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# ON A DISTANCE FUNCTION BETWEEN DIFFERENTIABLE STRUCTURES*

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## Introduction

In this note, we investigate a relation between the connected sum of manifolds and the distance of manifolds ([2]). Since the smoothing of a piecewise linear equivalence is given by connected sum of exotic spheres ([1]), we have a certain estimate of the smoothing obstruction using the distance of manifolds (Proposition 3). In § 3, an application is given to show the impossibility of the 0.64-pinching of an exotic sphere.

**1.** Let $M, N$ be smooth orientable manifolds with boundary so that the boundaries $\partial M, \partial N$ are diffeomorphic each other through a diffeomorphism $f$. Denote by $C(\partial M), C(\partial N)$ the collar neighbourhoods of $\partial M, \partial N$, respectively, and let

$$\alpha : \partial M \times [0,1) \to C(\partial M)\ , \qquad \beta : \partial N \times [0,1) \to C(\partial N)$$

be the diffeomorphisms. Then the map which sends $\alpha(x,t)(x \in \partial M, t \in [0,1))$ into $\beta(f(x), 1-t)$ defines a diffeomorphism $F = F(f)$ between $C(\partial M), C(\partial N)$ and the identified space $M \bigcup_F N$ turns out to be a smooth manifold.

LEMMA 1. *Let $M_i, N_i$ $(i = 1,2)$ be smooth manifolds with boundary and let $f_1$ be a diffeomorphism between $\partial M_1$ and $\partial N_1$. If homeomorphisms $g_1 : M_1 \to M_2$ and $g_2 : N_1 \to N_2$ are diffeomorphic on some neighbourhoods of the closures of collar neighbourhoods $C(\partial M_1)$, $C(\partial N_1)$, then there are collar neighbourhoods $C(\partial M_2), C(\partial N_2)$ and a diffeomorphism $F_2$ of $C(\partial M_2)$ onto $C(\partial N_2)$ so that $M_2 \bigcup_{F_2} N_2$ is homeomorphic to $M_1 \bigcup_{F(f_1)} N_1$ by a homeomorphism $g_1 \cup g_2$ defined by*

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$$g_1 \cup g_2(x) = \begin{cases} g_1(x) , & \text{if } x \in M_1 \\ g_2(x) , & \text{if } x \in N_1 \end{cases}$$

PROPOSITION 1. *Let $M_i, N_i, g_i$ $(i = 1, 2), f_1$, be as in Lemma 1. Suppose moreover that with respect to Riemannian metrics $\rho_i, \sigma_i$ $(i = 1, 2)$ on $M_i, N_i$ respectively, the homeomorphisms $g_i$ $(i = 1, 2)$ satisfy that*

$$\begin{cases} \rho_1(x, y)/k_1 \le \rho_2(g_1(x), g_1(y)) \le k_1\rho_1(x, y) & \text{for } x, y \in M_1 , \\ \sigma_1(x, y)/k_2 \le \sigma_2(g_2(x), g_2(y)) \le k_2\sigma_1(x, y) & \text{for } x, y \in N_1 , \end{cases}$$

*then there exist Riemannian metrics $\tau_i$ on $M_i \bigcup_{F_i} N_i$ $(i = 1, 2)$ such that*

$$\tau_1(x, y)/\max(k_1, k_2) \le \tau_2(g_1 \cup g_2(x), g_1 \cup g_2(y)) \le \max(k_1, k_2)(\tau_1(x, y)) .$$

*Proof.* Take a real valued smooth function $\varphi$ such that

$$\begin{aligned} &0 \le \varphi(t) \le 1 , \quad \varphi(t) = 0 \quad \text{for } t \le 0, \varphi(t) = 1 \quad \text{for } t \ge 1 , \\ &0 \le \varphi'(t) , \quad \varphi'(t) = 0 \text{ for } t \le 0 \text{ or } t \ge 1 , \\ &\varphi(1 - t) = 1 - \varphi(t) \end{aligned}$$

and let

$$\alpha_1 : M_1 \times [0, 1) \to C(\partial M_1) , \qquad \beta_1 : N_1 \times [0, 1) \to C(\partial N_1)$$

be diffeomorphisms onto the collar neighbourhoods. Then

$$\alpha_2 = g_1 \circ \alpha_1((g_1^{-1}|_{\partial M_2}), \mathrm{id}) , \qquad \beta_2 = g_2 \circ \beta_1((g_2^{-1}|_{\partial N_2}), \mathrm{id})$$

