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ON SOME DEGENERATE PARABOLIC EQUATIONS II

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§ 1. **Introduction.** In the article I: [8], we have proved the hypoellipticity of a degenerate parabolic equation of the form:

$$Pu = \frac{\partial u}{\partial t} - a(x, t)\frac{\partial^2 u}{\partial x^2} + b(x, t)\frac{\partial u}{\partial x} + c(x, t)u = f\ , \tag{1.1}$$

where the coefficients $a(x, t)$, $b(x, t)$ and $c(x, t)$ are complex valued smooth functions. The fundamental assumption on the coefficients is that $\mathscr{R}_e\, a(x, t)$ satisfies the condition of Nirenberg and Treves ([8], (1.5)). To prove the hypoellipticity we have constructed recurcively the parametrices as pseudodifferential operators with parameter. This method may be viewed as an improvement of that of [9] and [7]. We have analyzed the properties of these parametrices by estimating the symbols with parameter associated with the given operator. We shall summerize these results in § 3.

The aim of this paper is to solve the Cauchy problem for an equation of the type (1.1). To this end, we shall apply the Levi method, which is described in detail in the book [2], Ch. 9.

The first step is to construct the fundamental solution starting with the above mentioned parametrices (§ 4, § 5). There we need to obtain the precise estimates of the singularity of parametrices. Those estimates are derived by observing the order of the corresponding symbols more precisely than [8]. The difficulty arises in constructing the fundamental solution since the singularity of the derivative of the first parametrix can no longer be integrable. We shall be able to obtain the fundamental solution as a solution of an integral equation by using the μ-th parametrix with μ sufficiently large (§ 5). As soon as the fundamental solution is obtained, we can prove the existence of the solution of the Cauchy problem by forming the integral expression (§ 6).

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The second step is to prove the uniqueness of the solution. The estimates of the derivatives of the fundamental solution are not available yet, so, in § 7, we shall make use of the method of [1] to prove the uniqueness of the solution, where it is necessary to assume all the coefficients of P are real valued. We shall therefore assume that the coefficients a, b, c are real valued smooth functions throughout this paper, while such an assumption is not necessary in the first step, namely in constructing the fundamental solution. The precise condition and the formulation of the Cauchy problem will be prescribed in § 2. These two steps will prove Theorem 2.1 which is our main result.

Finally, we remark that the degenerate Cauchy problem for a wide class of equations has been treated in [11], [12] by the method of elliptic regularization, and in [6] by using the method of semi-group theory. Our approach is based on the application of the theory of pseudo-differential operators for a classical treatment of the parabolic equations.

§ 2. Main results.

Let $a(x,t)$, $b(x,t)$ and $c(x,t)$ be real valued infinitely differentiable functions defined in $R \times I$, $I = [0, T]$, $T > 0$, and satisfy the following conditions:

(2.1) For any integer $m \geqq 0$, $|D_x^m a(x,t)|$, $|D_x^m b(x,t)|$ and $|D_x^m c(x,t)|$ are bounded in $R \times I$ where $D_x^m = \dfrac{\partial^m}{\partial x^m}$;

$$a(x,t) \geqq 0 \qquad \text{in } R \times I; \tag{2.2}$$

(2.3) there exist an integer $\sigma \geqq 1$ and a real number $\delta > 0$ such that $\int_{t'}^{t} a(x,\tau)d\tau \geqq \delta(t-t')^{\sigma} \quad x \in R, \quad 0 \leqq t' \leqq t \leqq T$;

$$|a_x(x,t)| \leqq Ca(x,t)^{1/2} \qquad \text{in } R \times I; \tag{2.4}$$

(2.5) $|b(x,t)| \leqq Ca(x,t)^{1/2}$ in $R \times I$, where C denotes a positive constant

Under these conditions, we shall solve the following Cauchy problem:

$$Pu = \frac{\partial u}{\partial t} - a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial t} + c(x,t)u = f(x,t) \qquad \text{in } R \times (0,T)\,, \tag{2.6}$$

$$u(x,0) = \varphi(x) \qquad \text{on } R\,, \tag{2.7}$$

where $f(x,t)$ is a measurable function in $R \times I$ and $\varphi(x)$ is a continuous function in R.

By a solution of (2.6), (2.7) we mean a function $u = u(x,t)$ in $C(R \times I)$ which satisfies the equation (2.6) in the distribution sense in $R \times (0,T)$ and satisfies the initial condition (2.7) in the usual sense. Our main result is the following.

THEOREM 2.1. *Let $f(x,t)$ be a measurable function in $R \times I$ and $\varphi(x)$ be a continuous function in R and assume that*

$$(2.8) \qquad |f(x,t)| \leqq M \exp[k|x|^2] \quad \text{a.e. } in\ R \times I\ ,$$

$$(2.9) \qquad |\varphi(x)| \leqq M \exp[k|x|^2] \quad in\ R$$

for some positive constants M and k. Then there exists a solution $u(x,t)$ of the Cauchy problem (2.6), (2.7) *in the strip $R \times [0 \leqq t \leqq t_0]$, where t_0 $(0 < t_0 \leqq T)$ is a constant depending on the operator P and where*

$$(2.10) \qquad |u(x,t)| \leqq \text{const.} \exp[k'|x|^2] \quad in\ R \times [0 \leqq t \leqq t_0]$$

for some constant k'. The solution is unique in the class of continuous functions satisfying (2.10) *with some constant k'. Furthermore the solution $u(x,t)$ is in $C^\infty(R \times [0 < t \leqq t_0])$ if $f(x,t)$ is in $C^\infty(R \times I)$.*

In § 6, we shall prove the existence part of this theorem, and in § 7 we shall prove the uniqueness of the solution. We remark that the solution exists in the whole strip $R \times [0,T]$ if we replace the condition (2.8), (2.9) by

$$(2.8') \qquad |f(x,t)| \leqq M \exp[k|x|^{2-\varepsilon}] \quad \text{a.e. in } R \times I\ ,$$

$$(2.9') \qquad |\varphi(x)| \leqq M \exp[k|x|^{2-\varepsilon}] \quad \text{in } R, \qquad \varepsilon > 0\ .$$

EXAMPLE: The coefficients of the operator

$$P_k = \frac{\partial}{\partial t} - (t^k + e^{-1/|x|})\frac{\partial^2}{\partial x^2} + t^{[(k+1)/2]}\frac{\partial}{\partial x} + \text{const.},\ k \text{ integer} > 0$$

satisfy the conditions (2.1)~(2.5) in $R_x \times [0,\mathrm{T}]$ $(\mathrm{T} > 0)$ with $\sigma = k + 1$.

§ 3. Application of the results of the article I: [8].

We can construct the parametrices for the operator P as in [8] under the conditions (2.1)~(2.5). We note that we have treated the case where $\sigma = 2\ell + 1$, $\ell \geqq 0$ and the coefficients a, b, c have been assumed to be complex valued functions (in [8]). We recall the procedure in the following. We set

$$L_1=\frac{\partial}{\partial t}+a(x,t)\xi^2\ ,$$

$$L_2=-i\xi a(x,t)\frac{\partial}{\partial x}-a(x,t)\frac{\partial^2}{\partial x^2}+b(x,t)\Big(\frac{\partial}{\partial x}+i\xi\Big)+c(x,t)\ ,$$

for $(x, t, \xi)\in R_x\times I\times R_\xi$. Consider the problem:

$$L_1K_0=\Big(\frac{\partial}{\partial t}+a(x,t)\xi^2\Big)K_0(x,\xi\,;t,t')=0 \qquad \text{in } R_x\times R_\xi\times \varDelta\ , \tag{3.1}$$

$$\varDelta\equiv\{(t,t')\,;0\leqq t'<t\leqq T\}\ ,$$

$$K_0(x,\xi\,;t,t')|_{t=t'}=1\ , \tag{3.2}$$

$$K_0(x,\xi\,;t,t')=0 \qquad \text{if } 0\leqq t<t'\leqq T\ . \tag{3.3}$$

Then we have the solution

$$K_0(x,\xi\,;t')=\begin{cases}\exp\Big[-\int_{t'}^{t}a(x,\tau)d\tau\cdot\xi^2\Big]\ , & (x,\xi,t,t')\in R\times R_\xi\times\bar{\varDelta}\ ,\\ 0 \qquad \text{when } 0\leqq t<t'\leqq T\ . & \end{cases} \tag{3.4}$$

