J. T. Gresser
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RESTRICTED PRINCIPAL CLUSTER SETS OF A CERTAIN HOLOMORPHIC FUNCTION

JOHN T. GRESSER

Let D be the open unit disk and let K be the unit circle. We say that α is an arc at $\zeta \in K$ if α is contained in D and is the image of a continuous function $z = z(t)$ $(0 \leq t < 1)$ such that $z(t) \to \zeta$ as $t \to 1$. We call α a segment as ζ if the function $z = z(t)$ is linear in t. If P is a property which is meaningful for each point of K, we say that nearly every point of K has property P if the exceptional set is a set of first Baire category in K. We assume that the reader is familiar with the rudiments of cluster set theory, and in particular with the terms ambiguous point, Meier point, and Plessner point of a function (cf. [4] or [7]). If f is a function which maps D into the Riemann sphere, if $\zeta \in K$ and α is an arc at ζ, then $C(f, \zeta, \alpha)$ will denote the arc-cluster set of f at ζ along α. The principal cluster set of f at ζ is defined to be the set

$$\Pi(f, \zeta) = \bigcap_{\alpha} C(f, \zeta, \alpha) ,$$

where α ranges over all arcs of ζ. The nontangential principal cluster set of f at ζ is defined to be the set

$$\Pi^*(f, \zeta) = \bigcap_{\alpha} C(f, \zeta, \alpha) ,$$

where α ranges over all arcs at ζ for which there is a Stolz angle at ζ containing α. Finally, if Δ is a Stolz angle at ζ, we define the set

$$\Pi_{\Delta}(f, \zeta) = \bigcap_{\alpha} C(f, \zeta, \alpha) ,$$

where α ranges over all arcs at ζ such that $\alpha \subseteq \Delta$.

Let Δ be a Stolz angle at $\zeta_0 = 1$, and for each $\zeta \in K$ let $\Delta(\zeta)$ be the Stolz angle at ζ which is obtained by rotating Δ about the origin. We

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mention as background for this paper two well-known results by E. F. Collingwood. Let f be an arbitrary complex-valued function defined in D. Then

$$C_{\Delta(\zeta)}(f,\zeta) = C(f,\zeta)$$

for nearly every point $\zeta \in K$, and

$$C_B(f,\zeta) = C(f,\zeta)$$

for all but a countable number of points $\zeta \in K$. Here $C_B(f,\zeta)$ denotes the boundary cluster set of f at ζ (cf. [4, pp. 80, 82]). More recently, using analogous definitions and techniques, the boundary principal cluster set of f at ζ, denoted by $B\Pi(f,\zeta)$, has been investigated [5]. This set describes the behavior of $\Pi(f,\xi)$ at points ξ which are near and distinct from ζ. It has been shown that if f is continuous in D, then

$$B\Pi(f,\zeta) = \Pi(f,\zeta)$$

for nearly every point $\zeta \in K$ [5, Theorem 9]. This naturally suggests the question of whether for a continuous function f in D it follows that

$$\Pi_{\Delta(\zeta)}(f,\zeta) = \Pi(f,\zeta)$$

for nearly every point $\zeta \in K$. The purpose of this paper is to answer this question in the negative, even for a holomorphic function; we show, in fact, that an even stronger inequality is possible. We use the symbol $\subset$ in the sequel to denote proper set inclusion.

THEOREM. *Let Δ be a Stolz angle at $\zeta_0 = 1$. Then there exists a holomorphic function f in D such that*

$$\Pi^*(f,\zeta) \subset \Pi_{\Delta(\zeta)}(f,\zeta)$$

for nearly every point $\zeta \in K$.

Proof. Our construction is a modification of a method presented by F. Bagemihl in [2]. If ρ is a segment at ζ, we let $\theta(\rho)$ denote the angle $(0 < \theta(\rho) < \pi)$ between ρ and the forward tangent to the unit circle at ζ, and we let $\ell(\rho)$ denote the length of ρ.

