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ON p-ADIC L-FUNCTIONS AND CYCLOTOMIC FIELDS

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1. Introduction.

Let p be a prime. If one adjoins to $\boldsymbol{Q}$ all p^n-th roots of unity for $n = 1, 2, 3, \cdots$, then the resulting field will contain a unique subfield $\boldsymbol{Q}_\infty$ such that $\boldsymbol{Q}_\infty$ is a Galois extension of $\boldsymbol{Q}$ with $\mathrm{Gal}(\boldsymbol{Q}_\infty/\boldsymbol{Q}) \cong \boldsymbol{Z}_p$, the additive group of p-adic integers. We will denote $\mathrm{Gal}(\boldsymbol{Q}_\infty/\boldsymbol{Q})$ by Γ. Let $\boldsymbol{Q}_p$ be the field of p-adic numbers and let Ω_p be an algebraic closure of $\boldsymbol{Q}_p$. Let χ be a Dirichlet character which takes its values in Ω_p. We will say that χ is of the first kind if its conductor is not divisible by q, where $q = p$ if p is odd and $q = 4$ if $p = 2$. For each such χ of the first kind, we will define in section 2 a certain representation space V_χ for Γ over Ω_p. The space V_χ will be constructed from certain Γ-modules that occur in Iwasawa's work on cyclotomic fields.

Our main purpose in this paper is to study the relationship between the p-adic L-functions defined by Kubota and Leopoldt in [11] and the structure of the representation spaces V_χ. We will describe these functions more fully later, but let us just say here that with each primitive Ω_p-valued Dirichlet character ψ one can associate a certain Ω_p-valued function $L_p(s, \psi)$ defined for all $s \in \boldsymbol{Z}_p$, except $s = 1$ if ψ is the principal character ψ^0. If ψ is odd (i.e. if $\psi(-1) = -1$), then $L_p(s, \psi)$ is identically zero. However, if ψ is even (i.e. if $\psi(-1) = +1$) and of the first kind, then Iwasawa has constructed a certain power series $G_\psi(T)$ whose coefficients are integers in the field $\boldsymbol{Q}_p(\psi)$ generated by the values of ψ and which has the property that

$$L_p(s, \psi) = \frac{G_\psi(\kappa_0^s - 1)}{(\kappa_0^s - \kappa_0)^\delta},$$

where κ_0 is a certain p-adic unit that we will define later and where

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$\delta = 1$ or 0 according to whether ψ is principal or non-principal. The p-adic unit κ_0 will be a principal unit so that κ_0^s is defined and $\equiv 1 \pmod p$ for all $s \in \boldsymbol{Z}_p$. Thus the above power series converges and we see that $L_p(s, \psi)$ is analytic for $s \in \boldsymbol{Z}_p$ except for a possible pole (which in fact exists) at $s = 1$ when $\psi = \psi^0$. The p-adic unit κ_0 is somewhat arbitrary and its choice amounts to choosing a fixed topological generator γ_0 of Γ. The power series $G_\psi(T)$ also depends on this choice.

Now one can associate in a rather natural way a certain monic polynomial $g_\psi(T)$ with the power series $G_\psi(T)$ (see section (4)). Let $f_\chi(T)$ denote the characteristic polynomial of $\gamma_0 - 1$ acting on V_χ. We will now state a certain conjecture relating the $g_\psi(T)$'s and $f_\chi(T)$'s which was suggested to us by recent work of Iwasawa and of Coates and Lichtenbaum. In fact our conjecture is essentially a slightly weaker form of conjecture 2.3 of [2]. We will say that two Dirichlet characters χ and ψ (Ω_p-valued and primitive, as we will usually assume without mention) are dual if they are of the first kind and if

$$\chi\psi = \omega ,$$

where ω is the unique Dirichlet character with conductor q such that

$$\omega(a) \equiv a \pmod q$$

for all $a \in \boldsymbol{Z}$, $(a, p) = 1$. (The product of two Dirichlet characters is the primitive character corresponding to their pointwise product.) The precise relationship between the above sets of polynomials should be as follows.

CONJECTURE. Let χ and ψ be dual characters with χ odd. Then

$$f_\chi(T) = g_\psi(T) .$$

In this paper, we will show that $g_\psi(\gamma_0 - 1)$ annihilates V_χ. This will follow from a classical theorem of Stickelberger and Iwasawa's construction of p-adic L-functions. Furthermore, Iwasawa has proven the following relationship between the degrees of the f_χ's and g_ψ's. Let $f = nq$, where $(n, p) = 1$. As ψ varies over all even characters whose conductor divides f, the dual character χ will vary over all such odd characters. Then as we will explain more fully in section 5,

$$(1) \qquad \sum_\chi \deg(f_\chi(T)) = \sum_\psi \deg(g_\psi(T)).$$

Consider the following assumption concerning V_χ:

(C) V_χ is spanned by $\{\gamma_0^i(v) \mid i = 0, 1, 2, \cdots\}$ for some v in V_χ.

