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BÄCKLUND TRANSFORMATIONS OF THE FIRST KIND ASSOCIATED WITH MONGE-AMPÈRE'S EQUATIONS

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Due to Clairin and Goursat, a Bäcklund transformation of the first kind can be associated with Monge-Ampère's equation. We shall consider Monge-Ampère's equation of the form $s + f(x, y, z, p, q) + g(x, y, z, p, q)t = 0$, where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$. The following theorems will be obtained:

1. The transformed equation takes on the same form $s' + f' + g't' = 0$ if and only if the given equation can be transformed to a Teixeira equation $s + L(x, y, z, q)t + M(x, y, z, q)p + N(x, y, z, q) = 0$ by a contact transformation.

2. Teixeira equation $s + tL + pM + N = 0$ is solved by integrable systems of order n if and only if the transformed equation is solved by integrable systems of order $n - 1$.

Here, the method of solving $s + f + gt = 0$ by integrable systems of higher order is a generalization of that for solving $s + f = 0$ given by the author [8]. In [8] and ([9], [10]) respectively, he proved the second theorem for a Laplace transformation associated with $s + a(x, y)p + b(x, y)q + c(x, y)z = 0$ and for an Imschenetsky transformation associated with $s + M(x, y, z, q)p + N(x, y, z, q) = 0$. These two transformations are special ones of the Bäcklund transformations of the first kind.

1. Introduction. Bäcklund transformations were classified by Clairin and Goursat to the three kinds ([3], [5]). Given Monge-Ampère's equation

$$Rr + 2Ss + Tt + U + V(rt - s^2) = 0\,,$$

two Bäcklund transformations of the first kind can be associated with it according to its two characteristics ([5, p. 27]). Here, $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$, and R, S, T, U, V are

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functions of x, y, z, p, q. We shall consider Monge-Ampère's equation of the form

$$(1) \qquad s + f(x, y, z, p, q) + g(x, y, z, p, q)t = 0 .$$

Monge-Ampère's equation whose two characteristics are different can be transformed to an equation of this type by a contact transformation if and only if it has an intermediate integral of the first order with respect to one of the characteristics.

Consider a Pfaffian system Σ generated by

$$dz - pdx - qdy = dq + fdx - t(dy - gdx) = 0 ,$$

where x, y, z, p, q, t are independent variables and f, g are functions of x, y, z, p, q; $f_t = g_t = 0$. Let $\omega = 0$ be an element of Σ. Then an integral vector field ξ of Σ is called a Monge characteristic of Σ with respect to $\omega = 0$ if it satisfies $d\omega(\xi, \eta) = 0$ for every integral vector field η of Σ. The element $\omega = 0$ is called a singular equation of Σ if there exists a nontrivial Monge characteristic of Σ with respect to $\omega = 0$, and in this case the Pfaffian system $C(\Sigma; \omega)$ for defining the Monge characteristics is called a Monge characteristic system of Σ with respect to $\omega = 0$. Let θ denote the Pfaffian form $dz - pdx - qdy$. Then $\theta = 0$ is a singular equation of Σ, and the Monge characteristic system $C(\Sigma; \theta)$ is generated by

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{q} = \frac{-dp}{f + tg} = \frac{dq}{t} .$$

Suppose that $z = \phi(x, y)$, $p = p(x, y)$, $q = q(x, y)$, $t = t(x, y)$ give an integral surface of Σ. Then $z = \phi$ is a solution of (1), and $p = \phi_x$, $q = \phi_y$, $t = \phi_{yy}$. Conversely suppose that $z = \phi(x, y)$ is a solution of (1). Then $z = \phi$, $p = \phi_x$, $q = \phi_y$, $t = \phi_{yy}$ give an integral surface of Σ. Equation (1) is called the resolvent equation of Σ with respect to $\theta = 0$.

Equation (1) is the resolvent equation of the other Pfaffian system Σ' generated by

$$\theta = dp + fdy + gdq - vdx = 0$$

with respect to $\theta = 0$, where x, y, z, p, q, v are independent variables. Let us say that Σ and Σ' are the first and second associated systems with equation (1) respectively.

Consider the first associated system Σ with equation (1). Let $\omega = 0$ be an element of Σ of the form

$$dq + fdx - t(dy - gdx) + \lambda\theta = 0\,.$$

Then it will be proved that it is a singular equation of Σ if and only if λ takes on the value $f_p + tg_p$ (Proposition 2). Let $\theta' = 0$ be this singular equation of Σ different from $\theta = 0$. Then it will be shown that the Pfaffian equation $\theta' = 0$ is of class 5 if and only if the resolvent equation (1) of Σ with respect to $\theta = 0$ is not Monge integrable (Proposition 3). Suppose that $\theta' = 0$ is of class 5. Then it can be expressed in the form $dz' - p'dx' - q'dy' = 0$, where x', y', z', p', q' are functionally independent. The resolvent equation of Σ with respect to $\theta' = 0$ is in general an equation of the second order

$$F'(x', y', z', p', q', r', s', t') = 0 \tag{2}$$

([5, p. 20]). Through an integral surface of Σ a solution of the resolvent equation (1) is transformed to a solution of the other resolvent equation (2). This is the Bäcklund transformation of the first kind $B_1(\Sigma)$ associated with Σ. The resolvent equations (1) and (2) are called the original and transformed equations of $B_1(\Sigma)$ respectively. If $f = a(x, y)p + b(x, y)q + c(x, y)z$ and $g = 0$, then $B_1(\Sigma)$ is a Laplace transformation. In this case, the original equation is $s + ap + bq + cz = 0$, and the transformed equation takes on the same form $s' + a'p' + b'q' + c'z' = 0$. We shall prove the following (Theorem 1):

Suppose that equation (1) is not Monge integrable with respect to each of its characteristics. Then, the transformed equation of $B_1(\Sigma)$ takes on the same form $s' + f' + g't' = 0$ if and only if the original equation (1) can be transformed by a contact transformation to a Teixeira equation

$$s + L(x, y, z, q)t + M(x, y, z, q)p + N(x, y, z, q) = 0\,. \tag{3}$$

Here, we assumed that Σ is the second associated system with the transformed equation. This assumption is satisfied in the case of the Laplace transformation, and we are going to generalize a result on this transformation obtained by the author [8].

Let us try to solve the Cauchy problem of equation (1) by integrating ordinary differential equations. The author [7] gave a method by integrable systems for solving the problem in the space of (x, y, z, p, q), which can be applied to Monge-Ampère's equation of any type. Consider the problem in the space of $(x, y, z, p, q_1, \cdots, q_n)$ involving the derivatives of higher order $q_i = \partial^i z/\partial y^i (q_1 = q)$ with respect to y, where $n > 1$. Then it requires us to find an integral surface of the Pfaffian system generated by n equations

$$(4) \qquad \begin{cases} dz - pdx - qdy = 0 \, , \\ dq_i + F_{i-1}dx - q_{i+1}(dy - gdx) = 0 \quad (1 \leqq i \leqq n-1) \end{cases}$$

which contains a given integral curve of (4). Here, $F_i (i \geqq 0)$ is a function of $x, y, z, p, q_1, \cdots, q_{i+1}$ defined inductively by

$$(5) \qquad F_i = Y_i F_{i-1} + q_{i+1} Y_1 g \, , \quad F_0 = f \, ,$$

and $Y_i (i \geqq 1)$ is an operator defined by

$$(6) \qquad Y_i = \frac{d}{dy} + \sum_{j=1}^{i} q_{j+1} \frac{\partial}{\partial q_j} - (f + q_2 g) \frac{\partial}{\partial p} \, , \quad \frac{d}{dy} = \frac{\partial}{\partial y} + q \frac{\partial}{\partial z} \, .$$