We note that the notion of the set $\varDelta$ is slightly different from that of [8]. For $j=0,1,2,\cdots$ we define recurcively the symbol functions $K_j(x,\xi\,;t,t')$ as the solution of the problem:

$$L_1K_{j+1}(x,\xi\,;t,t')=-L_2K_j(x,\xi\,;t,t') \qquad \text{in } R_x\times R_\xi\times\varDelta\ , \tag{3.5}$$

$$K_{j+1}(x,\xi\,;t,t')|_{t=t'}=0\ , \tag{3.6}$$

$$K_{j+1}(x,\xi\,;t,t')=0 \qquad \text{if } 0\leqq t<t'\leqq T\ . \tag{3.7}$$

The K_j's satisfy.

$$K_{j+1}(x,\xi\,;t,t')=\begin{cases}-\int_{t'}^{t}K_0(x,\xi\,;t,s)L_2K_j(x,\xi\,;s,t')ds\ , & \\ \qquad\qquad (x,\xi,t,t')\in R_x\times R_\xi\times\bar{\varDelta} & \\ 0\ , \qquad \text{when } 0\leqq t<t'\leqq T\ . & \end{cases} \tag{3.8}$$

As in [8], we have the following proposition:

PROPOSITION 3.1. (*cf.* [8], *Prop.* 4.1, *Prop.* 5.1) *For any* $\varepsilon>0$ *and for any integer* α *and* $\beta\geqq 0$, *we have*

$$(3.9)\quad K_j(x,\xi;t,t')\in\mathscr{E}(\varDelta;S^{-\infty}(R_x\times R_\xi))\cap\bigcap_{p\geqq 0}\mathscr{E}^p(\bar{\varDelta};S^{\varepsilon+2p-j/\sigma}_{1,(\sigma-1)/\sigma}(R_x\times R_\xi))\,,\quad j=0,1,2,\cdots,$$

$$(3.10)\quad |(K_0(x,\xi;t,t')-1)(1+|\xi|)^{-\varepsilon}\rightrightarrows 0\quad in\ R_x\times R_\xi\quad as\ t\downarrow t'\,,$$

$$(3.11)\quad |(D^p_{t,t'}D^\beta_x D^\alpha_\xi K_j)(1+|\xi|)^{(-\beta(\sigma-1))/\sigma-2p+\alpha+j/\sigma-\varepsilon}|\rightrightarrows 0\quad in\ R_x\times R_\xi\quad as\ t\downarrow t'\quad for\ 0\leqq p<j\,.$$

For the notations in the proposition we refer to the article [8].

By virtue of Proposition 3.1, for every $j\geqq 0$, we can define a distribution $\mathscr{K}_j=\mathscr{K}_j(x,y,t,t')\in\mathscr{D}'(R_x\times R_y\times I_t\times I_{t'})$ by an oscillatory integral:

$$(3.12)\quad \mathscr{K}_j(x,y,t,t')=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{i(x-y)\xi}K_j(x,\xi;t,t')d\xi\,.$$

As in [8], we can show that

(i) $P_{x,t}[\mathscr{K}_0+\cdots+\mathscr{K}_\mu]=\delta(x-y),t-t')+(2\pi)^{-1/2}\int_{-\infty}^{\infty}e_i^{(x-y)\xi}L_2K_\mu(x,\xi;t,t')d\xi$ in $R_x\times R_y\times I_t\times I_{t'}$,

(ii) $\mathscr{K}_j(x,y,t,t')\in C^\infty(W)$, $j=0,1,2,\cdots,$
$W=\{(x,y,t,t')\in R_x\times R_y\times I\times I;|x-y|+|t-t'|>0\}$

(iii) $\mathscr{K}_j(x,y,t,t')$ is very regular in the sense of Schwartz [15], $j=0,1,2,\cdots,$ that is to mean the mappings

$$\psi(y,t')\mapsto\iint_{R\times I}\mathscr{K}_j(x,y,t,t')\psi(y,t')dydt'\,,$$

$$\varphi(x,t)\mapsto\iint_{R\times I}\mathscr{K}_j(x,y,t,t')\varphi(x,t)dxdt$$

define linear continuous transformations from $\mathscr{D}(R\times I)$ into $\mathscr{E}(R\times I)$. These are extended to the linear continuous transformation from $\mathscr{E}'(R\times I)$ into $\mathscr{D}'(R\times I)$ by continuity (cf. [8], Prop. 4.2).

(iv) the second term in the right of (i) is also very regular and becomes smoother in $R_x\times R_y\times I\times I$ according as μ becomes larger.

From these facts we obtain the following theorem.

THEOREM 3.1. (*cf.* [8], *Theorem* 1.1). *The operator P is hypoelliptic in the strip $R \times I$.*

§ 4. Precise estimates of $D_x^m \mathscr{K}_j(x, y, t, t'), m, j = 0, 1, 2, \cdots$.

We set

$$A = A(x, t, t') = \int_{t'}^{t} a(x, \tau) d\tau \qquad (x, t, t') \in R \times \bar{\Delta} ,$$
$$\Delta = \{(t, t') | 0 \leqq t' < t \leqq T\} .$$

Then by the assumptions (2.1)~(2.4) we have

$$\delta(t - t')^{\sigma} \leqq A(x, t, t') \leqq C(t - t') \qquad (x, t, t') \in R \times \bar{\Delta} , \tag{4.1}$$

$$\left| \int_{t'}^{t} a_x(x, \tau) d\tau \right| \leqq C_1(t - t')^{1/2} A^{1/2} \leqq C_2(t - t') \qquad (x, t, t') \in R \times \bar{\Delta} . \tag{4.2}$$

In the following we use the symbols $C, C_1, C_2, \cdots$ to express the different positive constants.

Now we recall that

$$K_0(x, \xi; t, t') = \begin{cases} \exp\left[-\int_{t'}^{t} a(x, \tau) d\tau \cdot \xi^2\right] & (x, \xi, t, t') \in R_x \times R \times \bar{\Delta} , \\ 0 \quad 0 \leqq t < t' \leqq T . \end{cases}$$

Substituting ξ by $\zeta = \xi + i\eta \in C$, we can extend the domain of K_0 to the set $R_x \times \boldsymbol{C} \times I \times I$:

$$K_0(x, \zeta; t, t') = \begin{cases} \exp\left[\int_{t'}^{t} a(x, \tau) d\tau \cdot (\xi + i\eta)^2\right] , & (x, \zeta, t, t') \in R \times \boldsymbol{C} \times \bar{\Delta} , \\ 0 \quad 0 \leqq t < t' \leqq T . \end{cases}$$

LEMMA 4.1. ((*cf.* [8], *Prop.* 4.1) *For every integer $m \geqq 0$ we have*

$$|D_x^m K_0(x, \zeta; t, t')| \leqq C_m(1 + |\xi| + |\eta|)^{(m(\sigma-1))/\sigma} \exp[-\delta_1 A \xi^2 + \delta_2 A \eta^2] , \tag{4.3}$$
$$(x, \zeta, t, t') \in R \times \boldsymbol{C} \times \bar{\Delta} ,$$

where C_m, δ_1, δ_2 are positive constants depending only on m.