This assumption implies that $f_\chi(T)$ is actually the minimal polynomial for $\gamma_0 - 1$ acting on V_χ. It would then follow that $f_\chi(T)$ divides $g_\psi(T)$. If (C) holds for all odd χ with conductor dividing f, we could then conclude from (1) that in fact $f_\chi(T) = g_\psi(T)$. Thus (C) implies the above conjecture and even implies that the p-adic L-functions completely describe, in a sense, the structure of the representation spaces V_χ for odd χ.

It is possible that assumption (C) is always valid. Some computations of Iwasawa and Sims [10] show that if $p \leq 4001$ and if χ has conductor p, then V_χ is at most one dimensional so that (C) certainly holds. Moreover, Iwasawa proved in [6] that if p is properly irregular (i.e. if p does not divide the class number of $\boldsymbol{Q}(\zeta_p + \zeta_p^{-1})$), then (C) is again valid for those χ of conductor p. At the end of section 2, we will describe a much more general result due to Coates and Lichtenbaum which gives sufficient conditions for (C) to hold. Coates and Lichtenbaum have verified their conditions and hence (C) in many cases.

In a recent paper [5], we have discussed in some detail the following question: If k is a totally real number field, is $\lambda(k) = \mu(k) = 0$? Although the evidence does not seem sufficient to conjecture that this question always has an affirmative answer, we know of no examples to the contrary. This question is closely related to (C). In fact, as we also explain at the end of section 2, theorem 5 of [5] implies immediately that the vanishing of λ for totally real abelian extensions of $\boldsymbol{Q}$ is a sufficient condition for (C) to hold. Thus many of the examples we give at the end of [5] provide new cases where (C) is known to be valid.

The conjecture above would have some important consequences. If F is a finite totally real abelian extension of $\boldsymbol{Q}$ and if $\zeta(s, F)$ denotes the Dedekind zeta function of F, then $\zeta(1-n, F)$ is a non-zero rational number for every positive even integer n (this follows from (4) and (5)). These rational numbers seem to be closely connected with the arithmetic of cyclotomic extensions of F (e.g. Kummer's criterion) and it would be interesting to describe this connection precisely. Of course, it is sufficient to focus on one prime at a time. Now, if one could describe for each ψ the integer m_ψ and the polynomial $g_\psi(T)$ which are defined in section 4, then one could clearly determine the exact power of p dividing $\zeta(1-n, F)$ for each n. As we explain in section 4, the vanishing of the Iwasawa μ-invariant for abelian number fields (which is extremely probable) would imply that $m_\psi = 0$ for all ψ. Thus the above conjecture (together with the vanishing of μ) would give, in principle, a complete description of the p-part of $\zeta(1-n, F)$ for $n = 2, 4, \cdots$. One could then deduce some of Lichtenbaum's conjectures and in particular the Birch-Tate conjecture (but only for abelian ground fields). This is in fact essentially done in [2].

It seems reasonable to expect that p-adic L-functions should exist for Dirichlet characters of an arbitrary totally real ground field. Recent work of Serre [12] and of Coates and Sinnott [3] show that in certain cases such functions do in fact exist. There is some evidence that these functions are also connected with Iwasawa theory, but one essential link is missing, namely a suitable generalization of Stickelberger's theorem.

Our work in this paper is based considerably on earlier work of other authors. Iwasawa ([6] and [7]), Brumer [1], and, in greater generality, Coates and Lichtenbaum [2] have studied a somewhat stronger version of the conjecture stated earlier. Instead of the vector spaces that we consider, they consider certain compact Γ-modules. Our approach, although parallel, seems simpler in some respects. On the other hand, we lose some hold on the torsion subgroup of these Γ-modules and hence on the μ-invariant. To a certain extent, this could be studied independently (for example, see theorem 2). In any case, our hope is that this approach will clarify the problems we study here.

Finally I would like to thank J. Coates and K. Iwasawa for many stimulating discussions concerning the topic of this paper. I also want to thank the Institute for Advanced Study for its hospitality while this work was done.

2. The representation space V_χ.

Let K be a number field and let $K_\infty = K\boldsymbol{Q}_\infty$. For each $n \geq 0$, there is a unique cyclic extension K_n of K of degree p^n contained in K_∞. For $m \leq n$, we have $K_m \subseteq K_n$. Let A_n denote the p-primary subgroup of the ideal class group of K_n. For $n \geq m$, let $N_{n,m}: A_n \to A_m$ be the norm map. We let

$$X_K = \varprojlim A_n .$$

Clearly, X_K can be considered as a Z_p-module since that A_n's are finite abelian p-groups. We let

$$V_K = X_K \otimes_{Z_p} \Omega_p .$$

Obviously, $\mathrm{Gal}(K_\infty/K)$ acts in a natural way on both X_K and V_K. The module X_K can also be defined in the following way. Let L denote the maximal abelian unramified p-extension of K_∞. Then $X_K \cong \mathrm{Gal}(L/K_\infty)$ canonically, the isomorphism being defined by class field theory.

