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ON LIE ALGEBRAS OF VECTOR FIELDS WITH INVARIANT SUBMANIFOLDS

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§ 0. Introduction.

It is known (Pursell and Shanks [9]) that an isomorphism between Lie algebras of infinitesimal automorphisms of C^∞ structures with compact support on manifolds M and M' yields an isomorphism between C^∞ structures of M and M'.

Omori [5] proved that this is still true for some other structures on manifolds. More precisely, let M and M' be Hausdorff and finite dimensional manifolds without boundary. Let α be one of the following structures:

(1) C^∞-structures, $(\alpha = \phi)$

(2) SL-structure, i.e. a volume element (positive n-form) with a non-zero constant multiplicative factor, $(\alpha = dV)$

(3) Sp-(symplectic) structure, i.e. symplectic 2-form with a non-zero constant multiplicative factor, $(\alpha = \Omega)$

(4) Contact structure, i.e. contact 1-form with a non-zero C^∞-function as a multiplicative factor, $(\alpha = \omega)$

(5) Fibring with compact fibre, $(\alpha = \mathscr{F})$

Let α (resp. α') be one of the above structures on M (resp. M'). Let $\Gamma_\alpha(T_M)$ be the Lie algebra of all C^∞, α-preserving infinitesimal transformations with compact support. We denote by $\mathscr{D}_\alpha(M)$ the group of all C^∞, α-preserving diffeomorphisms on M with compact support, that is, identity outside a compact subset. Then we have the following theorem

THEOREM (Omori [5]). *$\Gamma_\alpha(T_M)$ is algebraically isomorphic to $\Gamma_{\alpha'}(T_{M'})$, if and only if (M,α) is isomorphic to (M',α'). Especially, $\mathscr{D}_\alpha(M)$ is isomorphic to $\mathscr{D}_{\alpha'}(M')$.*

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Omori [6, 7, 8] defined the notion of I.L.H.-Lie group and proved that the group $\mathscr{D}_\alpha(M)$ stated above is an I.L.H.-Lie group. As a matter of fact, $\mathscr{D}_\alpha(M)$ is a (strong) I.L.H.-Lie group with the Lie algebra $\boldsymbol{\Gamma}_\alpha(T_M)$. So we can say that the I.L.H.-Lie group $\mathscr{D}_\alpha(M)$ is determined by its Lie algebra.

Let (M, N) be a pair of paracompact C^∞ manifolds such that N is a closed submanifold of M (may be $\dim N = 0$). We denote by $\boldsymbol{\Gamma}_N(T_M)$ the Lie algebra of all C^∞, N-preserving, i.e. tangent to N, infinitesimal transformations with compact support. By $\mathscr{D}(M, N)$ we denote the group of all C^∞, N-preserving diffeomorphisms on M with compact support. The purpose of this paper is to prove the following theorem.

THEOREM. *$\boldsymbol{\Gamma}_N(T_M)$ is algebraically isomorphic to $\boldsymbol{\Gamma}_{N'}(T_{M'})$, if and only if there exists a C^∞ diffeomorphism $\varphi: M \to M'$ such that $\varphi(N) = N'$. Especially $\mathscr{D}(M, N)$ is isomorphic to $\mathscr{D}(M', N')$.*

If M is compact, then $\mathscr{D}(M, N)$ becomes an I.L.H.-Lie subgroup of $\mathscr{D}(M)$ with the Lie algebra $\boldsymbol{\Gamma}_N(T_M)$ (Ebin and Marsden [2]). So in this case we can say that $\mathscr{D}(M, N)$ is determined as an I.L.H.-Lie group by its Lie algebra.

The proof of our theorem is parallel to that of Pursell and Shanks. Main parts of our proof are § 2 and § 3. We denote by $\boldsymbol{\Gamma}_0(T_M)$ instead of $\boldsymbol{\Gamma}_N(T_M)$ for the case $N = \{p_0\}$, where $p_0 \in M$ is an arbitrary but fixed point. Since the structure of $\boldsymbol{\Gamma}_0(T_M)$ is different from that of $\boldsymbol{\Gamma}_N(T_M)$ for $\dim N \geqq 1$, we will investigate $\boldsymbol{\Gamma}_0(T_M)$ and $\boldsymbol{\Gamma}_N(T_M)$ separately, that is, in § 2 we will study maximal ideals of $\boldsymbol{\Gamma}_0(T_M)$ and in § 3 that of $\boldsymbol{\Gamma}_N(T_M)$.

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§ 1. Preliminaries.

Let $\boldsymbol{R}^n \times \boldsymbol{R}^\ell$ be the euclidean space with coordinates $\{x^1, \cdots, x^n, y^1, \cdots, y^\ell\}$. Let $\mathscr{F} = C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^\ell)$ be the set of all C^∞ functions on $\boldsymbol{R}^n \times \boldsymbol{R}^\ell$. Let $\mathscr{G} = C^\infty(\boldsymbol{R}^n \times 0) = C^\infty(\boldsymbol{R}^n)$ be the set of all C^∞ functions on $\boldsymbol{R}^n$. $\mathscr{G}$ is naturally identified with the subset of $C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^\ell)$ by the projection $\boldsymbol{R}^n \times \boldsymbol{R}^\ell \to \boldsymbol{R}^n$. Let $\mathscr{I}$ be the ideal of $\mathscr{F}$ of functions vanishing on $\boldsymbol{R}^n \times 0$, i.e.

$$\mathscr{I} = \{f \in \mathscr{F} \mid f|_{\boldsymbol{R}^n \times 0} = 0\} .$$

Clearly $x^i \in \mathscr{G}$ $(i = 1, \cdots, n)$ and $y^\alpha \in \mathscr{I}$ $(\alpha = 1, \cdots, \ell)$.

LEMMA 1.1. *For any $f \in \mathscr{F}$ there exist $g_0 \in \mathscr{G}$ and $f_\alpha \in \mathscr{F}$ $(1 \leqq \alpha \leqq \ell)$ such that $f = g_0 + y^1 f_1 + \cdots + y^\ell f_\ell$.*

Proof. Easy computation. (see, for example, [1])

COROLLARY 1.2. *If $f \in \mathscr{I}$, then $g_0 = 0$.*

Let M be a C^∞ manifold of dimension m, and N be a closed submanifold of dimension n such that $n \geqq 0$. We set $\ell = m - n$.

LEMMA 1.3. *The subset $\Gamma_N(T_M)$ of $\Gamma(T_M)$ is a Lie subalgebra of $\Gamma(T_M)$.*

Proof. Let $(U; x^1, \cdots, x^n, y^1, \cdots, y^\ell)$ be a coordinate system at $p \in N$ such that $U \cap N = \{y^1 = \cdots = y^\ell = 0\}$. Let $X = \xi^i(\partial/\partial x^i) + \xi^\alpha(\partial/\partial y^\alpha)$ and $Y = \eta^i(\partial/\partial x^i) + \eta^\alpha(\partial/\partial y^\alpha)$ be in $\Gamma_N(T_M)$. Then by Corollary 1.2 ξ^α and η^α are written as

$$\xi^\alpha = y^1\xi_1^\alpha + \cdots + y^\ell\xi_\ell^\alpha \quad \text{and} \quad \eta^\alpha = y^1\eta_1^\alpha + \cdots + y^\ell\eta_\ell^\alpha \qquad (\alpha = 1, \cdots, \ell)\,,$$

where $\xi_s^\alpha, \eta_s^\alpha \in C^\infty(M)$ $(s = 1, \cdots, \ell)$. We have then

$$[X, Y] = \left(\xi^i \frac{\partial \eta^j}{\partial x^i} - \eta^i \frac{\partial \xi^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} \qquad \text{on } U \cap N\,.$$

Hence $[X, Y] \in \Gamma_N(T_M)$.

LEMMA 1.4. *For each $X \in \Gamma_N(T_M)$ $r_*(X)$ denotes the restriction of X to N. Then r_* is a Lie algebra homomorphism of $\Gamma_N(T_M)$ onto $\Gamma(T_N)$, that is, $r_*[X, Y] = [r_*X, r_*Y]$.*

Proof. Easy computation.

LEMMA 1.5. *Let $X \in \Gamma_N(T_M)$ such that $X(p) \neq 0$ at $p \in M$. Then there is a local coordinate system $(U; x^1, \cdots, x^n, y^1, \cdots, y^\ell)$ such that $X = \partial/\partial x^1$ on U and if $p \in N$ and $\dim N \geqq 1$ then $U \cap N = \{y^1 = \cdots = y^\ell = 0\}$.*

Proof. Easy computation.

§ 2. Characterization of maximal ideals of $\Gamma_0(T_M)$.

We denote by $\Gamma(T_M)$ the Lie algebra of all C^∞ vector fields on M with compact support, and $\Gamma_0(T_M) = \{X \in \Gamma(T_M) \mid X(p_0) = 0\}$ is a Lie subalgebra of $\Gamma(T_M)$, where $p_0 \in M$ is an arbitrary but fixed point. We set

$$\boldsymbol{\Gamma}_0^k(T_M) = \{X \in \boldsymbol{\Gamma}_0(T_M) \,|\, \mathfrak{j}^r(X)(p_0) = 0 \quad \text{for all } r \leqq k\}\,,$$

where $\mathfrak{j}^r(X)(p_0)$ is the r-jet of X at p_0.