Proof. For $m = 0$, we easily have

$$|K_0(x, \zeta; t, t')| = \exp[-A\xi^2 + A\eta^2] , \qquad (t, t') \in \bar{\Delta} . \tag{4.4}$$

For a positive integer m, $D_x^m K_0(x, \zeta; t, t')$ is expressed as a linear combination of terms

$$\left[\prod_j \left(\int_{t'}^t D_x^{m^{(j)}} a(x,\tau)d\tau(\xi+i\eta)^2\right)\right] \exp\left[-\int_{t'}^t a(x,\tau)d\tau(\xi+i\eta)^2\right] \tag{4.5}$$
$$0 < m^{(j)}, m^{(l)} + \cdots + m^{(j)} + \cdots = m .$$

We are now ready to analyse each factor in (4.5). For a factor with $m^{(j)} = 1$ we see by (4.2)

$$\left|\int_{t'}^t a_x(x,\tau)d\tau(\xi+i\eta)^2\right| \leqq C(t-t')^{1/2}A^{1/2}(\xi^2+\eta^2) , \qquad (t,t') \in \bar{\Delta} .$$

Then by (4.1) we have

$$\begin{aligned} &(t-t')^{1/2}A^{1/2}\xi^2 \exp[-\delta_1 A\xi^2] \\ &\leqq |\xi|^{(\sigma-1)/\sigma}((t-t')^\sigma\xi^2)^{1/2\sigma}(A\xi^2)^{1/2}\exp[-\delta_1 A\xi^2] \\ &\leqq C_\varepsilon |\xi|^{(\sigma-1)/\sigma}\exp[-(\delta_1-\varepsilon)A\xi^2] , \end{aligned} \tag{4.6}$$

for any ε such that $0 < \varepsilon < \delta_1$. Similarly we have

$$(t-t')^{1/2}A^{1/2}\eta^2 \exp[\delta_2 A\eta^2] = C_1|\eta|^{(\sigma-1)/\sigma}\exp[(\delta_2+C_2)A\eta^2] \tag{4.7}$$

with some positive constants C_1 and C_2. Next, for a factor with $m^{(j)} \geqq 2$ we have

$$\begin{aligned} \left|\int_{t'}^t D_x^{m^{(j)}} a(x,\tau)d\tau(\xi+i\eta)^2\right| &\leqq C(t-t')(\xi^2+\eta^2) \\ &\leqq C(|\xi|^{(2(\sigma-1))/\sigma} + |\eta|^{(2(\sigma-1))/\sigma})((t-t')^\sigma(\xi^2+\eta^2))^{1/\sigma} . \end{aligned}$$

Hence we have, as above, for $m^{(j)} \geqq 2$

$$\begin{aligned} &\left|\int_{t'}^t D_x^{m^{(j)}} a(x,\tau)d\tau(\xi+i\eta)^2\right| \cdot \exp[-\delta_1 A\xi^2 + \delta_2 A\eta^2] \\ &\leqq C_1(|\xi|^{(2(\sigma-1))/\rho} + |\eta|^{(2(\sigma-1))/\sigma})\exp[-(\delta_1-\varepsilon)A\xi^2 + (\delta_2+C_2)A\eta^2] , \\ &\qquad 0 < \varepsilon < \delta_1 . \end{aligned} \tag{4.8}$$

The assumption (2.1) has been assumed throughout the proof of Lemma 4.1, while we have used the same symbols $C_1, C_2, \delta_1, \delta_2, \cdots$ although they differ in different cases. Combining (4.6), (4.7) and (4.8) we prove Lemma 4.1.

Now we recall that

$$K_{j+1}(x,\xi;t,t') = \begin{cases} -\int_{t'}^t K_0(x,\xi;t,s)L_2K_j(x,\xi;s,t')ds , & (t,t') \in \bar{\Delta} , \\ 0 \quad 0 \leqq t' < t \leqq T , \end{cases}$$

where

$$L_2 = -i\xi a(x,t)\frac{\partial}{\partial x} - a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\left(\frac{\partial}{\partial x} + i\xi\right) + c(x,t) .$$

We can also extend the domain of $K_{j+1}(x,\xi;t,t')$ by substituting ξ be $\zeta = \xi + i\eta$.

LEMMA 4.2. (*cf.* [8], *Prop.* 5.1) *We have*

$$(4.9)\quad \begin{aligned} |D_x^m K_j(x,\zeta;t,t')| &\leqq C_{m,j}(1+|\xi|+|\eta|)^{(m(\sigma-1))/\sigma - j/\sigma}\exp[-\delta_1 A\xi^2 + \delta_2 A\eta^2] , \\ &(x,\zeta,t,t') \in R\times C\times \bar{\Delta} , \qquad j,m=0,1,2,\cdots, \end{aligned}$$

where the constants $C_{m,j}$, δ_1 and δ_2 depend only on m and j.

Proof. We shall use mathematical induction in j. By Lemma 4.1 we have the result in the case $j=0$. Assume the inequality (4.9) for some $j \geqq 0$ to estimate $D_x^m K_{j+1}$ for $m=0,1,2,\cdots$:

$$(4.10)\quad \begin{aligned} D_x^m K_{j+1}(x,\zeta;t,t') &= -D_x^m \int_{t'}^t K_0(x,\zeta;t,s)L_2 K_j(x,\zeta;s,t')ds \\ &= -\sum_{\alpha=0}^m \binom{m}{\alpha} \\ &\quad \cdot \int_{t'}^t D_x^{m-\alpha}K_0(x,\zeta;t,s)D_x^\alpha(L_2K_j(x,\zeta;s,t'))ds . \end{aligned}$$

It suffices to estimate some typical terms in the last expression, and other terms will be treated similarly:

(i) By Lemma 4.1 and by the assumption on K_j we have

$$\begin{aligned} I_{m,\alpha}^{(1)} &\equiv \left|\int_{t'}^t D_x^{m-\alpha}K_0(x,\zeta;t,s)D_x^\alpha(i\zeta a(x,s)D_xK_j(x,\zeta;s,t')ds\right| \\ &\leqq \sum_{\beta=0}^\alpha \binom{\alpha}{\beta}\int_{t'}^t |D_x^{m-\alpha}K_0(x,\zeta;t,s)D_x^{\alpha-\beta+1}K_j(x,\zeta;s,t')D_x^\beta(x,s)|\,ds(|\xi|+|\eta|) \\ &\leqq C\sum_{\beta=0}^\alpha (1+|\xi|+|\eta|)^{(m-\beta+1)(\sigma-1)/\sigma - j/\sigma}\exp[-\delta_1 A\xi^2 + \delta_2 A\eta^2] \\ &\qquad\qquad \cdot \int_{t'}^t |D_x^\beta a(x,s)|\,ds(|\xi|+|\eta|) . \end{aligned}$$

As in the proof of Lemma 4.1 we will estimate each term in the last summation. For the term with $\beta=0$ we have

$$(1+|\xi|+|\eta|)^{(m+1)(\sigma 1)/\sigma - j/\sigma}\exp[-\delta_1 A\xi^2 + \delta_2 A\eta^2]\cdot\int_{t'}^t a(x,\tau)d\tau(|\xi|+|\eta|)$$

$$\leqq C_1(1+|\xi|+|\eta|)^{(m+1)(\sigma-1)/\sigma-j/\sigma-1}\exp[-(\delta_1-\varepsilon)A\xi^2+(\delta_2+C_2)A\eta^2] \qquad (0<\varepsilon<\delta_1)$$
$$=C_1(1+|\xi|+|\eta|)^{(m(\sigma-1)/\sigma-(j+1)/\sigma}\exp[-(\delta_1-\varepsilon)A\xi^2+(\delta_2+C_2)A\eta^2]\,.$$

For the term with $\beta=1$, applying (4.2) we have

$$(1+|\xi|+|\eta|)^{(m(\sigma-1))/\sigma-j/\sigma}\exp[-\delta_1A\xi^2+\delta_2A\eta^2]\int_{t'}^{t}|a_x(x,\tau)|\,d\tau(|\xi|+|\eta|)$$
$$\leqq C_1(1+|\xi|+|\eta|)^{(m(\sigma-1))-(j+1)/\sigma}\exp[-(\delta_1{}^{-}\varepsilon)A\xi^2+(\delta_2+C_2)A\eta^2]\,.$$

We use (4.8), then the terms with $\beta\geqq 2$ are bounded by

$$(1+|\xi|+|\eta|)^{(m-\beta+1)(\sigma-1)/\sigma-j/\sigma+(2(\sigma-1))/\sigma}\exp[-(\delta_1-\varepsilon)A\xi^2+(\delta_2+C_2)A\eta^2]$$
$$\leqq(1+|\xi|+|\eta|)^{(m(\sigma-1))/\sigma-(j+1)/\sigma}\exp[-(\delta_1-\varepsilon)A^2+(\delta_2+C_2)A\eta^2]\,.$$

Thus we have

$$I^{(1)}_{m,\alpha}\leqq C(1+|\xi|+|\eta|)^{(m(\sigma-1))/\sigma-(j+1)/\sigma}\exp[-\delta_1A\eta^2+\delta_2A\eta^2] \tag{4.11}$$

with some constants C, δ_1 and δ_2 depending only on m, j.