One of Iwasawa's basic results concerning the structure of X_K is that $X_K \cong Z_p^\lambda \times T$ as a Z_p-module, where T is a torsion group and $\lambda = \lambda_p(K) < \infty$ (see [9]). It follows that V_K is a finite dimensional vector space over Ω_p of dimension λ. The structure of T can be described somewhat more precisely in terms of another invariant $\mu = \mu_p(K)$. However, it seems likely that T is always finite (or equivalently $\mu = 0$). Another basic result of Iwasawa states that $|A_n| = p^{e_n}$, where $e_n = \lambda n + \mu p^n + \nu$ for all sufficiently large n, where $\nu = \nu_p(K)$ is an integer. The integers λ, μ, and ν are called the Iwasawa invariants for K and are clearly uniquely determined by the above formula for $|A_n|$.

We will assume now that K is an abelian extension of Q and that the conductor of K/Q is not divisible by qp. (Just as for characters, such K will be said to be of the first kind.) This implies that $K \cap Q_\infty = Q$. Let $\Delta = \mathrm{Gal}(K/Q)$. Then we have a canonical isomorphism (induced by restriction):

$$\mathrm{Gal}(K_\infty/Q) \cong \Delta \times \Gamma .$$

We identify Δ with $\mathrm{Gal}(K_\infty/Q_\infty)$ and Γ with $\mathrm{Gal}(K_\infty/K)$. In particular, if $K = Q(\zeta_q)$, then K_∞ contains all p-power roots of unity and we can then define the p-adic unit κ_0 mentioned in the introduction by the equation:

$$\gamma_0(\zeta) = \zeta^{\kappa_0}$$

for every root of unity ζ of order a power of p. Clearly, $\kappa_0 \equiv 1 \pmod{q}$ since $\zeta_q \in K$.

Now let χ be any primitive Ω_p-valued Dirichlet character of the first kind. Let K_χ be the cyclic extension of Q corresponding to χ by class field theory and let K be any abelian extension of Q of the first kind which contains the field K_χ. If $\Delta = \mathrm{Gal}(K/Q)$, then χ can be considered as a character of Δ and since Δ acts naturally on V_K we may define

$$V_\chi = \{v \in V_K \,|\, \delta(v) = \chi(\delta)v \quad \text{for all } \delta \in \Delta\} .$$

Obviously $\gamma(V_\chi) = V_\chi$ for all $\gamma \in \Gamma$ and so V_χ is a representation space for Γ. If K$'$ is another abelian extension of $\boldsymbol{Q}$ such that $K \subseteq K'$ and if $\Delta' = \mathrm{Gal}(K'/\boldsymbol{Q})$, then the Dirichlet character χ corresponds to a character χ' of Δ'. However, a simple argument shows that the norm mapping $N_{K'/K}$ induces a homomorphism of Γ-modules from $X_{K'}$ to X_K and that one then gets a canonical isomorphism

$$V_{\chi'} \cong V_\chi .$$

Hence the representation of Γ on V_χ does not depend essentially on the choice of K above.

Clearly $f_\chi(T)$, the characteristic polynomial of $\gamma_0 - 1$ acting on V_χ, has coefficients in $\boldsymbol{Q}_p(\chi)$. Let $\mathcal{O}_\chi$ denote the ring of integers in $\boldsymbol{Q}_p(\chi)$. Since Γ acts continuously and Γ is a pro-p-group, it follows that $f_\chi(T) \in \mathcal{O}_\chi[T]$ and that its coefficients except for the leading one are in the maximal ideal of $\mathcal{O}_\chi$. Thus $f_\chi(T)$ is a so-called distinguished polynomial.

We will continue to assume throughout the remainder of this section that K is abelian over $\boldsymbol{Q}$ and of the first kind. It is clear that

$$V_K \cong \sum_\chi V_\chi ,$$

a direct sum over the characters χ belonging to K. Assume also that K is totally imaginary. We then define

$$V_K^+ = \sum_{\text{even } \chi} V_\chi \quad \text{and} \quad V_K^- = \sum_{\text{odd } \chi} V_\chi .$$

(We could equivalently define $V_K^+ = (1 + J)V_K$ and $V_K^- = (1 - J)V_K$, where J denotes complex conjugation. This definition is meaningful provided K is a totally imaginary quadratic extension of a totally real number field.) Let K^+ denote the maximal real subfield of K. Suppose λ, μ, ν and λ^+, μ^+, ν^+ are the Iwasawa invariants of K and K^+. We clearly have $V_K^+ \cong V_{K^+}$ and so $\dim(V_K^+) = \lambda^+$. Thus, $\dim(V_K^-) = \lambda - \lambda^+$. We put $\lambda^- = \lambda - \lambda^+$, and similarly $\mu^- = \mu - \mu^+$, $\nu^- = \nu - \nu^+$. If h_n^- denotes the first factor of the class number of K_n (i.e. the ratio of the class numbers of K_n and K_n^+), then it follows from one of Iwasawa's results mentioned earlier that the power of p dividing h_n^- is given by p^{e_n}, where

$e_n = \lambda^- n + \mu^- p^n + \nu^-$ for all sufficiently large n. This formula, which obviously characterizes λ^-, μ^-, and ν^-, will be important later.