also are diffeomorphism of $\partial M_2 \times [0, 1), \partial N_2 \times [0, 1)$ onto collar neighbourhoods $C(\partial M_2), C(\partial N_2)$, respectively, moreover the identification map $F_2$ obtained from $\alpha_2, \beta_2$, and $(g_2|_{\partial N_1}) \circ f_1 \circ (g_1^{-1}|_{\partial M_2})$ satisfies that

$$g_2 \circ F_1 = F_2 \circ g_1 \qquad \text{on } C(\partial M_1) .$$

Define quadratic forms $\tilde{\tau}_i$ on $M_i \bigcup_{F_i} N_i$ $(i = 1, 2)$ by

$$(\tilde{\tau}_i)_x = \begin{cases} (\tilde{\rho}_i) & , \quad x \in M_i - C(\partial M_i) , \\ \varphi(t(x))(\tilde{\rho}_i)_x + (1 - \varphi(t(x)))(F_i^* \tilde{\sigma}_i)_x , & \quad x \in C(\partial M_i) , \\ (\tilde{\sigma}_i)_x & , \quad x \in N_i - C(\partial N_i) . \end{cases}$$

where $t(x)$ denotes the $t$-coordinate of $x$ in the collor neighbourhood and $(\tilde{\ })$ indicates the quadratic form of a metric ( ). Then it is easy to see that the well defined quadratic forms $\tilde{\tau}_i$ $(i = 1, 2)$ give Riemannian metrics $\tau_i$ on $M_i \bigcup_{F_i} N_i$. Since

$$\rho_1(x, y)/k_1 \leq \rho_2(g_1(x), g_1(y)) \leq k_1\rho_1(x, y)$$
$$\sigma_1(F_1(x), F_1(y))/k_2 \leq \sigma_2(g_2F_1(x), g_2F_1(y)) \leq k_2\sigma_1(F_1(x), F_1(y)) \text{ ,}$$

it holds that

$$\tilde{\rho}_1/k_1 \prec g_1^*\tilde{\rho}_2 \prec k_1\tilde{\rho}_1 \text{ ,}$$
$$F_1^*\tilde{\sigma}_1/k_2 \prec g_1^*(F_2^*\tilde{\sigma}_2) = (g_2F_1)^*\tilde{\sigma}_2 \prec k_2F_1^*\tilde{\sigma}_1 \text{ .}$$

Therefore the metrics $\tau_i$ satisfy that

$$\tilde{\tau}_1/\max(k_1, k_2) \prec g_1^*\tilde{\tau}_2 \prec \max(k_1, k_2)\tilde{\tau}_1$$

on $C(\partial M_i)$, thus from the construction of $g_1 \cup g_2$ we may conclude that

$$\tau_1(x, y)/\max(k_1, k_2) \leq \tau_2((g_1 \cup g_2)(x), (g_1 \cup g_2)(y)) \leq (\max(k_1, k_2))(\tau_1(x, y)) \text{ .}$$

Let $M_i$ $(i = 1, 2)$ be smooth manifolds with metrics $\rho_i$ $(i = 1, 2)$ and $f$ be a map of $M_1$ into $M_2$, then we define $\ell(f : \rho_1, \rho_2)$ by

$$\ell(f : \rho_1, \rho_2) = \inf\{k \geq 1/ \rho_1(x, y)/k \leq \rho_2(f(x), f(y)) \leq k\rho_1(x, y), \text{ for any } x, y \in M\}$$

Definition. Let $\Sigma_i$ $(i = 1, 2)$ be differential structures on a combinatorial manifold $X$ represented by smooth manifolds $M_i$ $(i = 1, 2)$ with Riemannian metrics $\rho_i$ $(i = 1, 2)$. The distance $d(\Sigma_1, \Sigma_2)$ between the differential structures is defined to be

$$d(\Sigma_1, \Sigma_2) = \log(\inf \ell(f : \rho_1, \rho_2)) \text{ ,}$$

where the infimum is taken over all the piecewise linear equivalences $f$ of $M_1$ onto $M_2$ and all the Riemannian metrics $\rho_1, \rho_2$. It is known ([2]) that $d$ is actually a distance function.

