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SIEGEL DOMAINS OVER SELF-DUAL CONES AND THEIR AUTOMORPHISMS

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Introduction

The Lie algebra $\mathfrak{g}_h$ of all infinitesimal automorphisms of a Siegel domain in terms of polynomial vector fields was investigated by Kaup, Matsushima and Ochiai [6]. It was proved in [6] that $\mathfrak{g}_h$ is a graded Lie algebra; $\mathfrak{g}_h = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1$ and the Lie subalgebra $\mathfrak{g}_a$ of all infinitesimal affine automorphisms is given by the graded subalgebra; $\mathfrak{g}_a = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0$. Nakajima [9] proved without the assumption of homogeneity that the non-affine parts $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ can be determined from the affine part $\mathfrak{g}_a$.

The main purpose of the present paper is to determine explicitly the Lie algebras $\mathfrak{g}_h$ for Siegel domains over self-dual cones. In § 2 we will prove that if the adjoint representation ρ of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is irreducible, then $\mathfrak{g}_h$ is simple or $\mathfrak{g}_h = \mathfrak{g}_a$ (Theorem 2.1). Moreover using Nakajima's result we will give sufficient conditions of the vanishing of $\mathfrak{g}_{1/2}$ (Proposition 2.3 and Corollary 2.7) and a method of calculating $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ (Propositions 2.6 and 2.8). Using the results in § 2, we determine in § 3 (Theorems 3.3–3.6) infinitesimal automorphisms of most of the homogeneous Siegel domains over self-dual cones (other than circular cones) which were constructed by Pjateckii-Sapiro [10].

The *circular cone* $C(n)$ of dimension n ($n \geq 3$) is defined to be the set $\{{}^t(x_1, x_2, \cdots, x_n) \in \boldsymbol{R}^n ; x_1 > 0, x_1x_2 - x_3^2 - \cdots - x_n^2 > 0\}$. Pjateckii-Sapiro [10] found all the homogeneous Siegel domains over circular cones which are constructed by using the representation theory of Clifford algebras. But it was shown by Kaneyuki and Tsuji [5] that there exists a homogeneous Siegel domain over a circular cone which does not appear in Pjateckii-Sapiro's construction. In view of this fact the purpose in § 4 is to give a method of constructing all homogeneous Siegel domains over

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circular cones (Theorem 4.4) by making use of the considerations analogous to [5].

Pjateckii-Sapiro [10] pointed out without proof that the exceptional bounded symmetric domain in $\boldsymbol{C}^{16}$ is realized as a Siegel domain over the cone C (8). In § 5 we consider a certain homogeneous Siegel domain D over C (8), which is implicitly given in [10], and by means of results in § 2 and § 4 we prove that D is isomorphic to the above exceptional symmetric domain (Theorem 5.4).

Finally, in § 6 we determine infinitesimal automorphisms of homogeneous Siegel domains over circular cones (Theorem 6.1, Propositions 6.2 and 6.3).

Some of results of the present paper were announced in the note [15].

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§ 1. Preliminaries

In this section, after introducing notations which are used throughout this paper, we recall some of results of [6] and [9].

1.1. Let R be a real vector space of dimension n and W be a complex vector space of dimension m. Let $D(V,F)$ denote a Siegel domain of type I or type II in $R^C \times W$ associated with a convex cone V in R and a V-hermitian form F on W, which is defined by Pjateckii-Sapiro [10], where R^C is the complexification of R. Throughout this paper we will employ the following notations;

$\mathfrak{g}_h$ (resp. $\mathfrak{g}_a$); the Lie algebra of all infinitesimal holomorphic (resp. affine) automorphisms of $D(V,F)$.

$\mathfrak{g}(V)$; the Lie algebra of the automorphism group $G(V) = \{g \in GL(R);\ gV = V\}$ of the cone V.

$\{e_1, \cdots, e_n\}$ (resp. $\{f_1, \cdots, f_m\}$); a base of R (resp. W).

$(z_1, \cdots, z_n, w_1, \cdots, w_m)$; the complex coordinate system of $R^C \times W$ associated with the base $\{e_1, \cdots, e_n, f_1, \cdots, f_m\}$.

The following ranges of indices will be taken in each summation: $1 \leq j, k, l, \cdots \leq n$, $1 \leq \alpha, \beta, \gamma, \cdots \leq m$.

For a positive integer p, $U(p)$ (resp. $O(p)$) denotes the unitary (resp. real orthogonal) group of degree p and E_p denotes the unit matrix of degree p. And for two positive integers p and q, we denote by $M(p,q;\boldsymbol{F})$ the real (resp. complex) vector space of all real (resp. complex) $p \times q$-matrices and by $\mathfrak{gl}(p, \boldsymbol{F})$ the real (resp. complex) general linear Lie algebra of degree p, where $\boldsymbol{F} = \boldsymbol{R}$ (resp. $\boldsymbol{C}$).

1.2. Put $\partial = \sum z_k \partial/\partial z_k + \frac{1}{2} \sum w_\alpha \partial/\partial w_\alpha$ and $\partial' = i \sum w_\alpha \partial/\partial w_\alpha$. Then the following results (1.4)–(1.6) are known in [6].

(1.1) The vector field ∂ belongs to $\mathfrak{g}_h$ and $\mathfrak{g}_h$ is a graded Lie algebra; $\mathfrak{g}_h = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1$, where $\mathfrak{g}_\lambda$ is the λ-eigenspace of $\mathrm{ad}(\partial)$ $(\lambda = \pm 1, \pm \frac{1}{2}, 0)$. Furthermore $\mathfrak{g}_a$ is the graded subalgebra; $\mathfrak{g}_a = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0$.

(1.2) $$\mathfrak{g}_{-1} = \{\sum a^k \partial/\partial z_k \,;\, a^k \in \boldsymbol{R}\} .$$

(1.3) $\mathfrak{g}_{-1/2} = \{2i \sum F^k(w, c)\partial/\partial z_k + \sum c^\alpha \partial/\partial w_\alpha \,;\, c = \sum c^\alpha f_\alpha \in W\}$, where $F(w, c) = \sum F^k(w, c) e_k$.

(1.4) $\mathfrak{g}_0 = \{\sum a_{kl} z_l \partial/\partial z_k + \sum b_{\alpha\beta} w_\beta \partial/\partial w_\alpha \,;\, A = (a_{kl}) \in \mathfrak{g}(V),\ B = (b_{\alpha\beta}) \in \mathfrak{gl}(W),$ $AF(u, u) = F(Bu, u) + F(u, Bu)$ for each $u \in W\}$.

Let $\mathfrak{r}$ be the radical of $\mathfrak{g}_h$. Then

(1.5) $\mathfrak{r}$ is a graded ideal of $\mathfrak{g}_h$ such that $\mathfrak{r} = \mathfrak{r}_{-1} + \mathfrak{r}_{-1/2} + \mathfrak{r}_0$, where $\mathfrak{r}_{-\lambda} = \mathfrak{r} \cap \mathfrak{g}_{-\lambda}$ $(\lambda = 1, \frac{1}{2}, 0)$.

(1.6) $$\dim \mathfrak{g}_\lambda = \dim \mathfrak{g}_{-\lambda} - \dim \mathfrak{r}_{-\lambda} \quad (\lambda = 1, \tfrac{1}{2}) .$$

Considering (1.1) we denote by ρ (resp. σ) the adjoint representation of the subalgebra $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_{-1/2}$). Let us define real linear isomorphisms φ_{-1} and $\varphi_{-1/2}$ as follows;

$$\varphi_{-1} \ : a = \sum a^k e_k \in R \mapsto \varphi_{-1}(a) = \sum a^k \partial/\partial z_k \in \mathfrak{g}_{-1} ,$$
$$\varphi_{-1/2} : c = \sum c^\alpha f_\alpha \in W \mapsto \varphi_{-1/2}(c) = 2i \sum F^k(w, c)\partial/\partial z_k + \sum c^\alpha \partial/\partial w_\alpha \in \mathfrak{g}_{-1/2} .$$

Then by easy computations we can see that the following (1.7) and (1.8) are valid; for $a \in R$, $c, c' \in W$ and $X = \sum a_{kl} z_l \partial/\partial z_k + \sum b_{\alpha\beta} w_\beta \partial/\partial w_\alpha \in \mathfrak{g}_0$,

(1.7) $\rho(X)(\varphi_{-1}(a)) = -\varphi_{-1}(Aa)$ and $\sigma(X)(\varphi_{-1/2}(c)) = -\varphi_{-1/2}(Bc)$, where $A = (a_{kl})$ and $B = (b_{\alpha\beta})$. In particular $\sigma(\partial')(\varphi_{-1/2}(c)) = -\varphi_{-1/2}(ic)$.

(1.8) $$[\varphi_{-1/2}(c), \varphi_{-1/2}(c')] = 4\varphi_{-1}(\mathrm{Im}\, F(c', c)) .$$

By the facts stated above we can identify $\rho(\mathfrak{g}_0)$ with a subalgebra of $\mathfrak{g}(V)$.

The following results (1.9) and (1.10) are due to Nakajima (Proposition 2.6 in [9]).

(1.9) The subspace $\mathfrak{g}_{1/2}$ of $\mathfrak{g}_h$ consists of all polynomial vector fields $X = \sum p^k_{1,1}\partial/\partial z_k + \sum (p^\alpha_{1,0} + p^\alpha_{0,2})\partial/\partial w_\alpha$ satisfying the condition $[\mathfrak{g}_{-1/2}, X] \subset \mathfrak{g}_0$, where $p^k_{\lambda,\mu}$ and $p^\alpha_{\lambda,\mu}$ are polynomials of homogeneous degree λ in $z_1, \cdots, z_n$ and homogeneous degree μ in $w_1, \cdots, w_m$.

(1.10) The subspace $\mathfrak{g}_1$ of $\mathfrak{g}_h$ consists of all polynomial vector fields $X = \sum p^k_{2,0}\partial/\partial z_k + \sum p^\alpha_{1,1}\partial/\partial w_\alpha$ satisfying the following conditions; $[\mathfrak{g}_{-1/2}, X] \subset \mathfrak{g}_{1/2}$, $[\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_0$ and $\operatorname{Im} \operatorname{Tr} \sigma([Y, X]) = 0$ for each $Y \in \mathfrak{g}_{-1}$.

§ 2. Lie algebras of infinitesimal automorphisms

2.1. Kaneyuki and Sudo [4] proved that if $D(V, F)$ is an irreducible symmetric domain (or equivalently $\mathfrak{g}_h$ is simple), then the representation ρ is irreducible. Conversely without the assumption of homogeneity of $D(V, F)$ we have

THEOREM 2.1. *If the representation ρ is irreducible, then $\mathfrak{g}_h$ is simple or $\mathfrak{g}_h = \mathfrak{g}_a$.*

Proof. By our assumption we have $\mathfrak{r}_{-1} = (0)$ or $\mathfrak{r}_{-1} = \mathfrak{g}_{-1}$, since $\mathfrak{r}_{-1}$ is a subspace of $\mathfrak{g}_{-1}$ invariant under $\rho(\mathfrak{g}_0)$. First we suppose $\mathfrak{r}_{-1} = (0)$. Then it follows from (1.5), (1.7) and (1.8) that $\mathfrak{r}_{-1/2} = \mathfrak{r}_0 = (0)$ and $\mathfrak{r} = (0)$ (this fact was proved more generally in [9]). So $\mathfrak{g}_h$ is semi-simple. Suppose that $\mathfrak{g}_h$ is not simple. Then the Siegel domain $D(V, F)$ is reducible and the cone V is decomposed into irreducible factors (cf. [9], Corollaries 4.8 and 4.9), which means that ρ is not irreducible. This contradicts to our assumption. Thus $\mathfrak{g}_h$ is simple.

Now we consider the case $\mathfrak{r}_{-1} = \mathfrak{g}_{-1}$. It follows from (1.6) that $\mathfrak{g}_1 = (0)$. We will show that $\mathfrak{g}_{1/2} = (0)$. By (1.9) every $X \in \mathfrak{g}_{1/2}$ is represented as $X = \sum p^k_{1,1}\partial/\partial z_k + \sum (p^\alpha_{1,0} + p^\alpha_{0,2})\partial/\partial w_\alpha$. Put $Z = [X, [\partial', X]]$. Then from the direct verification it follows that Z is represented as

$$Z = 2i \sum p^\alpha_{1,0} \frac{\partial p^k_{1,1}}{\partial w_\alpha} \partial/\partial z_k + 2i \sum \left(p^\beta_{1,0} \frac{\partial p^\alpha_{0,2}}{\partial w_\beta} - p^k_{1,1} \frac{\partial p^\alpha_{1,0}}{\partial z_k} \right) \partial/\partial w_\alpha .$$

By (1.1) and the fact $\partial' \in \mathfrak{g}_0$, the vector field Z belongs to $\mathfrak{g}_1 = (0)$. Hence we have

$$\sum p^\alpha_{1,0} \frac{\partial p^k_{1,1}}{\partial w_\alpha} = 0 \qquad (1 \leq k \leq n) . \tag{2.1}$$

Since $[\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_{-1/2}$, there exist $c_l = \sum c_l^\alpha f_\alpha \in W$ $(1 \leq l \leq n)$ such that $[\partial/\partial z_l, X] = 2i \sum F^k(w, c_l)\partial/\partial z_k + \sum c_l^\alpha \partial/\partial w_\alpha$ $(1 \leq l \leq n)$. On the other hand, $[\partial/\partial z_l, X] = \sum \frac{\partial p_{1,1}^k}{\partial z_l} \partial/\partial z_k + \sum \frac{\partial p_{1,0}^\alpha}{\partial z_l} \partial/\partial w_\alpha$ $(1 \leq l \leq n)$, which implies $\frac{\partial p_{1,1}^k}{\partial z_l} = 2i\, F^k(w, c_l)$ and $\frac{\partial p_{1,0}^\alpha}{\partial z_l} = c_l^\alpha$. Hence we have

$$p_{1,1}^k = 2i \sum F^k(w, c_l) z_l \quad \text{and} \quad p_{1,0}^\alpha = \sum c_l^\alpha z_l \quad (1 \leq k \leq n,\ 1 \leq \alpha \leq \mathrm{m}) .$$

In view of (2.1) we obtain $\sum F^k(c_j, c_l) z_j z_l = 0$ $(1 \leq k \leq n)$. So we get $F^k(c_l, c_l) = 0$ $(1 \leq k, l \leq n)$. Therefore $c_l = 0$ and $p_{1,1}^k = p_{1,0}^\alpha = 0$ $(1 \leq k \leq n,\ 1 \leq \alpha \leq m)$. Thus X is written as $X = \sum p_{0,2}^\alpha \partial/\partial w_\alpha$. It is easily seen that $[\partial', X] = iX$. So both X and iX are contained in $\mathfrak{g}_h$. This means $X = 0$ by the well-known theorem of H. Cartan. Consequently we have $\mathfrak{g}_{1/2} = (0)$ and by (1.1) we conclude that $\mathfrak{g}_h = \mathfrak{g}_a$. q.e.d.

The above theorem will be used to determine the Lie algebras $\mathfrak{g}_h$ of certain Siegel domains in the following sections.

A Siegel domain $D(V, F)$ in $R^C \times W$ is said to be *non-degenerate* if the linear closure of the set $\{F(u, u) ; u \in W\}$ in R coincides with R (cf. [4]). Otherwise $D(V, F)$ is called *degenerate.*

Without the assumptions of irreducibility of ρ and homogeneity of $D(V, F)$, we have

PROPOSITION 2.2. *If $D(V, F)$ is non-degenerate and $\mathfrak{g}_{1/2} = (0)$, then $\mathfrak{g}_h = \mathfrak{g}_a$.*

Proof. From (1.7) and (1.8) it follows that $D(V, F)$ is non-degenerate if and only if $[\mathfrak{g}_{-1/2}, \mathfrak{g}_{-1/2}] = \mathfrak{g}_{-1}$. For $X \in \mathfrak{g}_1$, we have $[X, \mathfrak{g}_{-1/2}] \subset \mathfrak{g}_{1/2} = (0)$ and so $[X, \mathfrak{g}_{-1}] = [X, [\mathfrak{g}_{-1/2}, \mathfrak{g}_{-1/2}]] = (0)$. On the other hand, the condition $[X, \mathfrak{g}_{-1}] = (0)$ implies $X = 0$ (see [9], Lemma 3.1). By (1.1) we have $\mathfrak{g}_h = \mathfrak{g}_a$. q.e.d.

2.2. We now discuss sufficient conditions of the vanishing of $\mathfrak{g}_{1/2}$ of a Siegel domain $D(V, F)$ of type II in $R^C \times W$. Let $X = \sum p_{1,1}^k \partial/\partial z_k + \sum (p_{1,0}^\alpha + p_{0,2}^\alpha)\partial/\partial w_\alpha$ be a polynomial vector field on $R^C \times W$. Then it is known in [9] that X is contained in $\mathfrak{g}_{1/2}$ if and only if there exist $c_l = \sum c_l^\alpha f_\alpha \in W$ $(1 \leq l \leq n)$ and $b_{\beta\gamma}^\alpha \in \boldsymbol{C}$ $(b_{\beta\gamma}^\alpha = b_{\gamma\beta}^\alpha,\ 1 \leq \alpha, \beta, \gamma \leq m)$ satisfying the following (2.2), (2.3) and (2.4) (see (3.2) and (3.5) in [9]);

(2.2) X is represented as

$$X = 2i \sum F^k(w, c_l) z_l \partial/\partial z_k + \sum c_l^\alpha z_l \partial/\partial w_\alpha + \sum b_{\beta\gamma}^\alpha w_\beta w_\gamma \partial/\partial w_\alpha .$$

(2.3) $$\sum_{\alpha} b^{\alpha}_{\beta\gamma} F^k_{\alpha\delta} = i \sum_{\alpha,l} (F^l_{\beta\delta} \bar{c}^{\alpha}_l F^k_{\gamma\alpha} + F^l_{\gamma\delta} \bar{c}^{\alpha}_l F^k_{\beta\alpha})$$

for $1 \leq k \leq n$, $1 \leq \beta, \gamma, \delta \leq m$, where $F^k_{\alpha\beta} = F^k(f_\alpha, f_\beta)$.

(2.4) For each $d \in W$, the matrix $A(d) = (A(d)_{kl})$ belongs to $\mathfrak{g}(V)$, where $A(d)_{kl} = \operatorname{Im} F^k(c_l, d)$.

PROPOSITION 2.3. *If a vector field $X \in \mathfrak{g}_{1/2}$ satisfies the condition $\rho([\mathfrak{g}_{-1/2}, X]) = (0)$, then $X = 0$.*

Proof. By (2.2) there exist $c_l \in W$ $(1 \leq l \leq n)$ and $b^{\alpha}_{\beta\gamma} \in \boldsymbol{C}$ $(1 \leq \alpha, \beta, \gamma \leq m)$ such that X is represented as $X = 2i \sum F^k(w, c_l) z_l \partial/\partial z_k + \sum c^{\alpha}_l z_l \partial/\partial w_\alpha + \sum b^{\alpha}_{\beta\gamma} w_\beta w_\gamma \partial/\partial w_\alpha$. For each $d \in W$, we can verify that the matrix $\rho([\varphi_{-1/2}(d), X])$ coincides with $(4 \operatorname{Im} F^k(c_l, d))$. From our assumption it follows that $F^k(c_l, d) = 0$ for every $d \in W$ $(1 \leq k, l \leq n)$. Therefore $c_l = 0$ $(1 \leq l \leq n)$ and X is written as $X = \sum p^{\alpha}_{0,2} \partial/\partial w_\alpha$. By the same consideration as in the proof of Theorem 2.1 we have $X = 0$. q.e.d.

Now we suppose that W is the direct sum of subspaces W_i $(i = 1, 2)$ satisfying the condition $F(W_1, W_2) = (0)$. Let F_i denote the restriction of the V-hermitian form F to $W_i \times W_i$. Then F_i is a V-hermitian form on W_i. We denote by $\mathfrak{g}^{(i)}_h = \mathfrak{g}^{(i)}_{-1} + \mathfrak{g}^{(i)}_{-1/2} + \mathfrak{g}^{(i)}_0 + \mathfrak{g}^{(i)}_{1/2} + \mathfrak{g}^{(i)}_1$ the Lie algebra of all infinitesimal automorphisms of the Siegel domain $D(V, F_i)$ in $R^C \times W_i$. We can assume that $\{f_1, \cdots, f_{m_1}\}$ (resp. $\{f_{m_1+1}, \cdots, f_m\}$) is a base of W_1 (resp. W_2), where $m_1 = \dim W_1$.

We define a linear map Φ of the Lie algebra of all polynomial vector fields on $R^C \times W$ into that of all polynomial vector fields on $R^C \times W_1$ by

(2.5) $$\Phi\Big(\sum_{1 \leq k \leq n} p^k_{\lambda,\mu} \partial/\partial z_k\Big) = \sum_{1 \leq k \leq n} (p^k_{\lambda,\mu} \circ \iota) \partial/\partial z_k\,,$$
$$\Phi\Big(\sum_{1 \leq \alpha \leq m} p^{\alpha}_{\lambda,\mu} \partial/\partial w_\alpha\Big) = \sum_{1 \leq \alpha \leq m_1} (p^{\alpha}_{\lambda,\mu} \circ \iota) \partial/\partial w_\alpha\,,$$

where ι is the injection $(z, w_1) \in R^C \times W_1 \mapsto (z, w_1 + 0) \in R^C \times W$.