We will now describe certain conditions under which the V_χ's satisfy assumption (C). We first make one observation. Let $G = \mathrm{Gal}(K_\infty/\boldsymbol{Q})$ and let R denote the group ring of G over $\boldsymbol{Z}_p$ as defined in the theory of profinite groups.

Consider the following assumption:

(C′) $\qquad V_K^-$ is spanned by $\{gv \mid g \in G\}$ for some v.

It is easy to see that (C′) implies (C) for all odd χ belonging to K.

Let $X_K^- = (1 - J)X_K$. In [2], Coates and Lichtenbaum give a sufficient condition for X_K^- to be cyclic as an R-module. Of course, this implies (C′) immediately. Let p be odd and assume that no prime of K^+ dividing p splits in K. Let $\Delta = \mathrm{Gal}(K/\boldsymbol{Q})$ and let $R_0 = \boldsymbol{Z}_p[\Delta]$. Then theorem 2.8 of [2] states (essentially) that X_K^- is cyclic as an R-module if and only if A_0 is cyclic as an R_0-module. Coates and Lichtenbaum also give several examples of K satisfying these assumptions.

Let p now be an arbitrary prime and assume that $\zeta_q \in K$. According to theorem 5 of [5], if $\lambda(K^+) = 0$, then X_K^- contains a cyclic R-submodule Y_K^- such that $p^b X_K^- \subseteq Y_K^-$ for some integer b. Clearly (C′) follows. Note that the assumption $\lambda(K^+) = 0$ is equivalent to $V_K^+ = 0$. We also give in [5] several examples where $\lambda(K^+) = 0$. Some of these examples do not satisfy the Coates-Lichtenbaum conditions that we described above. We have not been able to find any examples for which $\lambda(K^+) > 0$ and it seems possible that no such examples exist. We should add however that there are many K (including some satisfying the Coates-Lichtenbaum conditions) for which we don't know whether or not $\lambda(K^+) = 0$.

3. An annihilator for V_χ.

We will now construct a certain annihilator of V_χ by making use of a classical theorem of Stickelberger [14]. Let $m \geq 1$ and let $F = \boldsymbol{Q}(\zeta_m)$, where ζ_m is a primitive m^{th} root of unity. Let $G = \mathrm{Gal}(F/\boldsymbol{Q})$ and, for $(a, m) = 1$, let $\sigma(a) \in G$ be defined by $\sigma(a)(\zeta_m) = \zeta_m^a$. Thus $\sigma(a) = \left(\frac{F/\boldsymbol{Q}}{a}\right)$, the Artin symbol. Let

$$\theta = \frac{1}{m} \sum_{\substack{a=1 \\ (a,m)=1}}^{m} a\sigma(a)^{-1} .$$

Thus $\theta \in \boldsymbol{Q}[G]$. Stickelberger's theorem asserts that every element of

$$I = \boldsymbol{Z}[G]\theta \cap \boldsymbol{Z}[G]$$

annihilates the ideal class group of F.

Let f be a positive integer divisible by q but not by qp. Let $K = \boldsymbol{Q}(\zeta_f)$. Then $K_n = \boldsymbol{Q}(\zeta_{f_n})$, where $f_n = fp^n$. Let

$$\theta_n = \frac{1}{f_n} \sum_{\substack{a=1 \\ (a,f)=1}}^{f_n} a\sigma_n(a)^{-1} ,$$

where $\sigma_n(a) = \left(\frac{K_n/\boldsymbol{Q}}{a}\right)$. Let $G_n = \mathrm{Gal}(K_n/\boldsymbol{Q})$. For $n \geq m$, the restriction homomorphism $G_n \to G_m$ induces a homomorphism from $\boldsymbol{Q}[G_n]$ to $\boldsymbol{Q}[G_m]$ which sends θ_n to

$$\theta_m + \frac{1}{2}(p^{n-m} - 1) \sum_{\substack{a=1 \\ (a,f)=1}}^{f_m} \sigma_m(a)^{-1} .$$

If we let $\alpha_n = (1 - \sigma_n(-1))\theta_n$ for $n \geq 0$, then α_n is mapped precisely to α_m for $n \geq m$. Note that $\sigma_n(-1)$ is just complex conjugation. Let b be any integer prime to f. Let

$$\beta_n = (\sigma_n(b) - b)\cdot\alpha_n .$$

Then one checks easily that $\beta_n \in \boldsymbol{Z}[G_n]$. Thus β_n annihilates A_n for each $n \geq 0$. Also β_n is mapped to β_m for $n \geq m$ and hence the sequence of β_n's defines an element

$$\beta = \varprojlim \beta_n$$

in

$$R = \varprojlim R_n ,$$

where $R_n = \boldsymbol{Z}_p[G_n]$. One can describe R in another way. Let $G = \mathrm{Gal}(K_\infty/\boldsymbol{Q}) = \Delta \times \Gamma$, where $\Delta = \mathrm{Gal}(K/\boldsymbol{Q})$. Then R is just the group ring of G over $\boldsymbol{Z}_p$ as defined in the theory of profinite groups. Now the group ring of Γ over $\boldsymbol{Z}_p$ can be identified with the power series ring

$$\Lambda = \boldsymbol{Z}_p[[T]]$$

by identifying T with $\gamma_0 - 1$. Thus R can be identified with $\Lambda[\Delta]$. Therefore, β can be expressed in the form

$$\beta = \sum_{\delta \in \Delta} B_\delta(T)\delta ,$$

where $B_\delta(T) \in \Lambda$.