LEMMA 2.1. *If $X \in \boldsymbol{\Gamma}_0(T_M)$ does not vanish at $p \in M$ ($p \neq p_0$), then for any $Z \in \boldsymbol{\Gamma}_0(T_M)$ there are a neighborhood U of p in M and a vector field $Y \in \boldsymbol{\Gamma}_0(T_M)$ such that $[X, Y] = Z$ on U.*

Proof. By Lemma 1.5 there exists a local coordinate system $(V; x^1, \cdots, x^m)$ at p such that $X = \partial/\partial x^1$ on V. For any $Z = \zeta^i(\partial/\partial x^i) \in \boldsymbol{\Gamma}_0(T_M)$, i.e. $Z \in \boldsymbol{\Gamma}_0(T_M)$ such that $Z|_V = \zeta^i(\partial/\partial x^i)$, we consider the differential equations

$$\frac{\partial \eta^i}{\partial x^1} = \zeta^i \qquad (i = 1, \cdots, m)\,.$$

These equations have solutions on some neighborhood $U \subset V$ of p. Set $Y = \eta^i(\partial/\partial x^i)$, then Y is a C^∞ vector field on U and satisfies the equation $[X, Y] = Z$ on U. Here we may assume that U is relatively compact in V and dose not contain p_0. Then an appropriate extension of Y is contained in $\boldsymbol{\Gamma}_0(T_M)$.

LEMMA 2.2. *Let $\mathfrak{gl}(m)$ be the Lie algebra of all $m \times m$ real matrices. Then we have the following results.*

(i) *$s\ell(m) = \{A \in \mathfrak{gl}(m) \,|\, \text{trace}\, A = 0\}$ is an ideal of $\mathfrak{gl}(m)$.*

(ii) *The center of $\mathfrak{gl}(m)$ is $\mathfrak{z} = \{\lambda I \,|\, I$ is the unit matrix and λ is a real number.$\}$, and $\mathfrak{z}$ is an ideal of $\mathfrak{gl}(m)$.*

(iii) *If $m \geqq 2$, then $\mathfrak{gl}(m) = \mathfrak{z} \oplus s\ell(m)$ (direct sum),* i.e. $\mathfrak{z} \cap s\ell(m) = 0$. *If $m = 1$, then $\mathfrak{gl}(m) = \mathfrak{z}$.*

(iv) *If $m \geqq 2$, then $s\ell(m)$ is a simple Lie algebra, that is, $s\ell(m)$ does not admit any non-trivial ideals.*

(v) *$\mathfrak{z}$ and $s\ell(m)$ are maximal ideals of $\mathfrak{gl}(m)$.*

Proof. These results are well known, and we omit the proofs. (see, for example, [3])

LEMMA 2.3. *For each point $p \in M$ such that $p \neq p_0$ we denote by $\mathscr{I}_p$ the subset $\{X \in \boldsymbol{\Gamma}_0(T_M) \,|\, X(p) = 0$ and $\mathfrak{j}^r(X)(p) = 0$ for all $r \geqq 1\}$ of $\boldsymbol{\Gamma}_0(T_M)$. Then for each $p \in M$, $\mathscr{I}_p$ is an ideal of $\boldsymbol{\Gamma}_0(T_M)$.*

Proof. The proof is direct computation.

LEMMA 2.4. *Let $p \in M$ be a given point such that $p \neq p_0$. If $\mathscr{I}$ is a proper ideal of $\boldsymbol{\Gamma}_0(T_M)$,* i.e. *$\mathscr{I} \subsetneqq \boldsymbol{\Gamma}_0(T_M)$, such that $X(p) = 0$ for all $X \in \mathscr{I}$. Then $\mathscr{I} \subset \mathscr{I}_p$.*

Proof. Since $p \neq p_0$, there is a local coordinate system $(U; x^1, \cdots, x^m)$ at p such that $\bar{U} \not\ni p_0$. Hence appropriate extensions of $\partial/\partial x^j$ $(j = 1, \cdots, m)$ are contained in $\boldsymbol{\Gamma}_0(T_M)$. We also denote the extended vector fields by the same letters. For any $X = \xi^i(\partial/\partial x^i) \in \mathscr{I}$ we have $[\partial/\partial x^j, X] = \partial\xi^i/\partial x^j \cdot \partial/\partial x^i$ for all $j = 1, \cdots, m$. Since $\mathscr{I}$ is an ideal, $[\partial/\partial x^j, X] \in \mathscr{I}$. From the assumption for $\mathscr{I}$, $(\partial\xi^i/\partial x^j)(p) = 0$ for all $i, j = 1, \cdots, m$. By induction on r, we have $\mathfrak{j}^r(X)(p) = 0$ for all $r \geqq 1$. Therefore $\mathscr{I} \subset \mathscr{I}_p$.

LEMMA 2.5. *Let A be an arbitrary Lie algebra. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of A such that $\mathfrak{a} \supset \mathfrak{b}$. Then $(A/\mathfrak{b})/(\mathfrak{a}/\mathfrak{b}) \cong A/\mathfrak{a}$.*

Proof. The result is well known, and we omit the proof.

LEMMA 2.6. *The subset $\boldsymbol{\Gamma}_0^1(T_M) = \{X \in \boldsymbol{\Gamma}_0(T_M) \mid \mathfrak{j}^1(X)(p_0) = 0\}$ is a proper ideal of $\boldsymbol{\Gamma}_0(T_M)$.*

Proof. Easy computation.

LEMMA 2.7. *Let $\pi: \boldsymbol{\Gamma}_0(T_M) \to \boldsymbol{\Gamma}_0(T_M)/\boldsymbol{\Gamma}_0^1(T_M) \cong \mathfrak{gl}(m)$ be the natural projection. We define $\mathscr{I}_{\mathfrak{z}}$ and $\mathscr{I}_{s\ell}$ by $\mathscr{I}_{\mathfrak{z}} = \pi^{-1}(\mathfrak{z})$ and $\mathscr{I}_{s\ell} = \pi^{-1}(s\ell(m))$. Then both $\mathscr{I}_{\mathfrak{z}}$ and $\mathscr{I}_{s\ell}$ are proper ideals of $\boldsymbol{\Gamma}_0(T_M)$.*

Proof. Since $\pi: \boldsymbol{\Gamma}_0(T_M) \to \mathfrak{gl}(m)$ is an onto Lie algebra homomorphism, we have the desired result.

PROPOSITION 2.8. *If $\mathfrak{m}$ is a maximal of ideal $\boldsymbol{\Gamma}_0(T_M)$ such that $\mathfrak{m} \supset \boldsymbol{\Gamma}_0^1(T_M)$, then $\mathfrak{m} = \mathscr{I}_{\mathfrak{z}}$ or $\mathscr{I}_{s\ell}$.*

Proof. Let $\mathfrak{m} \subsetneqq \boldsymbol{\Gamma}_0(T_M)$ be a maximal ideal such that $\mathfrak{m} \supset \boldsymbol{\Gamma}_0^1(T_M)$. Then by Lemma 2.5 $\mathfrak{m}/\boldsymbol{\Gamma}_0^1(T_M)$ is a proper ideal of $\boldsymbol{\Gamma}_0(T_M)/\boldsymbol{\Gamma}_0^1(T_M)$. By Lemma 2.2, $\boldsymbol{\Gamma}_0(T_M)/\boldsymbol{\Gamma}_0^1(T_M) \cong \mathfrak{gl}(m) = \mathfrak{z} \oplus s\ell(m)$ and both $\mathfrak{z}$ and $s\ell(m)$ are simple Lie algebras. Hence $\mathfrak{m}/\boldsymbol{\Gamma}_0^1(T_M)$ should be equal to either $\mathfrak{z}$ or $s\ell(m)$. Therefore we have $\mathfrak{m} = \pi^{-1}(\mathfrak{z}) = \mathscr{I}_{\mathfrak{z}}$ or $\mathfrak{m} = \pi^{-1}(s\ell(m)) = \mathscr{I}_{s\ell}$.

LEMMA 2.9. *If $\mathfrak{m}$ is a maximal ideal of $\boldsymbol{\Gamma}_0(T_M)$ such that $\mathfrak{m} \supset \boldsymbol{\Gamma}_0^\infty(T_M)$, then for any point $p \neq p_0$, there exists an element $X \in \mathfrak{M}$ such that $X(p) \neq 0$, where*

$$\boldsymbol{\Gamma}_0^\infty(T_M) = \{X \in \boldsymbol{\Gamma}_0(T_M) \mid \mathfrak{j}^r(X)(p_0) = 0 \ \textit{for all} \ r \geqq 1\} .$$

Proof. Assume that there exists a point $p \in M$ $(p \neq p_0)$ such that $X(p) = 0$ for all $X \in \mathfrak{m}$. By Lemma 2.4 $\mathfrak{m} \subset \mathscr{I}_p$. Since $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m} = \mathscr{I}_p$. On the other hand, since $p \neq p_0$, there exists $Y \in \boldsymbol{\Gamma}_0^\infty(T_M)$ such that $Y(p) \neq 0$. Hence $\mathfrak{m} = \mathscr{I}_p \not\ni Y$, contradicting the condition $\mathfrak{m} \supset \boldsymbol{\Gamma}_0^\infty(T_M)$.

