

IDEALS WITH SLIDING DEPTH

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Introduction

We study here a class of ideals of a Cohen-Macaulay ring $\{R, \mathfrak{m}\}$ somewhat intermediate between complete intersections and general Cohen-Macaulay ideals. Its definition, while a bit technical, rapidly leads to the development of its elementary properties. Let $I = (x_1, \dots, x_n) = (\mathbf{x})$ be an ideal of R and denote by $H_*(\mathbf{x})$ the homology of the ordinary Koszul complex $K_*(\mathbf{x})$ built on the sequence \mathbf{x} . It often occurs that the depth of the module H_i , $i > 0$, increases with i (as usual, we set $\text{depth}(0) = \infty$). We shall say that I satisfies *sliding depth* if

$$(SD) \quad \text{depth } H_i(\mathbf{x}) \geq \dim(R) - n + i, \quad i \geq 0.$$

This definition depends solely on the number of elements in the sequence \mathbf{x} . This property localizes (cf. [9]) and is an invariant of even linkage (cf. [10]).

An extreme case of this property is given by a complete intersection. A more general instance of it is that where all the modules H_i are Cohen-Macaulay, a situation that was dubbed *strongly* Cohen-Macaulay ideals (cf. [11]).

These ideals have appeared earlier in two settings:

(i) The investigation of arithmetical properties of the Rees algebra of I

$$S = \mathcal{R}(I) = \bigoplus I^s,$$

and of the associated graded ring

$$G = \text{gr}_I(R) = \bigoplus I^s / I^{s+1}.$$

It was shown in [7], [8] and [16] that for ideals satisfying (SD) and such that for each prime P containing I , $\text{height}(P) = \text{ht}(I) \geq \nu(I_P) =$ minimum number of generators of the localization I_P , both S and G are Cohen-Macaulay. In addition, if R is a Gorenstein ring, G will be Goren-

Received June 27, 1984.

(*) Partially supported by NSF grant DMS-8301870.

stein precisely when I is strongly Cohen-Macaulay ([9, (6.5)]).

(ii) The other context is that of a generalization and corrections by Huneke ([11]) of a result of Artin-Nagata on residual Cohen-Macaulayness ([1]), i.e. conditions under which for a subideal $J \subset I$, $J:I$ is Cohen-Macaulay, $(J:I) \cap I = J$ and $\text{ht}((J:I) + I) > \text{ht}(I)$. It connects with the notion of linkage—when J is a complete intersection—by requiring that I be a strongly Cohen-Macaulay ideal. In turn our extension shows that the assertions of the theorem are intertwined with the sliding depth condition.

Our goals here are the following:

(i) In Section 1 we demark more precisely the distinction between strongly Cohen-Macaulay ideals and ideals with (SD). This is more conveniently done if I is generated by a d -sequence—for ideals with (SD) this is essentially equivalent to requiring that $v(I_p) \leq \text{ht}(P)$, for prime ideals $P \supset I$. If one further assumes that R is Gorenstein, and $v(I_p) < \text{ht}(P) - 1$ for primes with $\text{ht}(P) > \text{ht}(I) + 2$, then I is strongly Cohen-Macaulay. This was proved by Huneke ([11]) using the duality of [6]. We reinforce this result by replacing the last inequality by $v(I_p) < \text{ht}(P)$. It still follows from [6] but depends on some quirks of the Koszul complex. The next case—i.e. $v(I_p) \leq \text{ht}(P)$ —is however critical. What precisely overcomes it is not well-known. Some conditions we impose involve the conormal module I/I^2 .

(ii) In Section 2 we discuss examples of Cohen-Macaulay prime ideals of codimension three in a regular local ring R , that have (SD), but are not strongly Cohen-Macaulay. It will rely on properties of the divisor class group of R/I . In particular we shall see that if I is the ideal generated by the $n - 1$ sized minors of a generic, symmetric, $n \times n$ matrix then I is syzygetic (cf. [7]). For $n = 3$ we have the desired example. Its Rees algebra $\mathcal{R}(I)$ is even integrally closed.

We also record an extension of a result of Serre asserting that Gorenstein ideals of codimension two are complete intersections. More generally, one can show that if I is a Cohen-Macaulay of codimension two, then the canonical module of R/I cannot have 2-torsion.