It is clear from what has been said that β annihilates X_K. Thus β also annihilates V_K. If χ is any Dirichlet character whose conductor divides f, then β also annihilates V_χ. Now χ can be considered as a character of Δ and if we let

$$C_{\chi,b}(T) = \sum_{\delta \in \Delta} \chi(\delta) B_\delta(T) ,$$

then $C_{\chi,b}(T) \in \mathcal{O}_\chi[[T]]$ and $C_{\chi,b}(T)$ annihilates V_χ (where of course T is $\gamma_0 - 1$). The decomposition $G = \Delta \times \Gamma$ gives us a corresponding decomposition $G_n = \Delta_n \times \Gamma_n$ with $\Gamma_n = \mathrm{Gal}(K_n/K)$. For $(a, f) = 1$, we write $\sigma_n(a) = \delta_n(a)\gamma_n(a)$. Then χ can be considered as a character of Δ_n such that $\chi(a) = \chi(\delta_n(a))$ for $(a, f) = 1$. Also,

$$\mathcal{O}_\chi[[T]] \cong \varprojlim \mathcal{O}_\chi[\Gamma_n] .$$

One easily sees that $C_{\chi,b}(T)$ corresponds in this isomorphism to

$$\beta(\chi, b) = \varprojlim \beta_n(\chi) ,$$

where

$$\beta_n(\chi) = (\chi(b)\gamma_n(b) - b)\cdot(1 - \chi(-1))\cdot\frac{1}{f_n} \sum_{\substack{a=1 \\ (a,f)=1}}^{f_n} a\chi(a)^{-1}\gamma_n(a)^{-1} . \tag{2}$$

In the next section we will see that the power series $C_{\chi,b}(T)$ is closely related to p-adic L-functions. The essential difference will be the factor

$$\chi(b)\gamma(b) - b = \varprojlim (\chi(b)\gamma_n(b) - b) .$$

We will make use of the freedom that we have for b in the following way. Assume first that $\chi \neq \omega$. Choose $\delta \in \Delta$ such that $\chi(\delta) \neq \omega(\delta)$. Let b_i, $i = 1, 2, \cdots$, be a sequence of integers prime to f such that

$$\delta(b_i) = \delta \quad \text{and} \quad \gamma(b_i) \to 1$$

as $i \to \infty$. It is clear that $b_i \to \omega(\delta)$ in $\mathbf{Z}_p$ and so

$$\chi(b_i)\gamma(b_i) - b_i \to \chi(\delta) - \omega(\delta) \tag{3}$$

as $i \to \infty$. It will be important that $\chi(\delta) - \omega(\delta)$ is non-zero. If $\chi = \omega$, then we remark that $C_{\omega,b}(T)$ could have been defined where b is any p-adic unit. Choosing $b = \kappa_0$, the above factor becomes $\gamma_0 - \kappa_0$.

4. p-adic L-functions.

Let ψ be a complex-valued Dirichlet character. The Dirichlet L-function $L(s, \psi)$ assumes algebraic values at the negative integers. More precisely, as shown in [8], if f is the conductor of ψ, then for $n \geq 1$,

$$L(1 - n, \psi) = \sum_{a=1}^{f} r_n(a)\psi(a) , \tag{4}$$

where the $r_n(a)'s$ are rational numbers independent of ψ (but depending on f). Moreover, one deduces immediately from the functional equation for $L(s, \psi)$ that for $n \geq 2$

$$L(1 - n, \psi) \neq 0 \text{ if and only if } \psi(-1) = (-1)^n . \tag{5}$$

This is also true for $n = 1$ unless ψ is the principal character.

If ψ is a Dirichlet character of conductor f with values in Ω_p, we take equation (4) as the definition of $L(1 - n, \psi)$. It is then clear that (5) remains valid. We also define

$$L^*(1 - n, \psi) = (1 - \psi(p)p^{n-1})L(1 - n, \psi) \tag{6}$$

(which for complex-valued ψ would amount to omitting the Euler factor for p from $L(1 - n, \psi)$). Now we can describe the Kubota-Leopoldt p-adic L-functions. Let ω be as before. We assume from here on that ψ is Ω_p-valued and we denote for brevity $\psi\omega^{-n}$ by ψ_n. Then the p-adic L-function $L_p(s, \psi)$ is the unique Ω_p-valued continuous function defined for $s \in \mathbf{Z}_p$ (except at $s = 1$ if $\psi = \psi^0$) such that

$$L_p(1 - n, \psi) = L^*(1 - n, \psi_n) \qquad \text{for all } n \geq 1 . \tag{7}$$

If ψ is odd, then from (5) we see that $L_p(s, \psi) \equiv 0$. If ψ is even, the existence of these functions is related to certain congruences similar to the well-known Kummer congruences. In this case, (5) shows that for all $n \geq 2$,

$$L_p(1 - n, \psi) \neq 0 . \tag{8}$$

We also see that

$$L_p(0, \psi) = 0 \text{ if and only if } \psi_1(p) = 1 . \tag{9}$$

For more details concerning the existence and properties of these functions, one should see [8] and [11].