The author [8] gave a method by integrable systems of higher order for solving the Cauchy problem of equation $s + f(x, y, z, p, q) = 0$ in this space. We shall generalize this method as follows; Consider a system of ordinary differential equations

$$(7) \qquad \frac{dx}{0} = \frac{dy}{1} = \frac{dz}{q} = \frac{-dp}{f + q_2 g} = \frac{dq_1}{q_2} = \cdots = \frac{dq_{n-1}}{q_n} = \frac{dq_n}{u} \, ,$$

where u is a function of $x, y, z, p, q_1, \cdots, q_n$. Then it will be said to be integrable if u is a solution of the system of two partial differential equations of the first order

$$(8) \qquad \frac{\partial u}{\partial p} = \frac{du}{dx} - \sum_{i=1}^{n} F_{i-1} \frac{\partial u}{\partial q_i} + g \frac{du}{dy} + \left(\frac{\partial F_{n-1}}{\partial q_n} + Y_1 g \right) u + Y_{n-1} F_{n-1} = 0 \, ,$$

where $F_i (0 \leqq i < n)$ is the function defined by (5). Equation (1) will be said to be solved by integrable systems of order n if the rank of the system (8) is greater than zero. In this case, it will be proved that the Cauchy problem is solved by integrating the system of ordinary differential equations (7) (Proposition 1). We shall obtain the following (Theorem 2):

Suppose that Teixeira equation (3) is not Monge integrable with respect to the characteristic $dy - Ldx = \theta = dq + (pM + N)dx = 0$. Then, it is solved by integrable systems of order n if and only if the transformed equation $s' + f' + g't' = 0$ is solved by integrable systems of order $n-1$.

Here, n may be any positive integer, in particular may be one. An equation solved by integrable systems of order 0 is by definition a Monge integrable equation ([7]). In [8] and ([9], [10]) respectively, the author proved this theorem for the Laplace transformation, and for an Imschenetsky transformation. The latter is the Bäcklund transformation of the first kind associated with $s + M(x, y, z, q)p + N(x, y, z, q) = 0$. The transformed equation takes on the form $s' + M'(x', y', z', p')q' + N'(x', y', z', p') = 0$.

Remark 1. Suppose that equation (1) is transformed to a Monge-Ampère equation (E^*) by a contact transformation $x^* = x^*(x, y, z, p, q)$, $\cdots$, $q^* = q^*(x, y, z, p, q)$. Then we have

$$dz^* - p^*dx^* - q^*dy^* = \rho\theta ,$$

$$\omega_1^* = \alpha_1(dq + fdx) + \beta_1(dy - gdx) + \gamma_1\theta ,$$

$$\omega_2^* = \alpha_2(dq + fdx) + \beta_2(dy - gdx) + \gamma_2\theta ,$$

where $dz^* - p^*dx^* - q^*dy^* = \omega_1^* = \omega_2^* = 0$ is one of the characteristics of (E^*), and $\rho \neq 0$, $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. Take

$$t^* = (t\alpha_2 + \beta_2)^{-1}(t\alpha_1 + \beta_1) .$$

Then Σ is transformed to Σ^* generated by

$$dz^* - p^*dx^* - q^*dy^* = \omega_1^* + t^*\omega_2^* = 0 .$$

The resolvent equation of Σ^* with respect to $dz^* - p^*dx^* - q^*dy^* = 0$ is the transformed equation (E^*).

Remark 2. The general form of Teixeira equation is

$$s + M(x, y, z, q)p + T(x, y, z, q, t) = 0 ,$$

with which we can associate a Bäcklund transformation of the first kind ([5, pp. 29–30]). For the general theory of Bäcklund transformations of the first kind, see Goursat [5, pp. 6–17].

2. Characteristic systems. Let Ω be a Pfaffian system. Then an integral vector field ξ of Ω is called a characteristic of Ω if it satisfies $d\omega(\xi, \eta) = 0$ for each element $\omega = 0$ of Ω and for every integral vector field η of Ω. The Pfaffian system for defining the characteristics is called the characteristic system of Ω. This notion of a characteristic system was obtained by E. Cartan [1]. Suppose that Ω is generated by

$$\omega_i = dx_i - \sum_{j>r}^{m} a_{ij} dx_j = 0 \quad (1 \leqq i \leqq r),$$

where $x_1, \cdots, x_m$ are independent variables and $a_{ij}(1 \leqq i \leqq r < j \leqq m)$ is a function of them. Then the characteristic system of Ω is generated by $\omega_i = 0(1 \leqq i \leqq r)$ and

$$\omega_{ij} = \sum_{k>r}^{m} b_{ijk} dx_k = 0 \quad (1 \leqq i \leqq r < j \leqq m),$$

where

$$b_{ijk} = -b_{ikj} = \frac{\partial a_{ik}}{\partial x_j} - \frac{\partial a_{ij}}{\partial x_k} + \sum_{h=1}^{r} \left(\frac{\partial a_{ik}}{\partial x_h} a_{hj} - \frac{\partial a_{ij}}{\partial x_h} a_{hk} \right).$$

Given a nontrivial vector field $\xi = \sum_{a=1}^{m} \xi_a \partial / \partial x_a \neq 0$ and a curve C defined by $x_a = x_a^0(s)(1 \leqq a \leqq m)$, let us define $S(\xi; C)$ as a surface $x_a = x_a(s, t)(1 \leqq a \leqq m)$ obtained by integrating $\partial x_a / \partial t = \xi_a (1 \leqq a \leqq m)$ under the initial condition that $x_a(s, 0) = x_a^0(s)(1 \leqq a \leqq m)$. Then, due to E. Cartan, we have the following ([2, p. 55]):

LEMMA 1. *Suppose that ξ is a nontrivial characteristic of Ω, and that C is an integral curve of Ω. Then, $S(\xi; C)$ is an integral surface of Ω.*

The characteristic system of Ω is completely integrable for any Ω ([2, p. 52]). A Pfaffian system is completely integrable if and only if the characteristic system is itself. Suppose that Ω is generated by a single equation $\omega = 0$. Then the Pfaffian equation $\omega = 0$ is said to be of class s if the rank of the characteristic system is s. This number s is odd. Here, the rank of a Pfaffian system is the number of independent equations contained in it. Suppose that $\omega = \rho(dx_r + x_{r+1}dx_1 + \cdots + x_{2r-1}dx_{r-1})$, $\rho \neq 0$, where $x_1, \cdots, x_{2r-1}$ are assumed to be functionally independent. Then, for each $t(1 \leqq t \leqq r)$, the characteristic system of $\omega = dx_1 = \cdots = dx_{t-1} = 0$ is generated by $dx_1 = \cdots = dx_r = dx_{r+t} = \cdots = dx_{2r-1} = 0$. Hence, the rank of this system is $2r - t$. In particular,

$\omega = 0$ is of class $2r-1$. Conversely, suppose that $\omega = 0$ is of class $2r-1$, and that $\phi_i = c$ is an integral of the characteristic system of $\omega = d\phi_1 = \cdots = d\phi_{i-1} = 0$ for each $i (1 \leqq i \leqq t)$, where $\phi_1, \cdots, \phi_t$ are assumed to be functionally independent. Then the rank of the characteristic system of $\omega = d\phi_1 = \cdots = d\phi_t = 0$ is $2r - t - 1$ ([2, p. 57], [4, Chap. XI]). Hence, if $\omega = 0$ is of class $2r-1$, then it is expressed in such a form $\psi_1 d\phi_1 + \cdots + \psi_r d\phi_r$ that $\phi_1, \cdots, \phi_r$ are functionally independent and $\phi_i = c$ is an integral of the characteristic system of $\omega = d\phi_1 = \cdots = d\phi_{i-1} = 0 (1 \leqq i \leqq r)$. Here, for each $i (1 \leqq i \leqq r)$, we have $\psi_i \neq 0$, and $\psi_i^{-1}\psi_j, \phi_k (1 \leqq j \leqq r, j \neq i, 1 \leqq k \leqq r)$ are functionally independent. They give $2r-1$ integrals of the characteristic system of $\omega = 0$.