For

$$X = 2i \sum F^k(w, c_l) z_l \partial/\partial z_k + \sum c^{\alpha}_l z_l \partial/\partial w_\alpha + \sum b^{\alpha}_{\beta\gamma} w_\beta w_\gamma \partial/\partial w_\alpha \in \mathfrak{g}_{1/2}$$

(cf. (2.2)), we define two vector fields $X^{(1)}$ and $X^{(2)}$ by

$$X^{(1)} = 2i \sum F^k_1(w_1, c_{l,1}) z_l \partial/\partial z_k + \sum_{1 \leq \alpha \leq m_1} c^{\alpha}_l z_l \partial/\partial w_\alpha$$

$$(2.6)\quad \begin{aligned} &+ \sum_{1\le\alpha,\beta,\gamma\le m_1} b^\alpha_{\beta\gamma} w_\beta w_\gamma \partial/\partial w_\alpha\,, \\ X^{(2)} &= 2i \sum F^k_2(w_2, c_{l,2}) z_l \partial/\partial z_k + \sum_{m_1<\alpha\le m} c^\alpha_l z_l \partial/\partial w_\alpha \\ &+ \sum_{m_1<\alpha,\beta,\gamma\le m} b^\alpha_{\beta\gamma} w_\beta w_\gamma \partial/\partial w_\alpha\,, \end{aligned}$$

where $w = w_1 + w_2$, $c_l = c_{l,1} + c_{l,2} \in W = W_1 + W_2$. Then we get

Lemma 2.4. *For each $X \in \mathfrak{g}_{1/2}, X^{(i)}$ belongs to $\mathfrak{g}^{(i)}_{1/2}$ $(i = 1, 2)$ and $\Phi(X) = X^{(1)}$.*

Proof. We will show that the polynomial vector field $X^{(1)}$ (resp. $X^{(2)}$) on $R^C \times W_1$ (resp. $R^C \times W_2$) satisfies the conditions (2.2), (2.3) and (2.4). In fact, by (2.6) $X^{(1)}$ (resp. $X^{(2)}$) satisfies the condition (2.2). By using the equalities $F(W_1, W_2) = (0), F^k_1(f_\alpha, f_\beta) = F^k_{\alpha\beta}$ $(1 \le \alpha, \beta \le m_1)$, $F^k_2(f_\alpha, f_\beta) = F^k_{\alpha\beta}$ $(m_1 < \alpha, \beta \le m)$ and the fact $X \in \mathfrak{g}_{1/2}$, we have

$$\begin{aligned} \sum_{1\le\alpha\le m_1} b^\alpha_{\beta\gamma} F^k_{\alpha\delta} &= \sum_{1\le\alpha\le m} b^\alpha_{\beta\gamma} F^k_{\alpha\delta} = i \sum_{\substack{1\le l\le n\\ 1\le\alpha\le m}} (F^l_{\beta\delta} \bar{c}^\alpha_l F^k_{\gamma\alpha} + F^l_{\gamma\delta} \bar{c}^\alpha_l F^k_{\beta\alpha}) \\ &= i \sum_{\substack{1\le l\le n\\ 1\le\alpha\le m_1}} (F^l_{\beta\delta} \bar{c}^\alpha_l F^k_{\gamma\alpha} + F^l_{\gamma\delta} \bar{c}^\alpha_l F^k_{\beta\alpha}) \\ &\qquad (1 \le k \le n,\ 1 \le \beta, \gamma, \delta \le m_1)\,, \end{aligned}$$

which implies that $X^{(1)}$ satisfies the condition (2.3). For each $d_1 \in W_1$ the matrix $(\operatorname{Im} F^k_1(c_{l,1}, d_1))$ belongs to $\mathfrak{g}(V)$, since the matrix $(\operatorname{Im} F^k(c_l, d_1))$ belongs to $\mathfrak{g}(V)$ and $F^k(c_l, d_1) = F^k_1(c_{l,1}, d_1)$. Thus we showed that $X^{(1)}$ satisfies the condition (2.4). Therefore $X^{(1)}$ is contained in $\mathfrak{g}^{(1)}_{1/2}$. Analogously we can see that $X^{(2)}$ belongs to $\mathfrak{g}^{(2)}_{1/2}$. From (2.5), (2.6) and the condition $F(W_1, W_2) = (0)$ it follows immediately that $\Phi(X) = X^{(1)}$. q.e.d.

Lemma 2.5. *For each $X \in \mathfrak{g}_0, \Phi(X)$ belongs to $\mathfrak{g}^{(1)}_0$.*

Proof. We put $\sigma(X) = \begin{pmatrix} \sigma_1(X) & \sigma_3(X) \\ \sigma_2(X) & \sigma_4(X) \end{pmatrix}$, where $\sigma_1(X)$ is the submatrix of degree m_1. Then it can be easily seen that $\Phi(X)$ is represented by

$$\Phi(X) = \sum_{1\le k,l\le n} a_{kl} z_l \partial/\partial z_k + \sum_{1\le\alpha,\beta\le m_1} b_{\alpha\beta} w_\beta \partial/\partial w_\alpha\,,$$

where the matrices (a_{kl}) and $(b_{\alpha\beta})$ coincide with $\rho(X)$ and $\sigma_1(X)$, respectively. From the condition $F(W_1, W_2) = (0)$ and (1.4) it follows that for each $u_1 \in W_1$,

$$\rho(X) F_1(u_1, u_1) = \rho(X) F(u_1, u_1)$$

$$= F(\sigma(X)u_1, u_1) + F(u_1, \sigma(X)u_1)$$
$$= F(\sigma_1(X)u_1 + \sigma_2(X)u_1, u_1) + F(u_1, \sigma_1(X)u_1 + \sigma_2(X)u_1)$$
$$= F_1(\sigma_1(X)u_1, u_1) + F_1(u_1, \sigma_1(X)u_1) \ .$$

So, by (1.4) $\Phi(X)$ belongs to $\mathfrak{g}_0^{(1)}$. q.e.d.

We now denote by Φ_λ the map Φ restricted to the subspace $\mathfrak{g}_\lambda$ of $\mathfrak{g}_h$ ($\lambda = \pm 1, \pm\frac{1}{2}, 0$). Then we have

PROPOSITION 2.6. *If* $\mathfrak{g}_{1/2}^{(2)} = (0)$, *then the map* Φ *induces a grade-preserving linear map of* $\mathfrak{g}_h$ *into* $\mathfrak{g}_h^{(1)}$ *satisfying the following conditions:*

(1) *The subspace* $\mathfrak{g}_{-1}$ *of* $\mathfrak{g}_h$ *coincides with* $\mathfrak{g}_{-1}^{(1)}$ *and* Φ_{-1} *is an identity. Furthermore* $\Phi_{-1/2}$ *is a surjection of* $\mathfrak{g}_{-1/2}$ *onto* $\mathfrak{g}_{-1/2}^{(1)}$.

(2) *The map* $\Phi_{1/2}$ *is an injection of* $\mathfrak{g}_{1/2}$ *into* $\mathfrak{g}_{1/2}^{(1)}$.

(3) *The subspace* $\mathfrak{g}_1$ *of* $\mathfrak{g}_h$ *is contained in* $\mathfrak{g}_1^{(1)}$ *and* Φ_1 *is an identity.*

(4) *The maps* Φ_λ *satisfy the condition;* $\Phi_0([X, Y]) = [\Phi_{-\lambda}(X), \Phi_\lambda(Y)]$ *for* $X \in \mathfrak{g}_{-\lambda}, Y \in \mathfrak{g}_\lambda$ ($\lambda = 1, \frac{1}{2}$).

Proof. By (1.2) it is obvious that $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^{(1)}$ and $\Phi_{-1}(\partial/\partial z_k) = \partial/\partial z_k$. Now we show $\Phi(\mathfrak{g}_{-1/2}) = \mathfrak{g}_{-1/2}^{(1)}$. In fact, from (1.3) and the condition $F(W_1, W_2) = (0)$ it follows that $\Phi(\varphi_{-1/2}(c)) = \varphi_{-1/2}(c_1)$ for $c = c_1 + c_2 \in W = W_1 + W_2$. Thus we have $\Phi(\mathfrak{g}_{-1/2}) = \mathfrak{g}_{-1/2}^{(1)}$ and the assertion (1) was proved.

By Lemma 2.4 we have $\Phi(\mathfrak{g}_{1/2}) \subset \mathfrak{g}_{1/2}^{(1)}$. For $X \in \mathfrak{g}_{1/2}$ we suppose that $\Phi_{1/2}(X) = 0$. Then from the assumption $\mathfrak{g}_{1/2}^{(2)} = (0)$ and Lemma 2.4 it follows that $X^{(1)} = X^{(2)} = 0$ and X is represented as $X = \sum p_{0,2}^\alpha \partial/\partial w_\alpha$. Therefore, (as we stated before,) $X = 0$. Thus the assertion (2) was proved.

Now we show that $\Phi_1(X) = X$ for each $X \in \mathfrak{g}_1$. In fact, let $X = \sum A_{jl}^k z_j z_l \partial/\partial z_k + \sum B_{l\beta}^\alpha z_l w_\beta \partial/\partial w_\alpha \in \mathfrak{g}_1$ ($A_{jl}^k = A_{lj}^k, B_{l\beta}^\alpha \in \boldsymbol{C}$, cf. (1.10)). Then from the condition $[\mathfrak{g}_{-1/2}, X] \subset \mathfrak{g}_{1/2}$ it follows that for each $c \in W$,

$$\begin{aligned}[\varphi_{-1/2}(c), X] = 2i \textstyle\sum (2F^j(w, c)A_{jl}^k - B_{l\beta}^\alpha F^k(f_\alpha, c)w_\beta)z_l\partial/\partial z_k \\ + \textstyle\sum c^\beta B_{l\beta}^\alpha z_l \partial/\partial w_\alpha + 2i \textstyle\sum B_{k\beta}^\alpha F^k(w, c)w_\beta \partial/\partial w_\alpha\end{aligned} \tag{2.7}$$

belongs to $\mathfrak{g}_{1/2}$. On the other hand, by (2.2) there exist $c_l \in W$ ($1 \leq l \leq n$) and $b_{\beta\gamma}^\alpha \in \boldsymbol{C}$ ($1 \leq \alpha, \beta, \gamma \leq m$) such that

$$[\varphi_{-1/2}(c), X] = 2i \textstyle\sum F^k(w, c_l)z_l\partial/\partial z_k + \sum c_l^\alpha z_l \partial/\partial w_\alpha + \sum b_{\beta\gamma}^\alpha w_\beta w_\gamma \partial/\partial w_\alpha \ .$$

By the assumption $\mathfrak{g}_{1/2}^{(2)} = (0)$ and Lemma 2.4 we have $[\varphi_{-1/2}(c), X]^{(2)} = 0$. Therefore by (2.6) c_l is contained in W_1 (i.e., $c_l^\alpha = 0$ if $m_1 < \alpha \leq m$).

By (2.7) we have

$$B^{\alpha}_{l\beta} = 0 \ (1 \leq l \leq n,\ m_1 < \alpha \leq m,\ 1 \leq \beta \leq m)$$

and

$$F^k(w_1, c_l) = 2 \sum_{1\leq j\leq n} F^j(w, c)A^k_{jl} - \sum_{\substack{1\leq\alpha\leq m_1 \\ 1\leq\beta\leq m}} B^{\alpha}_{l\beta}F^k(f_\alpha, c)w_\beta \ .$$

By the condition $F(W_1, W_2) = (0)$ we get

$$2 \sum_{1\leq j\leq n} F^j(w_2, c_2)A^k_{jl} - \sum_{\substack{1\leq\alpha\leq m_1 \\ m_1<\beta\leq m}} B^{\alpha}_{l\beta}F^k(f_\alpha, c_1)w_\beta = 0 \ .$$

As $c = c_1 + c_2$ is an arbitrary element in $W = W_1 + W_2$, so

$$\sum_{\substack{1\leq\alpha\leq m_1 \\ m_1<\beta\leq m}} B^{\alpha}_{l\beta}F^k(f_\alpha, c_1)w_\beta = 0 \ .$$

By putting $c_1 = \sum_{1\leq\alpha\leq m_1} B^{\alpha}_{l\beta}f_\alpha$ we have $F^k\left(\sum_{1\leq\alpha\leq m_1} B^{\alpha}_{l\beta}f_\alpha, \sum_{1\leq\alpha\leq m_1} B^{\alpha}_{l\beta}f_\alpha\right) = 0.$ Therefore

$$B^{\alpha}_{l\beta} = 0 \qquad (1 \leq l \leq n,\ 1 \leq \alpha \leq m_1 < \beta \leq m) \ ,$$

and X is written as

$$X = \sum_{1\leq j,k,l\leq n} A^k_{jl}z_jz_l\partial/\partial z_k + \sum_{\substack{1\leq l\leq n \\ 1\leq\alpha,\beta\leq m_1}} B^{\alpha}_{l\beta}z_lw_\beta\partial/\partial w_\alpha \ . \tag{2.8}$$

By (2.5) we conclude that $\Phi_1(X) = X$.

We want to show $\mathfrak{g}_1 \subset \mathfrak{g}_1^{(1)}$. It is enough to show that each element $X \in \mathfrak{g}_1$ considered as a polynomial vector field on $R^C \times W_1$ satisfies the conditions in (1.10).

For each $c_1 \in W_1$, by (2.7) and (2.8) we have

$$\Phi_{1/2}([\varphi_{-1/2}(c_1), X]) = [\varphi_{-1/2}(c_1), X] \ .$$

From the facts $[\varphi_{-1/2}(c_1), X] \in \mathfrak{g}_{1/2}$ and $\Phi_{1/2}(\mathfrak{g}_{1/2}) \subset \mathfrak{g}_{1/2}^{(1)}$ it follows that $[\varphi_{-1/2}(c_1), X]$ belongs to $\mathfrak{g}_{1/2}^{(1)}$. We put $Y_k = [\partial/\partial z_k, X]$ $(1 \leq k \leq n)$. Then by (2.8) $\Phi_0(Y_k) = Y_k$. From the fact $[\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_0$ and Lemma 2.5 it follows that Y_k is contained in $\mathfrak{g}_0^{(1)}$. By (2.8) we can see that

$$\sigma(Y_k) = \begin{pmatrix} \sigma_1(Y_k) & 0 \\ 0 & 0 \end{pmatrix} .$$

Thus, $\operatorname{Im} \operatorname{Tr} \sigma_1(Y_k) = \operatorname{Im} \operatorname{Tr} \sigma(Y_k) = 0$. Therefore by (1.10) we conclude that X belongs to $\mathfrak{g}_1^{(1)}$. The assertion (3) was proved.

By (1) and (3) we have $[X, Y] \in \mathfrak{g}_0^{(1)}$ for $X \in \mathfrak{g}_{-1}, Y \in \mathfrak{g}_1$. Therefore we get $\Phi_0([X, Y]) = [\Phi_{-1}(X), \Phi_1(Y)]$. Let

$$X = 2i \sum_{1 \leq k, l \leq n} F^k(w_1, c_l) z_l \partial/\partial z_k + \sum_{\substack{1 \leq l \leq n \\ 1 \leq \alpha \leq m_1}} c_l^\alpha z_l \partial/\partial w_\alpha$$
$$+ \sum_{1 \leq \alpha, \beta, \gamma \leq m} b_{\beta\gamma}^\alpha w_\beta w_\gamma \partial/\partial w_\alpha \in \mathfrak{g}_{1/2} \qquad (c_l \in W_1) .$$

Then for each $d = d_1 + d_2 \in W = W_1 + W_2$ we have

$$[\Phi_{-1/2}(\varphi_{-1/2}(d)), \Phi_{1/2}(X)] = [\varphi_{-1/2}(d_1), \Phi_{1/2}(X)] .$$

We can verify that $\rho([\varphi_{-1/2}(d_1), \Phi_{1/2}(X)] = (4 \operatorname{Im} F^k(c_l, d_1))$ and the (α, β)-component of the matrix $\sigma_1([\varphi_{-1/2}(d_1), \Phi_{1/2}(X)])$ is

$$2 \sum_{\substack{1 \leq k \leq n \\ 1 \leq \gamma \leq m_1}} (i F_{\beta\gamma}^k \bar{d}^\gamma c_k^\alpha + b_{\beta\gamma}^\alpha d^\gamma) \qquad (1 \leq \alpha, \beta \leq m_1) .$$

On the other hand, by the conditions $c_l \in W_1$ and $F(W_1, W_2) = (0)$ we have

$$[\varphi_{-1/2}(d), X] = 4 \sum_{1 \leq k, l \leq n} \operatorname{Im} F^k(c_l, d_1) z_l \partial/\partial z_k$$
$$+ 2 \sum_{1 \leq \alpha, \beta, \gamma \leq m} \Big(i \sum_{1 \leq k \leq n} F_{\beta\gamma}^k \bar{d}^\gamma c_k^\alpha + b_{\beta\gamma}^\alpha d^\gamma\Big) w_\beta \partial/\partial w_\alpha .$$

We can see that $b_{\beta\gamma}^\alpha = 0$ if $1 \leq \alpha, \beta \leq m_1 < \gamma \leq m$. In fact, by (2.3) and the condition $F(W_1, W_2) = (0)$ it follows that $\sum_{1 \leq \alpha \leq m_1} b_{\beta\gamma}^\alpha F_{\alpha\delta}^k = 0$ $(1 \leq \delta \leq m_1)$, which implies $F^k\Big(\sum_{1 \leq \alpha \leq m_1} b_{\beta\gamma}^\alpha f_\alpha, f_\delta\Big) = 0$ $(1 \leq k \leq n,\ 1 \leq \delta \leq m_1)$. So, $\sum_{1 \leq \alpha \leq m_1} b_{\beta\gamma}^\alpha f_\alpha = 0$ and $b_{\beta\gamma}^\alpha = 0$ $(1 \leq \alpha, \beta \leq m_1 < \gamma \leq m)$. Therefore by (2.5) we have

$$\Phi_0([\varphi_{-1/2}(d), X]) = 4 \sum_{1 \leq k, l \leq n} \operatorname{Im} F^k(c_l, d_1) z_l \partial/\partial z_k$$
$$+ 2 \sum_{1 \leq \alpha, \beta, \gamma \leq m_1} \Big(i \sum_{1 \leq k \leq n} F_{\beta\gamma}^k \bar{d}^\gamma c_k^\alpha + b_{\beta\gamma}^\alpha d^\gamma\Big) w_\beta \partial/\partial w_\alpha ,$$

which implies that $\Phi_0([\varphi_{-1/2}(d), X]) = [\Phi_{-1/2}(\varphi_{-1/2}(d)), \Phi_{1/2}(X)]$. q.e.d.

By (2) in the above proposition we get

COROLLARY 2.7. *If* $\mathfrak{g}_{1/2}^{(i)} = (0)$ $(i = 1, 2)$, *then* $\mathfrak{g}_{1/2} = (0)$.

2.3. Let $D(V, F)$ be a Siegel domain of type II in $R^C \times W$. Let D' denote the *associated tube domain* with $D(V, F)$, i.e.,

(2.9) $$D' = D(V, F) \cap (R^C \times \{0\}) ,$$

which is isomorphic to the Siegel domain $D(V)$ of type I in R^C. It was proved by Kaup, Matsushima and Ochiai [6] that the subalgebra $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ of $\mathfrak{g}_h$ is the Lie subalgebra corresponding to the subgroup of all automorphisms of $D(V,F)$ leaving the domain D' invariant. Let $\mathfrak{g}'_h = \mathfrak{g}'_{-1} + \mathfrak{g}'_0 + \mathfrak{g}'_1$ be the Lie algebra of all infinitesimal automorphisms of D'. Then there exists a grade-preserving Lie algebra homomorphism ξ of $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ into $\mathfrak{g}'_h = \mathfrak{g}'_{-1} + \mathfrak{g}'_0 + \mathfrak{g}'_1$;

$$\xi : X \in \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \mapsto \xi(X) \in \mathfrak{g}'_h , \tag{2.10}$$

where $\xi(X)$ is the vector field which is the restriction of X to D'.

As a corollary to Proposition 2.6 we have the following proposition which will be used in order to determine the subspace $\mathfrak{g}_1$ of $\mathfrak{g}_h$.

PROPOSITION 2.8. *If $\mathfrak{g}_{1/2} = (0)$, then $\mathfrak{g}_1$ is a subspace of $\mathfrak{g}'_1$ and the map ξ restricted to $\mathfrak{g}_1$ is an identity.*

Proof. We put $W_1 = (0)$ and $W_2 = W$. Then the Siegel domains $D(V,F_1)$ and $D(V,F_2)$ coincide with D' and $D(V,F)$, respectively. Therefore $\mathfrak{g}_h^{(1)} = \mathfrak{g}'_h$ and $\mathfrak{g}_h^{(2)} = \mathfrak{g}_h$. It is easy to see that the map Φ restricted to $\mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ coincides with the map ξ (cf. (2.5)). Thus our assertions follow from (3) of Proposition 2.6. q.e.d.