We will now briefly describe Iwasawa's construction of p-adic L-functions (see [7] or [8]). Let ψ be an even character of the first kind. Assume first of all that ψ is non-principal. Let f be the least common multiple of the conductor of ψ and q. As in section 2, let $f_n = fp^n$. Let

$$\xi_n = -\frac{1}{2f_n} \sum_{\substack{a=1 \\ (a,f)=1}}^{f_n} a\psi_1(a)\gamma_n(a)^{-1} . \tag{10}$$

Thus $\xi_n \in \boldsymbol{Q}_p(\psi)[\Gamma_n]$. However, it is shown in [8] that $\xi_n \in \mathcal{O}_\psi[\Gamma_n]$ (for ψ non-principal) and also that $\xi_n \to \xi_m$ for $n \geq m$ under the natural map. Let

$$R_\psi = \varprojlim \mathcal{O}_\psi[\Gamma_n]$$

and let

$$\xi = \xi(\psi) = \varprojlim \xi_n .$$

Under the isomorphism $R_\psi \cong \mathcal{O}_\psi[[T]]$ defined by sending γ_0 to $1 + T$, $2\xi(\psi)$ is mapped to a power series $G_\psi(T)$ such that

$$L_p(s, \psi) = G_\psi(\kappa_0^s - 1)$$

for all $s \in \boldsymbol{Z}_p$. Iwasawa proves this for $\gamma_0 = \gamma(1 + f_0)$ but it obviously follows immediately for any choice of the topological generator γ_0.

If ψ is the principal character ψ^0, let $\eta_n = (\gamma_{0,n} - \kappa_0)\xi_n$, where $\gamma_{0,n}$ is the image of γ_0 in Γ_n. Then $\eta_n \in \boldsymbol{Z}_p[\Gamma_n]$ and also $\eta_n \to \eta_m$ for $n \geq m$. In this case, if $\eta = \varprojlim \eta_n$, Iwasawa has shown that 2η corresponds to a power series $G_{\psi^0}(T)$ such that

$$(\kappa_0^s - \kappa_0)L_p(s, \psi^0) = G_{\psi^0}(\kappa_0^s - 1)$$

for all $s \in \boldsymbol{Z}_p$, $s \neq 1$.

If θ is an arbitrary Ω_p-valued Dirichlet character, then θ can be uniquely expressed in the form $\theta = \psi\phi$, where ψ is a character of the first kind and ϕ is of the second kind (i.e. the class field corresponding to ϕ is a subfield of $\boldsymbol{Q}_\infty$). If θ is even, ψ will be also. Then Iwasawa shows in [8] that

$$L_p(s,\theta) = \frac{G_\psi(\zeta\kappa_0^s - 1)}{(\zeta\kappa_0^s - \kappa_0)^\delta}\,, \tag{11}$$

where $\zeta = \theta(\gamma_0)^{-1} = \phi(\gamma_0)^{-1}$ is a certain p-power root of unity and $\delta = 1$ or 0 according to whether ψ is principal or nonprincipal.

Note that the coefficients of $G_\psi(T)$ are always divisible by 2. Also, by letting $s = 1 - (p-1)$ (or $s = -1$ if $p = 2$) and using the well-known fact that $\zeta(1-n) = -\dfrac{B_n}{n}$ for $n \geq 2$ (where B_n is the n^{th} Bernoulli number) together with von Staudt's theorem, one sees that $\frac{1}{2}G_{\psi^0}(T)$ is an invertible power series over $\boldsymbol{Z}_p$.

Let π be a prime element of $\mathcal{O}_\psi$. Now the power series $G_\psi(T)$ is non-zero and so can be uniquely expressed in the form

$$G_\psi(T) = 2\pi^{m_\psi}\cdot g_\psi(T)\cdot u_\psi(T)\,, \tag{12}$$

where m_ψ is a non-negative integer, $u_\psi(T)$ is an invertible element in $\mathcal{O}_\psi[[T]]$, and $g_\psi(T)$ is a monic polynomial in $\mathcal{O}_\psi[T]$ whose non-leading coefficients are multiples of π. For example, it is clear that for $\psi = \psi^0$, we must have $g_\psi(T) = 1$ and $m_\psi = 0$.