LEMMA 2.10. *If* $\mathfrak{m}$ *is a maximal ideal of* $\boldsymbol{\Gamma}_0(T_M)$ *such that* $\mathfrak{m} \supset \boldsymbol{\Gamma}_0^\infty(T_M)$, *then* $\mathfrak{j}^1(\mathfrak{m})(p_0)$ *is a proper ideal of* $\mathfrak{gl}(m)$, *where* $\mathfrak{j}^1(\mathfrak{m})(p_0)$ *is the image of* $\mathfrak{m}$ *under the natural projection*

$$\pi : \boldsymbol{\Gamma}_0(T_M) \to \boldsymbol{\Gamma}_0(T_M)/\boldsymbol{\Gamma}_0^1(T_M) \cong \mathfrak{gl}(m) .$$

Proof. Assume $\mathfrak{j}^1(\mathfrak{m})(p_0) = \mathfrak{gl}(m)$. Then by Sternberg's linearization theorem [4], there exist a vector field $X \in \mathfrak{m}$ and a smooth local coordinate system $(U; x^1, \cdots, x^m)$ at p_0 such that $X|_U = \sum_i x^i(\partial/\partial x^i)$. On the other hand, for any $Z \in \boldsymbol{\Gamma}_0^1(T_M)$, there exists a sequence of neighborhoods $V \supset V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ of p_0 such that $V \subset U$ and

$$Z = \sum_i \Big(\sum_{|\alpha| \geqq 2} \phi_\alpha^i(x) \cdot a_\alpha^i x^\alpha\Big) \frac{\partial}{\partial x^i} + \tilde{Z} \qquad \text{on } V ,$$

where $\phi_\alpha^i(x)$ is a C^∞ function on U such that

$$\phi_\alpha^i(x) = \begin{cases} 1 & \text{on } V_i \subset V \\ 0 & \text{outside some neighborhood of } V , \end{cases}$$

$\sum_{|\alpha| \geqq 2} \phi_\alpha^i(x) \cdot a_\alpha^i \cdot x^\alpha$ is a power series which converges on V and $\tilde{Z}$ is a C^∞ vector field on M such that $\tilde{Z}(p_0) = 0$ and $\mathfrak{j}^r(\tilde{Z})(p_0) = 0$ for all $r \geqq 1$ (see [4] p. 35). Now we consider the following power series

$$\sum_j \Big(\sum_{|\alpha| \geqq 2} \phi_\alpha^j(x) \cdot \frac{a_\alpha^j}{|\alpha| - 1} \cdot x^\alpha\Big) \cdot \frac{\partial}{\partial x^j} .$$

This series converges on V and becomes a C^∞ vector field on V. Hence a suitable extension Y of this vector field, i.e.

$$Y|_V = \sum_j \Big(\sum_{|\alpha| \geqq 2} \phi_\alpha^j(x) \cdot \frac{a_\alpha^j}{|\alpha| - 1} \cdot x^\alpha\Big) \cdot \frac{\partial}{\partial x^j} ,$$

is contained in $\boldsymbol{\Gamma}_0^1(T_M)$. Since $X \in \mathfrak{m}$ and $\mathfrak{m}$ is an ideal of $\boldsymbol{\Gamma}_0(T_M)$, we obtain $[X, Y] \in \mathfrak{m}$. Furthermore we have $[X, Y] = A^j \cdot \partial/\partial x^j$, where

$$A^j = \sum_i x^i\Big(\sum_{|\alpha| \geqq 2} \frac{\partial \phi_\alpha^j}{\partial x^i} \cdot \frac{a_\alpha^j}{|\alpha| - 1} \cdot x^\alpha\Big) + \sum_{|\alpha| \geqq 2} \phi_\alpha^j \cdot a_\alpha^j \cdot x^\alpha .$$

By the definition of ϕ^j_α, we have

$$\frac{\partial^{|\beta|}\phi^j_\alpha}{\partial x_\beta} = 0 \qquad \text{on } V_j, \text{ for all multiple indices } \beta \text{ with } |\beta| \geqq 1\,.$$

Therefore the Taylor expansion of $[X, Y]$ at $p_0 =$ the Taylor expansion of $(Z - \tilde{Z})$ at p_0. Hence $Z - \tilde{Z} - [X, Y] \in \boldsymbol{\Gamma}^\infty_0(T_M) \subset \mathfrak{m}$. Then $Z \in \mathfrak{m}$, hence $\boldsymbol{\Gamma}^1_0(T_M) \subset \mathfrak{m}$. Therefore, by Proposition 2.8, $\mathfrak{m} = \mathscr{I}_{\mathfrak{z}}$ or $\mathscr{I}_{s\ell}$. We have then $\mathfrak{j}^1(\mathfrak{m})(p_0) \subsetneqq \mathfrak{gl}(m)$, contradicting the assumption.

PROPOSITION 2.11. *If* $\mathfrak{m}$ *is a maximal ideal of* $\Gamma_0(T_M)$ *such that* $\mathfrak{m} \supset \boldsymbol{\Gamma}^\infty_0(T_M)$, *then* $\mathfrak{m} = \mathscr{I}_{\mathfrak{z}}$ *or* $\mathscr{I}_{s\ell}$.

Proof. By Lemma 2.10, $\mathfrak{j}^1(\mathfrak{m})(p_0)$ is a proper ideal of $\mathfrak{gl}(m)$. By Lemma 2.2, $\mathfrak{j}^1(\mathfrak{m})(p_0)$ should be equal to either $\mathfrak{z}$ or $s\ell(m)$. If $\mathfrak{j}^1(\mathfrak{m})(p_0) = \mathfrak{z}$ (resp. $s\ell(m)$), then $\mathfrak{m} \subset \mathscr{I}_{\mathfrak{z}}$ (resp. $\mathfrak{m} \subset \mathscr{I}_{s\ell}$). By the maximality of $\mathfrak{m}$, $\mathfrak{m} = \mathscr{I}_{\mathfrak{z}}$ (resp. $\mathfrak{m} = \mathscr{I}_{s\ell}$).

LEMMA 2.12. *If* $\mathfrak{m}$ *is a maximal ideal of* $\boldsymbol{\Gamma}_0(T_M)$ *such that* $\mathfrak{m} \not\supset \boldsymbol{\Gamma}^\infty_0(T_M)$, *then* $\mathfrak{j}^1(\mathfrak{m})(p_0) = \mathfrak{gl}(m)$.

Proof. Assume $\mathfrak{j}^1(\mathfrak{m})(p_0)$ be a proper ideal of $\mathfrak{gl}(m)$. Then there occur three cases. If $\mathfrak{j}^1(\mathfrak{m})(p_0) = \{0\}$, then $\mathfrak{m} \subset \boldsymbol{\Gamma}^1_0(T_M)$, contradicting the assumption. If $\mathfrak{j}^1(\mathfrak{m})(p_0) = \mathfrak{z}$ (resp. $\mathfrak{j}^1(\mathfrak{m})(p_0) = s\ell(m)$), $\mathfrak{m} \supset \mathscr{I}_{\mathfrak{z}} \supset \boldsymbol{\Gamma}^\infty_0(T_M)$ (resp. $\mathfrak{m} \supset \mathscr{I}_{s\ell} \supset \boldsymbol{\Gamma}^\infty_0(T_M)$), contradicting the assumption. Hence $\mathfrak{j}^1(m)(p_0)$ should be equal to $\mathfrak{gl}(m)$.

LEMMA 2.13. *Let* $\mathfrak{m}$ *be a maximal ideal of* $\boldsymbol{\Gamma}_0(T_M)$ *such that* $\mathfrak{j}^1(\mathfrak{m})(p_0) = \mathfrak{gl}(m)$. *If for any* $p \in M$ *with* $p \neq p_0$ *there exists* $Y \in \mathfrak{m}$ *such that* $Y(p) \neq 0$, *then* $\mathfrak{m} \supset \boldsymbol{\Gamma}^\infty_0(T_M)$.

Proof. We set $\mathscr{I}^C_{p_0} = \{X \in \boldsymbol{\Gamma}^\infty_0(T_M) \mid \operatorname{supp} X \not\ni p_0\}$. First of all we prove that $\mathscr{I}^C_{p_0} \subset \mathfrak{m}$.

(Remark that the assumption $\mathscr{I}^C_{p_0} \subset \mathfrak{m}$ has identical meaning with that of Lemma 1 of Pursell and Shanks [9], but unfortunately their proof contains a mistake about the argument of supports of the vector fields denoted by N_i. A complete proof for Lemma 1 is given in [5]. We use here the method used in [5].)

Let X be an arbitrary element of $\mathscr{I}^C_{p_0}$. From the assumption of Lemma 2.13, for any $p \in \operatorname{supp} X$ there are a vector field $Y \in \mathfrak{m}$ and a local coordinate system $(V; x^1, \cdots, x^m)$ such that $Y|_V = \partial/\partial x^1$. Since

$\operatorname{supp} X$ is compact, there are $Y_i \in \mathfrak{m}$, $X_i \in \mathscr{I}^C_{p_0}$ and $(V_i; x_i^1, \cdots, x_i^m)$, $i = 1, \cdots, r$, such that $Y_i|_{V_i} = \partial/\partial x_i^1$, $X = X_1 + \cdots + X_r$, $\operatorname{supp} X_i \subset V_i$ and $X_i = \sum_k \xi^k(\partial/\partial x_i^k)$ on V_i.