§ 3. Automorphisms of Siegel domains over self-dual cones

In this section we calculate infinitesimal automorphisms of the homogeneous Siegel domains over self-dual cones (except circular cones) which were constructed by Pjateckii-Sapiro [10].

3.1. We will use the following notations and well-known results for irreducible self-dual cones.

1) The cone $H^+(p,\boldsymbol{R})$.

Let $R = H(p,\boldsymbol{R})$ be the real vector space of all real symmetric matrices of degree p. We denote by $H^+(p,\boldsymbol{R})$ the cone of all positive definite matrices in R. Then $\dim R = \frac{1}{2}p(p+1)$. Let E_{ij} denote a square matrix of degree p whose (i,j)-component is one and others are zero. We define a base $\{e_{ij}\}_{1\le i\le j\le p}$ of R by $e_{ii} = E_{ii}$ $(1 \le i \le p)$ and $e_{ij} = E_{ij} + E_{ji}$ $(1 \le i < j \le p)$. $(z_{ij})_{1\le i\le j\le p}$ denotes the coordinate system of R^C associated with the base $\{e_{ij}\}_{1\le i\le j\le p}$.

It is known in [17] that the Lie algebra $\mathfrak{g}(H^+(p,\boldsymbol{R}))$ consists of all linear endomorphisms $\tilde{A}$ of the form;

$$\tilde{A}: X \in R \mapsto AX + X^t A \in R\,, \tag{3.1}$$

where A is an element of $\mathfrak{gl}(p, \boldsymbol{R})$.

2) The cone $H^+(p, \boldsymbol{C})$.

Let $R = H(p, \boldsymbol{C})$ be the real vector space of all hermitian matrices of degree p. We denote by $H^+(p, \boldsymbol{C})$ the cone of all positive definite matrices in R. Then $\dim R = p^2$. We define a base $\{e_{ii}(1 \le i \le p), e_{ij,s}$ $(1 \le i < j \le p,\ s = 1, 2)\}$ of R by $e_{ii} = E_{ii}$ $(1 \le i \le p)$, $e_{ij,1} = E_{ij} + E_{ji}$ and $e_{ij,2} = i(E_{ij} - E_{ji})$ $(1 \le i < j \le p)$. $(z_{ii}$ $(1 \le i \le p), z_{ij,s}$ $(1 \le i < j \le p,\ s = 1, 2))$ denotes the coordinate system of R^C associated with the base $\{e_{ii}, e_{ij,s}\}$.

It is known in [17] that the Lie algebra $\mathfrak{g}(H^+(p, \boldsymbol{C}))$ consists of all linear endomorphisms $\tilde{A}$ of the form;

$$\tilde{A}: X \in R \mapsto AX + X^t \bar{A} \in R\,, \tag{3.2}$$

where A is an element of $\mathfrak{gl}(p, \boldsymbol{C})$.

3) The cone $H^+(p, \boldsymbol{K})$.

Let $R = H(p, \boldsymbol{K})$ be the real vector space of all hermitian matrices X of degree $2p$ satisfying the condition; $XJ = J\bar{X}$, where

$$J = \begin{pmatrix} j & & 0 \\ & \ddots & \\ 0 & & j \end{pmatrix} \quad \text{and} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We denote by $H^+(p, \boldsymbol{K})$ the cone of all positive definite matrices in R. Let $X = (X_{kl})$ be a hermitian matrix of degree $2p$, where X_{kl} is a 2×2-minor matrix of X $(1 \le k, l \le p)$. Then X belongs to R if and only if X_{kl} is represented as follows;

$$X_{kk} = \begin{pmatrix} x_{kk} & 0 \\ 0 & x_{kk} \end{pmatrix} (1 \le k \le p)\,, \qquad X_{kl} = \begin{pmatrix} x_{kl} & y_{kl} \\ -\bar{y}_{kl} & \bar{x}_{kl} \end{pmatrix} (1 \le k < l \le p)\,,$$

where $x_{kk} \in \boldsymbol{R}$ and $x_{kl}, y_{kl} \in \boldsymbol{C}$. Thus we have $\dim R = p(2p-1)$. We define a base $\{e_{ii}$ $(1 \le i \le p)$, $e_{ij,s}$ $(1 \le i < j \le p, 1 \le s \le 4)\}$ of R by $e_{ii} = E_{2i-1\,2i-1} + E_{2i\,2i}$ $(1 \le i \le p)$, $e_{ij,1} = E_{2i-1\,2j-1} + E_{2i\,2j}$, $e_{ij,2} = i(E_{2i-1\,2j-1} - E_{2i\,2j})$, $e_{ij,3} = E_{2i-1\,2j} - E_{2i\,2j-1}$, $e_{ij,4} = i(E_{2i-1\,2j} + E_{2i\,2j-1})$ $(1 \le i < j \le p)$, where E_{ij} is the square matrix of degree $2p$ whose (i, j)-component is one and others are zero. $(z_{ii}$ $(1 \le i \le p)$, $z_{ij,s}$ $(1 \le i < j \le p, 1 \le s \le 4))$ denotes the coordinate system of R^C associated with the base $\{e_{ii}, e_{ij,s}\}$.

It is known in [17] that the Lie algebra $\mathfrak{g}(H^+(p, \boldsymbol{K}))$ consists of all linear endomorphisms $\tilde{A}$ of the form;

(3.3) $$\tilde{A}: X \in R \mapsto AX + X^t\bar{A} \in R ,$$

where A is an element of $\mathfrak{gl}(2p, \boldsymbol{C})$ satisfying the condition $AJ = J\bar{A}$.

3.2. As an application of Theorem 2.1 we have

LEMMA 3.1. *For each of the homogeneous Siegel domains $D(V, F)$ given in the following* (1), (2) *and* (3), *the Lie algebra $\mathfrak{g}_h$ coincides with the subalgebra $\mathfrak{g}_a$.*

(1) $V = H^+(p, \boldsymbol{R})$, $W = M(p, q; \boldsymbol{C})$ $(p \geq 2)$,

$$F(u, v) = \tfrac{1}{2}(u^t\bar{v} + \bar{v}^t u) \qquad \textit{for } u, v \in W .$$

(2) $V = H^+(p, \boldsymbol{C})$, $W = M(p, q_1; \boldsymbol{C}) + M(p, q_2; \boldsymbol{C})$ (*direct sum*, $p \geq 2$),

$$F(u, v) = \tfrac{1}{2}(u^{(1)t}\bar{v}^{(1)} + \bar{v}^{(2)t}u^{(2)})$$
$$\textit{for } u = u^{(1)} + u^{(2)},\ v = v^{(1)} + v^{(2)} \in W .$$

(3) $V = H^+(p, \boldsymbol{K})$, $W = M(2p, q; \boldsymbol{C})$ $(p, q \geq 2)$,

$$F(u, v) = \tfrac{1}{2}(u^t\bar{v} + J\bar{v}^t u^t J) \qquad \textit{for } u, v \in W .$$

Proof. First we show that for each Siegel domain $D(V, F)$ in (1), (2) and (3), the subalgebra $\rho(\mathfrak{g}_0)$ of $\mathfrak{g}(V)$ coincides with $\mathfrak{g}(V)$.

Case (1): For each $\tilde{A} \in \mathfrak{g}(V)$ $(A \in \mathfrak{gl}(p, \boldsymbol{R}))$ we define a complex linear endomorphism B of W by

$$B: u \in W \mapsto Au \in W ,$$

where Au means a usual matrix multiplication of A and u. Then by (3.1) we have

$$\tilde{A}F(u, u) = F(Bu, u) + F(u, Bu)$$

for every $u \in W$. Hence by (1.4) $\tilde{A}$ is contained in $\rho(\mathfrak{g}_0)$. Therefore we have $\rho(\mathfrak{g}_0) = \mathfrak{g}(V)$.

Case (2): For each $\tilde{A} \in \mathfrak{g}(V)$ $(A \in \mathfrak{gl}(p, \boldsymbol{C}))$ we define a complex linear endomorphism B of W by

$$B: u = u^{(1)} + u^{(2)} \in W \mapsto Au^{(1)} + \bar{A}u^{(2)} \in W .$$

Then by using (3.2) we can verify

$$\tilde{A}F(u, u) = F(Bu, u) + F(u, Bu)$$

for every $u \in W$. It follows from (1.4) that $\tilde{A}$ belongs to $\rho(\mathfrak{g}_0)$. Thus, we have $\rho(\mathfrak{g}_0) = \mathfrak{g}(V)$.

Case (3): For each $\tilde{A} \in \mathfrak{g}(V)$ $(A \in \mathfrak{gl}(2p, \boldsymbol{C}), AJ = J\bar{A})$ we define a complex linear endomorphism B of W by

$$B : u \in W \mapsto Au \in W .$$

Then by (3.3) we have

$$\tilde{A}F(u, u) = F(Bu, u) + F(u, Bu)$$

for every $u \in W$. Hence by (1.4) $\tilde{A}$ belongs to $\rho(\mathfrak{g}_0)$ and $\rho(\mathfrak{g}_0) = \mathfrak{g}(V)$.

Each cone V in (1), (2) and (3) is an irreducible homogeneous self-dual cone. On the other hand, it was proved by Rothaus [11] that for an irreducible homogeneous self-dual cone V, the Lie algebra $\mathfrak{g}(V)$ is irreducible. Therefore the representation ρ is irreducible. Furthermore each domain $D(V, F)$ in (1), (2) and (3) is non-symmetric (cf. [10]). Thus, from Theorem 2.1 we conclude that $\mathfrak{g}_h = \mathfrak{g}_a$. q.e.d.

Now we consider degenerate Siegel domains over the cones $V = H^+(p, \boldsymbol{F})$ $(p \geq 2)$, where $\boldsymbol{F}$ is $\boldsymbol{R}$ or $\boldsymbol{C}$ or $\boldsymbol{K}$. Let F be a V-hermitian form on a complex vector space W of dimension m $(m > 0)$. Then we get

LEMMA 3.2. *If there exists a positive integer q $(q < p)$ such that the linear closure of the set $\{F(u, u); u \in W\}$ in R coincides with the proper subspace* $\begin{pmatrix} H(q, \boldsymbol{F}) & 0 \\ 0 & 0 \end{pmatrix}$ *of R, then* $\mathfrak{g}_{1/2} = (0)$.

Proof. Case $\boldsymbol{F} = \boldsymbol{R}$: We show that if a linear endomorphism $\tilde{A} \in \mathfrak{g}(V)$ belongs to $\rho(\mathfrak{g}_0)$, then A must be of the form;

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} , \tag{3.4}$$

where $a \in \mathfrak{gl}(q, \boldsymbol{R})$, $b \in M(q, p-q; \boldsymbol{R})$ and $c \in \mathfrak{gl}(p-q, \boldsymbol{R})$. In fact, let $\tilde{A} \in \rho(\mathfrak{g}_0)$, $A = \begin{pmatrix} a & b \\ d & c \end{pmatrix}$. Then by (1.4) there exists $B \in \mathfrak{gl}(W)$ such that $(\tilde{A}, B)$ satisfies the condition; $\tilde{A}F(u, u) = F(Bu, u) + F(u, Bu)$ for every $u \in W$. Therefore A must satisfy the following; for each $Y \in H(q, \boldsymbol{R})$,

$$A\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}{}^tA \text{ belongs to } \begin{pmatrix} H(q, \boldsymbol{R}) & 0 \\ 0 & 0 \end{pmatrix} ,$$

which implies $d = 0$.

Now we want to show $\mathfrak{g}_{1/2} = 0$. For each $X \in \mathfrak{g}_{1/2}$, by (2.2) and (2.4) there exist $c_{kl} \in W$ $(1 \leq k \leq l \leq p)$ such that

$$\tfrac{1}{4}\rho([\varphi_{-1/2}(d), X]) = (\operatorname{Im} F^{ij}(c_{kl}, d))$$

for every $d \in W$. From our assumption we can see that $F^{ij} = 0$ if $j > q$. Therefore, the linear endomorphism $\rho([\varphi_{-1/2}(d), X])$ maps the space $R = H(p, \boldsymbol{R})$ into the proper subspace $\begin{pmatrix} H(q, \boldsymbol{R}) & 0 \\ 0 & 0 \end{pmatrix}$ of R. On the other hand, from (3.4) there exists $A \in \mathfrak{gl}(p, \boldsymbol{R})$ of the form: $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ satisfying $\rho([\varphi_{-1/2}(d), X]) = \tilde{A}$. Thus, for each $Y_1 \in H(q, \boldsymbol{R})$, $Y_2 \in M(q, p - q; \boldsymbol{R})$ and $Y_3 \in H(p - q, \boldsymbol{R})$,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} Y_1 & Y_2 \\ {}^tY_2 & Y_3 \end{pmatrix} + \begin{pmatrix} Y_1 & Y_2 \\ {}^tY_2 & Y_3 \end{pmatrix}\begin{pmatrix} {}^ta & 0 \\ {}^tb & {}^tc \end{pmatrix} \text{ belongs to } \begin{pmatrix} H(q, \boldsymbol{R}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence we get $aY_2 + Y_2^t c + bY_3 = 0$ and $cY_3 + Y_3^t c = 0$, which implies $b = 0$. We can see that $a = 0$ and $c = 0$ by taking Y_2 and Y_3 suitably. So, $\tilde{A} = 0$ and $\rho([\varphi_{-1/2}(d), X]) = 0$. By Proposition 2.3 we conclude that $\mathfrak{g}_{1/2} = (0)$.

Case $F = \boldsymbol{C}$: We proceed analogously as in the above case. Let $\tilde{A} \in \mathfrak{g}(V)$ belong to $\rho(\mathfrak{g}_0)$. Then by (1.4) it can be easily verified that A must be of the form;

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \tag{3.5}$$

where $a \in \mathfrak{gl}(q, \boldsymbol{C})$, $b \in M(q, p - q; \boldsymbol{C})$ and $c \in \mathfrak{gl}(p - q, \boldsymbol{C})$.

Now we show $\mathfrak{g}_{1/2} = (0)$. Let $X \in \mathfrak{g}_{1/2}$. Then by (2.2) and (2.4) there exist c_{kk} $(1 \leq k \leq p)$, $c_{kl,t}$ $(1 \leq k < l \leq p,\ t = 1, 2) \in W$ such that

$$\tfrac{1}{4}\rho([\varphi_{-1/2}(d), X]) = (\operatorname{Im} F^{ij,s}(c_{kl,t}, d))$$

for each $d \in W$, where we put $F^{ii,s} = F^{ii}, c_{ii,s} = c_{ii}$ and $F(u, v) = \sum F^{ij,s}(u, v)e_{ij,s}$. From our assumption it follows that $F^{ij,s} = 0$ if $j > q$. Therefore the linear endomorphism $\rho([\varphi_{-1/2}(d), X])$ maps the space $R = H(p, \boldsymbol{C})$ into the proper subspace $\begin{pmatrix} H(q, \boldsymbol{C}) & 0 \\ 0 & 0 \end{pmatrix}$ of R. On the other hand, there exists $A \in \mathfrak{gl}(p, \boldsymbol{C})$ of the form (3.5) such that $\rho([\varphi_{-1/2}(d), X]) = \tilde{A}$. Thus for each $Y_1 \in H(q, \boldsymbol{C})$, $Y_2 \in M(q, p - q; \boldsymbol{C})$ and $Y_3 \in H(p - q, \boldsymbol{C})$,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} Y_1 & Y_2 \\ {}^t\bar{Y}_2 & Y_3 \end{pmatrix} + \begin{pmatrix} Y_1 & Y_2 \\ {}^t\bar{Y}_2 & Y_3 \end{pmatrix}\begin{pmatrix} {}^t\bar{a} & 0 \\ {}^t\bar{b} & {}^t\bar{c} \end{pmatrix} \text{ belongs to } \begin{pmatrix} H(q, \boldsymbol{C}) & 0 \\ 0 & 0 \end{pmatrix},$$

that is, $aY_2 + Y_2^t\bar{c} + bY_3 = 0$ and $cY_3 + Y_3^t\bar{c} = 0$. Taking Y_2 and Y_3 suitably we have $b = 0$, $a = i\theta E_q$ and $c = i\theta E_{p-q}$, where θ is a real number. By considering (3.2) we get $\tilde{A} = 0$. Therefore $\rho([\varphi_{-1/2}(d), X]) = 0$ for every $d \in W$. So, by Proposition 2.3, $\mathfrak{g}_{1/2} = (0)$.

Case $\boldsymbol{F} = \boldsymbol{K}$: By the same considerations as in the above, we can see that if $\tilde{A} \in \mathfrak{g}(V)$ belongs to $\rho(\mathfrak{g}_0)$, then A must be of the form;

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \tag{3.6}$$

where $a \in \mathfrak{gl}(2q, \boldsymbol{C})$, $b \in M(2q, 2(p-q); \boldsymbol{C})$ and $c \in \mathfrak{gl}(2(p-q), \boldsymbol{C})$ satisfying $aJ_1 = J_1\bar{a}$, $cJ_2 = J_2\bar{c}$, $bJ_2 = J_1\bar{b}$, $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$ (cf. (3.3)).

Now we want to show $\mathfrak{g}_{1/2} = (0)$. For each $X \in \mathfrak{g}_{1/2}$, by (2.2) and (2.4) there exist c_{kk} $(1 \leq k \leq p)$, $c_{kl,t}$ $(1 \leq k < l \leq p,\ 1 \leq t \leq 4) \in W$ such that

$$\tfrac{1}{4}\rho([\varphi_{-1/2}(d), X]) = (\operatorname{Im} F^{ij,s}(c_{kl,t}, d))$$

for every $d \in W$, where we put $F^{ii,s} = F^{ii}$, $c_{ii,s} = c_{ii}$ and $F(u, v) = \sum F^{ij,s}(u, v)e_{ij,s}$. By our assumption, $F^{ij,s} = 0$ if $j > q$. Therefore the linear endomorphism $\rho([\varphi_{-1/2}(d), X])$ maps the space $R = H(p, \boldsymbol{K})$ into the proper subspace $\begin{pmatrix} H(q, \boldsymbol{K}) & 0 \\ 0 & 0 \end{pmatrix}$ of R. On the other hand, there exists $\tilde{A} \in \rho(\mathfrak{g}_0)$ of the form (3.6) such that $\rho([\varphi_{-1/2}(d), X]) = \tilde{A}$. Thus, for each $Y_1 \in H(q, \boldsymbol{K})$, $Y_2 \in M(2q, 2(p-q); \boldsymbol{C})$ and $Y_3 \in H(p-q, \boldsymbol{K})$ satisfying $Y_2J_2 = J_1\bar{Y}_2$,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} Y_1 & Y_2 \\ {}^t\bar{Y}_2 & Y_3 \end{pmatrix} + \begin{pmatrix} Y_1 & Y_2 \\ {}^t\bar{Y}_2 & Y_3 \end{pmatrix}\begin{pmatrix} {}^t\bar{a} & 0 \\ {}^t\bar{b} & {}^t\bar{c} \end{pmatrix} \text{ belongs to } \begin{pmatrix} H(q, \boldsymbol{K}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence we have

$$aY_2 + Y_2^t\bar{c} + bY_3 = 0 \quad \text{and} \quad cY_3 + Y_3^t\bar{c} = 0\,.$$

Taking Y_2 and Y_3 suitably we get $a = 0$, $b = 0$ and $c = 0$. So, $\tilde{A} = 0$ and $\rho([\mathfrak{g}_{-1/2}, X]) = (0)$. From Proposition 2.3 it follows that $\mathfrak{g}_{1/2} = (0)$. q.e.d.

3.3. In this paragraph we calculate infinitesimal automorphisms of all homogeneous Siegel domains of type II over the cone $V = H^+(p, \boldsymbol{R})$ $(p \geq 2)$.

Let s be a positive integer and $r(t)$ be a non-decreasing integer valued function defined on an interval $[1, s]$ such that $1 \leq r(1)$, $r(s) \leq p$. Let W be the complex vector space of all complex $p \times s$-matrices $u = (u_{ij})$ such that $u_{ij} = 0$ if $i > r(j)$. We put $F(u, v) = \frac{1}{2}(u{}^t\bar{v} + \bar{v}{}^t u)$ for $u, v \in W$. Then it is known in [10] that F is a V-hermitian form on W and the Siegel domain $D(V, F)$ is homogeneous. We note that every homogeneous Siegel domain of type II over the cone $H^+(p, \boldsymbol{R})$ $(p \geq 2)$ is isomorphic to the one given here (cf. [10], [13]). It was proved by Kaneyuki and Sudo [4] that the Siegel domain $D(V, F)$ is non-degenerate if and only if $r(s) = p$.

THEOREM 3.3.[1] *For a Siegel domain $D(V, F)$ mentioned above, the subspaces $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of $\mathfrak{g}_h$ are given as follows;*

$\mathfrak{g}_{1/2} = (0)$,

$\mathfrak{g}_1$ is isomorphic to the vector space $H(p - r(s), \boldsymbol{R})$.

Proof. First we suppose that $D(V, F)$ is degenerate. Then $r(s) < p$ and the linear closure of the set $\{F(u, u); u \in W\}$ in R coincides with the proper subspace $\begin{pmatrix} H(q, \boldsymbol{R}) & 0 \\ 0 & 0 \end{pmatrix}$ of R, where $q = r(s)$ (cf. [4]). Hence, by Lemma 3.2 we have $\mathfrak{g}_{1/2} = (0)$.