We can now draw some important conclusions concerning the structure of the representation spaces V_χ for odd χ. Let χ be an odd character of the first kind and let us assume at first that $\chi \neq \omega$. If $\chi\psi = \omega$, then $\psi \neq \psi^0$ and $\chi^{-1} = \psi_1$. It follows from (2) and (10) that

$$C_{\chi,b}(T) = -2(\chi(b)\gamma(b) - b)\cdot G_\psi(T)\,.$$

Now $C_{\chi,b}(\gamma_0 - 1)$ annihilates V_χ and, by choosing a sequences of b's so that (3) holds, we see that in fact $G_\psi(\gamma_0 - 1)$ and therefore $g_\psi(\gamma_0 - 1)$ annihilate V_χ. Now if $\chi = \omega$, we have

$$C_{\omega,\kappa_0}(T) = -2G_{\psi^0}(T)$$

and so $g_{\psi^0}(\gamma_0 - 1)$ annihilates V_ω. We have proved the following theorem.

THEOREM 1. *Let χ be an odd character of the first kind and let ψ be defined by the equation $\chi\psi = \omega$. Then $g_\psi(\gamma_0 - 1)$ annihilates V_χ.*

There are several interesting corollaries to Theorem 1. First of all since $g_{\psi^0}(T) = 1$, we have

COROLLARY 1. $V_\omega = 0$.

This could also be proved quite directly without using Stickelberger's theorem and p-adic L-functions as we have done here.

The following corollaries also follow immediately, using (8) for Corollary 3.

COROLLARY 2. *If χ and ψ are as in Theorem* 1, *then every root of $f_\chi(T)$ is a root of $g_\psi(T)$.*

COROLLARY 3. *If χ is an odd Dirichlet character of the first kind and if n is a positive integer, then $f_\chi(\kappa_0^{-n}-1)\neq 0$.*

The conjecture that we stated in the introduction is of course equivalent to the assertion that $f_\chi(T)$ and $g_\psi(T)$ have precisely the same roots, counting multiplicity. It is instructive to consider $T=0$. Because of (9), it is clear that $g_\psi(T)$ vanishes at $T=0$ if and only if $\psi_1(p)=1$ (i.e. $\chi(p)=1$). However, $\chi(p)=1$ means that p splits completely in the cyclic extension K_χ of $\boldsymbol{Q}$ corresponding to χ. One can deduce easily from this (essentially by a genus theory type argument) that $(V_\chi)^\Gamma=\{v\in V_\chi \mid \gamma(v)=v \text{ for all } \gamma\in\Gamma\}$ is non-zero if and only if χ is an odd character for which $\chi(p)=1$ (see [2], [4], or [9]). Thus, $f_\chi(0)=0$ exactly when $g_\psi(0)=0$. It is proved in [4] that $f_\chi(T)$ has at most a simple zero at $T=0$. It would be interesting to establish the same result for $g_\psi(T)$.

5. Results concerning the degree of $g_\psi(T)$.

Let K be a totally imaginary abelian extension of $\boldsymbol{Q}$ which is of the first kind. We define

$$G_K(T)=\prod_{\psi} G_\psi(T)\,, \tag{13}$$

where ψ varies over all even Dirichlet characters such that ψ_1 belongs to K. There are d such characters ψ, where $d=\frac{1}{2}[K:\boldsymbol{Q}]$. It is not hard to see that $G_K(T)\in \boldsymbol{Z}_p[[T]]$. We also find that

$$G_K(T)=2^d p^m g_K(T)u(T) \tag{14}$$

where m is a non-negative integer, $u(T)$ is an invertible power series over $\boldsymbol{Z}_p$, and

$$g_K(T)=\prod_{\psi} g_\psi(T)\,. \tag{15}$$

In [8], Iwasawa proves the following theorem:

THEOREM 2 (Iwasawa): *We have* $\deg(g_K(T)) = \lambda^-$ *and* $m = \mu^-$, *where* $\lambda^- = \lambda^-(K)$ *and* $\mu^- = \mu^-(K)$.

Let $f_K^-(T)$ denote the characteristic polynomial of $\gamma_0 - 1$ acting on V_K^- so that

$$f_K^-(T) = \prod_{\chi} f_\chi(T) ,$$

the product being taken over all odd characters χ belonging to K. The conjecture that we stated in the introduction of course would imply that $f_K^-(T) = g_K(T)$. Since $\lambda^- = \dim(V^-)$, Theorem 2 tells us that $\deg(f_K^-(T)) = \deg(g_K(T))$. Thus Theorem 2 clearly implies the equation (1) stated in the introduction.

We also want to mention here that if $\mu^- = 0$, then it would obviously follow that m and therefore all the m_ψ's are 0. There is now substantial support for the conjecture that $\mu_p(K) = 0$, where K is an arbitrary number field. The most convincing evidence is a recent (unpublished) result of L. Washington and B. Ferrero that $\mu_2(K)$ and $\mu_3(K)$ are zero when K is a finite abelian extension of $\boldsymbol{Q}$.