(ii) We have

$$I^{(2)}_{m,\alpha}\equiv\left|\int_{t'}^{t}D_x^{m-\alpha}K_0(x,\zeta;t,s)D_x^{\alpha}(i\zeta b(x,s)K_j(x,\zeta;s,t')ds\right|$$
$$\leqq\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_{t'}^{t}|D_x^{m-\alpha}K_0(x,\zeta;t,s)D_x^{\alpha-\beta}K_j(x,\zeta;s,t')D_x^{\beta}b(x,s)|\,ds(|\xi|+|\eta|)$$
$$\leqq C\sum_{\beta=0}^{\alpha}(1+|\xi|+|\eta|)^{(m-\beta)(\sigma-1)/\sigma-j/\sigma}\cdot\exp[-\delta_1A\xi^2+\delta_2A\eta^2]$$
$$\cdot\int_{t'}^{t}|D_x^{\beta}b(x,s)|\,ds(|\xi|+|\eta|)\,.$$

We will estimate the term with $\beta=0$:

$$I=(1+|\xi|+|\eta|)^{(m(\sigma-1))/\sigma-j/\sigma}\exp[-\delta_1A\xi^2+\delta_2A\eta^2]\cdot\int_{t'}^{t}|b(x,s)|\,ds(|\xi|+|\eta|)\,.$$

By the assumption (2.5) we have

$$\int_{t'}^{t}|b(x,s)|\,ds(|\xi|+|\eta|)\leqq C(t-t')^{1/2}A^{1/2}(|\xi|+|\eta|)$$
$$\leqq C(|\xi|^{-1/\sigma}+|\eta|^{-1/\sigma})((t-t')^{\sigma}(\xi^2+\eta^2)^{1/2\alpha}(A(\xi^2+\eta^2))^{1/2})\,.$$

Then we have

$$I\leqq C_1(1+|\xi|+|\eta|)^{(m(\sigma-1))-(j+1)/\sigma}\exp[-(\delta_1-\varepsilon)A\xi^2+(\delta_2+C_2)A\eta^2] \qquad (0<\varepsilon<\delta_1)\,.$$

To estimate the terms with $\beta \geqq 1$ we use the assumption (2.1).

$$II \equiv \sum_{\beta=1}^{\alpha} (1 = |\xi| + |\eta|)^{(m-\beta)(\sigma-1)/\sigma j/\sigma} \exp[-\delta_1 A\xi^2 + \delta_2 A\eta^2]$$
$$\cdot \int_{t'}^{t} |D_x^\beta b(x,s)|\, ds(|\xi| + |\eta|)$$
$$\leqq C(1 + |\xi| + |\eta|)^{(m-1)(\sigma-1)/\sigma - j/\sigma} \exp[-\delta_1 A\xi^2 + \delta_2 A\eta^2]\cdot(t-t')(|\xi| + |\eta|)\,,$$
$$(t-t')(|\xi| + |\eta|) \leqq C(|\xi| + |\eta|)^{1-(2/\sigma}((t-t')^\sigma(\xi^2+\eta^2))^{1/\sigma}\,.$$

Thus we have

$$II \leqq C_1(1 + |\xi| + |\eta|)^{(m(\sigma-1))/\sigma - (j+1)/\sigma} \exp[-(\delta_1 - \varepsilon)A\xi^2 + (\delta_2 + C_2)A\eta^2]\,.$$

Combining the above two estimates we have

$$I_{m,\alpha}^{(2)} \leqq C(1 + |\xi| + |\eta|)^{(m(\sigma-1))/\sigma - (j+)/\sigma} \exp[-\delta_1 A\xi^2 + \delta_2 A\eta^2]\,, \tag{4.12}$$

where C, δ_1 and δ_2 depend only on m and j.

To complete the proof of Lemma 4.2, it remains to estimate the terms:

$$D_x^m \int_{t'}^{t} K_0(x,\zeta;t,s)a(x,s)D_x^2 K_j(x,\zeta;s,t')ds\,,$$
$$D_x^m \int_{t'}^{t} K_0(x,\zeta;t,s)b(x,s)D_x K_j(x,\zeta;s,t')ds\,,$$
$$D_x^m \int_{t'}^{t} K_0(x,\zeta;t,s)c(x,s)K_j(x,\zeta;s,t')ds$$

which are treated in the same manner as above and we get Lemma 4.2.

Using Lemma 4.2 we shall obtain the precise estimate of $\mathscr{K}_j$:

$$\mathscr{K}_j(x,y,t,t') = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{i(x-y)\xi} K_j(x,\xi;t,t')d\xi \quad (3.12)\,.$$

THEOREM 4.3. *For every integers m, $j \geqq 0$ the following inequality holds*:

$$|D_x^m \mathscr{K}_j(x,y,t,t')| \leqq CA^{j/2\sigma-(1+m)/2} \exp\left[\frac{-c(x-y)^2}{A}\right], \tag{4.13}$$
$$(x,y,t,t') \in R \times R \times \Delta$$

where C and c are positive constants depending only on m and j.

Proof. We can easily see that $K_j(x,\zeta;t,t')$, m, $j = 0,1,\cdots,\cdot$ are entire functions in $\zeta, \zeta \in \boldsymbol{C}$. Using Cauchy's theorem and the inequality (4.9), we find that

$$\mathscr{K}_j(x,y,t,t') = (2\pi)^{-1/2}\int_{R_\xi} e^{i(x-y)(\xi+i\eta)} K_j(x,\xi+i\eta;t,t')d\xi \tag{4.14}$$

is independent of η, and hence, it coincides with the functions $\mathscr{K}_j$ defined in (3.12) for $0 \leqq t' < t \leqq T$. We have

$$D_x^m \mathscr{K}_j(x,y,t,t') = (2\pi)^{-1/2}\sum_{\alpha=0}^{m}\binom{m}{\alpha}$$

$$\cdot \int_{R_\xi} e^{i(x-y)(\xi+i\eta)}(i\xi-\eta)^{m-\alpha} D_x^\alpha K_j(x,\xi+i\eta;t,t')d\xi\,, \qquad 0 \leqq t' < t \leqq T\,.$$

We shall analyse each term of the right side. Using (4.9) again we set

$$\left|\int_{R_\xi} e^{i(x-y)(\xi+i\eta)}(i\xi-\eta)^{m-\alpha} D_x^\alpha K_j(x,\xi+i\eta;t,t')d\xi\right|$$

$$\leqq Ce^{-(x-y)\eta}\int_{R_\xi}(1+|\xi|+|\eta|)^{m-\alpha+(\alpha(\sigma-1))/\sigma-j/\sigma}\exp[-\delta_1 A\xi^2+\delta_2 A\eta^2]d\xi$$

$$\leqq C_1 e^{-(x-y)\eta}\exp[\delta_2 A\eta^2]\int_{R_\xi}(1+|\xi|)^{m-j/\sigma}\exp[-\delta_1 A\xi^2]d\xi$$

$$+\, C_2 e^{-(x-y)\eta}|\eta|^{m-j/\sigma}\exp[\delta_2 A\eta^2]\int_{R_\xi}\exp[-\delta_1 A\xi^2]d\xi$$

$$\equiv I + II\,.$$

Take

$$\eta = \frac{x-y}{2\delta_2 A(x,t,t')}\,, \qquad (x,y,t,t')\in R\times R\times \varDelta\,. \tag{4.15}$$

Then we obtain

$$I \leqq C_1' A^{j/2\sigma-1/2(1+m)}\exp\left[\frac{-(x-y)^2}{4\delta_2 A}\right].$$

As for II we have

$$II \leqq C' A^{-1/2}\exp\left[-\frac{(x-y)^2}{4\delta_2 A}\right]\left(\frac{x-y}{2\delta_2 A}\right)^{m-j/\sigma}$$

$$= C' A^{-1/2}\exp\left[\frac{(x-y)^2}{4\delta_2 A}\right]\left(\frac{x-y}{2\delta_2 A^{1/2}}\right)^{m-j/\sigma} A^{-1/2(m-j/\sigma)}$$

$$\leqq C'' A^{j/2\sigma-1/2(1+m)}\exp\left[-\frac{(x-y)^2}{5\delta_2 A}\right] \qquad \text{Q.E.D.}$$

§ 5. Construction of the fundamental solution.