Now we determine $\mathfrak{g}_1$.[2] We consider the associated tube domain D' with $D(V, F)$ (cf. (2.9)). It is known in [10] that D' is the classical domain of type (III) and the Lie algebra $\mathfrak{g}'_h = \mathfrak{g}'_{-1} + \mathfrak{g}'_0 + \mathfrak{g}'_1$ of all infinitesimal automorphisms of D' can be identified with $\mathfrak{sp}(p, \boldsymbol{R})$ as follows (cf. [10], Chap. 2, § 7);

$$\mathfrak{g}'_h = \mathfrak{sp}(p, \boldsymbol{R}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}; A \in \mathfrak{gl}(p, \boldsymbol{R}),\ B, C \in H(p, \boldsymbol{R}) \right\},$$

$$\mathfrak{g}'_{-1} = \begin{pmatrix} 0 & H(p, \boldsymbol{R}) \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{g}'_1 = \begin{pmatrix} 0 & 0 \\ H(p, \boldsymbol{R}) & 0 \end{pmatrix},$$

$$\mathfrak{g}'_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix}; A \in \mathfrak{gl}(p, \boldsymbol{R}) \right\}.$$

For each $g = \begin{pmatrix} E_p & 0 \\ C & E_p \end{pmatrix} \in \exp \mathfrak{g}'_1$, g acts on D' by

[1] If $s=1$, then this theorem was proved by Tanaka [14] and Murakami [8]. Nakajima [18] calculated the dimensions of $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of this theorem by using different methods.

[2] This idea of determining $\mathfrak{g}_1$ is due to Murakami [8].

$$g : z \in D' \mapsto z(Cz + E_p)^{-1} \in D' .$$

The image $\xi(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ is given by

$$\xi(\mathfrak{g}_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \in \mathfrak{g}'_0 ;\ \tilde{A} \in \rho(\mathfrak{g}_0) \right\} \qquad \text{(cf. (2.10))} .$$

We want to show that $\xi(\mathfrak{g}_1)$ coincides with the following subspace of $\mathfrak{g}'_1$;

$$\text{(3.7)} \qquad \left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \in \mathfrak{g}'_1 ;\ Y = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix},\ y \in H(p-q, \boldsymbol{R}) \right\} .$$

Let $X \in \mathfrak{g}_1$. Then, since $\xi(X) \in \mathfrak{g}'_1$, there exists $Y \in H(p, \boldsymbol{R})$ such that $\xi(X) = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$. By the conditions $\xi(\mathfrak{g}_{-1}) = \mathfrak{g}'_{-1}$ and $[\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_0$ we have $[\mathfrak{g}'_{-1}, \xi(X)] \subset \xi(\mathfrak{g}_0)$. Therefore, for each $B \in H(p, \boldsymbol{R})$, $\widetilde{BY}$ belongs to $\rho(\mathfrak{g}_0)$. So, BY must be of the form (3.4) for each $B \in H(p, \boldsymbol{R})$, which implies that Y must be of the form (3.7). Conversely let Y be an element in $H(p, \boldsymbol{R})$ of the form (3.7). We define the map g_t $(t \in \boldsymbol{R})$ of $D(V, F)$ into $R^C \times W$ by

$$g_t : (z, u) \in D(V, F) \mapsto (z(tYz + E_p)^{-1}, u) \in R^C \times W .$$

Then we can easily verify (cf. [8]) that

$$\operatorname{Im}(z(tYz + E_p)^{-1}) = {}^t\overline{(tYz + E_p)}^{-1} \operatorname{Im} z\, (tYz + E_p)^{-1}$$

and

$${}^t\overline{(tYz + E_p)}^{-1} F(u, u)(tYz + E_p)^{-1} = F(u, u)$$

for each $u \in W$.
Thus, g_t is a one-parameter group of transformations of $D(V, F)$ and g_t induces a vector field $X \in \mathfrak{g}_1$ such that $\xi(X) = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$. By the fact $\mathfrak{g}_{1/2} = (0)$ and Proposition 2.8 we conclude that $\mathfrak{g}_1$ is isomorphic to the vector space $H(p-q, \boldsymbol{R})$.

Now we suppose that $D(V, F)$ is non-degenerate. If $r(1) = p$, then W coincides with $M(p, s; \boldsymbol{C})$ and the Siegel domain $D(V, F)$ is the one given in (1) of Lemma 3.1. So, we can assume that $s \geq 2$ and $r(1) < p$. We put $t_0 = \min\{t \in [1, s];\ t$ is an integer such that $r(t) = p\}$ and define the complex subspaces W_i $(i = 1, 2)$ of W by

$$W_1 = \{u = (u_{ij}) \in W;\ u_{ij} = 0 \text{ if } j < t_0\}$$

and

$$W_2 = \{u = (u_{ij}) \in W;\ u_{ij} = 0 \text{ if } j \geq t_0\}\ .$$

Then it can be seen that

$$W = W_1 + W_2 \text{ (direct sum) and } F(W_1, W_2) = (0)\ .$$

We denote by F_i the restriction of F to the subspace W_i. Then the vector space W_1 is isomorphic to $M(p, s - t_0 + 1; \boldsymbol{C})$, and the Siegel domain $D(V, F_1)$ in $R^C \times W_1$ is isomorphic to the one given in (1) of Lemma 3.1. Thus $\mathfrak{g}_{1/2}^{(1)} = (0)$.
On the other hand, for the Siegel domain $D(V, F_2)$ in $R^C \times W_2$ we can see that the linear closure of the set $\{F_2(u, u); u \in W_2\}$ in R coincides with the proper subspace $\begin{pmatrix} H(q, \boldsymbol{R}) & 0 \\ 0 & 0 \end{pmatrix}$ of R, where $q = r(t_0 - 1)$. Hence by Lemma 3.2 we have $\mathfrak{g}_{1/2}^{(2)} = (0)$. From Corollary 2.7 it follows that $\mathfrak{g}_{1/2} = (0)$. Therefore by Proposition 2.2 we get $\mathfrak{g}_h = \mathfrak{g}_a$. q.e.d.

3.4. In this paragraph we consider the Siegel domains of type II over the cone $V = H^+(p, \boldsymbol{C})$ $(p \geq 2)$.

Let s_1 and s_2 be two positive integers. Let $r_i(t)$ be a non-decreasing integer valued function defined on an interval $[1, s_i]$ such that $0 \leq r_i(t)$ and $r_i(t) \leq p$ $(i = 1, 2)$. We denote by $W^{(i)}$ the complex vector space of all complex $p \times s_i$-matrices $u^{(i)} = (u_{kl}^{(i)})$ such that $u_{kl}^{(i)} = 0$ if $k > r_i(l)$. Let W be the direct sum of the vector spaces $W^{(1)}$ and $W^{(2)}$. We put $F(u, v) = \frac{1}{2}(u^{(1)}{}^t\bar{v}^{(1)} + \bar{v}^{(2)}{}^t u^{(2)})$ for $u = u^{(1)} + u^{(2)}, v = v^{(1)} + v^{(2)} \in W = W^{(1)} + W^{(2)}$. Then it is known in [10] that the map F is a V-hermitian form on W and the Siegel domain $D(V, F)$ is homogeneous. Furthermore it was proved in [4] that the Siegel domain $D(V, F)$ is non-degenerate if and only if $r_1(s_1) = p$ or $r_2(s_2) = p$.

THEOREM 3.4.[3)] (i) *If a Siegel domain $D(V, F)$ mentioned above is degenerate, then the subspaces $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of $\mathfrak{g}_h$ are given by*

$\mathfrak{g}_{1/2} = (0)$,

$\mathfrak{g}_1$ *is isomorphic to the vector space $H(p - q, \boldsymbol{C})$,*

where $q = \max(r_1(s_1), r_2(s_2))$.

(ii) *If $r_1(s_1) = r_2(s_2) = p$, then $\mathfrak{g}_h = \mathfrak{g}_a$.*

Proof. First we consider the case (i). The linear closure of the

3) Nakajima [18] calculated the dimensions of $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of this theorem by using different methods.

set $\{F(u,u)\,;\,u\in W\}$ in R coincides with the proper subspace $\begin{pmatrix} H(q,C) & 0 \\ 0 & 0 \end{pmatrix}$ of R (cf. [4]). Thus, by Lemma 3.2 it follows $\mathfrak{g}_{1/2}=(0)$.

Now we determine $\mathfrak{g}_1$. We consider the tube domain D' associated with $D(V,F)$ (cf. (2.9)). Then it is known in [10] that D' is the classical domain of type (I). The Lie algebra $\mathfrak{g}'_h=\mathfrak{g}'_{-1}+\mathfrak{g}'_0+\mathfrak{g}'_1$ of all infinitesimal automorphisms of D' can be identified with $\mathfrak{su}(p,p)$ as follows (cf. [10], Chap. 2, § 6);

$$\begin{aligned}\mathfrak{g}'_h &= \mathfrak{su}(p,p)\\ &= \left\{\begin{pmatrix} A & B \\ C & -{}^t\bar{A}\end{pmatrix};\, A\in\mathfrak{gl}(p,C),\ B,C\in H(p,C)\right\} \quad (\mathrm{mod}\ \{i\theta E_{2p}\,;\,\theta\in\boldsymbol{R}\})\ ,\end{aligned}$$

$$\mathfrak{g}'_{-1}=\begin{pmatrix} 0 & H(p,C) \\ 0 & 0\end{pmatrix},\qquad \mathfrak{g}'_1=\begin{pmatrix} 0 & 0 \\ H(p,C) & 0\end{pmatrix},$$

$$\mathfrak{g}'_0=\left\{\begin{pmatrix} A & 0 \\ 0 & -{}^t\bar{A}\end{pmatrix};\, A\in\mathfrak{gl}(p,C)\right\}\qquad (\mathrm{mod}\ \{i\theta E_{2p}\,;\,\theta\in\boldsymbol{R}\})\ .$$

Each $g=\begin{pmatrix} E_p & 0 \\ C & E_p\end{pmatrix}(\in\exp\mathfrak{g}'_1)$ acts on D' by

$$g: z\in D'\mapsto z(Cz+E_p)^{-1}\in D'\ .$$

The image $\xi(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ is the subalgebra of $\mathfrak{g}'_0$ given by

$$\xi(\mathfrak{g}_0)=\left\{\begin{pmatrix} A & 0 \\ 0 & -{}^t\bar{A}\end{pmatrix}\in\mathfrak{g}'_0\,;\ \tilde{A}\in\rho(\mathfrak{g}_0)\right\}\ .$$

We want to show that the subspace $\xi(\mathfrak{g}_1)$ of $\mathfrak{g}'_1$ coincides with the following subspace of $\mathfrak{g}'_1$;

$$\left\{\begin{pmatrix} 0 & 0 \\ Y & 0\end{pmatrix};\, Y=\begin{pmatrix} 0 & 0 \\ 0 & y\end{pmatrix},\ y\in H(p-q,C)\right\}\ . \tag{3.8}$$

In fact, let $X\in\mathfrak{g}_1$. Then $\xi(X)$ belongs to $\mathfrak{g}'_1$ and $\xi(X)$ is represented as

$$\xi(X)=\begin{pmatrix} 0 & 0 \\ Y & 0\end{pmatrix},\qquad Y\in H(p,C)\ .$$

From the condition $[\mathfrak{g}_{-1},X]\subset\mathfrak{g}_0$ and the fact $\xi(\mathfrak{g}_{-1})=\mathfrak{g}'_{-1}$, we have $[\mathfrak{g}'_{-1},\xi(X)]\subset\xi(\mathfrak{g}_0)$. Thus it can be seen that, for each $B\in H(p,C)$, BY must be of the form (3.5). It follows that Y must be of the form (3.8).

Conversely let Y be an element in $H(p,C)$ of the form (3.8). We define the map g_t $(t\in\boldsymbol{R})$ of $D(V,F)$ into $R^C\times W$ by

$$g_t : (z, u) \in D(V, F) \mapsto (z(tYz + E_p)^{-1}, u) \in R^C \times W .$$

Then we can easily verify that

$$\mathrm{Im}\,(z(tYz + E_p)^{-1}) = {}^t\overline{(tYz + E_p)}^{-1}\,\mathrm{Im}\, z\,(tYz + E_p)^{-1}$$

and

$${}^t\overline{(tYz + E_p)}^{-1} F(u, u)(tYz + E_p)^{-1} = F(u, u)$$

for each $u \in W$. Therefore the map g_t is a one-parameter group of transformations of $D(V, F)$ and the vector field X induced by g_t belongs to $\mathfrak{g}_1$. Furthermore we have $\xi(X) = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$. Considering Proposition 2.8 we can identify $\mathfrak{g}_1$ with the vector space $H(p - q, \boldsymbol{C})$.

Now we consider the case (ii). If $r_1(1) = r_2(1) = p$, then the Siegel domain $D(V, F)$ is the one given in (2) of Lemma 3.1. Thus we get $\mathfrak{g}_h = \mathfrak{g}_a$. We suppose that $r_1(1) = p$ and $r_2(1) < p$. We put $t_0 = \min\{t \in [1, s_2]$; t is an integer such that $r_2(t) = p\}$ and define the subspaces W_1 and W_2 of W by

$$\begin{aligned} W_1 &= \{u = u^{(1)} + u^{(2)} \in W ;\ u^{(2)}_{ij} = 0 \quad \text{if } j < t_0\} , \\ W_2 &= \{u = u^{(1)} + u^{(2)} \in W ;\ u^{(1)} = 0,\ u^{(2)}_{ij} = 0 \quad \text{if } j \geq t_0\} . \end{aligned}$$

Then we can see that

$$W \dot{=} W_1 + W_2 \text{ (direct sum)} \quad \text{and} \quad F(W_1, W_2) = (0) .$$

The Siegel domain $D(V, F_1)$ in $R^C \times W_1$ is isomorphic to the one given in (2) of Lemma 3.1. Thus we get $\mathfrak{g}^{(1)}_{1/2} = (0)$.

For the Siegel domain $D(V, F_2)$ in $R^C \times W_2$, it can be seen that the linear closure of the set $\{F_2(u, u) ; u \in W_2\}$ in R coincides with the proper subspace $\begin{pmatrix} H(q, \boldsymbol{C}) & 0 \\ 0 & 0 \end{pmatrix}$ of R, where $q = r_2(t_0 - 1)$ (cf. [4]). From Lemma 3.2 it follows that $\mathfrak{g}^{(2)}_{1/2} = (0)$. By Corollary 2.7 we have $\mathfrak{g}_{1/2} = (0)$. Applying Proposition 2.2 to the non-degenerate Siegel domain $D(V, F)$, we get $\mathfrak{g}_h = \mathfrak{g}_a$.

If $r_1(1) \neq p$ and $r_2(1) = p$, then the fact $\mathfrak{g}_h = \mathfrak{g}_a$ can be analogously obtained.

Now we suppose that $r_1(1) \neq p$ and $r_2(1) \neq p$. We put $t_i = \min\{t \in [1, s_i]$; t is an integer such that $r_i(t) = p\}$ $(i = 1, 2)$ and define the subspaces W_i $(i = 1, 2)$ of W by

$$W_1 = \{u = u^{(1)} + u^{(2)} \in W\,;\, u_{ij}^{(1)} = 0 \text{ if } j < t_1,\ u_{ij}^{(2)} = 0 \text{ if } j < t_2\}$$

and

$$W_2 = \{u = u^{(1)} + u^{(2)} \in W\,;\, u_{ij}^{(1)} = 0 \text{ if } j \geq t_1,\ u_{ij}^{(2)} = 0 \text{ if } j \geq t_2\}\ .$$

Then we have

$$W = W_1 + W_2 \text{ (direct sum)} \quad \text{and} \quad F(W_1, W_2) = (0)\ .$$

It is easy to see that the Siegel domain $D(V, F_1)$ in $R^C \times W_1$ is isomorphic to the one given in (2) of Lemma 3.1. Thus we have $\mathfrak{g}_{1/2}^{(1)} = (0)$. And for the Siegel domain $D(V, F_2)$ in $R^C \times W_2$, the linear closure of the set $\{F_2(u, u)\,;\, u \in W_2\}$ in R coincides with the proper subspace $\begin{pmatrix} H(q, C) & 0 \\ 0 & 0 \end{pmatrix}$ of R, where $q = \max(r_1(t_1 - 1),\ r_2(t_2 - 1))$ (cf. [4]). Hence by Lemma 3.2 we get $\mathfrak{g}_{1/2}^{(2)} = (0)$. From Corollary 2.7 it follows that $\mathfrak{g}_{1/2} = (0)$. Using Proposition 2.2 we conclude that $\mathfrak{g}_h = \mathfrak{g}_a$. q.e.d.

THEOREM 3.5.[4] *If $r_1(s_1) < p$ and $r_2(s_2) = p$, then the subspaces $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of $\mathfrak{g}_h$ are given as follows;*

$\mathfrak{g}_{1/2}$ is isomorphic to the real vector space $M(s_0, p - q\,;\, C)$,

$\mathfrak{g}_1$ is isomorphic to the vector space $H(p - q, C)$,

where $s_0 = s_2 - t_0 + 1$, $q = \max(r_1(s_1),\ r_2(t_0 - 1))$ and $t_0 = \min\{t \in [1, s_2]\,;$ t is an integer such that $r_2(t) = p\}$, and $r_2(t_0 - 1)$ means zero if $t_0 = 1$.

Proof. We define the subspaces W_1 and W_2 of W by

$$W_1 = \{u = u^{(1)} + u^{(2)} \in W\,;\, u^{(1)} = 0,\ u_{ij}^{(2)} = 0 \text{ if } j < t_0\}\ ,$$
$$W_2 = \{u = u^{(1)} + u^{(2)} \in W\,;\, u_{ij}^{(2)} = 0 \text{ if } j \geq t_0\}\ .$$

Then we can see that

$$W = W_1 + W_2 \text{ (direct sum)} \quad \text{and} \quad F(W_1, W_2) = (0)\ .$$

If $W_2 = (0)$, then $D(V, F)$ is the classical domain of type (I) (cf. [10], Chap. 2).[*] Therefore we consider the case $W_2 \neq (0)$.

The Siegel domain $D(V, F_2)$ in $R^C \times W_2$ is degenerate and the linear closure of the set $\{F_2(u, u)\,;\, u \in W_2\}$ in R coincides with the proper sub-

[4] Nakajima [18] calculated the dimensions of $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of this theorem by using different methods.

[*] By the following decomposition of the Lie algebra $\mathfrak{g}_h{}^{(1)}$, we can see that the theorem is valid for this case.

space $\begin{pmatrix} H(q,C) & 0 \\ 0 & 0 \end{pmatrix}$ of R (cf. [4]). Hence, by Lemma 3.2 we get $\mathfrak{g}_{1/2}^{(2)} = (0)$.

On the other hand, the Siegel domain $D(V,F_1)$ in $R^C \times W_1$ is the classical domain of type (I). The Lie algebra $\mathfrak{g}_h^{(1)}$ can be identified with $\mathfrak{su}(s_0 + p, p)$ as follows (cf. [10], Chap. 2, § 6);

$$\mathfrak{g}_h^{(1)} = \mathfrak{su}(s_0 + p, p)$$
$$= \left\{ \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} ;\ \begin{matrix} A_{33} = -{}^t\overline{A}_{11} \in \mathfrak{gl}(p,C),\ A_{22} \in \mathfrak{u}(s_0) \\ A_{12} = i\,{}^t\overline{A}_{23},\ A_{32} = -i\,{}^t\overline{A}_{21} \in M(p, s_0; C) \\ A_{13},\ A_{31} \in H(p,C) \end{matrix} \right\}$$
$$(\mathrm{mod}\ \{i\theta E_{2p+s_0};\ \theta \in \boldsymbol{R}\})\ .$$

$$\mathfrak{g}_{-1}^{(1)} = \begin{pmatrix} 0 & 0 & H(p,C) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\qquad \mathfrak{g}_{-1/2}^{(1)} = \left\{ \begin{pmatrix} 0 & C & 0 \\ 0 & 0 & i\,{}^t\overline{C} \\ 0 & 0 & 0 \end{pmatrix} ;\ C \in M(p, s_0; C) \right\}\ ,$$

$$\mathfrak{g}_{1}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ H(p,C) & 0 & 0 \end{pmatrix},\qquad \mathfrak{g}_{1/2}^{(1)} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -i\,{}^t\overline{D} & 0 \end{pmatrix} ;\ D \in M(s_0, p; C) \right\}\ ,$$

$$\mathfrak{g}_{0}^{(1)} = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -{}^t\overline{A} \end{pmatrix} ;\ A \in \mathfrak{gl}(p,C),\ B \in \mathfrak{u}(s_0) \right\}$$
$$(\mathrm{mod}\ \{i\theta E_{2p+s_0};\ \theta \in \boldsymbol{R}\})\ .$$

First we note that for

$$g = \begin{pmatrix} E_p & 0 & 0 \\ D & E_{s_0} & 0 \\ -\frac{1}{2}i\,{}^t\overline{D}D & -i\,{}^t\overline{D} & E_p \end{pmatrix} \in \exp \mathfrak{g}_{1/2}^{(1)}$$

and

$$h = \begin{pmatrix} E_p & 0 & 0 \\ 0 & E_{s_0} & 0 \\ Y & 0 & E_p \end{pmatrix} \in \exp \mathfrak{g}_{1}^{(1)}\ ,$$

g and h act on $D(V,F_1)$ as follows (cf. [10]);

$$g(z,u_1) = (z', u_1') \quad \text{and} \quad h(z,u_1) = (z(Yz + E_p)^{-1},\ {}^t(Yz + E_p)^{-1}u_1)\ ,$$

where

$$z' = z(-\tfrac{1}{2}i\,{}^t\overline{D}Dz - i\,{}^t\overline{D}{}^tu_1 + E_p)^{-1}$$

and

$$u_1' = {}^t(-\tfrac{1}{2}i\,{}^t\overline{D}Dz - i\,{}^t\overline{D}{}^tu_1 + E_p)^{-1}({}^tz{}^tD + u_1)$$

for each $(z, u_1) \in D(V, F_1)$.