Hence if we want to prove that $X \in \mathfrak{m}$, it suffices to prove that $X_i \in \mathfrak{m}$ for each i. Because the argument is local we may delete the indices, that is, we may assume that there is a local coordinate system $(V; x^1, \cdots, x^m)$ such that X is written as $X = \sum \xi^i(\partial/\partial x^i)$ on V with $\operatorname{supp} \xi^i \subset V$ for all $i = 1, \cdots, m$, and a suitable extension of $\partial/\partial x^1$ is contained in $\mathfrak{m}$. We use the same notation for the extended vector fields because all argument here is local. Since $\partial/\partial x^1 \in \mathfrak{m}$ and $\frac{1}{2}[\partial/\partial x^1, (x^1)^2(\partial/\partial x^1)] = x^1(\partial/\partial x^1), x^1(\partial/\partial x^1) \in \mathfrak{m}$. For $\xi^1(\partial/\partial x^1)$ we have the following formulae:

$$\left[\frac{\partial}{\partial x^1}, x^1\xi^1\frac{\partial}{\partial x^1}\right] = \left(\xi^1 + x^1\frac{\partial \xi^1}{\partial x^1}\right)\frac{\partial}{\partial x^1} \in \mathfrak{m}$$

and

$$\left[x^1\frac{\partial}{\partial x^1}, \xi^1\frac{\partial}{\partial x^1}\right] = \left(x^1\frac{\partial \xi^1}{\partial x^1} - \xi^1\right)\frac{\partial}{\partial x^1} \in \mathfrak{m} .$$

Hence we have $\frac{1}{2}([\partial/\partial x^1, x^1\xi^1(\partial/\partial x^1)] - [x^1(\partial/\partial x^1), \xi^1(\partial/\partial x^1)]) = \xi^1(\partial/\partial x^1) \in \mathfrak{m}$. On the other hand for $\xi^i(\partial/\partial x^i)$, $i \geqq 2$, we have the following formulae:

$$\left[\frac{\partial}{\partial x^1}, x^1\xi^i\frac{\partial}{\partial x^i}\right] = \left(\xi^i + x^1\frac{\partial \xi^i}{\partial x^1}\right)\frac{\partial}{\partial x^i} \in \mathfrak{m}$$

and

$$\left[x^1\frac{\partial}{\partial x^1}, \xi^i\frac{\partial}{\partial x^i}\right] = x^1\frac{\partial \xi^i}{\partial x^1}\frac{\partial}{\partial x^i} \in \mathfrak{m} .$$

Hence we have

$$\left[\frac{\partial}{\partial x^1}, x^1\xi^i\frac{\partial}{\partial x^i}\right] - \left[x^1\frac{\partial}{\partial x^1}, \xi^i\frac{\partial}{\partial x^i}\right] = \xi^i\frac{\partial}{\partial x^i} \in \mathfrak{m} .$$

Therefore we have $X = \sum \xi^i(\partial/\partial x^i) \in \mathfrak{m}$. Finally we obtain $\mathscr{I}^C_{p_0} \subset \mathfrak{m}$. Now we continue the proof of Lemma 2.13.

Since $\mathfrak{j}^1(\mathfrak{m})(p_0) = \mathfrak{gl}(m)$, by the Sternberg's linearization theorem there are a vector field $X \in \mathfrak{m}$ and a local coordidate system $(U; x^1, \cdots, x^m)$ at p_0 such that $X|_U = x^i(\partial/\partial x^i)$. For any $Z \in \Gamma_0^\infty(T_M)$ such that $Z|_U = \zeta^i(\partial/\partial x^i)$ we consider the following system of differential equations on a neighborhood of p_0:

$$x^i \frac{\partial \eta^j}{\partial x^i} - \eta^j = \zeta^j \qquad (j = 1, \cdots, m) .$$

By the polar coordinate system $x^i = r\phi_i(\theta^1, \cdots, \theta^{m-1})$ $(i = 1, \cdots, m)$, above equations are written as

$$r \frac{d\eta^j}{dr} - \eta^j = \zeta^j \qquad (j = 1, \cdots, m) ,$$

where $r^2 = \sum_i (x^i)^2$. By $r(d\eta^j/dr) - \eta^j = 0$ we have $\eta^j = C(r)\cdot r$, where $C(r)$ is a function of r. So we have $dC/dr = \zeta^j/r^2$. Since ζ^j is flat at $r = 0$,

$$C(r) = \int_0^r \frac{\zeta^j}{r^2} dr .$$

Hence we have

$$\eta^j = r \int_0^r \frac{\zeta^j}{r^2} dr$$

on some neighborhood $W \subset U$ of p_0. Clearly $\eta^j(0) = 0$ $(j = 1, \cdots, m)$. Therefore a suitable extension Y of $\eta^i(\partial/\partial x^i)$, i.e. $Y|_W = \eta^i(\partial/\partial x^i)$, is contained in $\boldsymbol{\Gamma}_0(T_M)$. Obviously $[X, Y]|_W = Z|_W$. On the other hand $[X, Y] \in \mathfrak{m}$. We set $A = Z - [X, Y]$. Then $A \in \boldsymbol{\Gamma}_0^\infty(T_M)$. Since $\operatorname{supp} A \not\ni p_0$, $A \in \mathscr{I}^C_{p_0} \subset \mathfrak{m}$. Then $Z = A + [X, Y]$, hence $Z \in \mathfrak{m}$. Therefore $\boldsymbol{\Gamma}_0^\infty(T_M) \subset \mathfrak{m}$.

PROPOSITION 2.14. *If* $\mathfrak{m}$ *is a maximal ideal of* $\boldsymbol{\Gamma}_0(T_M)$ *such that* $\mathfrak{m} \not\supset \boldsymbol{\Gamma}_0^\infty(T_M)$, *then there exists a unique point* $p \in M$ *such that* $p \neq p_0$ *and* $\mathfrak{m} = \mathscr{I}_p$.

Proof. By Lemma 2.12, $\mathfrak{j}^1(\mathfrak{m})(p_0) = \mathfrak{gl}(m)$. By Lemma 2.13, there exists a point $p \in M$ such that $p \neq p_0$ and $X(p) = 0$ for all $X \in \mathfrak{m}$. By Lemma 2.4, $\mathfrak{m} \subset \mathscr{I}_p$. Since $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m} = \mathscr{I}_p$. Furthermore the maximality of $\mathfrak{m}$ implies the uniqueness of the point p.

THEOREM 2.15. *Any maximal ideal of* $\boldsymbol{\Gamma}_0(T_M)$ *should be equal to one of the following ideals;*

(i) $\mathscr{I}_{\mathfrak{s}}$, (ii) $\mathscr{I}_{\mathfrak{sl}}$: *ideal with finite codimension and corresponding to* p_0,

(iii) $\mathscr{I}_p$: *ideal with infinite codimension and corresponding to* p $(p \neq p_0)$.

Proof. The result is an immediate consequence of Propositions 2.11 and 2.14.

§ 3. Characterization of maximal ideals of $\boldsymbol{\Gamma}_N(T_M)$ ($\dim N \geqq 1$).

LEMMA 3.1. *Let $X \in \boldsymbol{\Gamma}_N(T_M)$ such that $X(p) \neq 0$ at $p \in M$. Then for any $Z \in \boldsymbol{\Gamma}_N(T_M)$ there exist an element $Y \in \boldsymbol{\Gamma}_N(T_M)$ and a neighborhood U of p in M such that $[X, Y] = Z$ on U.*

Proof. The case $p \notin N$ was already proved in Lemma 2.1. Let p be a point in N. By Lemma 1.5 we can take a local coordinate system $(U; x^1, \cdots, x^n, y^1, \cdots, y^\ell)$ at p such that $U \cap N = \{y^1 = \cdots = y^\ell = 0\}$ and $X = \partial/\partial x^1$ on U. For any $Z = \zeta^i(\partial/\partial x^i) + \zeta^\alpha(\partial/\partial y^\alpha) \in \boldsymbol{\Gamma}_N(T_M)$ we consider the following differetial equations.

$$\begin{cases} \dfrac{\partial \eta^i}{\partial x^1} = \zeta^i & (i = 1, \cdots, n) \\[2ex] \dfrac{\partial \eta^\alpha}{\partial x^1} = \zeta^\alpha & (\alpha = 1, \cdots, \ell), \text{ where } \zeta^\alpha(x^1, \cdots, x^n, 0, \cdots, 0) = 0 \,. \end{cases}$$

These equations have solutions on U:

$$\begin{cases} \eta^i = \displaystyle\int \zeta^i dx^1 + C^i(x^2, \cdots, x^n, y^1, \cdots, y^\ell) \\[2ex] \eta^\alpha = \displaystyle\int \zeta^\alpha dx^1 + C^\alpha(x^2, \cdots, x^n, y^1, \cdots, y^\ell) \,. \end{cases}$$

Set $C^\alpha(x^2, \cdots, x^n, 0, \cdots, 0) = 0$ for $\alpha = 1, \cdots, \ell$.

Then $\eta^\alpha(x^1, \cdots, x^n, 0, \cdots, 0) = 0$. Let Y be an appropriate extension of $\eta^i(\partial/\partial x^i) + \eta^\alpha(\partial/\partial y^\alpha)$. Then $Y \in \boldsymbol{\Gamma}_N(T_M)$ and $[X, Y] = Z$ on U.

LEMMA 3.2. *For any proper ideal $\mathscr{I} \subset \boldsymbol{\Gamma}_N(T_M)$ there exists a point $p \in M$ such that $X(p) = 0$ for all $X \in \mathscr{I}$.*

Proof. The proof is done by the method which was used to prove $\mathscr{I}^C_{p_0} \subset \mathfrak{m}$ in Lemma 2.13, and omitted.