As in the book [2], we give the following definition.

DEFINITION 5.1. A fundamental solution of $Pu=0$ in $R_x \times I$ is a function $\Gamma(x, y, t, t')$ defined in $R_x \times R_y \times \Delta$ which satisfies the following conditions:

(i) for fixed (y, t'), Γ is, as a function of (x, t) $(x \in R, 0 \leqq t' < t \leqq T)$, a solution of the equation $Pu=0$;

(ii) it holds that

$$\lim_{t \downarrow t'} \int_{R_y} \Gamma(x, y, t, t')\varphi(y)dy = \varphi(x) , \qquad x \in R_x , \tag{5.1}$$

for every continuous function $\varphi(x)$ such that

$$|\varphi(x)| \leqq \text{const.} \exp[hx^2] , \qquad h > 0 . \tag{5.2}$$

LEMMA 5.1. *Let $\varphi(x)$ be a continuous function satisfying* (5.2) *with some positive constant h. Then we have*

$$\lim_{t \downarrow t'} \int_{R_y} \mathscr{K}_0(x, y, t, t')\varphi(y)dy = \varphi(x) , \qquad x \in R_x , \tag{5.3}$$

$$\lim_{t \downarrow t'} \int_{R_y} \mathscr{K}_j(x, y, t, t')\varphi(y)dy = 0, x \in R_x , \qquad j = 1, 2, \cdots . \tag{5.4}$$

Proof. We can easily see that

$$\mathscr{K}_0(x, y, t, t') = \frac{1}{2\sqrt{\pi A}} \exp\left[-\frac{(x-y)^2}{4A}\right] .$$

Then it follows that

$$\int_{-\infty}^{\infty} \mathscr{K}_0(x, y, t, t')dy = 1 .$$

Thus we have

$$\int_{-\infty}^{\infty} \mathscr{K}_0(x, y, t, t')\varphi(y)dy = \varphi(x) + \int_{-\infty}^{\infty} \mathscr{K}_0(x, y, t, t')(\varphi(y) - \varphi(x))dy$$
$$\equiv \varphi(x) + I .$$

The integral I is divided into two parts: I_1 with $|y-x| < \delta$ and I_2 with $|y-x| \geqq \delta$, and take δ such that $|f(y) - f(x)| < \varepsilon$ if $|y-x| < \delta$. Here ε is any fixed positive number, and δ is a fixed positive number depending on ε. If we use the estimate (4.13) we obtain

$$\int_{-\infty}^{\infty} |\mathscr{K}_0(x, y, t, t')| \, dy \leqq C \qquad 0 \leqq t' < t \leqq T , \tag{5.5}$$

where C is independent of t, t'. Similarly we see that

$$(5.6)\qquad \int_{|y-x|\geqq\delta} |\mathscr{K}_0(x, y, t, t')| \exp[2h(x-y)^2]dy \to 0 \qquad t \downarrow t' .$$

Using (5.4) it follows that

$$(5.7)\qquad |I_1| \leqq \varepsilon \int_{|y-x|<\delta} |\mathscr{K}_0(x, y, t, t')|\, d\xi \leqq C\cdot\varepsilon .$$

Using (5.6) and the inequality

$$(5.8)\qquad |\varphi(y)| \leqq \text{const.} \exp[2h(x-y)^2] ,$$

where the constant depends on x, it is proved that $I_2 \to 0$ as $t \downarrow t'$ (for each fixed x). Hence $|I_2| < \varepsilon$ if $t - t'$ is sufficiently small. Combining this with (5.7) we get $|I| \leqq (C+1)\varepsilon$ if $t - t'$ is sufficiently small. Since ε is arbitrary, (5.3) follows.

To prove (5.4), using the inequalities (4.13) and (5.8) we have, for any fixed $x \in R$,

$$\begin{aligned}\left|\int |\mathscr{K}_j(x, y, t, t')\varphi(y)dy\right| &\leqq C_x A^{j/2\sigma} \int A^{-1/2} \exp\left[\frac{c_x(x-y)^2}{A}\right]dy \\ &\leqq C'A(x, t, t')^{j/2\sigma} \to 0 \quad \text{as } t \downarrow t' , \quad j = 1, 2, \cdots ,\end{aligned}$$

Now we set

$$E_\mu = \sum_{j=0}^{\mu} \mathscr{K}_j(x, y, t, t') , \qquad \mu = 0, 1, 2, \cdots .$$

Then we have

$$\begin{aligned}P_{x,t}E_\mu &= \delta(x-y, t-t') + (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i(x-y)\xi} L_2 K_\mu(x, \xi; t, t')d\xi \\ &\equiv \delta(x-y, t-t') + F_\mu(x, y, t, t') , \qquad (\S\,3,\ (\text{i})) .\end{aligned}$$

LEMMA 5.2. *We have*

$$(5.9)\quad |F_\mu(x, y, t, t')| \leqq CA^{-3/2+\mu/2\sigma} \exp\left[-\frac{c(x-y)^2}{A}\right] , \qquad \mu = 0, 1, 2, \cdots,$$
$$(x, y, t, s') \in R \times R \times \varDelta ,$$

where the constants C and c depend only on μ.

Proof. If we substitute ξ by $\zeta = \xi + i\eta \in \boldsymbol{C}$, then $L_2K_\mu(x, \zeta; t, t')$ is an entire function of ζ. By Lemma 4.2 we have

$$|L_2K_\mu(x,\zeta;t,t')| = C(1+|\xi|+|\eta|)^{2-\mu/\sigma}\exp[-\delta_1A\xi^2+\delta_2A\eta^2],$$

where the constants C, δ_1 and δ_2 depend only on μ. Using Cauchy's theorem and setting $\eta=(x-y)/2\delta_2A(x,t,t')$ as in the proof of Theorem 4.3, we have the inequality (5.9).

We shall need the following lemmas.

LEMMA 5.3. *(cf. [2], Ch. 1. Lemma 1.) Let $f(x,y)$ be a continuous function of (x,y) when x,y bury in a compact domain S of $R_x^m\times R_y^m$ and $x\neq y$, and let*

$$\int_{S(x,\varepsilon)}|f(x,y)|dy\to 0 \qquad as\ \varepsilon\to 0,$$

uniformly with respect to x in S, where $S(x,\varepsilon)$ is the intersection of S with the ball with center x and radius ε. Then, for any bounded measurable function $g(y)$ in S, the (improper) integral

$$h(x)=\int_S f(x,y)g(y)dy$$

is a continuous function in S.