Now we show that if $\tilde{A}$ belongs to $\rho(\mathfrak{g}_0)(A \in \mathfrak{gl}(p, \boldsymbol{C}))$, then A must be of the form (3.5). In fact, there exists $B \in \mathfrak{gl}(W)$ such that $(\tilde{A}, B)$ satisfies the condition: $\tilde{A}F(u, u) = F(Bu, u) + F(u, Bu)$ for every $u \in W$. Putting $u = u_2 \in W_2$ we have

$$\tilde{A}F(u_2, u_2) = F(Bu_2, u_2) + F(u_2, Bu_2) \ ,$$

which implies

$$AF_2(u_2, u_2) + F_2(u_2, u_2)^t\bar{A} = F_2((Bu_2)_2, u_2) + F_2(u_2, (Bu_2)_2) \ .$$

Therefore by the same considerations as in Lemma 3.2 it follows that A must be of the form (3.5). By Proposition 2.6 we have

$$\Phi_{-\lambda}(\mathfrak{g}_{-\lambda}) = \mathfrak{g}_{-\lambda}^{(1)} \qquad (\lambda = 1, \tfrac{1}{2})$$

and

$$\Phi_0(\mathfrak{g}_0) = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -{}^t\bar{A} \end{pmatrix} \in \mathfrak{g}_0^{(1)} ; \tilde{A} \in \rho(\mathfrak{g}_0) \right\} .$$

Now we want to show that

$$\Phi_{1/2}(\mathfrak{g}_{1/2}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -i\,{}^t\bar{D} & 0 \end{pmatrix} \in \mathfrak{g}_{1/2}^{(1)} ; D = (0, D_1), D_1 \in M(s_0, p-q; \boldsymbol{C}) \right\} . \tag{3.9}$$

Let $X \in \mathfrak{g}_{1/2}$. Then by (2) of Proposition 2.6 $\Phi_{1/2}(X)$ belongs to $\mathfrak{g}_{1/2}^{(1)}$. Thus, there exists $D \in M(s_0, p; \boldsymbol{C})$ such that

$$\Phi_{1/2}(X) = \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -i\,{}^t\bar{D} & 0 \end{pmatrix} .$$

From (1) and (4) of Proposition 2.6 it follows that $[\mathfrak{g}_{-1/2}^{(1)}, \Phi_{1/2}(X)]$ belongs to $\Phi_0(\mathfrak{g}_0)$. So, for each $C \in M(p, s_0; \boldsymbol{C})$,

$$\left[\begin{pmatrix} 0 & C & 0 \\ 0 & 0 & i\,{}^t\bar{C} \\ 0 & 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -i\,{}^t\bar{D} & 0 \end{pmatrix} \right] \text{ belongs to } \Phi_0(\mathfrak{g}_0) \ .$$

Therefore $\widetilde{CD}$ is contained in $\rho(\mathfrak{g}_0)$. Thus CD must be of the form (3.5), which implies that D must be of the form (3.9).

Conversely let $D(\in M(s_0, p; \boldsymbol{C}))$ be of the form (3.9). We define the map g_t $(t \in \boldsymbol{R})$ of $D(V, F)$ into $R^C \times W$ by

$$g_t : (z, u_1 + u_2) \in D(V, F) \mapsto (z', u_1' + u_2') \in R^C \times W ,$$

where

$$\begin{aligned} z' &= z(-\tfrac{1}{2}it^2\, {}^t\bar{D}Dz - it\, {}^t\bar{D}{}^tu_1 + E_p)^{-1} , \\ u_1' &= {}^t(-\tfrac{1}{2}it^2\, {}^t\bar{D}Dz - it\, {}^t\bar{D}{}^tu_1 + E_p)^{-1}(t{}^tz{}^tD + u_1) , \\ u_2' &= u_2 . \end{aligned}$$

Then, by elementary calculations we can verify that

$$\operatorname{Im} z' - F(u', u') = {}^t\bar{Q}(\operatorname{Im} z - F(u, u))Q ,$$

where $Q = (-\frac{1}{2}it^2\, {}^t\bar{D}Dz - it\, {}^t\bar{D}{}^tu_1 + E_p)^{-1}$, $u = u_1 + u_2$ and $u' = u_1' + u_2'$. Therefore the map g_t is a one-parameter group of transformations of $D(V, F)$. Let X be the vector field induced by g_t. Then it is obvious that X belongs to $\mathfrak{g}_{1/2}$ and $\Phi_{1/2}(X) = \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & -i\,{}^t\bar{D} & 0 \end{pmatrix}$. By (2) of Proposition 2.6 we have proved that $\mathfrak{g}_{1/2}$ is isomorphic to the real vector space $M(s_0, p - q; \boldsymbol{C})$.

Now we determine $\mathfrak{g}_1$. We can show

$$\Phi_1(\mathfrak{g}_1) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y & 0 & 0 \end{pmatrix} \in \mathfrak{g}_1^{(1)} ;\ Y = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix},\ y \in H(p - q; \boldsymbol{C}) \right\} . \tag{3.10}$$

In fact, let $X \in \mathfrak{g}_1$. Then by (3) of Proposition 2.6 $\Phi_1(X)$ belongs to $\mathfrak{g}_1^{(1)}$. So, there exists $Y \in H(p, C)$ such that

$$\Phi_1(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y & 0 & 0 \end{pmatrix} .$$

From the condition $[\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_0$ and (4) of Proposition 2.6 it follows that for each $B \in H(p, C)$,

$$\left[\begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y & 0 & 0 \end{pmatrix} \right] \text{ belongs to } \Phi_0(\mathfrak{g}_0) .$$

Hence, $\widetilde{BY}$ belongs to $\rho(\mathfrak{g}_0)$, which implies that BY must be of the form

(3.5). Therefore Y must be of the form (3.10). Conversely let $Y(\in H(p, \boldsymbol{C}))$ be of the form (3.10). We define the map h_t $(t \in \boldsymbol{R})$ of $D(V, F)$ into $R^C \times W$ by

$$h_t: (z, u_1 + u_2) \in D(V, F) \mapsto (z', u_1' + u_2') \in R^C \times W,$$

where $z' = z(tYz + E_p)^{-1}$, $u_1' = {}^t(tYz + E_p)^{-1}u_1$ and $u_2' = u_2$. Then we can verify that

$$\operatorname{Im} z' - F(u', u') = {}^t(\overline{tYz + E_p})^{-1}(\operatorname{Im} z - F(u, u))(tYz + E_p)^{-1},$$

where $u = u_1 + u_2$, $u' = u_1' + u_2' \in W$. Therefore the map h_t is a one-parameter group of transformations of $D(V, F)$ and h_t induces a vector field $X \in \mathfrak{g}_1$ such that $\Phi_1(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y & 0 & 0 \end{pmatrix}$. Thus, by (3) of Proposition 2.6 we have proved that $\mathfrak{g}_1$ is isomorphic to the vector space $H(p-q, \boldsymbol{C})$.

q.e.d.

Remark. If $r_1(s_1) = p$ and $r_2(s_2) < p$, then the Siegel domain $D(V, F)$ is isomorphic to the one given in the above theorem. If $s_1 = s_2 = 1$, $r_1(1) = p-1$ and $r_2(1) = p$, then the fact $\dim \mathfrak{g}_{1/2} = 2$ was proved by Sudo [12] by using different methods.

3.5. In this paragraph we treat the Siegel domains of type II over the cone $V = H(p, \boldsymbol{K})$ $(p \geq 2)$.

Let s be a positive integer and $r(t)$ be a non-decreasing integer valued function defined on an interval $[1, s]$ such that $1 \leq r(1), r(s) \leq 2p$. We denote by W the complex vector space of all complex $2p \times s$-matrices $u = (u_{ij})$ such that $u_{ij} = 0$ if $i > r(j)$. We put $F(u, v) = \frac{1}{2}(u\,{}^t\bar{v} + J\bar{v}\,{}^tu{}^tJ)$ for $u, v \in W$. Then it is known in [10] that the map F is a V-hermitian form on W and the Siegel domain $D(V, F)$ is homogeneous. Furthermore it was proved in [4] that the domain $D(V, F)$ is non-degenerate if and only if $r(s) = 2p$ or $2p-1$.

THEOREM 3.6.[5] (i) *If a Siegel domain $D(V, F)$ mensioned above is degenerate, then the subspaces $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of $\mathfrak{g}_h$ are given by*

$\mathfrak{g}_{1/2} = (0)$,

$\mathfrak{g}_1$ *is isomorphic to the vector space $H(p-q, \boldsymbol{K})$,*

where $q = [(r(s)+1)/2]$.

[5] Nakajima [18] calculated the dimensions of $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of this theorem by using different methods.

(ii) *If* $s \geq 2$ *and* $r(1) = 2p$, *or if* $s \geq 3$ *and there exists an integer* t_0 *such that* $1 < t_0 \leq s-1, r(t_0) = 2p$ *and* $r(t_0 - 1) \leq 2p-2$, *then* $\mathfrak{g}_h = \mathfrak{g}_a$.

Proof. First we consider the case (i). The linear closure of the set $\{F(u,u); u \in W\}$ in R coincides with the proper subspace $\begin{pmatrix} H(q,K) & 0 \\ 0 & 0 \end{pmatrix}$ of R, where $q = [(r(s)+1)/2]$ (cf. [4]). Hence by Lemma 3.2 we have $\mathfrak{g}_{1/2} = (0)$.

We determine $\mathfrak{g}_1$. Now, we consider the tube domain D' associated with $D(V,F)$ (cf. (2.9)). Then it is known in [10] that D' is the classical domain of type (II). The Lie algebra $\mathfrak{g}'_h = \mathfrak{g}'_{-1} + \mathfrak{g}'_0 + \mathfrak{g}'_1$ of all infinitesimal automorphisms of D' can be identified with $\mathfrak{so}^*(4p)$ as follows (cf. [10], Chap. 2, § 7);

$$\begin{aligned}
\mathfrak{g}'_h &= \mathfrak{so}^*(4p) \\
&= \left\{ \begin{pmatrix} A & B \\ C & -{}^t\bar{A} \end{pmatrix}; A \in \mathfrak{gl}(2p, C), AJ = J\bar{A},\ B, C \in H(p,K) \right\}, \\
\mathfrak{g}'_{-1} &= \begin{pmatrix} 0 & H(p,K) \\ 0 & 0 \end{pmatrix}, \qquad \mathfrak{g}'_1 = \begin{pmatrix} 0 & 0 \\ H(p,K) & 0 \end{pmatrix}, \\
\mathfrak{g}'_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^t\bar{A} \end{pmatrix}; A \in \mathfrak{gl}(2p, C),\ AJ = J\bar{A} \right\}.
\end{aligned}$$

We note that $g = \begin{pmatrix} E_{2p} & 0 \\ Y & E_{2p} \end{pmatrix}$ ($\in \exp \mathfrak{g}'_1$) acts on D' by

$$g : z \in D' \mapsto z(Yz + E_{2p})^{-1} \in D'.$$

It can be easily seen that the image $\xi(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ (cf. (2.10)) is the following subalgebra of $\mathfrak{g}'_0$;

$$\xi(\mathfrak{g}_0) = \left\{ \begin{pmatrix} A & 0 \\ 0 & -{}^t\bar{A} \end{pmatrix} \in \mathfrak{g}'_0;\ \tilde{A} \in \rho(\mathfrak{g}_0) \right\}.$$

We want to show that $\xi(\mathfrak{g}_1)$ coincides with the following subspace of $\mathfrak{g}'_1$;

$$\left\{ \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \in \mathfrak{g}'_1;\ Y = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix},\ y \in H(p-q, K) \right\}. \tag{3.11}$$

In fact, let $X \in \mathfrak{g}_1$. Then $\xi(X)$ belongs to $\mathfrak{g}'_1$ and there exists $Y \in H(p,K)$ such that $\xi(X) = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$. On the other hand, $\xi(\mathfrak{g}_{-1}) = \mathfrak{g}'_{-1}$. So, by the condition $[\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_0$ we have $[\mathfrak{g}'_{-1}, \xi(X)] \subset \xi(\mathfrak{g}_0)$. Hence, for each $B \in H(p,K)$, $\widetilde{BY}$ must be contained in $\rho(\mathfrak{g}_0)$. Therefore BY must be of the form (3.6). Thus, Y must be of the form (3.11). Conversely let Y be an element in $H(p, \boldsymbol{K})$ of the form (3.11). We define the map g_t $(t \in \boldsymbol{R})$ of $D(V, F)$ into $R^C \times W$ by

$$g_t : (z, u) \in D(V, F) \mapsto (z(tYz + E_{2p})^{-1}, u) \in R^C \times W .$$

Then we can verify that

$$\operatorname{Im}(z(tYz + E_{2p})^{-1}) = {}^t\overline{(tYz + E_{2p})}^{-1} \operatorname{Im} z\,(tYz + E_{2p})^{-1}$$

and

$${}^t\overline{(tYz + E_{2p})}^{-1} F(u, u)(tYz + E_{2p})^{-1} = F(u, u) .$$

Therefore the map g_t is a one-parameter group of transformations of $D(V, F)$, and g_t induces a vector field $X \in \mathfrak{g}_1$ such that $\xi(X) = \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$. Thus, by the fact $\mathfrak{g}_{1/2} = (0)$ and Proposition 2.8 $\mathfrak{g}_1$ can be identified with the vector space $H(p - q, \boldsymbol{K})$.

Now we consider the case (ii). If $r(1) = 2p$, then the complex vector space W coincides with $M(2p, s; \boldsymbol{C})$ and the Siegel domain $D(V, F)$ is the one given in (3) of Lemma 3.1. So, we have $\mathfrak{g}_h = \mathfrak{g}_a$. We proceed to the second case. We define the subspaces W_1 and W_2 of W by

$$W_1 = \{u = (u_{ij}) \in W ; u_{ij} = 0 \text{ if } j < t_0\}$$

and

$$W_2 = \{u = (u_{ij}) \in W ; u_{ij} = 0 \text{ if } j \geq t_0\} .$$

Then we have

$$W = W_1 + W_2 \text{ (direct sum)} \quad \text{and} \quad F(W_1, W_2) = (0) .$$

The vector space W_1 is isomorphic to $M(2p, s - t_0 + 1; \boldsymbol{C})$ and the Siegel domain $D(V, F_1)$ in $R^C \times W_1$ is isomorphic to the one given in (3) of Lemma 3.1. Thus, we have $\mathfrak{g}_{1/2}^{(1)} = (0)$. For the Siegel domain $D(V, F_2)$ in $R^C \times W_2$, by our assumption $r(t_0 - 1) \leq 2p - 2$ the linear closure of the set $\{F_2(u, u) ; u \in W_2\}$ in R coincides with the proper subspace $\begin{pmatrix} H(q, \boldsymbol{K}) & 0 \\ 0 & 0 \end{pmatrix}$ of R, where $q = [(r(t_0 - 1) + 1)/2]$ (cf. [4]). Thus, by Lemma 3.2 we get $\mathfrak{g}_{1/2}^{(2)} = (0)$. It follows from Corollary 2.7 that $\mathfrak{g}_{1/2} = (0)$. Applying Proposition 2.2 to the non-degenerate Siegel domain $D(V, F)$, we conclude that $\mathfrak{g}_h = \mathfrak{g}_a$. q.e.d.

§ 4. Homogeneous Siegel domains over circular cones

In this section, we will study how to construct all homogeneous non-degenerate Siegel domains over circular cones and study their equivalence. We omit the terminology "of type II of rank 2", since we consider here exclusively N-algebras of type II of rank 2.

4.1. We will recall some of definitions and results about N-algebras and skeletons due to Kaneyuki and Tsuji [5] in the case of rank 2.

Let N be a finite dimensional algebra over the real number field. Suppose that N is the direct sum of the bigraded subspaces N_{ij} $(1 \leq i < j \leq 3)$ and that N is equipped with a positive definite inner product $\langle\ ,\ \rangle$. Let j be a linear endomorphism of the subspace $N_{13} + N_{23}$ of N. Then the triple $(N, \langle\ ,\ \rangle, j)$ is called an *N-algebra*[6] if the following conditions are satisfied;

$$N_{13} \neq (0) \quad \text{or} \quad N_{23} \neq (0)\ ,$$
$$N_{12}N_{23} \subset N_{13},\ N_{ij}N_{kl} = (0) \qquad \text{if } j \neq k\ ,$$
$$\langle N_{ij}, N_{kl} \rangle = 0 \qquad \text{if } i \neq k \text{ or } j \neq l\ ,$$

$$jN_{i3} = N_{i3} \qquad (i = 1, 2),\ j^2 = -1\ , \tag{4.1}$$

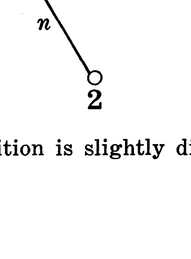
$$\langle ja, jb \rangle = \langle a, b \rangle \qquad \text{for } a, b \in N_{13} + N_{23}\ , \tag{4.2}$$

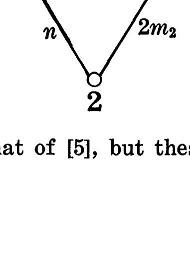
$$j(a_{12}a_{23}) = a_{12}j(a_{23})\ , \tag{4.3}$$

$$\begin{gathered} \text{for every } a_{12}, b_{12} \in N_{12} \text{ and } a_{23}, b_{23} \in N_{23}\ , \\ \langle a_{12}a_{23}, b_{12}b_{23} \rangle + \langle a_{12}b_{23}, b_{12}a_{23} \rangle = 2\langle a_{12}, b_{12} \rangle \langle a_{23}, b_{23} \rangle\ . \end{gathered} \tag{4.4}$$

Remark. Let $(N, \langle\ ,\ \rangle, j)$ be an N-algebra with $\dim N_{12} \cdot \dim N_{23} \neq 0$. Then the following condition is satisfied; $\max(\dim N_{12}, \dim N_{23}) \leq \dim N_{13}$ (cf. [5]).

A figure $\mathfrak{S}$ in the plane is called a *connected* 2-*skeleton* (*of type* II) if $\mathfrak{S}$ is one of the following $\mathfrak{S}_1$ or $\mathfrak{S}_2$;

$\mathfrak{S}_1$ 

, $\mathfrak{S}_2$ 

,

6) This definition is slightly different from that of [5], but these are equivalent.

where n and m_1 in $\mathfrak{S}_1$ are positive integers, and n, m_1, m_2 in $\mathfrak{S}_2$ are positive integers such that $\max(n, 2m_2) \leq 2m_1$.

Let $(N, \langle\ ,\ \rangle, j)$ be an N-algebra. Then it is said that $(N, \langle\ ,\ \rangle, j)$ *corresponds* to $\mathfrak{S}_1$ (resp. $\mathfrak{S}_2$) if $\dim N_{12} = n$, $\dim N_{23} = 0$ and $\dim N_{13} = 2m_1$ (resp. $\dim N_{12} = n$, $\dim N_{23} = 2m_2$ and $\dim N_{13} = 2m_1$). In this case, $\mathfrak{S}_1$ (resp. $\mathfrak{S}_2$) is called the *diagram* of $(N, \langle\ ,\ \rangle, j)$.

Let $(N, \langle\ ,\ \rangle, j)$ and $(N', \langle\ ,\ \rangle', j')$ be two N-algebras which correspond to the skeletons $\mathfrak{S}_1$ or $\mathfrak{S}_2$. Then $(N, \langle\ ,\ \rangle, j)$ is said to be *isomorphic* to $(N', \langle\ ,\ \rangle', j')$ if there exists a bigrade-preserving algebra isomorphism φ of N onto N' such that

$$\begin{aligned} &\langle \varphi(a), \varphi(b) \rangle' = \langle a, b \rangle, \ a, b \in N \ , \\ &\varphi \circ j = j' \circ \varphi \qquad \text{on } N_{13} + N_{23} \ . \end{aligned} \tag{4.5}$$

It follows immediately from the above definition that if two N-algebras which correspond to the skeletons $\mathfrak{S}_1$ or $\mathfrak{S}_2$ are isomorphic, then their diagrams are the same one.