Let x, y, z, p, q be independent variables. Then a transformation $x' = x'(x, y, z, p, q), \cdots,$ $q' = q'(x, y, z, p, q)$ is called a contact one if it satisfies

$$dz' - p'dx' - q'dy' = \rho(dz - pdx - qdy)\,, \qquad \rho \neq 0\,.$$

Let ϕ, ψ be two functions assumed to be functionally independent. Then there exists such a contact transformation that $x' = \phi$, $y' = \psi$ if and only if $[\phi, \psi] = 0$ (Lie [6, Chap. V]). Here $[\phi, \psi]$ is the Lagrange bracket:

$$[\phi, \psi] = \frac{\partial \phi}{\partial p}\frac{d\psi}{dx} + \frac{\partial \phi}{\partial q}\frac{d\psi}{dy} - \frac{\partial \psi}{\partial p}\frac{d\phi}{dx} - \frac{\partial \psi}{\partial q}\frac{d\phi}{dy}\,, \qquad \frac{d}{dx} = \frac{\partial}{\partial x} + p\frac{\partial}{\partial z}\,.$$

Hence we have the following:

LEMMA 2. *The Lagrange bracket $[\phi, \psi]$ vanishes if and only if $\psi = c$ is an integral of the characteristic system of $dz - pdx - qdy = d\phi = 0$.*

Consider a Pfaffian system $\Omega_n (n \geqq 1)$ generated by $n+1$ equations

$$dz - pdx - qdy = 0\,,$$

$$dq_i + F_{i-1}dx - q_{i+1}(dy - gdx) = 0 \quad (1 \leqq i < n)\,,$$

$$dq_n + F_{n-1}dx - u(dy - gdx) = 0$$

with independent variables $x, y, z, p, q_1, \cdots, q_n$, where $F_i (0 \leqq i < n)$ is the function defined by (5) and u is a function of $x, y, z, p, q_1, \cdots, q_n$. Then the characteristic system of $\Omega_n (n > 1)$ is generated by (7) and two equations

$$\frac{\partial u}{\partial p}dy = 0\,,$$

$$\left\{\frac{du}{dx} - \sum_{i=1}^{n} F_{i-1}\frac{\partial u}{\partial q_i} + g\frac{du}{dy} + \left(\frac{\partial F_{n-1}}{\partial q_n} + Y_1 g\right)u + Y_{n-1}F_{n-1}\right\}dy = 0\,,$$

where we assumed that $n > 1$. Hence $\Omega_n (n > 1)$ has a nontrivial characteristic if and only if u is a solution of (8), and in this case it is given by (7). Suppose that $n = 1$. Then the characteristic system of Ω_1 is generated by

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{q} = \frac{-dp}{f + ug} = \frac{dq}{u}$$

and two equations

$$\frac{\partial u}{\partial p}dy = 0\,,$$

$$\left\{\frac{du}{dx} - f\frac{\partial u}{\partial q} + g\frac{du}{dy} + \left(\frac{dg}{dy} - f\frac{\partial g}{\partial p} + \frac{\partial f}{\partial q} - g\frac{\partial f}{\partial p}\right)u + \frac{df}{dy} - f\frac{\partial f}{\partial p} + \left(\frac{\partial g}{\partial q} - g\frac{\partial g}{\partial p}\right)u^2\right\}dy = 0\,.$$

Hence Ω_1 has a nontrivial characteristic if and only if u is a solution of the system obtained by setting the two coefficients of dy equal to zero. If the rank of this system is greater than zero, then equation (1) is said to be solved by integrable systems of the first order ([7]).

Any system of partial differential equations of the first order with one unknown function can be prolonged either to a complete system or to an incompatible system by adding the compatibility conditions. If we get a complete system generated by r independent equations, then the original system is said to have the rank $m - r$, where m is the number of the independent variables.

PROPOSITION 1. *Suppose that equation* (1) *is solved by integrable systems of order n. Then for any initial curve C satisfying* (4) *we can make a solution of* (1) *from C integrating a characteristic of Ω_n. Here, n may be any positive integer.*

Proof. By the definition, the rank of the system for u to give a characteristic of Ω_n is greater than zero. Hence, we can find such a solution u of the system that it satisfies $dq_n + F_{n-1}dx + u(dy - gdx) = 0$ along C. Integrate the characteristic of Ω_n defined by this u starting from C. Then the surface thus obtained is a solution of (1), since it is an integral surface of Ω_n by Lemma 1.

3. Bäcklund transformations of the first kind. Let $\omega = 0$ be an element of Σ of the form

$$dq + fdx - t(dy - gdx) + \lambda(dz - pdx - qdy) = 0\,. \tag{9}$$

Then we have the following:

PROPOSITION 2. *The element $\omega = 0$ is a singular equation of Σ if and only if $\lambda = f_p + tg_p$.*

Proof. The Pfaffian system for defining the Monge characteristics of Σ with respect to $\omega = 0$ is generated by

$$\begin{aligned}(f_p + tg_p - \lambda)dx = dy - gdx = \theta = (f_p + tg_p - \lambda)dp \\ = dq + fdx = dt + edx = 0\,,\end{aligned}$$

where $\theta = dz - pdx - qdy$, and

$$e = \frac{df}{dy} - \lambda f + \left(f_q + \frac{dg}{dy} - \lambda g\right)t + g_q t^2\,.$$

Hence, the rank of this system is less than six if and only if $f_p + tg_p - \lambda = 0$.

Suppose that $\lambda = f_p + tg_p$. Let $\theta' = 0$ be the element (9) defined by this value of λ. Then the Monge characteristic system $C(\Sigma\,;\theta')$ is generated by

$$dy - gdx = \theta = dq + fdx = dt + edx = 0\,. \tag{10}$$

We notice that $e = F_1$ if q_2 is replaced by t, where F_1 is the function of x, y, z, p, q, q_2 defined by (5). A function $\phi(x, y, z, p, q, t)$ gives an integral $\phi = c$ of $C(\Sigma\,;\theta')$ if and only if it is a solution of

$$\zeta_0\phi = \frac{\partial\phi}{\partial p} = 0\,,$$

where

$$\zeta_0 = \frac{d}{dx} + g\frac{d}{dy} - f\frac{\partial}{\partial q} - e\frac{\partial}{\partial t}\,.$$

PROPOSITION 3. *The Pfaffian equation $\theta' = 0$ is of class 5 if and only if equation (1) is not Monge integrable with respect to the characteristic $\theta = dy - gdx = dq + fdx = 0$.*

Proof. The characteristic system of $\theta' = 0$ is generated by six equations

$$\begin{aligned} dy - gdx - g_p\theta = \lambda_p\theta = H\theta = \lambda_p dp + Hdx = \theta' \\ = dt + edx + \left(\frac{d\lambda}{dy} + t\lambda_q - \lambda^2\right)\theta = 0\,, \end{aligned}$$

where

$$H = \frac{d\lambda}{dx} - f\lambda_q + g\left(\frac{d\lambda}{dy} - \lambda^2\right) - (f_z - \lambda f_q) - (g_z - \lambda g_q)t - g_p e\,.$$