LEMMA 5.4. *Let $f(x,t)$ be a measurable function in $R\times I_0$, $I_0=[T_0,T_1]$, $0\leqq T_0<T_1\leqq T$, satisfying*

(5.10) $\quad |f(x,t)|\leqq const.\exp[\lambda x^2]$ *a.e. in $R\times I_0$ for $\lambda\leqq c(T_1-T_0)^{-\sigma}$*

where c is chosen depending only on μ.
Then

$$\Phi(x,t)=\int_{T_0}^t\int_{R_y}E_\mu(x,y,t,t')f(y,t')dydt', \qquad \mu=0,1,2,\cdots, \tag{5.11}$$

$$\Psi(x,t)=\int_{T_0}^t\int_{R_y}F_\mu(x,y,t,t')f(y,t')dydt', \qquad \mu\geqq 2\sigma \tag{5.12}$$

are continuous functions in $R\times I_0$ and

$$\lim_{t\downarrow T_0}\Phi(x,t)=0, \qquad x\in R_x. \tag{5.13}$$

Furthermore, for $\mu\geqq\sigma/2$ we have the following equality in the distribution sense:

$$P\Phi(x,t)=f(x,t)+\Psi(x,t) \qquad in\ R\times(T_0,T_1). \tag{5.14}$$

Proof. We note that $\Phi(x, t)$ is called the volume potential of f with respect to the parametrix E_μ. This is an improper integral, the integrand having a singularity at $y = x$, $t' = t$. However the singularity is integrable. Indeed, by Theorem 4.3, we have

$$(5.15) \qquad |E_\mu(x, y, t, t')| \leqq \text{const. } A^{-1/2} \exp\left[-\frac{c_\mu(x-y)^2}{A}\right] .$$

Then we have

$$\int_{-\infty}^{\infty} |E_\mu(x, y, t, t')|\, dy \leqq \text{const. } c_\mu{}^{1/2} .$$

Hence the singularity is integrable.
If

$$(5\cdot16) \qquad \lambda < \frac{c_\mu}{c(T_1 - T_0)}$$

then $\lambda < (c_\mu - \varepsilon)/A$ by the assumption (2.3), and the integral in (5.11) exists.

The continuity of $\Phi(x, t)$ follows by breaking the y-integral into two parts and treating each part separately. The continuity of the integral corresponding to the unbounded part D_1 of R_y (x is bounded away from D_1) follows by a standard theorem of calculus, whereas the continuity of the integral corresponding to the bounded part D_0 of R_y follows by employing Lemma 5.3.

The continuity of $\psi(x, t)$ is obtained similarly as above taking the bound of λ smaller than (5.16) if necessary. Indeed, we have by (5.9)

$$(5.17) \qquad |F_\mu(x, yt, t')| \leqq C'_\mu A^{-1/2} \exp\left[-\frac{c_\mu(x-y)^2}{A}\right]$$

if $\mu \geqq 2\sigma$ and the singularity is integrable as above.

Remembering that

$$E_\mu(x, y, t, t') = 0 \qquad t < t' ,$$

we have (5.13) by the same consideration as above.

It remains to prove the equality (5.14). By the property (i) of § 3 it follows that if $\mu \geqq 2\sigma$

$$P_{x,t}\int_{T_0}^{t}\int_{R_y} E_\mu(x, y, t, t')\varphi(y, t')dydt'$$
$$= \varphi(x, t) + \int_{T_0}^{t}\int_{R_y} F_\mu(x, y, t, t')\varphi(y, t')dydt'$$

for any $\varphi(x, t) \in C_0^\infty(R \times [T_0, T_1])$. By continuity this equality holds in the distribution sense for any bounded measurable function $\varphi(x, t)$ with compact support in $R \times I_0$. For $f(x, t)$ as given in Lemma 5.4, we set

$$f_n(x, t) = \begin{cases} f(x, t) & |x| \leqq n , \\ 0 & |x| > n , \end{cases} \qquad n = 1, 2, \cdots .$$

Then we find that

$$f_n(x, t) \to f(x, t) ,$$
$$\Phi_n(x, t) = \int_{T_0}^{t}\int_{R_y} E_\mu(x, y, t, t') f_n(y, t')dydt' \to \Phi(x, t) ,$$
$$\Psi_n(x, t) = \int_{T_0}^{t}\int_{R_y} F_\mu(x, y, t, t') f_n(y, t')dydt' \to \Psi(x, t)$$

in the distribution sense as $n \to \infty$. Thus we obtain the equality (5.14) and this completes the proof of Lemma 5.4.

Now applying the parametrix method (cf. [2], Ch. 9), we shall construct the fundamental solution Γ in the form

$$\Gamma(x, y, t, t') = E_\mu(x, y, t, t') + \int_{t'}^{t} ds \int_{R_z} E_u(x, z, t, s)\Phi(z, y, s, t')dz \tag{5.18}$$

when $\mu \geqq 3\sigma$.

If Φ is a function such that Lemma 5.4 can be applied to the integral on the right hand side of (5.18), then Γ satisfies the equation $P\Gamma = 0$ as a function of (x, t) (see (i) of Def. 5.1) if and only if

$$\begin{aligned} &\Phi(x, y, t, t') \\ &\quad = -F_\mu(x, y, t, t') - \int_{t'}^{t} ds \int_{R_z} F_\mu(x, z, t, s)\Phi(z, y, s, t')dz (t > t') . \end{aligned} \tag{5.19}$$

The following series is a formal solution of (5.19):

$$\Phi(x, y, t, t') = \sum_{m=1}^{\infty} \Phi_m(x, y, t, t') , \tag{5.20}$$

where $\Phi_1 = F_\mu$ and

$$\Phi_m(x, y, t, t') = \int_{t'}^{t} ds \int_{R_z} \Phi_1(x, z, t, s)\Phi_{m-1}(z, y, s, t')dz , \qquad m \geqq 2 . \tag{5.21}$$

Setting $a_\mu = \frac{1}{2}\left(\frac{\mu}{\sigma} - 3\right)$, $\mu \geqq 3\sigma$, we have by (5.9)

$$|F_\mu| \leqq c_1(t - t')^{a_\mu} \exp\left[\frac{-c_2(x - y)^2}{(t - t')}\right], \tag{5.22}$$

where the constants c_1 and c_2 depend only on μ. Thus $F_\mu = \Phi_1$ is a bounded function if $\mu \geqq 3\sigma$.

We shall prove by induction in m that

$$|\Phi_{m+1}(x, y, t, t')| \leqq C_1 A_1^m (t - t')^{m(a_\mu+1)} \frac{1}{m!} \exp\left[\frac{A_2(x - y)^2}{(t - t')}\right], \quad 0 < A_2 < c_2 . \tag{5.23}$$

Indeed, (5.22) implies (5.23) the case $m = 0$. Assume (5.23) for $m \geqq 0$, then we have

$$\begin{aligned} |\Phi_{m+2}| &= \frac{C_1^2 A_1^m}{m!} \int_{t'}^t \mathrm{ds} \int_{R_z} (t - s)^{a_\mu} \exp\left[\frac{-c_2(x - z)^2}{t - s}\right](s - t')^{m(a_\mu+1)} \\ &\qquad \cdot \exp\left[\frac{A_2(z - y)^2}{s - t'}\right] dz \\ &\leqq \frac{C_1^2 A_1^m}{m!}(t - t')^{a_\mu} \int_{t'}^t (s - t')^{m(a_\mu+1)} ds \int_{R_z} \exp\left[-\frac{(c_2 - A_2)(x - z)^2}{t - s}\right] \\ &\qquad \cdot \exp\left[-A_2\left(\frac{(x - z)^2}{t - s} + \frac{(z - y)^2}{s - t'}\right)\right] dz . \end{aligned}$$

We can easily see

$$\min_{s,t}\left[\frac{(x - z)^2}{t - s} + \frac{(z - y)^2}{s - t'}\right] = \frac{(x - y)^2}{t - t'} .$$

Thus we have

$$\begin{aligned} |\Phi_{m+2}| &\leqq \frac{C_1^2 A_1^m}{m!}(t - t')^{a_\mu} \cdot \exp\left[-\frac{A_2(x - y)^2}{t - t'}\right] \cdot \int_{t'}^t (s - t')^{m(a_\mu+1)} ds \int_{R_z} \\ &\qquad \cdot \exp\left[-\frac{(c_2 - A_2)(x - z)^2}{t - s}\right] dz \\ &\leqq \frac{C_1 A_1^m c_1 C T^{1/2}}{(m!(m(a_\mu + 1) + 1)}(t - t')^{(m+1)(a_\mu+1)} \exp\left[-\frac{A_2(x - y)^2}{t - t'}\right] . \end{aligned}$$

If we take $A_1 = \max(c_1, c_2 C T^{1/2})$, the proof of (5.23) is completed.