According to [5], [13], there is a one-to-one correspondence between the set of all (holomorphic) isomorphism classes of homogeneous Siegel domains of type II over circular cones and the set of all isomorphism classes of N-algebras whose diagrams are $\mathfrak{S}_1$ or $\mathfrak{S}_2$.
In what follows, for a Siegel domain $D(C(n+2), F)$ corresponding to an N-algebra whose diagram is $\mathfrak{S}_1$ (resp. $\mathfrak{S}_2$), we say that $D(C(n+2), F)$ *corresponds* to $\mathfrak{S}_1$ (resp. $\mathfrak{S}_2$).

It is known in [5] that for given positive integers n, m_1, there exists a unique homogeneous Siegel domain which corresponds to $\mathfrak{S}_1$. Furthermore the explicit forms of these domains are found in [5], [10].

4.2. By the facts stated above we will consider the case of $\mathfrak{S}_2$.

DEFINITION 4.1. Let $\{T_k\}_{1\leq k\leq n}$ be a system of $m_1 \times m_2$-complex matrices T_k $(1 \leq k \leq n)$ satisfying the condition;

$${}^t\overline{T}_k T_l + {}^t\overline{T}_l T_k = 2\delta_{kl} E_{m_2} \qquad (1 \leq k, l \leq n) \ . \tag{4.6}$$

Let $\{T'_k\}_{1\leq k\leq n}$ be another system of $m_1 \times m_2$-complex matrices satisfying (4.6). Then $\{T_k\}_{1\leq k\leq n}$ is said to be *equivalent* to $\{T'_k\}_{1\leq k\leq n}$ if there exists a triple $(O_1, U_1, U_2) \in O(n) \times U(m_1) \times U(m_2)$ such that

$$(T_1, \cdots, T_n) = U_1(T'_1, \cdots, T'_n)(O_1 \otimes U_2) \ , \tag{4.7}$$

for the $m_1 \times nm_2$-matrices $(T_1, \cdots, T_n)$ and $(T'_1, \cdots, T'_n)$.

From (4.7) it can be seen that the above "equivalence" is an equivalence relation in the set of all systems satisfying (4.6).

Let $\{T_k\}_{1\leq k\leq n}$ be a system of $m_1 \times m_2$-matrices satisfying (4.6). Let N_{12} be the euclidean space $\boldsymbol{R}^n$ with the inner product $(\, , \,)$ and N_{k3} be the complex euclidean space $\boldsymbol{C}^{m_k}$ $(k = 1, 2)$ with the hermitian inner product $(\, , \,)$. Let N be the direct sum of real vector spaces N_{ij} $(1 \leq i < j \leq 3)$. Then for a fixed orthonormal base $\{e_k\}_{1\leq k\leq n}$ of N_{12}, we define in N an inner product $\langle \, , \, \rangle$, a multiplication and a complex structure j as follows;

$$\begin{aligned} &\langle a_{12} + a_{23} + a_{13}, b_{12} + b_{23} + b_{13} \rangle \\ &\qquad = (a_{12}, b_{12}) + \mathrm{Re}\,(a_{23}, b_{23}) + \mathrm{Re}\,(a_{13}, b_{13}) \, , \\ &\qquad\quad a_{ij}, b_{ij} \in N_{ij} \qquad (1 \leq i < j \leq 3) \, . \end{aligned} \tag{4.8}$$

(4.9) $e_k a_{23} = T_k a_{23}$ holds in N_{13} $(1 \leq k \leq n)$ and $a_{ij} a_{st} = 0$ if $j \neq s$.

$$j a_{k3} = i a_{k3} \qquad (k = 1, 2) \, . \tag{4.10}$$

LEMMA 4.2. *With respect to* (4.8), (4.9) *and* (4.10) *the vector space N is an N-algebra which corresponds to $\mathfrak{S}_2$. Every N-algebra which corresponds to $\mathfrak{S}_2$ can be obtained in this way by taking some system satisfying* (4.6).

Proof. It can be easily seen that $(N, \langle \, , \, \rangle, j)$ satisfies all the conditions but (4.4). Using (4.6), (4.8) and (4.9), we obtain

$$\begin{aligned} &\langle e_k a_{23}, e_l b_{23} \rangle + \langle e_k b_{23}, e_l a_{23} \rangle \\ &\quad = \mathrm{Re}\,(T_k a_{23}, T_l b_{23}) + \mathrm{Re}\,(T_k b_{23}, T_l a_{23}) \\ &\quad = \mathrm{Re}\,(({}^t\overline{T}_k T_l + {}^t\overline{T}_l T_k) a_{23}, b_{23}) = 2\delta_{kl}\, \mathrm{Re}\,(a_{23}, b_{23}) \\ &\quad = 2\langle e_k, e_l \rangle \langle a_{23}, b_{23} \rangle \, , \end{aligned}$$

which implies (4.4). By Remark in the paragraph 4.1 it is obvious that $(N, \langle \, , \, \rangle, j)$ corresponds to $\mathfrak{S}_2$. Hence the first assertion was proved.

Conversely let $(N, \langle \, , \, \rangle, j)$ be an N-algebra which corresponds to $\mathfrak{S}_2$. Then by (4.1) and (4.2) we can identify N_{13} (resp. N_{23}) with $\boldsymbol{C}^{m_1}$ (resp. $\boldsymbol{C}^{m_2}$) as hermitian vector spaces. Let us identify N_{12} with $\boldsymbol{R}^n$ as euclidean vector spaces and put $\{e_k\}_{1\leq k\leq n}$ be an orthonormal base of $N_{12} = \boldsymbol{R}^n$. Let L_k denote the left multiplication by e_k in N (i.e., $L_k(x) = e_k x$ for $x \in N$) $(1 \leq k \leq n)$. Then L_k restricted to the subspace N_{23}

induces a complex linear mapping of N_{23} into N_{13} (cf. (4.3)). Hence, under the identification of N_{i3} with C^{m_i} $(i = 1, 2)$ L_k induces a complex $m_1 \times m_2$-matrix T_k such that $T_k a_{23} = e_k a_{23}$ $(1 \leq k \leq n)$. On the other hand, (4.4) implies

$$L_k^* L_l + L_l^* L_k = 2\delta_{kl} 1 ,$$

where $*$ is the adjoint with respect to the inner product $\langle\ ,\ \rangle$. Thus, it follows that the system $\{T_k\}_{1 \leq k \leq n}$ satisfies the condition (4.6). q.e.d.

In view of the above lemma the system $\{T_k\}_{1 \leq k \leq n}$ is called the *admissible system* of $(N, \langle\ ,\ \rangle, j)$ with respect to the orthonormal base $\{e_k\}_{1 \leq k \leq n}$.

LEMMA 4.3. *Let $(N, \langle\ ,\ \rangle, j)$ and $(N', \langle\ ,\ \rangle', j')$ be two N-algebras which correspond to $\mathfrak{S}_2$. Let $\{e_k\}_{1 \leq k \leq n}$ (resp. $\{e'_k\}_{1 \leq k \leq n}$) be an arbitrary orthonormal base of N_{12} (resp. N'_{12}) and let $\{T_k\}_{1 \leq k \leq n}$ (resp. $\{T'_k\}_{1 \leq k \leq n}$) be the admissible system of $(N, \langle\ ,\ \rangle, j)$ (resp. $(N', \langle\ ,\ \rangle', j')$) with respect to $\{e_k\}_{1 \leq k \leq n}$ (resp. $\{e'_k\}_{1 \leq k \leq n}$). Then $(N, \langle\ ,\ \rangle, j)$ is isomorphic to $(N', \langle\ ,\ \rangle', j')$ if and only if $\{T_k\}_{1 \leq k \leq n}$ is equivalent to $\{T'_k\}_{1 \leq k \leq n}$.*

Proof. Suppose that $(N, \langle\ ,\ \rangle, j)$ is isomorphic to $(N', \langle\ ,\ \rangle', j')$. Then from (4.5) it follows that there exists a triple (f, g, h) of linear isometries;

$$f : N_{12} \to N'_{12}, \quad g : N_{23} \to N'_{23}, \quad h : N_{13} \to N'_{13}$$

satisfying

$$f(e_k) g(a_{23}) = h(e_k a_{23}) \tag{4.11}$$

and

$$h \circ j = j' \circ h \text{ on } N_{13} \text{ and } g \circ j = j' \circ g \text{ on } N_{23} . \tag{4.12}$$

Let $O = (\alpha_{lk})$ be the orthogonal matrix of degree n defined by $f(e_k) = \sum \alpha_{lk} e'_l$ $(1 \leq k \leq n)$. Then (4.11) implies $\sum \alpha_{lk} e'_l g(a_{23}) = h(e_k a_{23})$. Hence, we have

$$\sum \alpha_{lk} L'_l \circ g = h \circ L_k \qquad (1 \leq k \leq n) . \tag{4.13}$$

From (4.12) it follows that g (resp. h) induces a unitary matrix G (resp. H) of degree m_2 (resp. m_1). Thus, (4.13) shows that $\sum \alpha_{lk} T'_l G = H T_k$ $(1 \leq k \leq n)$. From this we have

$$(T'_1, \cdots, T'_n)(O \otimes G) = H(T_1, \cdots, T_n) .$$

Hence, $\{T_k\}_{1\le k\le n}$ is equivalent to $\{T'_k\}_{1\le k\le n}$ (cf. Definition 4.1).

The converse of our assertion is analogously proved. q.e.d.

4.3. It was proved in [5] that homogeneous Siegel domains and N-algebras are in one-to-one correspondence. By considering the correspondence in detail in the rank 2 case, we will prove that every homogeneous non-degenerate Siegel domain $D(C(n+2), F)$ is constructed directly in terms of the system $\{T_k\}_{1\le k\le n}$.

Let $(N, \langle\ ,\ \rangle, j)$ be an N-algebra whose diagram is $\mathfrak{S}_2$ and let $\{T_k\}_{1\le k\le n}$ be the admissible system of $(N, \langle\ ,\ \rangle, j)$. Now we will construct the Siegel domain $D(C(n+2), F)$ which corresponds to $(N, \langle\ ,\ \rangle, j)$ in the sense of Corollary 2.7 in [5]. By Theorem 2.6 in [5] we can construct the T-algebra $(\mathfrak{A} = \sum_{1\le i,j\le 3} \mathfrak{A}_{ij}, *, j)$ which corresponds to $(N, \langle\ ,\ \rangle, j)$ as follows;

$$\mathfrak{A}_{ii} = \boldsymbol{R}\ (1 \le i \le 3),\ \mathfrak{A}_{ij} = N_{ij},\ \mathfrak{A}_{ji} = N_{ij}^*\ (1 \le i < j \le 3)\ ,$$

where $*$ is an involutive linear endomorphism of N_{ij} such that $*\circ j = j \circ *$ on $N_{13} + N_{23}$. And the multiplications in $\mathfrak{A}$ have the following properties;

$$\begin{aligned} a_{ij}a_{ji} &= \langle a_{ij}, a_{ji}^*\rangle \qquad (1 \le i < j \le 3)\ , \\ \langle a_{13}a_{32}, e_k\rangle &= \langle a_{13}, e_k a_{32}^*\rangle = \mathrm{Re}\,(a_{13}, T_k a_{32}^*)\ , \end{aligned} \tag{4.14}$$

where $a_{ij} \in \mathfrak{A}_{ij}$.

We denote by $R(\mathfrak{A})$ the direct sum $\mathfrak{A}_{11} + \mathfrak{A}_{22} + \mathfrak{A}_{12}$ and denote by $W(\mathfrak{A})$ the direct sum $\mathfrak{A}_{13} + \mathfrak{A}_{23}$ $(= \boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2})$. We define the subset $V(N)$ of $R(\mathfrak{A})$ as

$$V(N) = \{a = a_{11} + a_{22} + a_{12} \in R(\mathfrak{A})\,;\, a_{11} > 0,\ a_{11}a_{22} - \langle a_{12}, a_{12}\rangle > 0\}^{*)}\ .$$

Then we can see that $V(N)$ is a homogeneous convex cone and actually isomorphic to $C(n+2)$ under the following linear isomorphim f of $R(\mathfrak{A})$ onto $\boldsymbol{R}^{n+2}$;

$$f : a = a_{11} + a_{22} + a_{12} \in R(\mathfrak{A}) \mapsto {}^t(a_{11}, a_{22}, a_{12}^1, \cdots, a_{12}^n) \in \boldsymbol{R}^{n+2}\ , \tag{4.15}$$

where $a_{12} = \sum a_{12}^k e_k$.

We define the map $F : \boldsymbol{C}^{m_1+m_2} \times \boldsymbol{C}^{m_1+m_2} \mapsto \boldsymbol{C}^{n+2}$ by putting $F = {}^t(F^1, \cdots, F^{n+2})$, where

---

*) By $a_{11}a_{22}$ we mean a usual multiplication of real numbers $a_{ii} \in \mathfrak{A}_{ii} = \boldsymbol{R}\,(i = 1, 2)$.

$$(4.16)\quad \begin{aligned} &F^1(u,v)=(u_1,v_1)\ ,\qquad F^2(u,v)=(u_2,v_2)\ ,\\ &F^{k+2}(u,v)=\tfrac{1}{2}\{(u_1,T_kv_2)+(T_ku_2,v_1)\}\qquad (1\le k\le n)\end{aligned}$$

for $u=u_1+u_2,\ v=v_1+v_2\in \boldsymbol{C}^{m_1+m_2}=\boldsymbol{C}^{m_1}+\boldsymbol{C}^{m_2}$. Then we have

THEOREM 4.4.[7] (i) *For F above, the domain $D(C(n+2),F)$ is a homogeneous non-degenerate Siegel domain.*
(ii) *Conversely every homogeneous non-degenerate Siegel domain $D(C(n+2),F)$ is constructed in the above way* (4.16) *by taking some system $\{T_k\}_{1\le k\le n}$ satisfying* (4.6).
(iii) *Furthermore suppose that $D(C(n+2),F')$ is constructed by $\{T'_k\}_{1\le k\le n}$. Then $D(C(n+2),F)$ is holomorphically isomorphic to $D(C(n+2),F')$ if and only if $\{T_k\}_{1\le k\le n}$ is equivalent to $\{T'_k\}_{1\le k\le n}$.*

Proof. First we will show that the map F defined by (4.16) is a $C(n+2)$-hermitian form on $\boldsymbol{C}^{m_1}+\boldsymbol{C}^{m_2}$ and the Siegel domain $D(C(n+2),F)$ thus constructed is the one which corresponds to $(N,\langle\ ,\ \rangle,j)$ in the sense of [5]. By Theorem A in [13], the homogeneous Siegel domain which corresponds to the T-algebra $(\mathfrak{A},*,j)$ is given by the following $V(N)$-hermitian form $\tilde F=\sum_{1\le k\le l\le 2}F_{kl}$ on $W(\mathfrak{A})$;

$$F_{kl}(u,v)=\tfrac{1}{4}\{(u_{k3}v_{l3}^*+v_{k3}u_{l3}^*)+i(u_{k3}j(v_{l3}^*)+j(v_{k3})u_{l3}^*)\}$$

for $u=u_{13}+u_{23},\ v=v_{13}+v_{23}\in W(\mathfrak{A})$.
Hence, by (4.14) we have

$$\begin{aligned} F_{kk}(u,v)&=\tfrac{1}{4}\{2\langle u_{k3},v_{k3}\rangle+i(\langle u_{k3},j(v_{k3}^*)^*\rangle+\langle j(v_{k3}),u_{k3}\rangle)\}\\ &=\tfrac{1}{2}\{\langle u_{k3},v_{k3}\rangle+i\langle u_{k3},j(v_{k3})\rangle\}\ (\text{by } *\circ j=j\circ *)\\ &=\tfrac{1}{2}\{\mathrm{Re}\,(u_{k3},v_{k3})+i\,\mathrm{Re}\,(u_{k3},iv_{k3})\}\ (\text{by (4.8)})\\ &=\tfrac{1}{2}(u_{k3},v_{k3})\qquad (k=1,2)\ .\end{aligned}$$

And we have

$$\begin{aligned}\langle F_{12}(u,u),e_k\rangle&=\tfrac{1}{2}\langle u_{13}u_{23}^*,e_k\rangle+\tfrac{1}{4}i(\langle u_{13}j(u_{23})^*,e_k\rangle+\langle j(u_{13})u_{23}^*,e_k\rangle)\\ &=\tfrac{1}{2}\,\mathrm{Re}\,(u_{13},T_ku_{23})\ (\text{by (4.14)})\ ,\end{aligned}$$

which implies

$$F_{12}(u,v)=\tfrac{1}{4}\sum_{1\le k\le n}\{(u_{13},T_kv_{23})+(T_ku_{23},v_{13})\}e_k\ .$$

[7] If $m_1=m_2$ in $\mathfrak{S}_2$, then this construction is reduced to Pjateckii-Sapiro's [10].

We define the complex linear isomorphism g of $W(\mathfrak{A})$ onto $\boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2}$ by

$$g : u_{13} + u_{23} \in W(\mathfrak{A}) \mapsto \frac{1}{\sqrt{2}} u_{13} + \frac{1}{\sqrt{2}} u_{23} \in \boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2} .$$

Then we have

$$f(\tilde{F}(u, v)) = F(g(u), g(v)) \qquad (u, v \in W(\mathfrak{A}), \text{ cf. } (4.15)) .$$

Thus, it can be seen that the map F defined by (4.16) is a $C(n+2)$-hermitian form on $\boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2}$ and the Siegel domain $D(C(n+2), F)$ in $\boldsymbol{C}^{n+2} \times \boldsymbol{C}^{m_1+m_2}$ is linearly isomorphic to the Siegel domain $D(V(N), \tilde{F})$ in $R(\mathfrak{A})^C \times W(\mathfrak{A})$. Hence, the homogeneous Siegel domain $D(C(n+2), F)$ is the one which corresponds to $(N, \langle\ ,\ \rangle, j)$ in the sense of Corollary 2.7 in [5]. From Lemma 4.2 it follows that every homogeneous Siegel domain of type II over the cone $C(n+2)$ which corresponds to the skeleton $\mathfrak{S}_2$ is constructed by (4.16) by taking some system $\{T_k\}_{1 \le k \le n}$ satisfying (4.6).

Now we will show that a homogeneous Siegel domain $D(C(n+2), F)$ is non-degenerate if and only if $D(C(n+2), F)$ corresponds to $\mathfrak{S}_2$. Suppose that $D(C(n+2), F)$ corresponds to $\mathfrak{S}_2$. Then, as was proved above, $D(C(n+2), F)$ is constructed by (4.16) by some system $\{T_k\}_{1 \le k \le n}$ satisfying (4.6). The subset $\{F(u, u) ; u \in \boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2}\}$ of $\boldsymbol{R}^{n+2}$ contains $n+2$ linearly independent vectors in $\boldsymbol{R}^{n+2}$. In fact, take unit vectors $u_i \in \boldsymbol{C}^{m_i}$ $(i = 1, 2)$ and put

$$u^1 = u_1 + 0 , \quad u^2 = 0 + u_2 , \quad u^{k+2} = T_k u_2 + u_2 \in \boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2} \qquad (1 \le k \le n) .$$

Then we can verify that $\{F(u^1, u^1), F(u^2, u^2), \cdots, F(u^{n+2}, u^{n+2})\}$ spans $\boldsymbol{R}^{n+2}$. Suppose that $D(C(n+2), F)$ corresponds to $\mathfrak{S}_1$. Then it was proved in [5], [10] that the $C(n+2)$-hermitian form F on $\boldsymbol{C}^{m_1}$ is given by

$$(4.17) \qquad F(u, v) = {}^t((u, v), 0, \cdots, 0) \qquad (u, v \in \boldsymbol{C}^{m_1}) .$$

Hence $D(C(n+2), F)$ is degenerate.

Thus, the first and the second assertions of the theorem were proved. The last assertion follows immediately from Lemma 4.3. q.e.d.

§ 5. The exceptional bounded symmetric domain of type (V)

5.1. Let $\{T_1, T_2\}$ be a system satisfying the condition (4.6) and define an $m_1 \times 2m_2$-matrix B as $B = (T_1, T_2)$. Then it follows from (4.6) that ${}^t\bar{T}_1 T_2$ is a skew-hermitian matrix of degree m_2, and we have

$$(5.1) \qquad {}^t\bar{B}B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes {}^t\bar{T}_1 T_2 + E_{2m_2}\,.$$

LEMMA 5.1. *Let $\{T_1, T_2\}$ and $\{T'_1, T'_2\}$ be two systems satisfying* (4.6). *Suppose that ${}^t\bar{T}_1 T_2$ (resp. ${}^t\bar{T}'_1 T'_2$) has eigenvalues $\{i\lambda_1, \cdots, i\lambda_{m_2}\}$, $\lambda_1 \leq, \cdots, \leq \lambda_{m_2}$ (resp. $\{i\lambda'_1, \cdots, i\lambda'_{m_2}\}$, $\lambda'_1 \leq, \cdots, \leq \lambda'_{m_2}$). Then $\{T_1, T_2\}$ is equivalent to $\{T'_1, T'_2\}$ if and only if $(\lambda_1, \cdots, \lambda_{m_2}) = (\lambda'_1, \cdots, \lambda'_{m_2})$ or $(\lambda_1, \cdots, \lambda_{m_2}) = (-\lambda'_{m_2}, \cdots, -\lambda'_1)$.*

Proof. Suppose that $(\lambda_1, \cdots, \lambda_{m_2}) = (\lambda'_1, \cdots, \lambda'_{m_2})$ or $(\lambda_1, \cdots, \lambda_{m_2}) = (-\lambda'_{m_2}, \cdots, -\lambda'_1)$. Then there exists $U_2 \in U(m_2)$ such that ${}^t\bar{U}_2 {}^t\bar{T}'_1 T'_2 U_2 = \varepsilon {}^t\bar{T}_1 T_2$, $\varepsilon = \pm 1$. Putting $B'' = B'\left(\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \otimes U_2\right)$, we have ${}^t\bar{B}''B'' = {}^t\bar{B}B$. Hence, by an analogous consideration as in Lemma 4.3 in [5], there exists $U_1 \in U(m_1)$ satisfying $B = U_1 B''$, that is, $B = U_1 B'\left(\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \otimes U_2\right)$. Therefore $\{T_1, T_2\}$ is equivalent to $\{T'_1, T'_2\}$ (cf. Definition 4.1). By making use of (5.1) we can easily prove the "only if" part. q.e.d.

