stokes formula we obtain

$$E_\Omega((1-e_\Omega)u) = \int_\Omega (1-e_\Omega)^2 du \wedge^* du + \int_\Omega (1-e_\Omega(z))^2 u^2(z) P(z) dxdy ,$$

and since $0 \leq (1-e_\Omega) \leq 1$,

$$E_\Omega(u - e_\Omega u) \leq \int_\Omega du \wedge^* du + \int_\Omega u^2(z)P(z)dxdy = E_\Omega(u) .$$

Therefore we have

$$(4)\qquad E_\Omega(e_\Omega u)^{1/2} \leq 2E_\Omega(u)^{1/2} \leq 2E_R(u)^{1/2} .$$

Again by using the Stokes formula repeatedly we deduce

$$E_\Omega(e_\Omega g_\Omega) = \int_\Omega e_\Omega^2 dg_\Omega \wedge^* dg_\Omega .$$

Since $0 \leq e_\Omega \leq 1$, we see

$$(5)\qquad E_\Omega(e_\Omega g_\Omega) \leq D_\Omega(g_\Omega) \leq D_R(u) \leq E_R(u) .$$

On combining (3), (4), and (5) we conclude that

$$E_R(eh) \leq 9E_R(u) < \infty . \qquad \text{Q.E.D.}$$

2. The following is a reformulation of a result of Glasner-Katz [2] which is a sharpening of Royden [7]:

LEMMA. *In order that $T(PBE(R)) = HBD(R)$ it is necessary and sufficient that there exists a neighborhood V^* of the Royden harmonic boundary $\Delta(R)$ of R with $\int_V P(z)dxdy < \infty$ ($V = V^* \cap R$). In particular $\int_R P(z)dxdy < \infty$ implies $T(PBE(R)) = HBD(R)$.*

Proof. Suppose that $T(PBE(R)) = HBD(R)$. Let $u = T^{-1}1 \in PBE(R)$. Since $u = 1$ on $\Delta(R)$, $V^* = \{u > 1/2\}$ is a neighborhood of $\Delta(R)$. Observe that

$$\int_V P(z)dydy \leq 4\int_V u^2(z)P(z)dxdy \leq 4E_R(u) < \infty \qquad (V = V^* \cap R) .$$

Conversely suppose the existence of V^* described in the above statement. Take an $f \in C^\infty(R) \cap C(R^*)$, R^* being the Royden compactification of R, with $f = 1$ on $\Delta(R)$ and $f = 0$ on $R^* - V^*$. Then $E_R(f) < \infty$ and there exists $u \in PE(R)$ with $u = f = 1$ on $\Delta(R)$. From this it follows that $0 < u < 1$. Clearly $u < e_\Omega < 1$ on each regular subregion Ω of R and hence $u \leq e \leq 1$ on R. Therefore $e - u = 0$ on $\Delta(R)$ which in turn implies $u = e$ on R. Thus $E_R(e) < \infty$. Let h be any function in $HBD(R)$. Then $D_R(eh) < \infty$ and $\int_R e^2(z)h^2(z)P(z)dxdy \leq \sup_R h^2 \int e^2(z)P(z)dxdy < \infty$ and therefore $E_R(eh) < \infty$. By the proof of Lemma 1, $h \in T(PBE(R))$.

Q.E.D.

3. We denote by D the unit disk $|z| < 1$ in the complex plane C. Let P be any density on D. We maintain

LEMMA. *Let $u \in PB(D)$ and $u > 0$. If $\lim_{r\to 1} u(re^{i\theta}) = 1$ for almost every $\theta \in [0, 2\pi)$, then u is the P-unit e on D.*

Proof. Observe that $Tu = u + \dfrac{1}{2\pi}\int_D G_D(\cdot, \zeta)u(\zeta)P(\zeta)d\xi d\eta \in HB(D)$.

By the Littlewood theorem (cf. Tsuji [10]) $\lim_{r\to 1}\int_D G_D(re^{i\theta}, \zeta)u(\zeta)P(\zeta)d\xi d\eta = 0$ for almost every $\theta \in [0, 2\pi)$ and thus $\lim_{r\to 1}(Tu)(re^{i\theta}) = 1$ for almost every $\theta \in [0, 2\pi)$. Therefore $Tu \equiv 1$ and $0 < u < 1$ on D. Clearly $u \leq e_\Omega \leq 1$ on every regular subregion Ω of D and hence $u \leq e \leq 1$ on D. Then $\lim_{r\to 1} e(re^{i\theta}) = 1$ for almost every $\theta \in [0, 2\pi)$. We conclude, as above, that $Te \equiv 1$. The injectiveness of T and $Tu = Te$ imply $u = e$.

Q.E.D.

4. We are ready to prove Theorem. Let R be the simply connected hyperbolic Riemann surface. Besides the unit disk D there are infinitely many conformal representations of R as subregions of C. Here we take

$$R = \{z \in C; x > 10^{1/6}, \varphi(z) = x^{-4} - y^2 > 0\} \tag{6}$$

as the representation of R. Since φ is a bounded continuous function on ∂R (relative to C), i.e. $0 \leq \varphi \leq 10^{-2/3}$, we can consider the solution $H_\varphi^R(z)$ of the harmonic Dirichlet problem with the boundary values φ on ∂R in the sense of Constantinescu-Cornea [1]. Set

$$P(z) = -\Delta\varphi(z)/(1 - \varphi(z) + H_\varphi^R(z)) . \tag{7}$$

The denominator is of class C^∞ and not less than $1 - 10^{-2/3} > 0$ on R. Similarly $-\Delta\varphi(z) = 2(1 - 10x^{-6})$ is of class C^∞ and > 0 on R. Therefore $P(z)dxdy$ is a density on R. We assert that (R, P) given by (6) and (7) is the required pair.

Since $P(z) \leq 2(1 - 10^{-2/3})^{-1}$, we have

$$(8) \qquad \int_R P(z)dxdy \leq 4(1 - 10^{-2/3})^{-1}10^{-1/6} < \infty .$$

By Lemma 2, (8) implies

$$(9) \qquad T(PBE(R)) = HBD(R) .$$

The function

$$(10) \qquad e(z) = 1 - \varphi(z) + H_\varphi^R(z)$$

is certainly a solution of $\Delta u = Pu$ on R and continuous on $R \cup \partial R$ with $e|\partial R = 1$. Observe that there exists a homeomorphism from $R \cup \partial R$ onto $D \cup (\partial D - \{1\})$ which is a conformal mapping of R onto D. Therefore, by Lemma 3, $e(z)$ is the P-unit on R. Let $h(z) = x$. Since $\int_R dh \wedge {}^* dh = \int_R dxdy = 2\cdot 10^{-1/6} < \infty$, $h \in HD(R)$. However

$$(11) \qquad \begin{aligned} E_R(eh) &\geq \int_R e^2(z)h^2(z)P(z)dxdy \\ &\geq 2\int_R (1 - 10x^{-6})(1 - x^{-4})x^2dxdy = \infty . \end{aligned}$$

By Lemma 1, $h \notin T(PE(R))$, i.e.

$$(12) \qquad T(PE(R)) < HD(R) .$$

The proof of Theorem is herewith complete.

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