From (5.23) it follows that the series in (5.20) is convergent and that Φ satisfies (5.19). Furthermore, from the estimates of Φ_m ($m \geqq 1$) we see that

$$|\Phi(x, y, t, t')| \leqq A_1'(t-t')^{a_\mu} \exp\left[-\frac{A_2(x-y)^2}{t-t'}\right], \tag{5.24}$$

where the constants A_1' and A_2 depend only on μ. It is then clear that Lemma 5.4 is applied to the integral on the right hand side of (5.18) and thus Γ satisfies the equation $P_{x,t}\Gamma(x, y, t, t') = 0$ ($0 \leqq t' < t \leqq T$).

The second property of fundamental solution, namely (5.1), follows from Lemma 5.1 and the following estimate:

$$\left|\int_{t'}^{t} ds \int_{R_z} E_\mu(x, y, t, s)\Phi(z, y, s, t')dz\right| \leqq C_1(t-t')^{a_\mu} \exp\left[-\frac{C_2(x-y)^2}{t-t'}\right], \tag{5.25}$$

where the constants C_1 and C_2 depend only on μ. We have (5.25) as follows:

$$\begin{aligned}\left|\int_{t'}^{t} ds \int_{R_z} E_\mu \Phi dz\right| &\leqq \text{const.}\,(t-t')^{a_\mu}\cdot \int_{t'}^{t} ds \int_{R_z} (t-s)^{-\sigma/2} \\ &\quad \cdot \exp\left[-\frac{C'(x-y)^2}{t-s} - \frac{C''(z-y)^2}{s-t'}\right] \qquad \text{(Th. 4.3)} \\ &\leqq \text{const.}\,(t-t')^{a_\mu}\cdot \exp\left[-\frac{\text{const.}\,(x-y)^2}{t-t'}\right] \\ &\quad \cdot \int_{t'}^{t} (t-s)^{\sigma/2(\gamma-1)} ds \int_{R_z} (x-z)^{-\gamma} \exp\left[\frac{\text{const.}\,(x-z)^2}{t-s}\right] dz \\ &\qquad\qquad \left(1-\frac{2}{\sigma} < \gamma < 1\right) \\ &\leqq C_1(t-t')^{a_\mu} \exp\left[\frac{C_2(x-y)^2}{t-t'}\right].\end{aligned}$$

Thus we have obtained the following theorem.

THEOREM 5.5. *There exists a fundamental solution* $\Gamma(x, y, t, t')$ (*given in* (5.18)) *of* $Pu = 0$ *in* $R_x \times I$ *satisfying the inequality*:

$$|\Gamma(x, y, t, t')| \leqq CA^{-1/2} \exp\left[-\frac{c(x-y)^2}{A}\right] + C'(t-t')^{a_\mu} \cdot \exp\left[-\frac{c'(x-y)^2}{t-t'}\right], \tag{5.26}$$

where

$$A = \int_{t'}^{t} a(x, \tau) d\tau , \qquad a_\mu = \frac{1}{2}\Big(\frac{\mu}{\sigma} - 3\Big) ; \mu \geqq 3\sigma ,$$

and the constants C, c, C' and c' depend only on μ.

§ 6. Existence of solution.

THEOREM 6.1. (*Existence*) *Let $f(x, t)$ be a measurable function in $R \times I$ and $\varphi(x)$ be a continuous function in R and assume that*

$$(2.8) \qquad |f(x, t)| \leqq M \exp [kx^2] \quad a.e. \quad in\ R \times I ,$$

$$(2.9) \qquad |\varphi(x)| \leqq M \exp [kx^2] \qquad in\ R .$$

Then the function

$$(6.1) \quad u(x, t) = \int_0^t \int_{R_y} \Gamma(x, y, t, t') f(y, t') dy dt' + \int_{R_y} \Gamma(x, y, t, 0) \varphi(y) dy$$

is a solution of the Cauchy problem (2.6), (2.7) *in the strip $0 < t < t_0$ where $t_0 = \min\Big(T, \frac{\bar{c}}{k}\Big)$ and where $\bar{c}$ is a constant depending on P, and*

$$(6.2) \qquad |u(x, t)| \leqq const. \exp [k'x^2] \qquad x \in R , \qquad 0 \leqq t \leqq t_0 ,$$

for some constant k'. The solution $u(x, t)$ is in $C^\infty(R \times (0, t_0])$ if $f(x, t)$ is in $C^\infty(R \times I)$.

We prepare the following simple lemma whose proof is omitted.

LEMMA 6.2. *For any positive numbers A and B with $B < \frac{1}{3}A$, there exists a positive constant C such that, for all $x \in R$,*

$$(6.3) \qquad \int_{R_y} \exp [-A(x - y)^2] \exp [By^2] dy \leqq CA^{-1/2} \exp\Big[\frac{2}{3}Ax^2\Big] .$$

Proof of Theorem 6.1. First we consider the function

$$(6.4) \qquad u_1(x, t) = \int_0^t \int_{R_y} \Gamma(x, y, t, t') f(y, t') dy dt' .$$

By (5.18), decompose $u_1(x, t)$ into two parts;

$$\begin{aligned} u_1(x, t) &= \int_0^t \int_{R_y} E_\mu(x, y, t, t') f(y, t') dy dt' \\ &\quad + \int_0^t \int_{R_y} \Big[\int_{t'}^t \int_{R_z} E_\mu(x, z, t, s) \Phi(z, y, s, t') dz ds\Big] f(y, t') dy dt' \\ &\equiv I + II . \end{aligned}$$

By Lemma 5.4 we are given

$$P_{x,t}I = f(x, t) + \int_0^t\int_{R_y} F_\mu(x, y, t, t')f(y, t')dydt' \tag{6.5}$$

in a strip $0 < t < t_1$. And by using the estimates (5.10), (5.15) and Lemma 6.2 we have

$$|I| \leqq \text{const.} \exp[\text{const.}\ x^2]\ , \qquad 0 \leqq t \leqq t_1\ . \tag{6.6}$$

Next we analyse the integral II. Define $\Phi(x, y, t, t') = 0$ for $t < t'$. Then after changing the order of integration the integral II can be written in the form:

$$II = \int_0^t\int_{R_z} E_\mu(x, z, t, s)\left[\int_0^t\int_{R_y} \Phi(z, y, s, t')f(y, t')dydt'\right]dzds\ .$$

By Lemma 6.2 and by the estimate (5.23) we have

$$\left|\int_0^t\int_{R_y} \Phi(z, y, s, t')f(y, t')dydt'\right| \leqq C_1 \exp[C_2 z^2] \tag{6.7}$$

in a strip $0 < s < t_2$.
Thus by Lemma 5.4, the integral II has a meaning in some strip $0 \leqq t \leqq t_3$; in fact this is a continuous function in the strip $0 \leqq t \leqq t_3$ by the same reasoning as in the proof of Lemma 5.4, and we have the following equality in the distribution sense:

$$\begin{aligned} P_{x,t}II &= \int_0^t\int_{R_y} \Phi(x, y, t, t')f(y, t')dydt' \\ &\quad + \int_0^t\int_{R_y} F_\mu(x, z, t, s)\left[\int_0^t\int_{R_y} \Phi(z, y, s, t')f(y, t')dydt\right]dzds\ , \\ &\qquad 0 < t < t_4\ , \qquad t_4 = \min(t_1, t_2, t_3)\ . \end{aligned} \tag{6.8}$$

Furthermore, by (5.22) and by using Lemma 6.2 again, we have

$$|II| \leqq \text{const.} \exp[\text{const.}\ x^2] \qquad 0 \leqq t \leqq t_4\ . \tag{6.9}$$