For each ternary fraction

$$t = 0\cdot t_1 t_2 t_3 \cdots$$

where each t_j is zero or two, we denote by

$$b(t) = 0 \cdot b_1 b_2 b_3 \cdots$$

the binary fraction such that for each $j = 1, 2, 3, \cdots$

$$b_j = \begin{cases} 0 & \text{if} \quad t_j = 0 \\ 1 & \text{if} \quad t_j = 2 . \end{cases}$$

The set T of all such ternary fractions is the Cantor "middle thirds" set, and the set of all corresponding binary fractions is the closed unit interval. We set $T^* = T - \{1\}$, and for each $t \in T^*$ we let

$$\zeta_t = e^{2\pi i b(t)} .$$

We let θ_0 be any fixed number $(0 < \theta_0 < \pi)$ such that $\theta(\rho) > \theta_0$ for every segment ρ at $\zeta_0 = 1$ which is contained in Δ. Then for each $t \in T^*$ we let ρ_t be the segment at ζ_t defined by

$$\theta(\rho_t) = (1 - t)\theta_0/4, \quad \text{and} \quad \ell(\rho_t) = (1 - \sin(\pi t/2)) \sin(\theta_0/4) .$$

Since $\ell(\rho_t) < \sin(\theta(\rho_t))$ for $t \in T^*$, it follows that the line perpendicular to ρ_t and passing through the origin does not intersect ρ_t. In addition, $\theta(\rho_t)$ is a decreasing function of t which is always less than $\pi/4$. These observations easily imply that for each $t \in T^*$ the segments in the collection

$$\{\rho_s : s \in T^*, \ s \geq t, \quad \text{and} \quad \arg \zeta_t \leq \arg \zeta_s \leq \arg \zeta_t + \pi/2\}$$

are mutually nonintersecting. Thus in order to establish that the segments in the collection

$$P = \{\rho_t : t \in T^*\}$$

are mutually nonintersecting, it suffices to show that for each $t \in T^*$ such that ζ_t lies in the fourth quadrant, the corresponding segment ρ_t lies in the lower half of D. This follows from the observation that for each $t \in T^*$ with $t > 7/9$ we have $t > b(t) > 3/4$ and hence

$$\ell(\rho_t) < 1 - \sin(\pi b(t)/2) < 1 - \cos(2\pi b(t)) < |\sin(2\pi b(t))| .$$

There is a countably dense subset E of K such that for each $\zeta \in E$ there are two segments, say α_ζ and β_ζ, at ζ belonging to P, while for each $\zeta \in K - E$ there is exactly one segment at ζ belonging to P. For each $\zeta \in E$ we let τ_ζ be the segment at ζ defined by

$$\theta(\tau_\zeta) = (1/2)(\theta(\alpha_\zeta) + \theta(\beta_\zeta)) \quad \text{and} \quad \ell(\tau_\zeta) = (1/2)(\ell(\alpha_\zeta) + \ell(\beta_\zeta)) ,$$

and we let

$$Q = \{\tau_\zeta : \zeta \in E\}.$$

It follows easily from [3, Theorem 1, p. 187–8] that there is a holomorphic function f defined in D such that for every $\rho \in P$

$$f(z) \longrightarrow 0 \text{ as } |z| \longrightarrow 1 \text{ along } \rho,$$

and for every $\tau \in Q$

$$f(z) \longrightarrow \infty \text{ as } |z| \longrightarrow 1 \text{ along } \tau.$$

Because E, and $K - E$ are both dense in K, we have that $0, \infty \in C(f, \zeta)$ for every $\zeta \in K$, and hence no point of K is a Meier point of f. Therefore, nearly every point of K is a Plessner point of f [6, Theorem 6, p. 330]. We will show that

$$(1) \qquad \infty \in \Pi_{\Delta(\zeta)}(f, \zeta)$$

for every Plessner point ζ which is not an ambiguous point of f. Since there are at most countably many ambiguous points of f [1, Theorem 2, p. 380], and since $\Pi^*(f, \zeta) \subseteq \{0\}$ for every $\zeta \in K$, the verification of (1) will complete the proof.

Let ζ be a Plessner point of f which is not an ambiguous point of f, and suppose, to the contrary, that γ is an arc at ζ with $\gamma \subseteq \Delta(\zeta)$ such that $\infty \notin C(f, \zeta, \gamma)$. If ρ is the segment at ζ belonging to P, then $\infty \notin C(f, \zeta, \rho)$. Since the region bounded by γ, ρ, and any third convenient arc in D contains a Stolz angle, it follows that the full cluster set of f at ζ restricted to this subregion is total. By the Gross-Iversen theorem [4, Theorem 5.8, p. 101] it follows that ∞ is an asymptotic value of f at ζ, and this contradicts our assumption that ζ is not an ambiguous point of f. Thus (1) is established and the proof is complete.

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Bowling Green State University