LEMMA 3.3. *Let $\mathscr{I} \subsetneqq \boldsymbol{\Gamma}_N(T_M)$ be an ideal, and $p \in M$ be a point such that $X(p) = 0$ for all $X \in \mathscr{I}$.*

(Case $p \notin N$) Let $(U; x^1, \cdots, x^m)$ be a local coordinate system at p. Then for any $X = \xi^i(\partial/\partial x^i) \in \mathscr{I}$ we have

$$\frac{\partial^r \xi^i}{\partial x^{i_1} \cdots \partial x^{i_r}}(p) = 0 \qquad (1 \leqq i \leqq m\,; 1 \leqq r) \,.$$

(Case $p \in N$) Let $(U; x^1, \cdots, x^n, y^\ell \cdots, y^l)$ be a local coordinate system at p in M such that $U \cap N = \{y^1 = \cdots = y^\ell = 0\}$.
Then for any $X = \xi^i(\partial/\partial x^i) + \xi^\alpha(\partial/\partial y^\alpha) \in \mathscr{I}$ we have

$$\frac{\partial^r \xi^i}{\partial x^{i_1} \cdots \partial x^{i_r}}(p) = 0 \quad \text{and} \quad \frac{\partial^r \xi^\alpha}{\partial x^{i_1} \cdots \partial x^{i_r}}(p) = 0$$
$$(1 \leqq i \leqq n; 1 \leqq \alpha \leqq \ell; 1 \leqq r) .$$

Proof. The proof is all the same as that of Lemma 2.4.

LEMMA 3.4. *Let $\mathscr{I}$ be a proper ideal of $\Gamma_N(T_M)$ such that $X(p) = 0$ for all $X \in \mathscr{I}$ at a point $p \in M$. Then $p \notin N$, if and only if $\mathscr{I}$ does not contain* $\operatorname{Ker} r_*$, *where $r_*: \Gamma_N(T_M) \to \Gamma(T_N)$ is the Lie algebra homomorphism.*

Proof. Easy computation.

Let p be a point of M. We denote by $\mathscr{I}_p$ the ideal of $\Gamma_N(T_M)$ consisting of all element X such that X and its all derivatives vanish at the point p. Clearly if $p \notin N$, then $\mathscr{I}_p$ is a maximal ideal of $\Gamma_N(T_M)$.

For a given point $p \in N$ we denote by $\bar{\mathscr{I}}_p$ the ideal of $\Gamma(T_N)$ consisting of all element Y such that Y and its all derivatives vanish at the point p. $\bar{\mathscr{I}}_p$ is a maximal ideal of $\Gamma(T_N)$.

PROPOSITION 3.5. *For any maximal ideal $\mathscr{I}$ of $\Gamma_N(T_M)$, there exists a unique point $p \in M$ such that*

$$\mathscr{I} = \begin{cases} \mathscr{I}_p & (\text{if } \mathscr{I} \text{ does not contain } \operatorname{Ker} r_*) \\ r_*^{-1}\bar{\mathscr{I}}_p & (\text{if } \mathscr{I} \text{ contains } \operatorname{Ker} r_*) . \end{cases}$$

Proof. By Lemma 3.2 there is a point $p \in M$ such that $X(p) = 0$ for all $X \in \mathscr{I}$. If $\mathscr{I}$ does not contain $\operatorname{Ker} r_*$, then p is never contained in N. Hence by Lemma 3.3 $\mathscr{I}$ is contained in the proper ideal $\mathscr{I}_p$. Since $\mathscr{I}$ is maximal, $\mathscr{I} = \mathscr{I}_p$. If $\mathscr{I}$ contains $\operatorname{Ker} r_*$, by Lemma 3.3 $r_*(\mathscr{I}) \subset \bar{\mathscr{I}}_p$. By the maximality of $\mathscr{I}, r_*(\mathscr{I})$ is also maximal in $\Gamma(T_N)$. Hence $r_*(\mathscr{I}) = \bar{\mathscr{I}}_p$. Therefore $\mathscr{I} = r_*^{-1}\bar{\mathscr{I}}_p$. Furthermore the maximality of $\mathscr{I}$ implies the uniqueness of the point p.

LEMMA 3.6. *$\Gamma_N(T_M)/\mathscr{I}_p \cong \overbrace{R[[x^1, \cdots, x^m]] \times \cdots \times R[[x^1, \cdots, x^m]]}^{m}$ and $\Gamma_N(T_M)/r_*^{-1}\bar{\mathscr{I}}_p \cong \overbrace{R[[x^1, \cdots, x^n]] \times \cdots \times R[[x^1, \cdots, x^n]]}^{n}$ as Lie algebras, where $m = n + \ell$ and $R[[\cdots]]$ is the ring of formal power series.*

Proof. Let $(U; x^1, \cdots, x^m)$ be a local coordinate system at $p \in M$. Then the formal Taylor expansion of $X \in \Gamma_N(T_M)$ at p with respect to this coordinate is a homomorphism of $\Gamma_N(T_M)$ onto the product of the rings of formal power series, and its kernel is exactly $\mathscr{I}_p$.

For the case $p \in N$ we consider the following commutative diagram:

$$\begin{array}{ccc} \Gamma_N(T_M) & \xrightarrow{r_*} & \Gamma(T_N) \\ \downarrow \pi & & \downarrow \pi \\ \Gamma_N(T_M)/r_*^{-1}\bar{\mathscr{I}}_p & \xrightarrow[\bar{r}_*]{} & \Gamma(T_N)/\bar{\mathscr{I}}_p \cong R[[x^1, \cdots, x^n]] \times \cdots \times R[[x^1, \cdots, x^n]] \,. \end{array}$$

Since $\bar{r}_*$ is an isomorphism, we have the desired result.

§ 4. Stone topology of maximal ideal sets.

(Case $\Gamma_0(T_M)$) Let M and M' be C^∞ manifolds and p_0 (resp. p_0') be an arbitrary but fixed point of M (resp. M'). We define $\Gamma_0(T_M)$, $\Gamma_0(T_{M'})$, $\Gamma_0^1(T_M)$ and $\Gamma_0^1(T_{M'})$ as in § 2.

LEMMA 4.1. *If $\Phi: \Gamma_0(T_M) \to \Gamma_0(T_{M'})$ is a Lie algebra isomorphism, then $\Phi(\mathscr{I}_{\mathfrak{z}}) = \mathscr{I}_{\mathfrak{z}'}$, $\Phi(\mathscr{I}_{s\ell}) = \mathscr{I}_{s\ell'}$ and $\Phi(\mathscr{I}_p) = \mathscr{I}_{p'}$ (if $p \neq p_0$). Especially $\Phi(\Gamma_0^1(T_M)) = \Gamma_0^1(T_{M'})$.*

Proof. If $\mathfrak{m}' = \Phi(\mathfrak{m})$ is a maximal ideal, then $\Gamma_0(T_M)/\mathfrak{m}$ is isomorphic to $\Gamma_0(T_{M'})/\mathfrak{m}'$. Hence codim $\mathfrak{m}$ in $\Gamma_0(T_M) =$ codim $\mathfrak{m}'$ in $\Gamma_0(T_{M'})$. Since codim $\mathscr{I}_{\mathfrak{z}} = n^2 - 1$ and codim $\mathscr{I}_{s\ell} = 1$, we have $\Phi(\mathscr{I}_{\mathfrak{z}}) = \mathscr{I}_{\mathfrak{z}'}$ and $\Phi(\mathscr{I}_{s\ell}) = \mathscr{I}_{s\ell'}$. On the other hand, since each ideal $\mathscr{I}_p$ which has infinite codimension corresponds to a point p $(p \neq p_0)$ uniquely, $\Phi(\mathscr{I}_p) = \mathscr{I}_{p'}$ for some unique point p' $(p' \neq p_0')$. Moreover, since $\Gamma_0^1(T_M) = \mathscr{I}_{\mathfrak{z}} \cap \mathscr{I}_{s\ell}$, $\Phi(\Gamma_0^1(T_M)) = \Gamma_0^1(T_{M'})$.

We denote by M^* the set of all maximal ideals of $\Gamma_0(T_M)$, that is,

$$M^* = \{\mathscr{I} \mid \mathscr{I} \subset \Gamma_0(T_M): \text{maximal ideal}\} \,.$$

From now on, we denote both $\mathscr{I}_{\mathfrak{z}}$ and $\mathscr{I}_{s\ell}$ simply $\mathscr{I}_{p_0}$. Let $\sigma: M^* \to M$ be the natural correspondence defined by $\sigma(\mathscr{I}_p) = p$. (Note. $\sigma(\mathscr{I}_{p_0}) = \sigma(\mathscr{I}_{\mathfrak{z}}) = \sigma(\mathscr{I}_{s\ell}) = p_0$)

For any subset $A \subset M$ we set $A^* = \sigma^{-1}(A) = \{\mathscr{I}_p \in M^* \mid p \in A\}$. .

DEFINITION 4.2. (Stone topology of M^*) For any subset of M^* we define *a closure operator "$\mathscr{C}\ell$"* by the formulas:

(i) $\mathscr{C}\ell\phi = \phi$

(ii) If $B \neq \phi$ then $\mathscr{C}\ell B = \{\mathfrak{m} \mid \mathfrak{m}$ is a maximal ideal such that $\mathfrak{m} \supset \bigcap_{\mathscr{I} \in B} \mathscr{I}\}$.

DEFINITION 4.3. We call a subset $B \subset M^*$ is closed, if and only if $\mathscr{C}\ell B = B$.