We will just sketch the proof of Theorem 2. By comparing the residues of the Dedekind zeta functions of K_n and K_n^+ at $s = 1$ and by applying their functional equations one obtains the following formula:

$$\frac{2^{d_n} \cdot h_n^-}{w_n \cdot Q_n} = \prod_{\theta} L(0, \theta) ,$$

where θ varies over all odd complex-valued characters belonging to K_n. Here $d_n = \frac{1}{2}[K_n : \boldsymbol{Q}] = dp^n$, h_n^- is the first factor of the class number of K_n, w_n is the number of roots of unity in K_n, and Q_n is a certain unit group index which turns out to be 1 or 2. Obviously the above formula remains valid if θ varies over Ω_p-valued characters instead. We can express such a θ uniquely in the form $\theta = \chi\phi$, where χ belongs to K and ϕ is a character of the second kind. If ϕ is non-trivial, we have

$$L_p(0, \theta\omega) = L(0, \theta) .$$

Also if ψ equals $\chi\omega$, then it follows from (11) that

$$L_p(0, \theta\omega) = \frac{G_\psi(\zeta - 1)}{(\zeta - \kappa_0)^\delta} ,$$

where $\zeta = \phi(\gamma_0)^{-1}$ is a p^n-th root of unity. One then obtains

$$\frac{2^{d_n}h_n^-}{w_nQ_n} = \frac{2^d h_0^-}{w_0Q_0} \prod_\zeta G_K(\zeta - 1)/(\zeta - \kappa_0)^\delta \,, \tag{16}$$

where ζ varies over all p^n-th roots of unity except 1. Note that $\delta = 1$ exactly when ω belongs to K and in this case $w_n = p^n w_0$. If r is the degree of $g_K(T)$, then one deduces from (16) and (14) that the power of p dividing h_n^- is p^{e_n} where $|e_n - (rn + mp^n)|$ is bounded as $n \to \infty$. It clearly follows that $r = \lambda^-$ and $m = \mu^-$.

For certain χ, one can deduce from Theorem 2 that

$$\deg(f_\chi(T)) = \deg(g_\psi(T)) \,,$$

where $\chi\psi = \omega$, as usual. For example, if χ is an odd character of order 2 so that the corresponding field K_χ is imaginary quadratic, then these degrees certainly are equal. If K is an arbitrary totally imaginary cyclic extension of $\boldsymbol{Q}$, by applying Theorem 2 for K and its subfields one obviously obtains that

$$\sum_\chi \deg(f_\chi(T)) = \sum_\psi \deg(g_\psi(T)) \,,$$

where χ varies over all characters such that $K_\chi = K$ (these must be odd) and ψ varies over the corresponding dual characters. Now one can see easily that two characters χ and χ' have $K_\chi = K_{\chi'}$ if and only if $\chi = \sigma \circ \chi'$, where $\sigma \in \mathrm{Gal}(\boldsymbol{Q}(\chi)/\boldsymbol{Q})$. If $\chi = \sigma \circ \chi'$, where $\sigma \in \mathrm{Gal}(\boldsymbol{Q}_p(\chi)/\boldsymbol{Q}_p)$, then this same σ maps $f_{\chi'}(T)$ to $f_\chi(T)$ (allowing σ to act on coefficients). Thus, $\deg(f_\chi(T)) = \deg(f_{\chi'}(T))$ in this case. However, if ψ and ψ' are defined by $\chi\psi = \chi'\psi' = \omega$, then clearly $\psi = \sigma \circ \psi'$ and it follows from (4), (6) and (7) that

$$L_p(s, \psi) = \sigma(L_p(s, \psi'))$$

for all $s \in \boldsymbol{Z}_p$. Thus σ also maps $g_{\psi'}(T)$ to $g_\psi(T)$ and hence $\deg(g_\psi(T)) = \deg(g_{\psi'}(T))$. Combining these facts, we have the following proposition.

PROPOSITION 1. *Let χ be an odd Dirichlet character such that $[\boldsymbol{Q}(\chi) : \boldsymbol{Q}] = [\boldsymbol{Q}_p(\chi) : \boldsymbol{Q}_p]$ and let ψ be dual to χ. Then*

$$\deg(f_\chi(T)) = \deg(g_\psi(T)) \,.$$

It should also be clear from what we have said that if χ satisfies the assumptions of the above proposition, then one can easily determine the integer m_ψ occurring in (12) in terms of the μ-invariants for the various subfields of K_χ.

We will conclude this section by considering briefly the case where $\zeta_q \in K$. Then the character ω belongs to K and in (13) and (15), ψ simply varies over all even characters belonging to K. If we let K^+ denote the maximal real subfield of K, then the p-adic zeta function of K^+ is given by

$$\zeta_p(s, K^+) = \prod_{\psi} L_p(s, \psi) ,$$

where ψ varies as above. Our results show that

$$\zeta_p(s, K^+) = \frac{F(\kappa_0^s - 1)}{(\kappa_0^s - \kappa_0)} , \tag{17}$$

where $F(T) = G_K(T) = 2^d p^m g_K(T) u(T)$. Here $d = [K^+ : Q]$, $m = \mu^-(K)$, $u(T)$ is an invertible power series over Z_p, and $\deg(g_K(T)) = \lambda^-(K)$. Conjecturally, $g_K(T) = f^-(T)$, the characteristic polynomial of $\gamma_0 - 1$ acting on V_K^-. All of the above remarks remain valid if K is an arbitrary totally imaginary abelian extension of Q which contains ζ_q (but which is not necessarily of the first kind) and can be easily deduced by using (11). We will omit the details.

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