The rank of this system is less than five if and only if

$$\lambda_p = H = 0\,.$$

Equation (1) is Monge integrable with respect to the characteristic $\theta = dy - gdx = dq + fdx = 0$ if and only if the rank of the system of two equations

$$\frac{\partial u}{\partial p} = \frac{du}{dx} - g\frac{du}{dy} - f\frac{\partial u}{\partial q} = 0$$

is greater than one. The compatibility condition between them is given by

$$\frac{\partial u}{\partial z} + g_p\frac{du}{dy} - f_p\frac{\partial u}{\partial q} = 0\,.$$

These three equations form a complete system if and only if

$$f_{pp} = g_{pp} = H_1 = H_2 = 0\,,$$

where

$$H_1 = K_x + L\frac{dK}{dy} - NK_q + K(KN - LM) - \left(L_z + K\frac{dL}{dy} - ML_q\right),$$

$$H_2 = M_x + L\frac{dM}{dy} - NM_q + M(KN - LM) - \left(N_z + K\frac{dN}{dy} - MN_q\right),$$

and $f = pM + N$, $g = pK + L$, $M_p = N_p = K_p = L_p = 0$. Suppose that $f_{pp} = g_{pp} = 0$. Then we have $H = tH_1 + H_2$.

THEOREM 1. *Suppose that equation* (1) *is not Monge integrable with respect to each of its two characteristics* $\theta = dy - gdx = dq + fdx = 0$ *and* $\theta = dx = dp + fdy + gdq = 0$. *Then, the transformed equation of* $B_1(\Sigma)$ *takes on the form*

$$s' + f'(x', y', z', p', q') + g'(x', y', z', p', q')t' = 0 \tag{11}$$

if and only if the original equation (1) *can be transformed to a Teixeira equation* (3) *by a contact transformation. Here, we assumed that* Σ *is the second associated system with the transformed equation* (11).

Proof. By the assumption that equation (1) is not Monge integrable with respect to the characteristic $\theta = dy - gdx = dq + fdx = 0$, the Pfaffian equation $\theta' = 0$ is of class 5. Hence, it is expressed in the form $dz' - p'dx' - q'dy' = 0$, where x', y', z', p', q' are functionally independent. Equation (11) can be the transformed equation of $B_1(\Sigma)$ if and only if $C(\Sigma\,;\theta')$ is generated by

$$dy' - g'dx' = \theta' = dq' + f'dx' = dp' - v'dx' = 0\,, \tag{12}$$

where v' is a function of x, y, z, p, q, t: For we assumed that Σ is the second associated system with equation (11). In this case $x' = c$ is not an integral of $C(\Sigma\,;\theta')$. In the other case where Σ is the first associated system with equation (11), $x' = c$ is an integral of $C(\Sigma\,;\theta')$: For Σ is the first associated system with equation (11) if and only if $C(\Sigma\,;\theta')$ is generated by

$$dx' = \theta' = dp' + f'dy' + g'dq' = dq' - t'dy' = 0\,,$$

where t' is a function of x, y, z, p, q, t.

First let us assume that $\lambda_p \neq 0$, and prove that the transformed equation of $B_1(\Sigma)$ can not take on the form (11). The characteristic system of $\theta' = 0$ is generated by

$$dy - gdx = \theta = dq + fdx = dt + edx = dp + \mu dx = 0\,,$$

where

$$\mu = (\lambda_p)^{-1}H\,.$$

A function $\phi(x, y, z, p, q, t)$ gives an integral $\phi = c$ of this system if and only if $\zeta\phi = 0$, where

$$\zeta = \zeta_0 - \mu\frac{\partial}{\partial p} = \frac{d}{dx} + g\frac{d}{dy} - \mu\frac{\partial}{\partial p} - f\frac{\partial}{\partial q} - e\frac{\partial}{\partial t}\,.$$

Take such a solution x' of $\zeta x' = 0$ that $\partial x'/\partial p \neq 0$. The condition that $\partial x'/\partial p \neq 0$ is a necessary and sufficient one that the solution x' of $\zeta x' = 0$ may not give an integral $x' = c$ of $C(\Sigma\,;\theta')$. Starting from this x', let us express $\theta' = 0$ in the form $dz' - p'dx' - q'dy' = 0$. Then $y' = c$ is an integral of the characteristic system of $\theta' = dx' = 0$, and $z' = c$ is an integral of the completely integrable system $\theta' = dx' = dy' = 0$. The five functions x', y', z', p', q' are functionally independent solutions of $\zeta\phi = 0$. A function $\phi(x, y, z, p, q, t)$ takes on the form $\phi'(x', y', z', p', q')$ if and only if $\zeta\phi = 0$. The Monge characteristic system $C(\Sigma\,;\theta')$ is generated by

$$dy' - G'dx' = dz' - (p' + q'G')dx' = dq' + F'dx' = dp' - v'dx' = 0\,,$$

where

$$G' = \frac{\partial y'}{\partial p}\Big/\frac{\partial x'}{\partial p}\,,\quad F' = -\frac{\partial q'}{\partial p}\Big/\frac{\partial x'}{\partial p}\,,\quad v' = \frac{\partial p'}{\partial p}\Big/\frac{\partial x'}{\partial p}\,.$$

Hence, we have $\zeta G' = \zeta F' = 0$ if and only if every function of the form $\phi'(x', y', z', q')$ satisfies

$$\zeta\left(\frac{\partial \phi'}{\partial p}\Big/\frac{\partial x'}{\partial p}\right) = 0\,,$$

since

$$\frac{\partial z'}{\partial p}\Big/\frac{\partial x'}{\partial p} = p' + q'G'\,.$$

Let $[\phi', \psi']$ denote the Lagrange bracket with respect to (x', y', z', p', q'). Then we have $[\phi', x'] = \partial\phi'/\partial p'$ for $\phi'(x', y', z', p', q')$. Hence, by Lemma 2, a function $\phi(x, y, z, p, q, t)$ takes on the form $\phi'(x', y', z', q')$ if and only if $\phi = c$ is an integral of the characteristic system of $\theta' = dx' = 0$. This system is generated by four equations

$$dy - gdx = \left\{g_p - \lambda_p\left(\frac{\partial x'}{\partial p}\right)^{-1}\frac{\partial x'}{\partial t}\right\}\theta\,,$$

$$dp + \mu dx = -\left(\frac{\partial x'}{\partial p}\right)^{-1}\left\{\frac{\partial x'}{\partial z} + g_p\frac{dx'}{dy} - f_p\frac{\partial x'}{\partial q} - \frac{\partial x'}{\partial t}\left(\frac{d\lambda}{dy} + t\lambda_q - \lambda^2\right)\right\}\theta\,,$$

$$dq + fdx = -\left\{f_p + t\lambda_p\left(\frac{\partial x'}{\partial p}\right)^{-1}\frac{\partial x'}{\partial t}\right\}\theta ,$$

$$dt + edx = -\left\{\frac{d\lambda}{dy} + t\lambda_q - \lambda^2 - \lambda_p\left(\frac{\partial x'}{\partial p}\right)^{-1}\left(\frac{dx'}{dy} + t\frac{\partial x'}{\partial q}\right)\right\}\theta .$$