The following proposition is stated without proof in Pjateckii-Sapiro [10], but for the sake of completeness we prove it without using the theory of Clifford algebras.

PROPOSITION 5.2. *There exists a unique homogeneous Siegel domain (up to holomorphic equivalence) which corresponds to $\mathfrak{S}_2$ with $(n, m_1, m_2) = (6, 4, 4)$. Furthermore this Siegel domain is constructed by the following system $\{T_k\}_{1 \leq k \leq 6}$;*

$$T_1 = E_4\,, \quad T_2 = i\begin{pmatrix} -E_2 & 0 \\ 0 & E_2 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix},$$

$$(5.2) \qquad T_4 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$T_6 = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. It can be easily seen that the above $\{T_k\}_{1\le k\le 6}$ is a system satisfying (4.6) with $(n, m_1, m_2) = (6, 4, 4)$. Conversely let $\{S_k\}_{1\le k\le 6}$ be a system satisfying (4.6) with $(n, m_1, m_2) = (6, 4, 4)$. Then, by (4.6) S_k belongs to $U(4)$ $(1 \le k \le 6)$.

Now we will prove that $\{S_k\}_{1\le k\le 6}$ is equivalent to $\{T_k\}_{1\le k\le 6}$. Since $\{S_1, S_2\}$ is a system satisfying (4.6) with $(n, m_1, m_2) = (2, 4, 4)$, it follows from Lemma 5.1 that there exists a triple (O_1, U_1, U_2) in $O(2) \times U(4) \times U(4)$ such that

$$U_1(S_1, S_2)(O_1 \otimes U_2) = (E_4, S_2') , \tag{5.3}$$

where $S_2' = iE_4$, $i\begin{pmatrix} -1 & 0 \\ 0 & E_3 \end{pmatrix}$ or $i\begin{pmatrix} -E_2 & 0 \\ 0 & E_2 \end{pmatrix}$. Putting $O_2 = \begin{pmatrix} O_1 & 0 \\ 0 & E_4 \end{pmatrix} \in O(6)$, by (5.3) we have $U_1(S_1, \cdots, S_6)(O_2 \otimes U_2) = (E_4, S_2', U_1S_3U_2, \cdots, U_1S_6U_2)$. So, without loss of generality we can assume that $(S_1, \cdots, S_6) = (E_4, S_2, \cdots, S_6)$, where $S_2 = iE_4$ or $i\begin{pmatrix} -1 & 0 \\ 0 & E_3 \end{pmatrix}$ or $i\begin{pmatrix} -E_2 & 0 \\ 0 & E_2 \end{pmatrix}$. The case $S_2 = iE_4$ or $i\begin{pmatrix} -1 & 0 \\ 0 & E_3 \end{pmatrix}$ does not occur. In fact, suppose that $S_2 = iE_4$. Then it can be seen that $\{E_4, iE_4, S_3\}$ does not satisfy the condition (4.6). Furthermore suppose that $S_2 = i\begin{pmatrix} -1 & 0 \\ 0 & E_3 \end{pmatrix}$. Then it follows from the condition ${}^t\bar{S}_3S_k + {}^t\bar{S}_kS_3 = 0$ $(k = 1, 2)$ that S_3 is represented as

$$S_3 = \begin{pmatrix} 0 & z_1 & z_2 & z_3 \\ -\bar{z}_1 & 0 & 0 & 0 \\ -\bar{z}_2 & 0 & 0 & 0 \\ -\bar{z}_3 & 0 & 0 & 0 \end{pmatrix}, \quad z_k \in \boldsymbol{C} \qquad (1 \le k \le 3) .$$

This contradicts to the condition ${}^t\bar{S}_3S_3 = E_4$. Hence S_2 must be $T_2 = i\begin{pmatrix} -E_2 & 0 \\ 0 & E_2 \end{pmatrix}$. From (4.6) it follows that S_k $(3 \le k \le 6)$ is represented as

$$S_k = \begin{pmatrix} 0 & X_k \\ -{}^t\bar{X}_k & 0 \end{pmatrix}, \quad {}^t\bar{X}_kX_l + {}^t\bar{X}_lX_k = 2\delta_{kl}E_2 \qquad (3 \le k, l \le 6) . \tag{5.4}$$

We will show that $\{S_k\}_{1\le k\le 6}$ is equivalent to $\{S_k''\}_{1\le k\le 6}$, where $S_1'' = T_1$, $S_2'' = T_2$ and $S_3'' = T_3$. In fact, let $U_3 = \begin{pmatrix} {}^t\bar{X}_3 & 0 \\ 0 & E_2 \end{pmatrix}$. Then by (5.4) we have $U_3 \in U(4)$ and

$$\begin{aligned} U_3(S_1, \cdots, S_6)(E_6 \otimes {}^t\bar{U}_3) &= (U_3S_1{}^t\bar{U}_3, \cdots, U_3S_6{}^t\bar{U}_3) \\ &= (T_1, T_2, T_3, U_3S_4{}^t\bar{U}_3, U_3S_5{}^t\bar{U}_3, U_3S_6{}^t\bar{U}_3) . \end{aligned}$$

Thus, without loss of generality we can assume that

$$\{S_k\}_{1\le k\le 6} = \{T_1, T_2, T_3, S_4, S_5, S_6\} ,$$

where S_k $(4 \le k \le 6)$ is represented as follows;

$$S_k = \begin{pmatrix} 0 & Y_k \\ Y_k & 0 \end{pmatrix}, \quad {}^t\overline{Y}_k = -Y_k \in U(2) , \quad Y_kY_l + Y_lY_k = 0 \qquad (4 \le k \ne l \le 6) . \tag{5.5}$$

In view of (5.5) there exists $U_4 \in U(2)$ such that $U_4Y_4{}^t\overline{U}_4 = iE_2$ or $-iE_2$ or $i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Furthermore from the condition $Y_4Y_5 + Y_5Y_4 = 0$ it follows that $(U_4Y_4{}^t\overline{U}_4)(U_4Y_5{}^t\overline{U}_4) + (U_4Y_5{}^t\overline{U}_4)(U_4Y_4{}^t\overline{U}_4) = 0$. Therefore by the fact $U_4Y_5{}^t\overline{U}_4 \in U(2)$, $U_4Y_4{}^t\overline{U}_4$ must be $i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Putting $U_5 = \begin{pmatrix} U_4 & 0 \\ 0 & U_4 \end{pmatrix} \in U(4)$, we have

$$U_5(S_1, \cdots S_6)(E_6 \otimes {}^t\overline{U}_5) = (T_1, T_2, T_3, T_4, T_5', T_6') ,$$

where T_5' and T_6' are represented as follows;

$$T_k' = \begin{pmatrix} 0 & Z_k \\ Z_k & 0 \end{pmatrix}, \quad {}^t\overline{Z}_k = -Z_k \in U(2) \ (k = 5, 6) , \quad Z_5Z_6 + Z_6Z_5 = 0 .$$

On the other hand, by the condition ${}^t\overline{T}_4T_k' + {}^t\overline{T}_k'T_4 = 0$ $(k = 5, 6)$, Z_k is represented as

$$Z_5 = \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix}, \quad Z_6 = \begin{pmatrix} 0 & e^{i\eta} \\ -e^{-i\eta} & 0 \end{pmatrix} \qquad (\theta, \eta \in \boldsymbol{R}) .$$

And by the condition $Z_5Z_6 + Z_6Z_5 = 0$ we have $e^{i(\eta-\theta)} = \varepsilon i$, $\varepsilon = \pm 1$. Now we put

$$U_6 = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in U(4) \quad \text{and} \quad O_3 = \begin{pmatrix} E_5 & 0 \\ 0 & \varepsilon \end{pmatrix} \in O(6) .$$

Then the direct verification shows that

$${}^t\overline{U}_6(T_1, T_2, T_3, T_4, T_5', T_6')(O_3 \otimes U_6) = (T_1, \cdots, T_6) .$$

Hence, $\{S_k\}_{1\le k\le 6}$ is equivalent to $\{T_k\}_{1\le k\le 6}$. q.e.d.

5.2. We will investigate infinitesimal automorphisms of homogeneous Siegel domains over circular cones. The same notations as in the previous sections will be employed.

LEMMA 5.3. *Let $D(C(n+2),F)$ be a homogeneous Siegel domain which corresponds to the skeleton $\mathfrak{S}_2$. Then the representation ρ is irreducible if and only if $m_1 = m_2$ in $\mathfrak{S}_2$.*

Proof. As is known in Theorem 4.4, the $C(n+2)$-hermitian form $F = {}^t(F^1, \cdots, F^{n+2})$ is given by (4.16).

Suppose that $m_1 = m_2$ in $\mathfrak{S}_2$. Then it was proved by Pjateckii-Sapiro ([10], Chap. 5, § 18) that $\rho(\mathfrak{g}_0)$ coincides with $\mathfrak{g}(C(n+2))$. Since $C(n+2)$ is an irreducible homogeneous self-dual cone (cf. Vinberg [17]), $\mathfrak{g}(C(n+2))$ is irreducible (cf. Rothaus [11]). Thus it follows that ρ is irreducible.

Now we will show that if $m_1 \neq m_2$ in $\mathfrak{S}_2$, then ρ is not irreducible. It is known in [17] that the Lie algebra $\mathfrak{g}(C(n+2))$ consists of all matrices A of the form;

$$A = \begin{pmatrix} \lambda & 0 & 2a_1 & \cdots & 2a_n \\ 0 & \mu & 2b_1 & \cdots & 2b_n \\ b_1 & a_1 & & & \\ \vdots & \vdots & & \frac{1}{2}(\lambda+\mu)E_n + \alpha & \\ b_n & a_n & & & \end{pmatrix}, \tag{5.6}$$

where λ, μ, a_k and b_k are real numbers $(1 \leq k \leq n)$ and α is a real skew-symmetric matrix of degree n. Let $A \in \mathfrak{g}(C(n+2))$ and $B \in \mathfrak{gl}(W)$. Then (A, B) satisfies the condition; $AF(u,u) = F(Bu,u) + F(u,Bu)$ (for every $u \in W = \boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_2}$) if and only if B is represented as follows;

$$B = \begin{pmatrix} B_1 + \frac{1}{2}\lambda E_{m_1} & B_{12} \\ B_{21} & B_2 + \frac{1}{2}\mu E_{m_2} \end{pmatrix}, \tag{5.7}$$

where $B_{12} = \sum a_k T_k$, $B_{21} = \sum b_k {}^t\overline{T}_k$ and B_1 (resp. B_2) is a skew-hermitian matrix of degree m_1 (resp. m_2) satisfying the conditions

$$B_1(T_1, \cdots, T_n) = (T_1, \cdots, T_n)(\alpha \otimes E_{m_2} + E_n \otimes B_2) \tag{5.8}$$

and

$$2b_k E_{m_1} = T_k B_{21} + {}^t\overline{B}_{21}{}^t\overline{T}_k \qquad (1 \leq k \leq n)\,. \tag{5.9}$$

Now we suppose that $m_1 \neq m_2$. Then by (5.9) we have

$$2b_k E_{m_1} = \sum_{1\le l\le n} b_l(T_k\,{}^t\overline{T}_l + T_l\,{}^t\overline{T}_k) \qquad (1 \le k \le n)\ .$$

From the fact ${}^t\overline{T}_k T_k = E_{m_2}$ (cf. (4.6)) it follows that there exists $U \in U(m_1)$ satisfying $UT_k = \begin{pmatrix} E_{m_2} \\ 0 \end{pmatrix}$. By putting $UT_l = \begin{pmatrix} C_l \\ D_l \end{pmatrix}$ $(l \neq k)$, we have

$$\begin{aligned} 2b_k E_{m_1} &= \sum_l b_l U(T_k\,{}^t\overline{T}_l + T_l\,{}^t\overline{T}_k){}^t\overline{U} = \sum_l b_l\left\{\begin{pmatrix} E_{m_2} \\ 0 \end{pmatrix}({}^t\overline{C}_l, {}^t\overline{D}_l) + \begin{pmatrix} C_l \\ D_l \end{pmatrix}(E_{m_2}, 0)\right\} \\ &= \sum_l b_l\begin{pmatrix} C_l + {}^t\overline{C}_l & {}^t\overline{D}_l \\ D_l & 0 \end{pmatrix} \qquad (1 \le k \le n)\ , \end{aligned}$$

which implies that $b_1 = b_2 = \cdots = b_n = 0$. From (1.7) we conclude that if $m_1 \neq m_2$, then the representation ρ is not irreducible. q.e.d.

The following theorem is stated implicitly in Pjateckii-Sapiro [10], as we remarked in the introduction.

THEOREM 5.4. *The exceptional bounded symmetric domain in* $\boldsymbol{C}^{16}$ *of type (V) (in the sense of* E. Cartan) *is realized as* $D(C(8), F)$, *where* $F = {}^t(F^1, \cdots, F^8)$ *is the following* $C(8)$*-hermitian form on* $\boldsymbol{C}^8$;

$$\begin{aligned} &F^1(u,u) = \sum_{1\le k\le 4} |u_k|^2, \qquad F^2(u,u) = \sum_{1\le k\le 4} |u_{k+4}|^2\ , \\ &F^3(u,u) = \mathrm{Re}\,(u_1\overline{u}_5 + u_2\overline{u}_6 + u_3\overline{u}_7 + u_4\overline{u}_8)\ , \\ &F^4(u,u) = \mathrm{Im}\,(-u_1\overline{u}_5 - u_2\overline{u}_6 + u_3\overline{u}_7 + u_4\overline{u}_8)\ , \\ (5.10)\qquad &F^5(u,u) = \mathrm{Re}\,(u_1\overline{u}_7 + u_2\overline{u}_8 - u_3\overline{u}_5 - u_4\overline{u}_6)\ , \\ &F^6(u,u) = \mathrm{Im}\,(u_1\overline{u}_7 - u_2\overline{u}_8 + u_3\overline{u}_5 - u_4\overline{u}_6)\ , \\ &F^7(u,u) = \mathrm{Re}\,(u_1\overline{u}_8 - u_2\overline{u}_7 + u_3\overline{u}_6 - u_4\overline{u}_5)\ , \\ &F^8(u,u) = \mathrm{Im}\,(u_1\overline{u}_8 + u_2\overline{u}_7 + u_3\overline{u}_6 + u_4\overline{u}_5)\ , \end{aligned}$$

for $u = {}^t(u_1, \cdots, u_8) \in \boldsymbol{C}^8$.

Proof. We will show that the Lie algebra $\mathfrak{g}_h$ of all infinitesimal automorphisms of $D(C(8), F)$ is simple. It can be seen that $D(C(8), F)$ is constructed by the system $\{T_k\}_{1\le k\le 6}$ of (5.2) by using (4.16). Thus, $D(C(8), F)$ corresponds to the skeleton $\mathfrak{S}_2$ with $(n, m_1, m_2) = (6, 4, 4)$. Therefore, by Lemma 5.3 the representation ρ is irreducible.

Now we want to determine $\mathfrak{g}_0$. We define $A \in \mathfrak{g}(C(8))$ by putting

$$A = \begin{bmatrix} \lambda & 0 & 2a_1 & \cdots & 2a_6 \\ 0 & \mu & 2b_1 & \cdots & 2b_6 \\ b_1 & a_1 & & & \\ \vdots & \vdots & & \frac{1}{2}(\lambda+\mu)E_6 + \alpha & \\ b_6 & a_6 & & & \end{bmatrix}, \quad \alpha = (\alpha_{kl}) \in \mathfrak{gl}(6, \boldsymbol{R}), \quad {}^t\alpha = -\alpha .$$

Then by direct computations making use of (5.7), (5.8) and (5.9) we can verify that $B \in \mathfrak{gl}(8, \boldsymbol{C})$ satisfies the condition; $AF(u, u) = F(Bu, u) + F(u, Bu)$ (for every $u \in \boldsymbol{C}^8$) if and only if B is represented as follows;

$$B = \begin{pmatrix} B_1 + \frac{1}{2}\lambda E_4 & \sum_{1 \le k \le 6} a_k T_k \\ \sum_{1 \le k \le 6} b_k {}^t\overline{T}_k & B_2 + \frac{1}{2}\mu E_4 \end{pmatrix} + i\theta E_8 , \tag{5.11}$$

where $\theta \in \boldsymbol{R}$, and $B_1 = (a_{\alpha\beta})$ and $B_2 = (b_{\alpha\beta})$ are skew-hermitian matrices of degree 4 given by

$$\begin{aligned}
&a_{12} = b_{12} = \tfrac{1}{2}\{(-\alpha_{35} + \alpha_{46}) - i(\alpha_{36} + \alpha_{45})\} ,\\
&a_{13} = -\bar{b}_{24} = \tfrac{1}{2}\{-(\alpha_{13} + \alpha_{24}) - i(\alpha_{14} - \alpha_{23})\} ,\\
&a_{14} = \bar{b}_{23} = \tfrac{1}{2}\{-(\alpha_{15} + \alpha_{26}) - i(\alpha_{16} - \alpha_{25})\} ,\\
&a_{23} = \bar{b}_{14} = \tfrac{1}{2}\{(\alpha_{15} - \alpha_{26}) - i(\alpha_{16} + \alpha_{25})\} ,\\
&a_{24} = -\bar{b}_{13} = \tfrac{1}{2}\{(-\alpha_{13} + \alpha_{24}) + i(\alpha_{14} + \alpha_{23})\} ,\\
&a_{34} = b_{34} = \tfrac{1}{2}\{(\alpha_{35} + \alpha_{46}) + i(\alpha_{36} - \alpha_{45})\} ,\\
&a_{11} = i\alpha_{12} , \quad a_{22} = i(\alpha_{12} + \alpha_{34} + \alpha_{56}) , \quad a_{33} = i\alpha_{34} , \quad a_{44} = i\alpha_{56} ,\\
&b_{11} = 0 , \quad b_{22} = i(\alpha_{34} + \alpha_{56}) , \quad b_{33} = i(\alpha_{12} + \alpha_{34}) , \quad b_{44} = i(\alpha_{12} + \alpha_{56}) .
\end{aligned}$$

Hence, from this fact and (1.4) it follows that $\dim \mathfrak{g}_0 = \dim \mathfrak{g}(C(8)) + 1 = 30$.