LEMMA 4.4. *For each* $A^* = \sigma^{-1}(A)$, $\mathscr{C}\ell(A^*) = (\bar{A})^*$, *where* $\bar{A}$ *is the closure of* A *in* M.

Proof. First, we prove "$\subset$". For any $\mathfrak{m} \in \mathscr{C}\ell(A^*)$, since $\mathfrak{m}$ is a maximal ideal, there exists a unique point $p \in M$ such that $\mathfrak{m} = \mathscr{I}_p$ (may be $p = p_0$). Assume $p \notin \bar{A}$.

(Case $p \neq p_0$) $\mathfrak{m} = \mathscr{I}_p \supset \bigcap_{\mathscr{I} \in A^*} \mathscr{I}$. On the other hand, since $p \notin \bar{A}$, there is $X \in \bigcup_{\mathscr{I} \in A^*} \mathscr{I}$ such that $X(p) \neq 0$. Hence $X \notin \mathscr{I}_p \cdots$ contradiction.

(Case $p = p_0$) There are two cases, one is $\mathfrak{m} = \mathscr{I}_{\mathfrak{z}} \supset \bigcap_{\mathscr{I} \in A^*} \mathscr{I}$ and other is $\mathfrak{m} = \mathscr{I}_{s\ell} \supset \bigcap_{\mathscr{I} \in A^*} \mathscr{I}$. On the other hand there is $Y \in \boldsymbol{\Gamma}_0(T_M)$ such that $\mathfrak{j}^1(Y)(p_0) \notin \mathfrak{z} \cup s\ell(m)$ (set union). Let $\psi : M \to \boldsymbol{R}$ be a C^∞ function such that

$$\psi = \begin{cases} 1 & \text{in some neighborhood } U \text{ of } p_0 \text{ with } U \cap \bar{A} = \phi \\ 0 & \text{outside some neighborhood of } U. \end{cases}$$

Then $X = \psi Y \in \bigcap_{\mathscr{I} \in A^*} \mathscr{I}$ and $\mathfrak{j}^1(X)(p_0) \notin \mathfrak{z} \cup s\ell(m)$, that is, $X \notin \mathscr{I}_{\mathfrak{z}} \cup \mathscr{I}_{s\ell} \cdots$ contradiction. Therefore p should be contained in $\bar{A}$. So $\mathfrak{m} \in (\bar{A})^*$.

Next we prove "$\supset$". For any $\mathscr{I}_p \in (\bar{A})^*$ (may be $p = p_0$), $p \in \bar{A}$. If $p \in A$, then clearly $\mathscr{I}_p \in \mathscr{C}\ell(A^*)$. So we may assume $p \in \bar{A} - A$. For any $Y \in \bigcap_{\mathscr{I} \in A^*} \mathscr{I}$, $Y = 0$ on A. Since Y is a C^∞ vector field, $Y(p) = 0$ and $\mathfrak{j}^r(Y)(p) = 0$ for all $r \geqq 1$. Hence $Y \in \mathscr{I}_p$ (may be $p = p_0$). Therefore $\mathscr{I}_p \supset \bigcap_{\mathscr{I} \in A^*} \mathscr{I}$, that is, $\mathscr{I}_p \in \mathscr{C}\ell(A^*)$. This completes the proof of Lemma 4.4.

LEMMA 4.5. *The natural correspondence* $\sigma : M^* \to M$ *preserves the concept of closed subsets defined by Definition* 4.3, *that is,* A *is a closed subset of* M, *if and only if* $A^* = \sigma^{-1}(A)$ *is a closed subset of* M^*.

Proof. Let A be a closed subset of M. By Lemma 4.4, $\mathscr{C}\ell(A^*) = (\bar{A})^* = A^*$. Hence A^* is closed.

Conversely, let $A^* = \sigma^{-1}(A)$ be a closed subset of M^*, then by Lemma 4.4, $(\bar{A})^* = \mathscr{C}\ell(A^*) = A^*$. Hence $\bar{A} = \sigma((\bar{A})^*) = \sigma(A^*) = A$. So A is closed.

LEMMA 4.6. *Let $\Phi: \Gamma_0(T_M) \to \Gamma_0(T_{M'})$ be a Lie algebra isomorphism. Then A^* is a closed subset of M^*, if and only if $\Phi(A^*)$ is a closed subset of $(M')^*$, where $\Phi(A^*) = \{\Phi(\mathscr{I}) \mid \mathscr{I} \in A^*\}$.*

Proof. Since Φ is an isomorphism, $\Phi: M^* \to (M')^*$ is a one to one, onto correspondence. So we have

$$\Phi\Bigl(\bigcap_{\mathscr{I} \in A^*} \mathscr{I}\Bigr) = \bigcap_{\mathscr{I} \in A^*} \Phi(\mathscr{I}) = \bigcap_{\mathscr{I}' \in \Phi(A^*)} \mathscr{I}' .$$

Hence we have $\mathfrak{m} \supset \bigcap_{\mathscr{I} \in A^*} \mathscr{I}$, if and only if $\Phi(\mathfrak{m}) \supset \bigcap_{\mathscr{I}' \in \Phi(A^*)} \mathscr{I}'$. This completes the proof of Lemma 4.6.

Now we define a map $\varphi: M \to M'$ by the following formula.

$$\begin{cases} \varphi(p_0) = p'_0 \\ \varphi(p) = p' , & \text{if } p \neq p_0 \text{ and } \Phi(\mathscr{I}_p) = \mathscr{I}_{p'} . \end{cases}$$

PROPOSITION 4.7. *The natural map $\varphi: M \to M'$ is an onto homeomorphism.*

Proof. Clearly φ is a one to one and onto map. From the definition of φ, we have the following commutative diagram.

$$\begin{array}{ccc} M^* & \xrightarrow{\Phi} & (M')^* \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ M & \xrightarrow[\varphi]{} & M' , \end{array}$$

where σ_i is the natural correspondence. Let B be an arbitrary closed subset of M'. By Lemmas 4.5 and 4.6, $(\Phi^{-1} \circ \sigma_2^{-1})(B)$ is a closed subset of M^*. Since $\sigma_1^{-1}(\varphi^{-1}(B)) = (\Phi^{-1} \circ \sigma_2^{-1})(B)$, we see by Lemma 4.5 that $\varphi^{-1}(B)$ is a closed subset of M. Hence φ is a continuous map. By the same way we can prove that φ^{-1} is also continuous. Hence φ is a homeomorphism.

Next we study the case $\Gamma_N(T_M)$ with $\dim N \geqq 1$. Let M and M' be C^∞ manifolds and N (resp. N') be an arbitrary but fixed closed submanifold of M (resp. M').

PROPOSITION 4.8. *Let $\Phi: \Gamma_N(T_M) \to \Gamma_{N'}(T_{M'})$ be an isomorphism. Let $\mathscr{I}$ be the maximal ideal of $\Gamma_N(T_M)$ corresponding to p, and $\mathscr{I}' \doteq \Phi(\mathscr{I})$ be the maximal ideal of $\Gamma_{N'}(T_{M'})$ corresponding to p'. Then $p \in N$, if and only if $p' \in N'$.*

Proof. Since $\Phi: \boldsymbol{\Gamma}_N(T_M) \to \boldsymbol{\Gamma}_{N'}(T_{M'})$ is an isomorphism, $\boldsymbol{\Gamma}_N(T_M)/\mathscr{I}$ should be isomorphic to $\boldsymbol{\Gamma}_{N'}(T_{M'})/\mathscr{I}'$. By Lemma 3.6 this implies $p \in N \Leftrightarrow p' \in N'$.

LEMMA 4.9. *Let $\Phi: \boldsymbol{\Gamma}_N(T_M) \to \boldsymbol{\Gamma}_{N'}(T_{M'})$ be an isomorphism. Then $\Phi(\mathrm{Ker}\, r_*) = \mathrm{Ker}\, r'_*$, that is, Φ induces an isomorphism $\Psi: \boldsymbol{\Gamma}(T_N) \to \boldsymbol{\Gamma}(T_{N'})$, where $r_*: \boldsymbol{\Gamma}_N(T_M) \to \boldsymbol{\Gamma}(T_N)$* (resp. $r'_*: \boldsymbol{\Gamma}_{N'}(T_{M'}) \to \boldsymbol{\Gamma}(T_{N'})$) *is the homomorphism induced by the restriction of vector fields on M* (resp. M') *to N* (resp. N').

Proof. Obviously $\mathrm{Ker}\, r_* = \bigcap \{r_*^{-1}\bar{\mathscr{I}}_p \mid p \in N\}$. By Proposition 4.8, $\Phi(\mathrm{Ker}\, r_*) = \bigcap \{\Phi(r_*^{-1}\bar{\mathscr{I}}_p) \mid p \in N\} = \bigcap \{r'^{-1}_*\bar{\mathscr{I}}_p \mid q \in N'\} = \mathrm{Ker}\, r'_*$.

Let $\Phi: \boldsymbol{\Gamma}_N(T_M) \to \boldsymbol{\Gamma}_{N'}(T_{M'})$ be an isomorphism. Let $\mathscr{I}$ be the maximal ideal corresponding to $p \in M$. Then by Proposition 3.5 there exists a unique point $q \in M'$ such that the maximal ideal $\mathscr{I}' = \Phi(\mathscr{I})$ corresponds to q. We set $\varphi(p) = q$. Now we define the Stone topology of $M^* = \{\mathscr{I} \mid \mathscr{I} \subset \boldsymbol{\Gamma}_N(T_M)$: maximal ideal$\}$ as in the case $\boldsymbol{\Gamma}_0(T_M)$. Then we have the following proposition.