Hence, $\phi = c$ is an integral of this system if and only if ϕ is a solution of the system of two equations

$$(13)\qquad \begin{cases} \zeta\phi = 0 , \\ x'_p\left\{\dfrac{\partial\phi}{\partial z} + g_p\dfrac{d\phi}{dy} - f_p\dfrac{\partial\phi}{\partial q} - \left(\dfrac{d\lambda}{dy} + t\lambda_q - \lambda^2\right)\dfrac{\partial\phi}{\partial t}\right\} \\ \quad - \dfrac{\partial\phi}{\partial p}\left\{x'_z + g_p\dfrac{dx'}{dy} - f_p x'_q - \left(\dfrac{d\lambda}{dy} + t\lambda_q - \lambda^2\right)x'_t\right\} \\ \quad - \lambda_p\left\{x'_t\left(\dfrac{d\phi}{dy} + t\dfrac{\partial\phi}{\partial q}\right) - \dfrac{\partial\phi}{\partial t}\left(\dfrac{dx'}{dy} + tx'_q\right)\right\} = 0 . \end{cases}$$

This is complete, since the characteristic system of $\theta' = dx' = 0$ is completely integrable. A function $\phi(x, y, z, p, q, t)$ which satisfies $\zeta\phi = 0$ is a solution of $\zeta(\partial\phi/\partial p\big/\partial x'/\partial p) = 0$ if and only if ϕ is a solution of

$$(14)\qquad \begin{aligned} &x'_p\left(\frac{\partial\phi}{\partial z} + g_p\frac{d\phi}{dy} - f_p\frac{\partial\phi}{\partial q} - e_p\frac{\partial\phi}{\partial t}\right) \\ &\qquad - \frac{\partial\phi}{\partial p}\left(x'_z + g_p\frac{dx'}{dy} - f_p x'_q - e_p x'_t\right) = 0 . \end{aligned}$$

If a solution ϕ of (13) satisfies (14), then, by $\lambda_p \neq 0$ and

$$e_p = \frac{d\lambda}{dy} + t\lambda_q - \lambda^2 - (f + tg)\lambda_p ,$$

we have

$$\begin{aligned} &x'_t\left\{\frac{d\phi}{dy} + t\frac{\partial\phi}{\partial q} - (f + tg)\frac{\partial\phi}{\partial p}\right\} \\ &\qquad - \frac{\partial\phi}{\partial t}\left\{\frac{dx'}{dy} + tx'_q - (f + tg)x'_p\right\} = 0 . \end{aligned}$$

This equation is not a linear combination of the two equations (13) unless $x'_t = dx'/dy + tx'_q - (f + tg)x'_p = 0$. The two equation (13) are linearly independent with each other, and they form a complete system as it was noted above. By the assumption that equation (1) is not

Monge integrable with respect to the characteristic $dx = \theta = dp + fdy + gdq = 0$, we can not find such a solution x' of

$$\frac{\partial x'}{\partial t} = \frac{dx'}{dy} + t\frac{\partial x'}{\partial q} - (f + tg)\frac{\partial x'}{\partial p} = 0$$

that $\partial x'/\partial p \neq 0$. Hence, there exists a solution of the system (13) which does not satisfy (14). Therefore, we have $\zeta G' \neq 0$ or $\zeta F' \neq 0$, and $C(\Sigma\,; \theta')$ can not have such a system of generators as (12).

Secondly let us assume that $\lambda_p = g_p = 0$, and prove that the transformed equation takes on the form (11). Since $H \neq 0$, the characteristic system of $\theta' = 0$ is generated by

$$dx = dy = dz = dq = dt = 0 .$$

Take $x' = x$. Then $x' = c$ is not an integral of $C(\Sigma\,; \theta')$, since $\zeta_0 x' = 1$. The characteristic system of $\theta' = dx' = 0$ is generated by

$$dx = dy = dq + \lambda dz = dt + \left(\frac{d\lambda}{dy} + t\frac{\partial \lambda}{\partial q} - \lambda^2\right)dz = 0 .$$

Hence, we can take $y' = y$. The system $\theta' = dx' = dy' = 0$ is complete. Set $f = pM + N$, where $M_p = N_p = 0$. Then, $\lambda = M$, and a function $h(x, y, z, p, q, t)$ gives an integral $h = c$ of $\theta' = dx' = dy' = 0$ if and only if

$$\frac{\partial h}{\partial z} - M\frac{\partial h}{\partial q} = \frac{\partial h}{\partial p} = \frac{\partial h}{\partial t} = 0 .$$

Take a solution h of this system satisfying $h_q \neq 0$, and set $z' = h$. Then we have

$$\theta' = \rho'(dz' - p'dx' - q'dy') ,$$

where

$$p' = \frac{\partial h}{\partial x} - N\frac{\partial h}{\partial q} - tg\frac{\partial h}{\partial q} , \quad q' = \frac{dh}{dy} + t\frac{\partial h}{\partial q} , \quad \rho' = (h_q)^{-1} .$$

A function $\phi(x, y, z, p, q, t)$ takes on the form $\phi'(x', y', z', p', q')$ if and only if $\partial\phi/\partial p = 0$. The Monge characteristic system $C(\Sigma\,; \theta')$ is generated by

$$dy' - G'dx' = \theta' = dq' + F'dx' = dp' - v'dx' = 0 ,$$

where

$$G' = \zeta_0 y' = g , \quad F' = -\zeta_0 q' , \quad v' = \zeta_0 p' .$$

By our assumption that $g_p = 0$, we have $\partial G'/\partial p = g_p = 0$. The function e is given by

$$e = p\left(\frac{dM}{dy} - M^2 + M_q t\right) + \frac{dN}{dy} - MN + N_q t + \left(\frac{dg}{dy} - Mg + g_q t\right)t .$$

Hence,

$$\left[\frac{\partial}{\partial p}, \zeta_0\right] = \frac{\partial}{\partial z} - M\frac{\partial}{\partial q} - e_p\frac{\partial}{\partial t}, \quad e_p = \frac{dM}{dy} - M^2 + M_q t ,$$

$$\left[\frac{\partial}{\partial z} - M\frac{\partial}{\partial q} - e_p\frac{\partial}{\partial t},\ \frac{d}{dy} + t\frac{\partial}{\partial q}\right]$$
$$= -M\left(\frac{\partial}{\partial z} - M\frac{\partial}{\partial q}\right) + \left(\frac{de_p}{dy} + t\frac{\partial e_p}{\partial q}\right)\frac{\partial}{\partial t} .$$

Since $h_z - Mh_q = h_p = h_t = 0$, we have $\partial F'/\partial p = 0$. Therefore, $C(\Sigma ; \theta')$ is generated by (12), where f' and g' are functions of x', y', z', p', q'.