Theorem 1. *Let $\Sigma_{i,j}$ $(i, j = 1, 2,\ j = 1, 2)$ be differential structures on cominatorial manifolds $X_i$ $(i = 1, 2)$, respectively, then it holds that*

$$d(\Sigma_{1,1} \# \Sigma_{2,1}, \Sigma_{1,2} \# \Sigma_{2,2}) \leq \max(d(\Sigma_{1,1}, \Sigma_{1,2}), d(\Sigma_{2,1}, \Sigma_{2,2}))$$

*where $\Sigma_{i,1} \# \Sigma_{i,2}$ denotes the differential structure obtained by the connected sum.*

*Proof.* Represent $\Sigma_{i,j}$ by smooth manifolds $M_{i,j}$, and for $\varepsilon > 0$ take piecewise diffeomorphisms $g_i$ of $M_{i,1}$ into $M_{i,2}$ and Riemannian metrics $\rho_{i,j}$ on $M_{i,j}$ so that

$$\log \ell(g_i ; \rho_{i,1}, \rho_{i,2}) \leq d(\Sigma_{i,1}, \Sigma_{i,2}) + \varepsilon$$

Assume that $g_i$ are diffeomorphic on neighbourhoods of points $p_i \in M_{i,1}$. Let $M'_{i,1}$ (resp. $M'_{i,2}$) be the manifold obtained by cutting out a small imbedded disk around $p_i$ (resp. $g_i(p_i)$). Then $M'_{i,j}$ and $g_i$ turns out to satisfy the assumption of Proposition 1 with $k_i = \ell(g_i; \rho_{i,1}, \rho_{i,2})$. Since identified manifolds $M'_{1,j} \cup M'_{2,j}$ represent the connected sum $\Sigma_{i,j} \# \Sigma_{2,j}$, we have that

$$d(\Sigma_{1,1} \# \Sigma_{2,1}, \Sigma_{1,2} \# \Sigma_{2,2}) \leq \max(\log k_1, \log k_2)$$

finishing the proof.

COROLLARY 1. *Let $\Gamma_k$ be the group of k-dimensional homotopy spheres, then it holds that*

$$d(\Sigma_1 + \Sigma_3, \Sigma_2 + \Sigma_3) = d(\Sigma_1, \Sigma_2)$$

*for any $\Sigma_i \in \Gamma_k$ $(i = 1, 2, 3)$.*

COROLLARY 2. *The subset $\Gamma_k(a)$ of $\Gamma_k$ given by*

$$\Gamma_k(a) = \{\Sigma \in \Gamma_k / d(S^k, \Sigma) \leq a\}$$

*turns out to be a subgroup of $\Gamma_k$, where $S^k$ denotes the standard k-sphere.*

COROLLARY 3. *Let $M_i$ $(i = 1, 2)$ be k-dimensional manifolds such that $M_2 \approx M_1 \# \Sigma$ (diffeomorphic) with $\Sigma \in \Gamma_k(a)$, then*

$$d(M_1, M_2) \leq a .$$

COROLLARY 4. *Let* Diff $S^{k-1}$ *denote the set of orientation preserving diffeomorphisms onto itself and let $\pi$ denote the projection of* Diff $S^{k-1}$ *onto $\Gamma_k$, then taking the usual metric $|\ \ |$ on $S^{k-1}$ induced from that of $R^k \supset S^{k-1}$, it holds that*

$$d(S^k, \pi(f)) \leq \log \ell(f; |\ \ |, |\ \ |) .$$

*Proof.* Extend $f$ radially to a homeomorphism $g$ of disk $D^k$ onto itself which bounds the sphere $S^{k-1}$ and apply Lemma 1 to disks $D^k, g$, id and $f$:

$$\begin{array}{ccccccc} D^k & \supset & \partial D^k & \xrightarrow{f} & \partial D^k & \subset & D^k \\ \downarrow g & & & & & & \mathrm{id} \downarrow \\ D^k & & & & & & D^k \end{array}$$

to obtain a homeomorphism $g \cup \mathrm{id}$ and a diffeomorphism $F_2$ of $\partial D^k$ onto itself which can be chosen to be identity. Since it is obvious that

$$\ell(f\,;\,|\quad|,|\quad|) = \ell(g\,;\,|\quad|,|\quad|)\,,$$

Proposition 1 yields that

$$d\Big(S^{k-1} \bigcup_{F_2} S^{k-1}, \pi(f)\Big) \leq \log \ell(f\,;\,|\quad|,|\quad|)\,.$$

**2.** The partial converse to Corollary 3 holds as in the following:

PROPOSITION 2. *Let $f$ be a homeomorphism between $k$-dimensional manifolds $M_i$, $(i = 1, 2)$ with Riemannian metrics $\rho_i$ $(i = 1, 2)$ and assume that $f$ is diffeomorphic except finite number of points $P_1, \cdots P_m \in M_1$ then $M_2 \approx M_1 \# \Sigma$ (diffeomorphic) with $\Sigma \in \Gamma_k$ $(\log \ell(f\,;\,\rho_1, \rho_2))$.*