PROPOSITION 4.10. *The natural correspondence $\varphi: M \to M'$ is an onto homeomorphism such that $\varphi(N) = N'$.*

Proof. The proof for φ to be a homeomorphism is all the same as that of the case $\boldsymbol{\Gamma}_0(T_M)$. By Proposition 4.8, $\varphi(N) = N'$.

§ 5. Characterization of non-zero vector fields.

LEMMA 5.1. *Let $\mathscr{I}_p$ be the maximal ideal of $\boldsymbol{\Gamma}_0(T_M)$ corresponding to p ($p \neq p_0$). For any $X \in \boldsymbol{\Gamma}_0(T_M)$, $X(p) \neq 0$, if and only if $[X, \boldsymbol{\Gamma}_0(T_M)] + \mathscr{I}_p = \boldsymbol{\Gamma}_0(T_M)$.*

LEMMA 5.1′. *For any $X \in \boldsymbol{\Gamma}_N(T_M)$, $X(p) \neq 0$, if and only if*
(i) $[X, \boldsymbol{\Gamma}_N(T_M)] + \mathscr{I}_p = \boldsymbol{\Gamma}_N(T_M)$ *(for $p \notin N$) or*
(ii) $[r_*X, \boldsymbol{\Gamma}(T_N)] + \bar{\mathscr{I}}_p = \boldsymbol{\Gamma}(T_N)$ *(for $p \in N$).*

Proof. The proofs of these lemmas are all the same as that of Lemma 3 of Pursell and Shanks [9] (see also Omori (5]), and omitted.

LEMMA 5.2. *Let $\Phi: \boldsymbol{\Gamma}_0(T_M) \to \boldsymbol{\Gamma}_0(T_{M'})$ be a Lie algebra isomorphism and $\varphi: M \to M'$ be the induced homeomorphism. For any $p \in M$ ($p \neq p_0$) there are smooth local coordinate system $(U; x^1, \cdots, x^m)$ at p and $(V; y^1, \cdots, y^m)$ at $\varphi(p) = p'$ such that for any*

$$X = \xi^i \frac{\partial}{\partial x^i} \in \Gamma_0(T_M) , \qquad \Phi\left(\xi^i \frac{\partial}{\partial x^i}\right) = (\xi^i \circ \varphi^{-1}) \frac{\partial}{\partial y^i} .$$

Proof. Since $p \neq p_0$, there is a smooth local coordinate system $(U; x^1, \cdots, x^m)$ at p such that $p_0 \notin \overline{U}$. Hence suitable extensions of $\partial/\partial x^i$ $(i = 1, \cdots, m)$ are contained in $\Gamma_0(T_M)$. We also denote the extended vector fields by the same letters. Set $v_i = \Phi(\partial/\partial x^i)$ $(i = 1, \cdots, m)$. Then $v_i \in \Gamma_0(T_{M'})$ for all $i = 1, \cdots, m$. Since $(\partial/\partial x^i)(p) \neq 0$, by Lemma 5.1, $v_i(p') \neq 0$, where $p' = \varphi(p)$. Since Φ is a Lie algebra isomorphism, on some neighborhood of p', $[v_i, v_j] = \Phi([\partial/\partial x^i, \partial/\partial x^j]) = 0$ for all $i, j = 1, \cdots, m$. Hence there exists a smooth local coordinate system $(V; y^1, \cdots, y^m)$ at p' such that $v_i = \partial/\partial y^i$ on V. Let q be an arbitrary point in U. Now, for any $X = \xi^i(\partial/\partial x^i) \in \Gamma_0(T_M)$, a suitable extension of $\xi^i(q)(\partial/\partial x^i)$ is contained in $\Gamma_0(T_M)$. We denote it by X^*. Since $(X - X^*)(q) = 0$, by Lemma 5.1, $\Phi(X - X^*)(q') = 0$. Hence $\Phi(X)(q') = \Phi(X^*)(q') = \xi^i(q) \cdot v_i(q') = (\xi^i \circ \varphi^{-1}(q')) \cdot (\partial/\partial y^i)(q')$. Therefore $\Phi(\xi^i(\partial/\partial x^i)) = (\xi^i \circ \varphi^{-1})(\partial/\partial y^i)$ on V.

COROLLARY 5.3. *The induced homeomorphism $\varphi: M \to M'$ is linear with respect to the local coordinate systems defined in* Lemma 5.2, *that is, $\varphi^i(x^1, \cdots, x^m) = x^i$ $(i = 1, \cdots, m)$, where $\varphi^i = y^i \circ \varphi$.*

Proof. We use the same notations for the extended vector fields because all argument here is local. By Lemma 5.2, $\Phi(x^i(\partial/\partial x^j)) = (x^i \circ \varphi^{-1})(\partial/\partial y^j)$. On the other hand we have $[\partial/\partial y^k, (x^i \circ \varphi^{-1})(\partial/\partial y^j)] = (\partial/\partial y^k)(x^i \circ \varphi^{-1})(\partial/\partial y^j)$ and $[\partial/\partial y^k, (x^i \circ \varphi^{-1})(\partial/\partial y^j)] = \Phi([\partial/\partial x^k, x^i(\partial/\partial x^j)]) = \delta^i_k(\partial/\partial y^j)$, where δ^i_k is the Kronecker delta. So we have $(\partial/\partial y^k)(x^i \circ \varphi^{-1}) = \delta^i_k$. Hence $x^i \circ \varphi^{-1} = y^i + C$, where C is a constant of integration. Since $\varphi(0) = 0$, $C = 0$. Therefore $x^i \circ \varphi^{-1} = y^i$. Since φ is a homeomorphism, $y^i \circ \varphi = (x^i \circ \varphi^{-1}) \circ \varphi = x^i$.

PROPOSITION 5.4. *Let $\Phi: \Gamma_0(T_M) \to \Gamma_0(T_{M'})$ be a Lie algebra isomorphism and $\varphi: M \to M'$ be the induced homeomorphism. Then $\varphi|_{M_0}: M_0 \to M'_0$ is a C^∞ diffeomorphism, where $M_0 = M - \{p_0\}$ and $M'_0 = M' - \{p'_0\}$.*

Proof. Let $f \in C^\infty(M_0)$ be an arbitrary C^∞ function of M_0 and $p' \in M'_0$ be an arbitrary point, and set $p' = \varphi(p)$. Let $(U; x^1, \cdots, x^m)$ be a local coordinate system at p in M_0. Since $p_0 \notin U$, a suitable extension of $f \cdot \partial/\partial x^1$ is contained in $\Gamma_0(T_M)$. We denote the extended vector field by the same letter. By Lemma 5.2, $\Phi(f \cdot \partial/\partial x^1) = (f \circ \varphi^{-1}) \cdot \partial/\partial y^1$ on some coordinate neighborhood V of $p' \in M'_0$. Since $\Phi(f \cdot \partial/\partial x^1)$ is a C^∞ vector field, $f \circ \varphi^{-1}$ is a C^∞ function on V. Since p' and f are arbitrary, φ is a diffeomorphism.

COROLLARY 5.5. *Let $\varphi: M \to M'$ be the homeomorphism induced by the isomorphism Φ. Then $\Phi = d\varphi$ on $M - \{p_0\}$.*

Proof. For each point $p \in M - \{p_0\}$, by Lemma 5.2 and Corollary 5.3 $\varphi_i(x^1, \cdots, x^m) = y^i \circ \varphi(x^1, \cdots, x^m) = x^i$ in some neighborhood of p. Hence for any $X = \xi^i(\partial/\partial x^i) \in \boldsymbol{\Gamma}_0(T_M)$ we have

$$d\varphi(X) = (\xi^i \circ \varphi^{-1})\left(\frac{\partial \varphi_j}{\partial x^i}\right) \cdot \frac{\partial}{\partial y^j} = (\xi^i \circ \varphi^{-1}) \cdot \frac{\partial}{\partial y^i} .$$

On the other hand $\Phi(X) = (\xi^i \circ \varphi^{-1})\partial/\partial y^i$. Hence $d\varphi = \Phi$ on $M - \{p_0\}$.

§ 6. Proof of the theorem.

(Case $\boldsymbol{\Gamma}_0(T_M)$)

LEMMA 6.1. *For any $Y \in \boldsymbol{\Gamma}_0(T_{M'})$ and any $g \in C^\infty(M')$ we have*

$$\Phi^{-1}(gY) = (g \circ \varphi)\Phi^{-1}(Y) .$$

Proof. For the case $p \neq p_0$ we already proved in Lemma 5.2. Set $Z = gY - g(p_0') \cdot Y$, where $p_0' = \varphi(p_0)$. Clearly $Z(p_0') = 0$. Since $\Phi^{-1}: \boldsymbol{\Gamma}_0(T_{M'}) \to \boldsymbol{\Gamma}_0(T_M)$ is an isomorphism, $\Phi^{-1}(0) = 0$. Hence $\Phi^{-1}(Z)(p_0) = \Phi^{-1}(gY)(p_0) - g(p_0')\Phi^{-1}(Y)(p_0) = 0$. Hence we have $\Phi^{-1}(gY)(p_0) = g(p_0')\Phi^{-1}(Y)(p_0) = (g \circ \varphi)(p_0)\Phi^{-1}(Y)(p_0)$.