Lastly let us prove that equation (1) can be transformed to a Teixeira equation (3) by a contact transformation if $\lambda_p = 0$. Consider a contact transformation

$$\text{(15)} \qquad \begin{cases} x^* = x , \quad y^* = \phi(x, y, z, q) , \quad z^* = \psi(x, y, z, q) , \\ p^* = \dfrac{d\psi}{dx} - \dfrac{d\phi}{dx}\psi_q/\phi_q , \quad q^* = \psi_q/\phi_q , \end{cases}$$

where $\phi_q \neq 0$ and

$$\text{(16)} \qquad \phi_q\frac{d\psi}{dy} - \psi_q\frac{d\phi}{dy} = 0 , \quad \phi_q\psi_z - \psi_q\phi_z \neq 0 .$$

Then equation

$$\text{(17)} \qquad s + (pK + L)t + pM + N = 0 \quad (K_p = L_p = M_p = N_p = 0)$$

is transformed to

$$s^* + (p^*K^* + L^*)t^* + p^*M^* + N^* = 0 ,$$

where

$$K^* = \nu\left(K\frac{dy^*}{dy} + y^*_z - My^*_q\right) , \quad \nu = (\psi_z - \phi_z\psi_q/\phi_q)^{-1} ,$$

$$L^* = y^*_x + L\frac{dy^*}{dy} - Ny^*_q - \nu(\psi_x - \phi_x\psi_q/\phi_q)\left(K\frac{dy^*}{dy} + y^*_z - My^*_q\right)$$

and we have M^*, N^* replacing y^* by $-q^*$ in K^*, L^* respectively. They are functions of x^*, y^*, z^*, q^*, since $\partial/\partial p^* = \nu\partial/\partial p$. Take a solution ϕ of

$$K\frac{d\phi}{dy} + \phi_z - M\phi_q = 0\,, \quad \phi_q \neq 0\,,$$

and solve (16). Then the contact transformation (15) transforms equation (17) to $s^* + L^*t^* + p^*M^* + N^* = 0$, where L^*, M^*, N^* are functions of x^*, y^*, z^*, q^*. The transformed equation is not Monge integrable with respect to each of its characteristics if the original equation (17) has such a property, since this property is left invariant by the contact transformation.

Only in the first case where $\lambda_p \neq 0$, the assumption that equation (1) is not Monge integrable with respect to the characteristic $dx = \theta = dp + fdy + gdq = 0$ is necessary for our discussions.

COROLLARY 1. *Suppose that* $g = 0$. *Then the transformed equation takes on the form* $s' + f' = 0$ *if and only if the original equation* $s + f = 0$ *is transformed to an equation of Imschenetsky type* $s + pM + N = 0$ $(M_p = N_p = 0)$ *by a contact transformation.*

For details of Imschenetsky transformations, see [10].

Remark 3. Equation (11) can be the transformed equation of a Teixeira equation (3) by $B_1(\Sigma)$ if and only if it is transformed by a contact transformation to such an equation

$$s^* + f^*(x^*, y^*, z^*, p^*, q^*) + g^*(x^*, y^*, z^*, p^*, q^*)t^* = 0$$

that the coefficients f^*, g^* satisfy

$$\frac{\partial g^*}{\partial q^*} - g^*\frac{\partial g^*}{\partial p^*} = 0\,, \quad \frac{\partial \lambda^*}{\partial q^*} - g^*\frac{\partial \lambda^*}{\partial p^*} - \left(\frac{\partial g^*}{\partial z^*} - \lambda^*\frac{\partial g^*}{\partial p^*}\right) = 0\,,$$

$$\frac{\partial f^*}{\partial z^*} - \lambda^*\frac{\partial f^*}{\partial p^*} - \left(\frac{d\lambda^*}{dy^*} - f^*\frac{\partial \lambda^*}{\partial p^*}\right) \neq 0\,,$$

where

$$\lambda^* = \frac{\partial f^*}{\partial q^*} - g^*\frac{\partial f^*}{\partial p^*} - \left(\frac{dg^*}{dy^*} - f^*\frac{\partial g^*}{\partial p^*}\right).$$

Remark 4. Teixeira equation (3) can be transformed by a contact transformation to an equation of Imschenetsky type where the coefficient of t vanishes if and only if the rank of the system of two equations

$$\frac{\partial u}{\partial x} + L\frac{\partial u}{\partial y} - (N - qLM)\frac{\partial u}{\partial q} = \frac{\partial u}{\partial z} - M\frac{\partial u}{\partial q} = 0$$

with independent variables x, y, z, q is greater than zero.

4. Teixeira equation. Consider equation (1). Let $x, y, z, p, q_1, \cdots, q_n$ be independent variables, and q, t denote q_1, q_2 respectively. We shall assume that $n > 1$. Then, by the definition (5),

$$F_i = Y_i F_{i-1} + q_{i+1} Y_1 g \quad (1 \leqq i < n), \quad F_0 = f,$$

where $Y_i (1 \leqq i < n)$ is the operator defined by (6);

$$Y_i = \frac{d}{dy} + \sum_{j=1}^{i} q_{j+1}\frac{\partial}{\partial q_j} - (f + tg)\frac{\partial}{\partial p} \quad (1 \leqq i < n).$$

By the identities

$$\left[\frac{\partial}{\partial q_{i+1}}, Y_i\right] = \frac{\partial}{\partial q_i} \quad (i \geqq 2),$$

$$\frac{\partial F_1}{\partial t} = \frac{\partial f}{\partial q} - g\frac{\partial f}{\partial p} + t\left(\frac{\partial g}{\partial q} - g\frac{\partial g}{\partial p}\right) + Y_1 g,$$

we have

$$\begin{aligned} \text{(18)} \qquad \frac{\partial F_i}{\partial q_{i+1}} &= \frac{\partial F_{i-1}}{\partial q_i} + Y_1 g = \cdots = \frac{\partial F_1}{\partial t} + (i-1) Y_1 g \\ &= \frac{\partial f}{\partial q} - g\frac{\partial f}{\partial p} + t\left(\frac{\partial g}{\partial q} - g\frac{\partial g}{\partial p}\right) + i Y_1 g \quad (1 \leqq i < n). \end{aligned}$$

Suppose that $f = pM + N$, $g = L$, $M_p = N_p = L_p = 0$. Let us define the functions $A_i, B_i (0 \leqq i < n)$ of $x, y, z, q_1, \cdots, q_{i+1}$ inductively by

$$\text{(19)} \qquad \begin{cases} A_{i+1} = G_{i+1} A_i - M A_i, & A_0 = M, \\ B_{i+1} = G_{i+1} B_i - (N + tL) A_i + q_{i+2} G_1 L, & B_0 = N, \end{cases}$$

where G_i is the operator defined by

$$\text{(20)} \qquad G_i = \frac{d}{dy} + \sum_{j=1}^{i} q_{j+1}\frac{\partial}{\partial q_j} \quad (1 \leqq i < n).$$

Then we have

$$F_i = p A_i + B_i \quad (0 \leqq i < n).$$

Let us define the operators $Z_i, X_i (1 \leqq i \leqq n)$ by

$$Z_i = \frac{\partial}{\partial z} - \sum_{j=1}^{i} A_{j-1}\frac{\partial}{\partial q_j}, \tag{21}$$

$$X_i = \frac{\partial}{\partial x} + L\frac{\partial}{\partial y} - \sum_{j=1}^{i} (B_{j-1} - qLA_{j-1})\frac{\partial}{\partial q_j}. \tag{22}$$

Then, by (19), we have the identities

$$[Z_{i+1}, G_i] = -MZ_i + (G_iA_i)\frac{\partial}{\partial q_{i+1}} \quad (1 \leqq i < n), \tag{23}$$

$$\begin{aligned}[X_{i+1}, G_i] = -\{N - q(G_1L + LM)\}Z_i - (G_1L)G_i \\ + \{G_i(B_i - qLA_i)\}\frac{\partial}{\partial q_{i+1}} \quad (1 \leqq i < n).\end{aligned} \tag{24}$$

By (18), the second equation of (8) is written in the form

$$\begin{aligned}(p + qL)\left(Z_nu + \frac{\partial M}{\partial q}u + M_n\right) + X_nu \\ + \left(\frac{\partial N}{\partial q} - qL\frac{\partial M}{\partial q} - LM + t\frac{\partial L}{\partial q} + nG_1L\right)u + N_n - qLM_n = 0,\end{aligned}$$

where M_n and N_n are functions of $x, y, z, q_1, \cdots, q_n$ defined by

$$M_n = G_{n-1}A_{n-1} - MA_{n-1}, \tag{25}$$

$$N_n = G_{n-1}B_{n-1} - (N + tL)A_{n-1}. \tag{26}$$

LEMMA 3. *Teixeira equation* (3) *is solved by integrable systems of order n if and only if the rank of the system of two equations*

$$Z_nu + M_qu + M_n = 0, \tag{27}$$

$$X_nu + (N_q - qLM_q - LM + tL_q + nG_1L)u + N_n - qLM_n = 0 \tag{28}$$

with independent variables $x, y, z, q_1, \cdots, q_n$ is greater than zero, where $n > 1$.