*Proof.* Imbed small $k$-disks $D_i$ around $P_i$, then the images $f(D_i)$ turn out to be submanifolds in $M_2$. Apply Lemma 1 to manifolds $D_i, f(D_i)$, diffeomorphism $f|_{\partial D_i}$ and homeomorphisms $\mathrm{id}, f^{-1}$

$$\begin{array}{ccc} D_i \supset \partial D_i & \xrightarrow{f|\partial D_i} & \partial(f(D_i)) \subset f(D_i) \\ \big\downarrow \mathrm{id} & & f^{-1} \big\downarrow \\ D_i \supset \partial D_i & \xrightarrow{\mathrm{id}} & \partial D_i \subset D_i \end{array}$$

to obtain homotopy sheres $\Sigma_i = D_i \bigcup_{F_1} f(D_i)$ and a homeomorphism $\mathrm{id} \cup f^{-1}$ between the homotopy sphere and the sphere $S_i$. Because of Proposition 1 there are Riemannian metrics $\sigma_1^i, \sigma_2^i$ on $\Sigma_i, S_i$, respectively, so that

$$\ell(\mathrm{id} \cup f\,;\,\sigma_1^i, \sigma_2^i) \leq \ell(f : \rho_1, \rho_2)\,.$$

Therefore we have that

$$\Sigma_i \in \Gamma_k(\log \ell(f\,;\,\rho_1, \rho_2))\,.$$

On the other, since it is easy to see that

$$M_2 \approx M_1 \# \Sigma_1 \# \Sigma_2 \cdots \# \Sigma_m\,,$$

this finishes the proof.

In general, concerning the first obstruction of Munkres ([1]) to smoothing $f$, we obtain the following:

PROPOSITION 3. *Let $M_i$ $(i = 1, 2)$ be smoothly triangulated manifolds with Remannian metrics $\rho_i$ $(i = 1, 2)$ and let $L$ be an m-dimensional subcomplex of $M_1$. If a homeomorphism $f$ of $M_1$ onto $M_2$ is diffeomorphic* mod. $L$, *and if $\ell(f : \rho_1, \rho_2) < \ell_0 \doteqdot 1.32$ for the positive root $\ell_0$ of $x^3 - x - 1 = 0$, then the first obstruction chain $\lambda(f)$ of Munkres to smoothing $f$ lies in*

$$\Gamma_{k-m}(\log \ell(f)(1 - (\ell^3(f) - \ell(f))^2)^{-1/4})$$

*where $\ell(f) = \ell(f ; \rho_1, \rho_2)$*

*Proof.* Munkres obstruction is obtained as follows: Take an $m$-simplex $\sigma \in L$ and take trivializations of normal bundles as coordinate systems around $\sigma$ and $f(\sigma)$ so that the tubular neighbourhoods of $\sigma, f(\sigma)$ are diffeomorphic to $\sigma \times R^{k-m}$, $f(\sigma) \times R^{k-m}$, respectively, then if $\varepsilon > 0$ is sufficiently small, $\pi \circ f \circ i_p$ is a homeomorphism of the $\varepsilon$-disk $D_\varepsilon$ around 0 into $R^{k-m}$ for the inclusion $i_p : R^{k-m} \to p \times R^{k-m}$ and for the projection $\pi : f(\sigma) \times R^{k-m}$. Thus the obstruction $\lambda(f)(\sigma)$ is defined to be homotopy sphere obtained by glueing the boundaries of $D_\varepsilon$ and $\pi \circ f \circ i_p(D_\varepsilon)$ through $\pi \circ f \circ i_p$. Hence it is sufficient for the proof of Proposition 3 to compute $\ell(\pi \circ f \circ i_p ; \rho_1, \rho_2)$ (see Proposition 1) and because of the regularity of $f$ at $L$ ([1] p. 526 (4)) the computation is reduced to the following Assertion;

*Assertion.* Let $g$ be a map between manifolds $N_i$ $(i = 1, 2)$ with Riemannian metrics $\sigma_i$ $(i = 1, 2)$ satisfying that

$$\ell(g : \sigma_1, \sigma_2) < \kappa < \ell_0$$

then if $g$ is differentiable along any vector of an $m$ dimensional vector space $V \subset T_p(N_1)$, the angle $\theta$ between the vector $\overrightarrow{\exp_2^{-1} \circ g \circ \exp_1 (y), 0}$ and the plane dg $(V)$ is not too small, in fact $\theta$ satisfies that

$$\cos \theta < \kappa^3 - \kappa < 1 ,$$

for any $y$ in orthogonal linear subspace $W$ to $V$, provided $|y|$ is sufficiently small.