Summing up the above considerations we have

$$P_{x,t}u_1(x, t) = f(x, t) \qquad \text{in } R \times (0, t_4)\ , \tag{6.10}$$

$$|u_1(x, t)| \leqq \text{const.} \exp[\text{const.}\ x^2] \qquad \text{in } R \times [0, t_4]. \tag{6.11}$$

By (5.26) and by the similar way to the proof of Lemma 5.4 we have

$$\lim_{t \downarrow 0} u_1(x, t) = 0\ . \tag{6.12}$$

Now we consider the function

$$u_2(x,t)=\int_{R_y}\Gamma(x,y,t,0)\varphi(y)dy\ . \tag{6.13}$$

By using Lemma 6.2, we have

$$|u_2(x,t)|\leqq \text{const.}\ \exp[\text{const.}\ x^2] \tag{6.14}$$

in some strip $0\leqq t\leqq t_5$. By the properties of fundamental solution derived in § 5, we can easily see that $u_2(x,t)$ is a continuous function in the strip $0\leqq t\leqq t_3$ and

$$\lim_{t\downarrow 0} u_2(x,t)=\varphi(x)\qquad x\in R\ . \tag{6.15}$$

Furthermore, we can easily see that $u_2(x,t)$ satisfies the equation

$$Pu_2(x,t)=0\qquad \text{in } R\times(0,t_5) \tag{6.16}$$

in the distribution sense, hence in the usual sense by virtue of hypoellipticity of P. Combining (6.10), (6.12), (6.16) and (6.15) it follows that $u(x,t)$ defined in (6.1) is a solution of the Cauchy problem (2.6), (2.7) in a strip $0\leqq t\leqq t_0$, $t_0=\min(t_4,t_5)$. We note that the constants t_j, $1\leqq j\leqq 5$, are chosen in the form $\min\{T, C_j^*/k\}$ where C_j^* is a constant depending on P (and μ) so the same is true for t_0.

Finally, by the hypoellipticity of P we have $u(x,t)\in C^\infty\ (R\times(0,t_0])$ if $f(x,t)\in C^\infty(R\times I)$. In this case, we have $u(x,t)\in C(R\times[0,t_0])\cap C^\infty(R\times(0,t_0])$. Q.E.D.

§ **7. Uniqueness of solution.** As stated in the introduction we shall follow the method of [1]. Let Ω be an open finite interval in $R_x=\{x\,;\,-\infty<x<\infty\}$. We set $Q=\Omega\times(0,T)$, $\partial_pQ=\{\partial\Omega\times[0,T]\}\cup\{\Omega\times(t=0)\}$ and $\tilde{Q}=\overline{Q}/\partial_pQ$.

THEOREM 7.1. *(Maximum principle.) Let P be the parabolic operator given in* § 2. *Let $u(x,t)$ be a real valued function in $C^{2,1}_{x,t}(\tilde{Q})\cap C(Q)$ satisfying*

$$P_u\leqq 0\qquad in\ \tilde{Q}\ ,$$
$$u\leqq 0\qquad on\ \partial_pQ\ .$$

Then we have $u(x,t)\leqq 0$ in $\overline{Q}$.

Proof. (a) The case $c(x,t)>0$ in $\overline{Q}$. Assume that there were a point $(x_0, t_0)\in\tilde{Q}$ such that

$$\max_{\overline{Q}} u = u(x_0, t_0) > 0 .$$

Then the following must hold at (x_0, t_0):

$$\frac{\partial u}{\partial x} = 0\ , \quad \frac{\partial u}{\partial t} \geqq 0\ , \quad cu > 0\ .$$

And then

$$Pu|_{(x_0,t_0)} = \frac{\partial u}{\partial t}(x_0, t_0) - \alpha(x_0, t_0)\frac{\partial^2 u}{\partial x^2} + c(x_0, t_0)u(x_0, t_0) > 0\ ,$$

which is a contradiction.

(b) If $c(x,t)\geqq -\gamma$ in $\overline{Q}$ for some $\gamma>0$, we can reduce to the case $c(x,t)>0$. Setting $u(x,t)=e^{\gamma' t}v$ with any γ' larger than γ, we have

$$\begin{aligned} Pv + \gamma' v &\geqq 0 \qquad \text{in } \tilde{Q}\ , \\ v &\leqq 0 \qquad \text{on } \partial_p Q\ , \\ c(x,t) + \gamma' &> 0 \qquad \text{in } \overline{Q}\ , \end{aligned}$$

from which our assertion follows immediately.

COROLLARY 7.2. *Let $u(x,t)$ be a real valued function in $C^{2,1}_{x,t}(\tilde{Q})\cap C(Q)$ satisfying*

$$\begin{aligned} Pu &= 0 \qquad \textit{in } \tilde{Q}\ , \\ u &= 0 \qquad \textit{on } \partial_p Q\ . \end{aligned}$$

Then $u(x,t)=0$ in $\overline{Q}$.

THEOREM 7.3. (*cf.* [1], *Theorem* 1.) *Let $u(x,t)$ be a complex valued function in $C^{2,1}_{x,t}(R\times(0,T])\cap C(R\times[0,T])$ satisfying*

$$Pu = 0 \qquad \text{in } R\times(0,T]\ , \tag{7.1}$$

$$u(x,0) = 0 \qquad \text{on } R\ , \tag{7.2}$$

$$|u(x,t)| \leqq M\exp[k(x^2+1)] \quad \text{in } R\times[0,T]\ , \quad M, k > 0\ . \tag{7.3}$$

Then we have $u(x,t)=0$ in $R\times[0,T]$.

Proof. We can easily see that it is sufficient to prove the theorem in the case where $u(x,t)$ is a real value function.

We set $v = \exp[2ke^{\theta t}(x^2+1)]$ with $\theta > 0$ determined later. Then we have

$$\frac{Pv}{v} = 4ke^{\theta t}\theta(x^2+1) - 16k^2e^{2\theta t}x^2a(x,t) + 4ke^{\theta t}(a(x,t) + xb(x,)) + c(x,t)$$
$$\geqq 4ke^{\theta t}(x^2+1)(\theta - C_1ke^{\theta t} - C_2)$$
$$\geqq 4ke^{\theta t}(x^2+1)\theta - C_1ke - C_2)\ , \qquad 0 \leqq t \leqq \theta^{-1}\ ,$$

where the constants C_1 and C_2 depend only on $\max_{R\times I}(|a(x,t)| + |b(x,t)| + |c(x,t)|)$. If we put $\theta = 2H$, $H = C_1ke + C_2$, then we have

$$Pv > 0 \qquad \text{in } R \times [0, H^{-1}]\ .$$

Next, for any $\rho > 0$, we set

$$w = u - M\exp[2ke^{Ht}(x^2+1) - k(\rho^2+1)]$$
$$\text{in } (|x| < \rho) \times [0, H^{-1}] \equiv Q_\rho\ .$$

Then we have

$$Pw = Pu - M\exp[-k(\ ^2+1)Pv < 0 \quad \text{in } \tilde{Q}\ , \quad w(x,t) \leqq 0 \quad \text{on } \partial_p Q\ .$$

By Theorem 7.1, we have $w(x,t) \leqq 0$ in $\overline{Q}_\rho$. For any point $(x^*, t^*) \in R \times [0, H^{-1}]$, we have $(x^*, t^*) \in \overline{Q}_\rho$ if we take ρ sufficiently large, and hence we have $w(x^*, t^*) \leqq 0$. Thus we have

$$u(x^*, t^*) \leqq M\exp[-k(\rho^2+1)]v(x^*, t^*)\ .$$

Since ρ is arbitrary and the right hand side tends to zero as $\rho \to \infty$, we have

$$u(x,t) \leqq 0 \qquad \text{in } R \times [0, H^{-1}]\ .$$

Iterating this procedure finitely many times, we obtain

$$u(x,t) \leqq 0 \qquad \text{in } R \times [0, T)\ .$$

Similarly we have

$$u(x,t) \geqq 0 \qquad \text{in } R \times [0, T]\ .$$

Q.E.D.

Theorem 2.1 has now been proved combining Theorem 6.1 and Theorem 7.3.

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