We want to show that $\mathfrak{g}_{1/2} \neq (0)$. We define a polynomial vector field $X = \sum_{1 \le k \le 8} p^k_{1,1} \partial/\partial z_k + \sum_{1 \le \alpha \le 8} (p^\alpha_{1,0} + p^\alpha_{0,2}) \partial/\partial w_\alpha$ on $\boldsymbol{C}^{16}$ as follows;

$$\begin{aligned}
&p^1_{1,1} = 2z_1 w_1 , \quad p^2_{1,1} = 2\{(z_3 - iz_4)w_5 + (z_5 + iz_6)w_7 + (z_7 + iz_8)w_8\} ,\\
&p^3_{1,1} = z_1 w_5 + (z_3 - iz_4)w_1 + (z_5 + iz_6)w_3 + (z_7 + iz_8)w_4 ,\\
&p^4_{1,1} = -iz_1 w_5 + (iz_3 + z_4)w_1 + (-iz_5 + z_6)w_3 + (-iz_7 + z_8)w_4 ,\\
&p^5_{1,1} = z_1 w_7 + (-z_3 + iz_4)w_3 + (z_5 + iz_6)w_1 + (z_7 + iz_8)w_2 ,\\
&p^6_{1,1} = iz_1 w_7 + (-iz_3 - z_4)w_3 + (-iz_5 + z_6)w_1 + (iz_7 - z_8)w_2 ,\\
&p^7_{1,1} = z_1 w_8 + (-z_3 + iz_4)w_4 + (-z_5 - iz_6)w_2 + (z_7 + iz_8)w_1 ,\\
&p^8_{1,1} = iz_1 w_8 + (-iz_3 - z_4)w_4 + (-iz_5 + z_6)w_2 + (-iz_7 + z_8)w_1 ,
\end{aligned}$$

and

$$p^1_{1,0} = iz_1\,,\quad p^2_{1,0} = p^3_{1,0} = p^4_{1,0} = 0\,,\quad p^5_{1,0} = iz_3 - z_4\,,$$
$$p^6_{1,0} = 0\,,\quad p^7_{1,0} = iz_5 + z_6\,,\quad p^8_{1,0} = iz_7 + z_8\,,$$
$$p^1_{0,2} = 2w_1^2\,,\quad p^2_{0,2} = 2w_1w_2\,,\quad p^3_{0,2} = 2w_1w_3\,,$$
$$p^4_{0,2} = 2w_1w_4\,,\quad p^5_{0,2} = 2w_1w_5\,,$$
$$p^6_{0,2} = 2(w_2w_5 + w_3w_8 - w_4w_7)\,,\quad p^7_{0,2} = 2w_1w_7\,,\quad p^8_{0,2} = 2w_1w_8\,.$$

Then by elementary calculations, for each $c = {}^t(c^1, \cdots, c^8) \in \boldsymbol{C}^8$ we have

$$[\varphi_{-1/2}(c), X] = \sum a'_{kl} z_l \partial/\partial z_k + \sum b'_{\alpha\beta} w_\beta \partial/\partial w_\alpha\,,$$

where the matrices $A(c) = (a'_{kl})$ and $B(c) = (b'_{\alpha\beta})$ are given by

$$A(c) = 2\begin{pmatrix}
2\,\mathrm{Re}\,c^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\,\mathrm{Re}\,c^5 & 2\,\mathrm{Im}\,c^5 & 2\,\mathrm{Re}\,c^7 & -2\,\mathrm{Im}\,c^7 & 2\,\mathrm{Re}\,c^8 & -2\,\mathrm{Im}\,c^8 \\
\mathrm{Re}\,c^5 & 0 & \mathrm{Re}\,c^1 & \mathrm{Im}\,c^1 & \mathrm{Re}\,c^3 & -\mathrm{Im}\,c^3 & \mathrm{Re}\,c^4 & -\mathrm{Im}\,c^4 \\
\mathrm{Im}\,c^5 & 0 & -\mathrm{Im}\,c^1 & \mathrm{Re}\,c^1 & \mathrm{Im}\,c^3 & \mathrm{Re}\,c^3 & \mathrm{Im}\,c^4 & \mathrm{Re}\,c^4 \\
\mathrm{Re}\,c^7 & 0 & -\mathrm{Re}\,c^3 & -\mathrm{Im}\,c^3 & \mathrm{Re}\,c^1 & -\mathrm{Im}\,c^1 & \mathrm{Re}\,c^2 & -\mathrm{Im}\,c^2 \\
-\mathrm{Im}\,c^7 & 0 & \mathrm{Im}\,c^3 & -\mathrm{Re}\,c^3 & \mathrm{Im}\,c^1 & \mathrm{Re}\,c^1 & -\mathrm{Im}\,c^2 & -\mathrm{Re}\,c^2 \\
\mathrm{Re}\,c^8 & 0 & -\mathrm{Re}\,c^4 & -\mathrm{Im}\,c^4 & -\mathrm{Re}\,c^2 & \mathrm{Im}\,c^2 & \mathrm{Re}\,c^1 & -\mathrm{Im}\,c^1 \\
-\mathrm{Im}\,c^8 & 0 & \mathrm{Im}\,c^4 & -\mathrm{Re}\,c^4 & \mathrm{Im}\,c^2 & \mathrm{Re}\,c^2 & \mathrm{Im}\,c^1 & \mathrm{Re}\,c^1
\end{pmatrix},$$

$$B(c) = 2\begin{pmatrix}
c^1 & -\overline{c^2} & -\overline{c^3} & -\overline{c^4} & 0 & 0 & 0 & 0 \\
c^2 & \overline{c^1} & 0 & 0 & 0 & 0 & 0 & 0 \\
c^3 & 0 & \overline{c^1} & 0 & 0 & 0 & 0 & 0 \\
c^4 & 0 & 0 & \overline{c^1} & 0 & 0 & 0 & 0 \\
c^5 & 0 & -\overline{c^7} & -\overline{c^8} & 0 & -\overline{c^2} & 0 & 0 \\
0 & c^5 & c^8 & -c^7 & c^2 & -2i\,\mathrm{Im}\,c^1 & -c^4 & c^3 \\
c^7 & -\overline{c^8} & \overline{c^5} & 0 & 0 & \overline{c^4} & 0 & 0 \\
c^8 & \overline{c^7} & 0 & \overline{c^5} & 0 & -\overline{c^3} & 0 & 0
\end{pmatrix} + 4i\,\mathrm{Im}\,c^1 E_8\,.$$

Hence by (5.6), $A(c)$ belongs to $\mathfrak{g}(C(8))$. Considering (5.11) we can verify that $(A(c), B(c))$ satisfies the condition; $A(c)F(u, u) = F(B(c)u, u) + F(u, B(c)u)$ for every $u \in \boldsymbol{C}^8$. Therefore, by (1.4) $[\varphi_{-1/2}(c), X]$ belongs to $\mathfrak{g}_0$, and we have $[\mathfrak{g}_{-1/2}, X] \subset \mathfrak{g}_0$. From (1.9), thus it follows that X belongs to $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_{1/2} \neq (0)$.

So, as a consequence of Theorem 2.1, we conclude that $\mathfrak{g}_h$ is simple. By the well-known theorem of Borel-Koszul [1], [7], $D(C(8), F)$ is holomorphically isomorphic to an irreducible bounded symmetric domain in C^{16}.

This bounded symmetric domain is the exceptional domain of type (V). In fact, by using (1.6) we have $\dim \mathfrak{g}_h = 2(\dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-1/2}) + \dim \mathfrak{g}_0 = 78$. And there is no classical irreducible bounded symmetric domain in C^{16} whose Lie algebra of all infinitesimal automorphisms is of dimension 78 (cf. e.g., Helgason [2]). q.e.d.

Remark. The form F given by (5.10) is different from that of the note [15]. But it can be seen that this domain is isomorphic to that of [15] under a linear transformation (cf. Proposition 5.2).

§ 6. Automorphisms of Siegel domains over circular cones

In this section, we calculate infinitesimal automorphisms of homogeneous Siegel domains over circular cones.

The Lie algebra $\mathfrak{g}_h$ of a homogeneous non-degenerate Siegel domain $D(C(n+2), F)$ for which the representation ρ is irreducible is determined completely by the following theorem.

THEOREM 6.1. *The Lie algebra $\mathfrak{g}_h$ of all infinitesimal automorphisms of a homogeneous Siegel domain $D(C(n+2), F)$ which corresponds to the skeleton $\mathfrak{S}_2$ with $m_1 = m_2 (= m)$ is given as follows;*

(n, m)	$\mathfrak{g}_h$
$(2, m)$	(i) $\mathfrak{g}_h = \mathfrak{su}(m+2, 2)$ *provided that $D(C(4), F)$ is constructed by the system $\{T_1, T_2\}$ $(T_1, T_2 \in U(m))$ such that ${}^t\overline{T}_1 T_2$ has $\{i, \cdots, i\}$ or $\{-i, \cdots, -i\}$ as its eigenvalues.* (ii) $\mathfrak{g}_h = \mathfrak{g}_a$, *otherwise.*
(4,2)	$\mathfrak{g}_h = \mathfrak{so}^*(10)$
(6,4)	$\mathfrak{g}_h = \mathfrak{e}_6(-14)$
otherwise	$\mathfrak{g}_h = \mathfrak{g}_a$

Proof. Pjateckii-Sapiro ([10], Chap. 2) gave case by case the explicit realizations of all classical domains. From his realizations it follows that if $D(C(n+2), F)$ is classical, then $(n, m) = (2, m)$ or $(4, 2)$.

Suppose that $(n, m) = (2, m)$. Then it was proved in [10] that $D(C(4), F)$ is a symmetric domain if and only if ${}^t\overline{T}_1 T_2$ has $\{i, \cdots, i\}$ or

$\{-i, \cdots, -i\}$ as its eigenvalues and that in this case $D(C(4), F)$ is the classical domain in C^{4+2m} of type (I).

Suppose that $(n, m) = (4, 2)$. Then there exists a unique homogeneous Siegel domain which corresponds to the skeleton $\mathfrak{S}_2$ with $(n, m_1, m_2) = (4, 2, 2)$ (cf. [10], [16]). And it was proved in [10] that this domain is the classical domain in C^{10} of type (II).

Suppose that $(n, m) = (6, 4)$. Then there exists a unique homogeneous Siegel domain which corresponds to the skeleton $\mathfrak{S}_2$ with $(n, m_1, m_2) = (6, 4, 4)$ (Proposition 5.2) and this domain is the exceptional domain in C^{16} of type (V) (Theorem 5.4).

By the uniqueness theorem of realization (cf. Kaneyuki [3]), there exists no symmetric Siegel domain of type II over circular cones other than the domains listed above (cf. [10], and for the exceptional domain of type (VI), see e.g., Vinberg [17]). Thus, our assertion follows from Theorem 2.1 and Lemma 5.3. q.e.d.

Now we determine infinitesimal automorphisms of homogeneous degenerate Siegel domains of type II over $C(n+2)$. As we stated in section 4, every homogeneous degenerate Siegel domain $D(C(n+2), F)$ in $C^{n+2} \times C^m$ $(m > 0)$ can be constructed by the following $C(n+2)$-hermitian form F on C^m;

$$F(u, v) = {}^t((u, v), 0, \cdots, 0), \qquad u, v \in C^m \text{ (cf. (4.17))} .$$

PROPOSITION 6.2. *For the homogeneous degenerate Siegel domain $D(C(n+2), F)$ in $C^{n+2} \times C^m$ $(m > 0)$, the subspaces $\mathfrak{g}_{1/2}$ and $\mathfrak{g}_1$ of $\mathfrak{g}_h$ are given by*

$$\mathfrak{g}_{1/2} = (0) ,$$

$$\mathfrak{g}_1 = \left\{a\left(\sum_{1 \leq k \leq n} z_{k+2}^2 \partial/\partial z_1 + z_2^2 \partial/\partial z_2 + \sum_{1 \leq k \leq n} z_2 z_{k+2} \partial/\partial z_{k+2}\right); a \in \boldsymbol{R}\right\} .$$

Proof. First we will determine $\mathfrak{g}_0$. Let $A \in \mathfrak{g}(C(n+2))$ and $B \in \mathfrak{gl}(m, C)$. Then it can be easily verified that (A, B) satisfies the condition; $AF(u, u) = F(Bu, u) + F(u, Bu)$ (for each $u \in C^m$) if and only if (A, B) is represented as

$$(6.1) \qquad A = \begin{pmatrix} \lambda & 0 & 2a_1 & \cdots & 2a_n \\ 0 & \mu & 0 & \cdots & 0 \\ 0 & a_1 & & & \\ \vdots & \vdots & & \frac{1}{2}(\lambda + \mu)E_n + \alpha & \\ 0 & a_n & & & \end{pmatrix}, \qquad B + {}^t\overline{B} = \lambda E_m ,$$

where λ, μ, a_k $(1 \leq k \leq n)$ are real numbers and α is a real skew-symmetric matrix of degree n (cf. (5.6)). Thus, by (1.4) we have determined $\mathfrak{g}_0$.

Now we show $\mathfrak{g}_{1/2} = (0)$. In view of Corollary 2.7 we can assume that $m = 1$. Let $X \in \mathfrak{g}_{1/2}$. Then by (2.2), (2.3) and (2.4), there exist $c_l, b \in \boldsymbol{C}$ $(1 \leq l \leq n+2)$ satisfying the following conditions;

(6.2) X is represented as $X = 2i \sum \bar{c}_l z_l w \partial/\partial z_1 + \sum c_l z_l \partial/\partial w + b w^2 \partial/\partial w$,

(6.3) $b = 2i\bar{c}_1$,

(6.4) for each $d \in \boldsymbol{C}$, the matrix

$$\begin{pmatrix} \operatorname{Im}(c_1\bar{d}) & \operatorname{Im}(c_2\bar{d}) & \cdots & \operatorname{Im}(c_{n+2}\bar{d}) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

belongs to $\mathfrak{g}(C(n+2))$.

Hence, by (5.6) and (6.4), $\operatorname{Im}(c_l\bar{d}) = 0$ for each $d \in \boldsymbol{C}$ $(1 \leq l \leq n+2)$. So, $c_l = 0$ $(1 \leq l \leq n+2)$. From (6.2) and (6.3) it follows that $X = 0$. Thus, $\mathfrak{g}_{1/2} = (0)$ was proved.

Now we determine $\mathfrak{g}_1$. By (1.3) we have

$$\mathfrak{g}_{-1/2} = \{2i(w, c)\partial/\partial z_1 + \sum c^\alpha \partial/\partial w_\alpha ;\ c = \sum c^\alpha f_\alpha \in \boldsymbol{C}^m\} .$$

Let $X = \sum p^k_{2,0}\partial/\partial z_k + \sum p^\alpha_{1,1}\partial/\partial w_\alpha \in \mathfrak{g}_1$. Then by the condition $[\mathfrak{g}_{-1/2}, X] = (0)$, we get $\partial p^k_{2,0}/\partial z_1 = 0$ $(1 \leq k \leq n+2)$ and $p^\alpha_{1,1} = 0$ $(1 \leq \alpha \leq m)$. We write $p^k_{2,0} = \sum a^k_{ij} z_i z_j$ $(a^k_{ij} = a^k_{ji})$. Then we have

(6.5) $$a^k_{1j} = a^k_{j1} = 0 \qquad (1 \leq j,\ k \leq n+2) .$$

For each i $(1 \leq i \leq n+2)$, we define the $(n+2) \times (n+2)$-matrix A_i by

(6.6) $$A_i = \begin{pmatrix} a^1_{i1} & a^1_{i2} & \cdots & a^1_{in+2} \\ a^2_{i1} & a^2_{i2} & \cdots & a^2_{in+2} \\ \vdots & \vdots & & \vdots \\ a^{n+2}_{i1} & a^{n+2}_{i2} & \cdots & a^{n+2}_{in+2} \end{pmatrix} .$$

Then we have

$$\tfrac{1}{2}\rho([\partial/\partial z_i, X]) = A_i \quad \text{and} \quad \sigma([\partial/\partial z_i, X]) = 0 .$$

By (1.10) and (1.4), $(A_i, 0)$ must be of the form (6.1). Comparing (6.6) with (6.1), we can see that the real numbers a_{ij}^k $(1 \leq i, j, k \leq n+2)$ must satisfy the following relations;

(6.7) $\quad a_{ik+2}^1 = 2a_{i2}^{k+2} \qquad (1 \leq i \leq n+2,\ 1 \leq k \leq n)\,,$

(6.8) $\quad a_{i2}^1 = 0 \qquad (1 \leq i \leq n+2)\,,$

(6.9) $\quad a_{i2}^2 = 2a_{ik+2}^{k+2} \qquad (1 \leq i \leq n+2,\ 1 \leq k \leq n)\,,$

(6.10) $\quad a_{ik+2}^2 = 0 \qquad (1 \leq i \leq n+2,\ 1 \leq k \leq n)\,,$

(6.11) $\quad a_{il+2}^{k+2} = -a_{ik+2}^{l+2} \qquad (1 \leq i \leq n+2,\ 1 \leq k \neq l \leq n)\,.$

By (6.5) we have $a_{i1}^1 = a_{1i}^1 = 0$ $(1 \leq i \leq n+2)$. Applying (6.7) and (6.11) for $1 \leq k \neq l \leq n$, we get

$$a_{k+2l+2}^1 = 2a_{k+22}^{l+2} = 2a_{2k+2}^{l+2} = -2a_{2l+2}^{k+2} = -2a_{l+22}^{k+2} = -a_{l+2k+2}^1 = -a_{k+2l+2}^1\,,$$

which implies $a_{k+2l+2}^1 = 0$. Therefore, considering (6.8) we showed

(6.12) $\quad a_{ij}^1 = 0 \quad \text{if } 1 \leq i \leq 2 \quad \text{or} \quad 1 \leq j \leq 2 \quad \text{or} \quad 3 \leq i \neq j \leq n+2\,.$

By (6.5) and (6.10) we get

$$a_{ij}^2 = 0 \qquad \text{if } (i, j) \neq (2, 2)\,. \tag{6.13}$$

From (6.5) we have $a_{1i}^{k+2} = a_{i1}^{k+2} = 0$ $(1 \leq i \leq n+2)$ and by (6.7), (6.12) we can see $a_{2i}^{k+2} = a_{i2}^{k+2} = 0$ $(i = 2$ or $3 \leq i \neq k+2 \leq n+2)$. Furthermore if $1 \leq i \neq j \neq k \neq i \leq n$, then by (6.11) a_{i+2j+2}^{k+2} is skew-symmetric with respect to the indices j, k and symmetric with respect to the indices i, j. So, $a_{i+2j+2}^{k+2} = 0$ if $1 \leq i \neq j \neq k \neq i \leq n$. Hence by (6.9), (6.11) we have

(6.14) $\quad a_{ij}^{k+2} = 0 \qquad \text{if } (i, j) \neq (2, k+2) \text{ and } (i, j) \neq (k+2, 2)\ (1 \leq k \leq n)\,.$

On the other hand, we can see

$$\begin{aligned} a_{22}^2 &= 2a_{2k+2}^{k+2} && \text{(by (6.9))} \\ &= a_{k+2k+2}^1 && \text{(by (6.7))}\ (1 \leq k \leq n)\,. \end{aligned} \tag{6.15}$$

As a consequence of (6.12)–(6.15), it follows that X must be represented by

$$X = a_{22}^2\Big(\sum_{1 \leq k \leq n} z_{k+2}^2 \partial/\partial z_1 + z_2^2 \partial/\partial z_2 + \sum_{1 \leq k \leq n} z_2 z_{k+2} \partial/\partial z_{k+2}\Big)\,. \tag{6.16}$$

Conversely if X is a polynomial vector field of the form (6.16), then it can be easily seen that X satisfies all the conditions in (1.10). Thus, the subspace $\mathfrak{g}_1$ of $\mathfrak{g}_h$ consists of all polynomial vector fields of the form (6.16). q.e.d.

Finally we consider the homogeneous non-degenerate Siegel domains which correspond to the skeleton $\mathfrak{S}_2$ with $n \leq 2m_2 < 2m_1$. Let $\{T_k\}_{1\leq k\leq n}$ be a system of $m_2 \times m_2$-matrices satisfying the condition (4.6). We put $T'_k = \begin{pmatrix} T_k \\ 0 \end{pmatrix}$, where 0 means the $(m_1 - m_2) \times m_2$-zero matrix. Then it is easy to see that the system $\{T'_k\}_{1\leq k\leq n}$ satisfies the condition (4.6) and corresponds to this skeleton $\mathfrak{S}_2$. We denote by $D(C(n+2), F)$ the Siegel domain in $\boldsymbol{C}^{n+2} \times \boldsymbol{C}^{m_1+m_2}$ which is constructed by the system $\{T'_k\}_{1\leq k\leq n}$. Then, by (4.16) the $C(n+2)$-hermitian form F is given by

$$(6.17) \qquad \begin{aligned} &F^1(u,v) = (u_1, v_1) + (u_3, v_3)\,, \qquad F^2(u,v) = (u_2, v_2)\,, \\ &F^{k+2}(u,v) = \tfrac{1}{2}\{(u_1, T_k v_2) + (T_k u_2, v_1)\} \qquad (1 \leq k \leq n) \end{aligned}$$

for $u = (u_1 + u_3) + u_2$, $v = (v_1 + v_3) + v_2 \in \boldsymbol{C}^{m_1+m_2} = (\boldsymbol{C}^{m_2} + \boldsymbol{C}^{m_1-m_2}) + \boldsymbol{C}^{m_2}$.

PROPOSITION 6.3. *For the Siegel domain $D(C(n+2), F)$ given by (6.17), if $n \neq 2$, $(n, m_2) \neq (4, 2)$ and $(n, m_2) \neq (6, 4)$, then $\mathfrak{g}_h = \mathfrak{g}_a$. If $n = 2$ and ${}^t\bar{T}_1 T_2$ does not have $\{i, \cdots, i\}$ and $\{-i, \cdots, -i\}$ as its eigenvalues, then $\mathfrak{g}_h = \mathfrak{g}_a$.*

Proof. We put the subspaces W_1 and W_2 of $\boldsymbol{C}^{m_1+m_2} = (\boldsymbol{C}^{m_1} + \boldsymbol{C}^{m_1-m_2}) + \boldsymbol{C}^{m_2}$ by $W_1 = \boldsymbol{C}^{m_2} + \boldsymbol{C}^{m_2}$ and $W_2 = \boldsymbol{C}^{m_1-m_2}$, respectively. Then we can see that $F(W_1, W_2) = (0)$. The Siegel domain $D(C(n+2), F_2)$ in $\boldsymbol{C}^{n+2} \times W_2$ is the one given in Proposition 6.2. Therefore we have $\mathfrak{g}^{(2)}_{1/2} = (0)$. On the other hand, the Siegel domain $D(C(n+2), F_1)$ in $\boldsymbol{C}^{n+2} \times W_1$ is the one given in Theorem 6.1. Thus, by Theorem 6.1 we get $\mathfrak{g}^{(1)}_{1/2} = (0)$. From Corollary 2.7 it follows that $\mathfrak{g}_{1/2} = (0)$. Applying Proposition 2.2 to the non-degenerate Siegel domain $D(C(n+2), F)$, we conclude that $\mathfrak{g}_h = \mathfrak{g}_a$. q.e.d.

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