LEMMA 6.2. *Let $\boldsymbol{R}^1$ be the one dimensional Euclidean space with the standard coordinate x. If $f: \boldsymbol{R}^1 \to \boldsymbol{R}$ is a continuous function such that $g(x) = x \cdot f(x)$ is a C^{r+1} function, then $f(x)$ is a C^r function. Moreover if g is a C^∞ function, then f is also a C^∞ function.*

Proof. It suffices to prove that f is a C^1 function if g is a C^2 function. Clearly f is a C^2 function except the origin 0. We take the Taylor expansion of $g(x)$ at 0.

$$g(x) = g(0) + g'(0) \cdot x + \tfrac{1}{2} g''(\theta x) \cdot x^2 \qquad (0 < \theta < 1) .$$

Since $g(x) = x \cdot f(x)$, $g(0) = 0$. So $x \cdot f(x) = g'(0) \cdot x + \frac{1}{2} g''(\theta x) \cdot x^2$, and we have $f(x) = g'(0) + \frac{1}{2} x g''(\theta x)$ for $x \neq 0$. Since $f(x)$ is continuous, $f(0) = g'(0)$. Hence we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{1}{2} g''(\theta x) = \frac{1}{2} g''(0) .$$

Therefore $f(x)$ is differentiable at $x = 0$ and $f'(x)$ is continuous on $\boldsymbol{R}^1$, that is, $f(x)$ is a C^1 function. By induction on r, $f(x)$ becomes a C^r function.

COROLLARY 6.3. *Let $\boldsymbol{R}^m$ be the Euclidean m-space with the standard coordinate $(x^1, \cdots, x^m)$. If $f: \boldsymbol{R}^m \to \boldsymbol{R}$ is a continuous function such that $g(x) = x^1 \cdot f(x)$ is a C^{r+1} function, then $f(x)$ is a C^r function. Especially, if $g(x)$ is a C^∞ function then $f(x)$ is also a C^∞ function.*

Proof. We regard $x^2, \cdots, x^m$ as smooth parameters of $g(x)$, and take the Taylor expansion of $g(x)$ at the origin $0 \in \boldsymbol{R}^m$ with respect to the first coordinate x^1. Then we can easily prove the differentiability of $f(x)$.

THEOREM 6.4. *Let $\Phi: \boldsymbol{\Gamma}_0(T_M) \to \boldsymbol{\Gamma}_0(T_{M'})$ be a Lie algebra isomorphism. and $\varphi: M \to M'$ be the induced homeomorphism. Then φ becomes a C^∞ diffeomorphism.*

Proof. By Proposition 5.4, $\varphi: M - \{p_0\} \to M' - \{p_0'\}$ is a C^∞ diffeomorphism. So it suffices to prove the differentiability of φ at $p_0 \in M$. Let $(U; x^1, \cdots, x^m)$ be a local coordinate at $p_0 \in M$. Then suitable extension of $x^1 \cdot \partial/\partial x^1$ is contained in $\boldsymbol{\Gamma}_0(T_M)$. We denote the extended vector field by X. Set $Y = \Phi(X)$, then $Y \in \boldsymbol{\Gamma}_0(T_{M'})$. For any $g \in C^\infty(M')$ we set $Y_1 = gY$. Then $Y_1 \in \boldsymbol{\Gamma}_0(T_{M'})$. Hence $X_1 = \Phi^{-1}(Y_1)$ is contained in $\boldsymbol{\Gamma}_0(T_M)$. By Lemma 6.1,

$$X_1 = \Phi^{-1}(gY) = (g \circ \varphi)\Phi^{-1}(Y) = (g \circ \varphi)X .$$

Hence, on the neighborhood U, $X_1 = (g \circ \varphi) \cdot x^1(\partial/\partial x^1)$. Since X_1 is a C^∞ vector field, $(g \circ \varphi) \cdot x^1 \in C^\infty(U)$. By Proposition 4.7, φ is continuous. Therefore the composition $g \circ \varphi$ is continuous and, by Corollary 6.3, φ is C^∞ differentiable at $p_0 \in U \subset M$.

COROLLARY 6.5. *Let M and M' be compact manifolds without boundaries. If Lie algebras of $\mathscr{D}(M, p_0)$ and $\mathscr{D}(M', p_0')$ are isomorphic, then $\mathscr{D}(M, p_0) \cong \mathscr{D}(M', p_0')$ as* I.L.H.-*Lie groups.*

Proof. Since Lie algebras of $\mathscr{D}(M, p_0)$ and $\mathscr{D}(M', p_0')$ are exactly $\boldsymbol{\Gamma}_0(T_M)$ and $\boldsymbol{\Gamma}_0(T_{M'})$, by Theorem 6.4,

$$\mathscr{D}(M, p_0) \cong \mathscr{D}(M', p_0') .$$

COROLLARY 6.6. *Let $\varphi: M \to M'$ be the diffeomorphism induced by Φ. Then we have $d\varphi = \Phi$.*

Proof. Since $\Phi: \boldsymbol{\Gamma}_0(T_M) \to \boldsymbol{\Gamma}_0(T_{M'})$ is an isomorphism, for any $X \in \boldsymbol{\Gamma}_0(T_M)$, $\Phi(X) \in \boldsymbol{\Gamma}_0(T_{M'})$. Since $\varphi: M \to M'$ is a C^∞ diffeomorphism, also we have $d\varphi(X) \in \boldsymbol{\Gamma}_0(T_{M'})$. By Corollary 5.5 $d\varphi(X) = \Phi(X)$ on $M' - \{p_0'\}$ as C^∞ vector fields. By continuity of the vector fields we have $d\varphi(X)(p_0') = \Phi(X)(p_0')$. Hence $d\varphi = \Phi$.

COROLLARY 6.7. *Let $N = \{p_1, \cdots, p_s\}$ and $N' = \{p_1', \cdots, p_t'\}$ be zero dimensional manifolds consisting of finite number of points. If $\boldsymbol{\Gamma}_N(T_M)$ is isomorphic to $\boldsymbol{\Gamma}_{N'}(T_{M'})$, then $s = t$ and there exists a C^∞ diffeomorphism $\varphi: M \to M'$ such that $\varphi(N) = N'$.*

Proof. The proof is easy, and omitted.

(Case $\boldsymbol{\Gamma}_N(T_M)$ with $\dim N \geqq 1$)

LEMMA 6.8. *Let $\Phi: \boldsymbol{\Gamma}_N(T_M) \to \boldsymbol{\Gamma}_{N'}(T_{M'})$be a Lie algebra isomorphism. We have then, for any $f \in C^\infty(M)$ and $X \in \boldsymbol{\Gamma}_N(T_M)$, $\Phi(fX) = (f \circ \varphi^{-1})\Phi(X)$.*

Proof. The proof is all the same as that of Lemma 5.2, and omitted.

THEOREM 6.9. *Let $\Phi: \boldsymbol{\Gamma}_N(T_M) \to \boldsymbol{\Gamma}_{N'}(T_{M'})$ be an isomorphism and $\varphi: M \to M'$ be the induced homeomorphism. Then φ is a C^∞ diffeomorphism such that $\varphi(N) = N'$.*

Proof. Let g be an arbitrary function in $C^\infty(M')$, and $q = \varphi(p)$ be an arbitrarily fixed point. Let Y be any element of $\boldsymbol{\Gamma}_{N'}(T_{M'})$ such that $Y(q) \neq 0$. Actually we can take such Y, because of $\dim N' \geqq 1$. We set $X = \Phi^{-1}(Y)$, $Y_1 = gY$ and $X_1 = \Phi^{-1}(Y_1)$.

(Case $p \notin N$) By Lemma 5.1′, $[Y, \boldsymbol{\Gamma}_{N'}(T_{M'})] + \mathscr{I}_q' = \boldsymbol{\Gamma}_{N'}(T_{M'})$, where $\mathscr{I}_q'$ is the maximal ideal corresponding to q. Since Φ is a Lie algebra isomorphism, by operating Φ^{-1} to the above equality we have $[X, \boldsymbol{\Gamma}_N(T_M)] + \mathscr{I}_p = \boldsymbol{\Gamma}_N(T_M)$. Hence $X(p) \neq 0$.

(Case $p \in N$) By Lemma 5.1′, $[r_*'Y, \boldsymbol{\Gamma}(T_{N'})] + \bar{\mathscr{I}}_q' = \boldsymbol{\Gamma}(T_{N'})$. By operating the isomorphism $\Psi^{-1}: \boldsymbol{\Gamma}(T_{N'}) \to \boldsymbol{\Gamma}(T_N)$, we have $(r_*X)(p) \neq 0$. Hence $X(p) \neq 0$.

So we may assume that $X = \partial/\partial x^1$ on a some neighborhood U of p. On the other hand, $X_1 = \Phi^{-1}(Y_1) = \Phi^{-1}(gY) = (g \circ \varphi)\Phi^{-1}(Y) = (g \circ \varphi)X$.

Hence $X_1 = (g \circ \varphi)(\partial/\partial x^1)$ on U. This is an expression of the smooth vector field X_1 with respect to the local coordinate on U. Therefore $g \circ \varphi$ is contained in $C^\infty(M)$. So φ is a diffeomorphism.

COROLLARY 6.10. *Let $\varphi: M \to M'$ be the diffeomorphism induced by Φ. Then we have $d\varphi = \Phi$.*

Proof. The proof is same as that of Corollary 6.6.

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