Proof of Assertion. Taking an $\varepsilon$-disk $D_\varepsilon$ of 0 in $T_p(N_1)$, we may assume that $\tilde{g} = \exp_2^{-1} \circ g \circ \exp_1$ also satisfies that

$$\ell(\tilde{g} : |\quad|, |\quad|) < \kappa < \ell_0$$

on $D_\varepsilon$. Let $x \in V$ be such that $|x| = |y|$, then it holds that

$$\begin{aligned} 2\langle \tilde{g}(x), f(y)\rangle &= |\tilde{g}(x)|^2 + |f(y)|^2 - |\tilde{g}(x) - f(y)|^2 \\ &< \kappa(|x|^2 + |y|^2) - |x-y|^2/\kappa \\ &= 2|x|^2(\kappa - 1/\kappa) \end{aligned}$$

also it holds that

$$2\langle \tilde{g}(x), f(y)\rangle > 2|x|^2(1/\kappa - \kappa) ,$$

therefore we have that

$$|\cos(\overrightarrow{\tilde{g}(x)0}, \overrightarrow{f(y)0})| < \kappa^3 - \kappa ,$$

finishing the proof of Assertion.

Thus taking the regularity of $f$ into consideration, we may conclude that by an application of Assertion to $g = f \circ i_p$,

$$\kappa^{-1}(1 - (\kappa^3 - \kappa)^2)^{1/2} \leq \rho_2(\pi \circ f \circ i_p(x), \pi \circ f \circ i_p(y))/\rho_1(x, y) \leq \kappa$$

on a small disk around 0, completing the proof of Proposition 3.

**3.** The method in § 1, 2 applies to obtain a weak estimation of the pinching of an exotic sphere. Let $M_1, M_2$ be combinatorially equivalent compact manifolds, then according to the construction of Hirch-Munkres ([1]), we may have a sequence of compact manifolds $L_i$ $(i = 1 \cdots k)$ such that

i) $L_i$ are combinatorially equivalent to $M_1, M_2$.

ii) $L_1 = M_1, I_k = M_2$ (diffeomorphic).

iii) $L_{i+1}$ is obtained by attaching of $\Sigma^j \times I^{n-j}$ to $L_i$ through a certain attaching map, $(\Sigma^j \in \Gamma^j)$.

Now suppose $M_1, M_2$ have different (integral) Pontrjagin class, then for some $i$, $L_i$, $L_{i+1}$ have also different Pontrjagin classes. Since we know that manifolds having different Pontrjagin classes are of distance $\geqq 1/2 \log 3/2$ ([3]), we have that

$$\begin{aligned} 1/2 \log 3/2 &\leqq d(L_i, L_{i+1}) \\ &\leqq \max(d(L_i, L_i), d(S^j \times I^{n-j}, \Sigma^j \times I^{n-j})) \\ &\leqq d(S^j, \Sigma^j) . \end{aligned} \tag{1}$$

Here the last inequality follows from an easily proved Lemma below:

LEMMA 2. *If $M_i, N_i$ denote a pair of combinatorially equivalent compact manifolds $(i = 1, 2)$ then*

$$d(M_1 \times M_2, N_1 \times N_2) \leqq \max(d(M_1, N_1), d(M_2, N_2))$$

On the other as is improved by Karcher (unpublished, see also ([4])) $\delta$-pinched Riemannian manifold $M_\delta$ ($\delta \geqq 9/16$) has distance $\leq 4(1-\sqrt{\delta})$ from the standard sphere S, therefore if the exotic sphere $\Sigma^j$ in (1) is expressed as a $\delta$-pinched manifold $M_\delta$, $\delta$ must satisfy that

$$1/2 \log 3/2 \leqq 4(1-\sqrt{\delta}) .$$

hence

$$\delta \leqq 0.64 ,$$

thus we may conclude that a certain exotic sphere of dimension $\leqq 16$ which appears in the obstruction chain to smoothing a combinatorial equivalence can not be pinched by 0.64, because we know that there are compact 16 manifolds having different Pontrjagin classes.

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