Proof. The compatibility condition between the two equations of the system (8) is (27). Hence, the system (8) with $n + 4$ independent variables $x, y, z, p, q_1, \cdots, q_n$ has the rank greater than zero if and only if the system of (27) and (28) with $n + 3$ independent variables $x, y, z, q_1, \cdots, q_n$ has the rank greater than zero.

Teixeira equation (3) is Monge integrable with respect to the characteristic $dy - Ldx = \theta = dq + (pM + N)dx = 0$ if and only if the function H defined by

$$H = M_x - NM_q + L\left(\frac{dM}{dy} - M^2\right) - (N_z - MN_q) - t(L_z - ML_q)$$

vanishes identically. Suppose that $H \neq 0$. Then, as it was shown in the proof of Theorem 1, the solution $h(x, y, z, q)$ of $h_z - Mh_q = 0$, $h_q \neq 0$ defines the Bäcklund transformation of the first kind $B_1(\Sigma)$ by

$$\text{(29)} \qquad \begin{cases} x' = x\,, \quad y' = y\,, \quad z' = h\,, \quad p' = h_x - (N + tL)h_q\,, \\ q' = \dfrac{dh}{dy} + th_q = G_1 h \end{cases}$$

and $v' = \zeta_0 p'$. The relation (29) between x, y, z, q, t and x', y', z', p', q' is solved conversely with respect to x, y, z, q, t by $H \neq 0$. The transformed equation takes on the form (11), where $f' = -\zeta_0 q'$, $g' = L$. By the identities

$$\zeta_0 = (p + qL)Z_2 + X_2\,, \quad \zeta_0 h = p' + q'L$$

and

$$[\zeta_0, G_1] = -(pM + N + tL)\left(\frac{\partial}{\partial z} - M\frac{\partial}{\partial q}\right) + (G_1 e)\frac{\partial}{\partial t} - (G_1 L)G_1\,,$$

we have

$$\text{(30)} \qquad -f' = G_1 p' + LG_1 q' = \frac{dp'}{dy} + t\frac{\partial p'}{\partial q} + L\left(\frac{dq'}{dy} + t\frac{\partial q'}{\partial q}\right).$$

Since $L_t = 0$, $g' = L$ and

$$\frac{\partial}{\partial t} = \frac{\partial h}{\partial q}\left(\frac{\partial}{\partial q'} - L\frac{\partial}{\partial p'}\right),$$

we have

$$\text{(31)} \qquad \frac{\partial g'}{\partial q'} - g'\frac{\partial g'}{\partial p'} = 0\,.$$

LEMMA 4. *Suppose that $g_q - gg_p = 0$. Then equation* (1) *is solved by integrable systems of order n if and only if the rank of the system of two equations*

$$\frac{\partial u}{\partial p} = \frac{du}{dx} - \sum_{i=1}^{n} F_{i-1}\frac{\partial u}{\partial q} + g\frac{du}{dy} + \left(\frac{\partial f}{\partial q} - g\frac{\partial f}{\partial p} + nY_0 g\right)u + Y_{n-1}F_{n-1} = 0$$

with independent variables $x, y, z, p, q_1, \cdots, q_n$ is greater than zero, where Y_0 is the operator defined by

$$Y_0 = \frac{d}{dy} - f\frac{\partial}{\partial p}.$$

Here, n may be any positive integer, in particular may be one.

Proof. Suppose that $n = 1$. Then the system for u to give a characteristic of Ω_1 is written in the above form by the assumption that $g_q - gg_p = 0$. Suppose that $n > 1$. Then, by (18) and

$$Y_1 g = Y_0 g + t\left(\frac{\partial g}{\partial q} - g\frac{\partial g}{\partial p}\right) = Y_0 g,$$

the system (8) is written in the above form.

Change the independent variables $x, y, z, q_1, \cdots, q_n$ to $x', y', z', p', q'_1, \cdots, q'_{n-1}$ by (29) and

$$q'_i = G_i \cdots G_1 h = G_{i-1}q'_{i-1} + q_{i+1}h_q = G_i q'_{i-1} \quad (1 < i < n),$$

where $n > 1$. Then, these relations can be solved conversely with respect to $x, y, z, q_1, \cdots, q_n$ by $H \neq 0$ and $h_q \neq 0$.

LEMMA 5. *The operators $Z_n, X_n, G_i (1 \leqq i < n)$ are expressed in the following forms respectively:*

$$Z_n = h_q H\frac{\partial}{\partial p'}; \tag{32}$$

$$X_n = \frac{d}{dx'} - \sum_{j=1}^{n-1} F'_{j-1}\frac{\partial}{\partial q'_j} + g'\frac{d}{dy'} + X_n p'\frac{\partial}{\partial p'}; \tag{33}$$

$$G_i = Y'_{i-1} + \sum_{j=i}^{n-1} G_i q'_j \frac{\partial}{\partial q'_j} \quad (1 < i < n); \tag{34}$$

$$G_1 = Y'_0 + (G_1 q')\left(\frac{\partial}{\partial q'} - g'\frac{\partial}{\partial p'}\right) + \sum_{j=2}^{n-1} G_1 q'_j \frac{\partial}{\partial q'_j}. \tag{35}$$

Here, $F'_i (0 \leqq i < n-1)$ is the function of $x', y', z', p', q'_1, \cdots, q'_{i+1}$ defined by

$$F'_i = Y'_i F'_{i-1} + q'_{i+1} Y'_1 q' \quad (1 \leqq i < n-1), \quad F'_0 = f', \tag{36}$$

and $Y_i'(0 \leqq i < n-1)$ *is the operator defined by*

$$Y_i' = \frac{d}{dy'} + \sum_{j=1}^{i} q_{j+1}' \frac{\partial}{\partial q_j'} - (f' + q_2' g') \frac{\partial}{\partial p'} \quad (1 \leqq i < n-1),$$

$$Y_0' = \frac{d}{dy'} - f' \frac{\partial}{\partial p'} .$$

Proof. The $q_i'(1 \leqq i < n)$ is a function of $x, y, z, q_1, \cdots, q_{i+1}$. By (23) and $Z_1 h = h_z - M h_q = 0$, we have

$$Z_n q_i' = 0 \quad (1 \leqq i < n) . \tag{37}$$

By the identity

$$\left[Z_n, \frac{\partial}{\partial x} - (N + tL)\frac{\partial}{\partial q}\right] = H\frac{\partial}{\partial q} + \sum_{j=2}^{n} \left\{\frac{\partial A_{j-1}}{\partial x} - (N + tL)\frac{\partial A_{j-1}}{\partial q}\right\} \frac{\partial}{\partial q_j},$$

we have $Z_n p' = h_q H$. Hence, Z_n is expressed in the form (32). The identity (30) implies

$$G_i p' = -(f' + LG_i q') + q_3\left(\frac{\partial p'}{\partial t} + L\frac{\partial q'}{\partial t}\right) = -(f' + q_2' L) \quad (i > 1),$$

$$G_1 p' = -(f' + LG_1 q') .$$

Hence, $G_i(1 < i < n)$ and G_1 are expressed in the forms (34) and (35) respectively. Since $\zeta_0 = (p + qL)Z_2 + X_2$,

$$F_0' = -X_2 q' . \tag{38}$$

By (24) and (34),

$$X_{i+1} q_i' = X_{i+1} G_i q_{i-1}' = Y_{i-1}'(X_i q_{i-1}') - (G_1 L) q_i' \quad (1 < i < n) .$$

Hence, by (36), (38) and $G_1 L = Y_0' g' = Y_1' g'$, we obtain

$$X_{i+1} q_i' = -F_{i-1}' \quad (1 \leqq i < n) .$$

Therefore, X_n is expressed in the form (33) by $X_1 h = p' + g'q'$.

LEMMA 6. *Change the unknown function* u *to* u' *by*

$$u' = G_{n-1} q_{n-1}' + h_q u ,$$

where $n > 1$. *Then the two equations* (27) *and* (28) *are expressed in the following forms respectively:*

$$H\frac{\partial u'}{\partial p'} = 0\,; \tag{39}$$

$$(h_q)^{-1}\Big[\frac{du'}{dx'} - \sum_{j=1}^{n-1} F'_{j-1}\frac{\partial u'}{\partial q'_j} + g'\frac{du'}{dy'} + \Big\{(n-1)Y'_0 g' + \frac{\partial f'}{\partial q'} - g'\frac{\partial f'}{\partial p'}\Big\}u' + Y'_{n-2}F'_{n-2} + (X_n p')\frac{\partial u'}{\partial p'}\Big] = 0\,. \tag{40}$$

Proof. Since $Z_n h = 0$,

$$Z_n(h_q)^{-1} = -M_q(h_q)^{-1}\,. \tag{41}$$

By the definition, $\partial q'_{n-1}/\partial q_n = h_q$. Hence,

$$Z_n G_{n-1} q'_{n-1} = -M\Big(Z_n + A_{n-1}\frac{\partial}{\partial q_n}\Big)q'_{n-1} + (G_{n-1}A_{n-1})\frac{\partial q'_{n-1}}{\partial q_n} = h_q M_n \tag{42}$$

by (23) and (25). We have

$$\frac{\partial f'}{\partial q'} - L\frac{\partial f'}{\partial p'} = (h_q)^{-1}\frac{\partial f'}{\partial t} = -(h_q)^{-1}\Big(\frac{\partial p'}{\partial q} + L\frac{\partial q'}{\partial q}\Big) + G_1 L$$

by (30). Hence,

$$X_n(h_q)^{-1} = -(h_q)^{-1}\Big(N_q - LM - qLM_q + tL_q - \frac{\partial f'}{\partial q'} + L\frac{\partial f'}{\partial p'} + G_1 L\Big) \tag{43}$$

by $X_n h = p' + Lq'$. We get

$$G_1 F'_0 = Y'_0 F'_0 + (G_1 q')\Big(\frac{\partial f'}{\partial q'} - L\frac{\partial f'}{\partial p'}\Big)$$

by (35), and

$$\begin{aligned} G_{n-1}F'_{n-2} &= Y'_{n-2}F'_{n-2} + (G_{n-1}q'_{n-1})\frac{\partial F'_{n-2}}{\partial q'_{n-1}} \\ &= Y'_{n-2}F'_{n-2} + (G_{n-1}q'_{n-1})\Big\{(n-2)Y'_0 g' + \frac{\partial f'}{\partial q'} - g'\frac{\partial f'}{\partial p'}\Big\} \quad (n > 2) \end{aligned}$$

by (34) and (18). Hence, for any $n > 1$, we have

$$\begin{aligned} X_n G_{n-1} q'_{n-1} &= G_{n-1}(X_n q'_{n-1}) - \{N - q(G_1 L + LM)\}\Big(Z_n + A_{n-1}\frac{\partial}{\partial q_n}\Big)q'_{n-1} \\ &\quad - (G_1 L)G_{n-1}q'_{n-1} + \{G_{n-1}(B_{n-1} - qLA_{n-1})\}\frac{\partial q'_{n-1}}{\partial q_n} \\ &= -G_{n-1}F'_{n-2} - \{N - q(G_1 L + LM)\}A_{n-1}h_q - (G_1 L)G_{n-1}q'_{n-1} \end{aligned} \tag{44}$$

$$+\{G_{n-1}(B_{n-1}-qLA_{n-1})\}h_q = h_q(N_n - qLM_n)$$
$$-Y'_{n-2}F'_{n-2} - \left\{(n-1)Y'_0g' + \frac{\partial f'}{\partial q'} - L\frac{\partial f'}{\partial p'}\right\}G_{n-1}q'_{n-1}$$

by (24), (26) and $G_1L = Y'_0g'$. These identities (41)–(44) allow us to express the two equations (27) and (28) in the forms (39) and (40) respectively by Lemma 5.

THEOREM 2. *Suppose that Teixeira equation* (3) *is not Monge integrable with respect to the characteristic* $dy - Ldx = \theta = dq + (pM + N)dx = 0$. *Then, it is solved by integrable systems of order* n *if and only if the transformed equation* (11) *is solved by integrable systems of order* $n-1$.

Proof. By (31) we can apply Lemma 4 to the transformed equation (11). Hence, by Lemma 3 and Lemma 6, we have our theorem for $n > 1$. Suppose that $n = 1$. Then equation (3) is solved by integrable systems of order 1 if and only if the rank of the system of two equations

$$\frac{\partial u}{\partial z} - M\frac{\partial u}{\partial q} + \frac{dM}{dy} - M^2 + M_q u = 0\,,$$

$$\frac{\partial u}{\partial x} + L\frac{\partial u}{\partial y} - (N - qLM)\frac{\partial u}{\partial q} + \frac{dN}{dy} - MN - qL\left(\frac{dM}{dy} - M^2\right)$$
$$+\left(N_q + \frac{dL}{dy} - LM - qLM_q\right)u + L_q u^2 = 0$$

with independent variables x, y, z, q is greater than zero. It is possible if and only if the rank of the system of two linear equations

$$\frac{\partial u}{\partial z} - M\frac{\partial u}{\partial q} - \left(\frac{dM}{dy} - M^2 + M_q t\right)\frac{\partial u}{\partial t} = 0\,, \tag{45}$$

$$\frac{\partial u}{\partial x} + L\frac{\partial u}{\partial y} - (N - qLM)\frac{\partial u}{\partial q} - \left\{\frac{dN}{dy} - MN - qL\left(\frac{dM}{dy} - M^2\right)\right. \tag{46}$$
$$\left. + \left(N_q + \frac{dL}{dy} - LM - qLM_q\right)t + L_q t^2\right\}\frac{\partial u}{\partial t} = 0$$

with independent variables x, y, z, q, t is greater than one. Let us identify t with q_2. Then the two equations (45) and (46) are $Z_2u = 0$ and $X_2u = 0$ respectively. Change the independent variables x, y, z, q, t to x', y', z', p', q' by (29). Then, by Lemma 5, the two equations (45) and (46) are expressed in the following forms respectively:

$$h_q H \frac{\partial u}{\partial p'} = 0 \; ; \tag{47}$$

$$\frac{du}{dx'} - g' \frac{du}{dy'} - f' \frac{\partial u}{\partial q'} + X_1 p' \frac{\partial u}{\partial p'} = 0 \, . \tag{48}$$

The transformed equation is solved by integrable systems of order 0 if and only if the rank of the system of (47), (48) is greater than one ([7]). Hence, we obtain our theorem for $n = 1$.

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