(iii) In Section 3 the generalization of Huneke's theorem to ideals with sliding depth is given. Some of its elements may be used to construct ideals with sliding depth of a fixed height and various projective dimensions.

We thank Craig Huneke and Aron Simis for several conversations, and also Giuseppe Valla for raising one of our motivating questions.

§ 1. Strongly Cohen-Macaulay ideals

The rings considered throughout will be Noetherian, commutative with an identity. For notation, terminology and basic results—especially those dealing with Koszul complexes and Cohen-Macaulay rings—we shall use [13].

It is convenient to rephrase the condition (SD) for an ideal I in terms of the depths of the cycles and boundaries of the associated Koszul complex. Assume that R is a Cohen-Macaulay local ring of dimension d and that I is generated by the sequence $\mathbf{x} = \{x_1, \dots, x_n\}$; put $g = \text{ht}(I)$. Denote by Z_i and B_i the modules of cycles and boundaries of the associated Koszul complex K_* . If one uses the defining exact sequences

$$\begin{aligned} 0 \longrightarrow Z_{i+1} \longrightarrow K_{i+1} \longrightarrow B_i \longrightarrow 0 \\ 0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0 \end{aligned}$$

the depth conditions (SD) and (SCM = strongly Cohen-Macaulay) translate as follows:

$$\text{depth}(Z_i) \geq \begin{cases} \min\{d, d - n + i + 1\}, & \text{for (SD)} \\ \min\{d, d - g + 2\}, & \text{for (SCM)}. \end{cases}$$

We look at the case $i = n - g$ to examine the role of duality. From now on we assume that R is a Gorenstein ring.

PROPOSITION 1.1. *Let R be a Gorenstein local ring of dimension d and I be a Cohen-Macaulay ideal of height g generated by n elements. Then $\text{depth}(Z_{n-g}) \geq \min\{d, d - g + 2\}$.*

Proof. If $g = 0$, $Z_n = 0 : I = \text{Hom}_R(R/I, R)$ is Cohen-Macaulay since R/I is a Cohen-Macaulay module and R is Gorenstein.

If $g = 1$, the exact sequence

$$0 \longrightarrow B_{n-1} \longrightarrow Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0$$

yields ($*E$ denotes the R -dual $\text{Hom}(E, R)$):

$$0 \longrightarrow Z_{n-1}^* \longrightarrow B_{n-1}^* \longrightarrow \text{Ext}^1(H_{n-1}, R) \longrightarrow \text{Ext}^1(Z_{n-1}, R) \longrightarrow 0$$

Since $B_{n-1}^* = R$ and $\text{Ext}^1(H_{n-1}, R) = R/I$ by duality, we get an exact sequence

$$0 \longrightarrow R/Z_{n-1}^* \xrightarrow{\phi} R/I \longrightarrow \text{Ext}^1(Z_{n-1}, R) \longrightarrow 0.$$

Since Z_{n-1} is a second syzygy module, the last module has support at primes of height greater than two. In the identification $B_{n-1}^* = R$, ϕ maps Z_{n-1}^* maps exactly onto I : To see this it suffices to localize at any prime P (necessarily of height 1) associated to either Z_{n-1}^* or I . Thus ϕ is essentially the multiplication of R/I into itself via a regular element of the Cohen-Macaulay ring R/I . By the remark above on the support of $\text{Ext}^1(Z_{n-1}, R)$, ϕ is an isomorphism.

If $g > 1$, consider the sequence

$$0 \longrightarrow B_{n-g} \longrightarrow Z_{n-g} \longrightarrow H_{n-g} \longrightarrow 0.$$

Here B_{n-g} has depth $d - g + 1$ while H_{n-g} has depth $d - g$ being the canonical module of R/I . The exact sequence says that $\text{depth}(Z_{n-g}) \geq d - g$. We now test the vanishing of the modules $\text{Ext}^i(Z_{n-g}, R)$ for $i = g, g - 1$. From above we obtain the homology sequence

$$\begin{aligned} \text{Ext}^{g-1}(H_{n-g}, R) &\longrightarrow \text{Ext}^{g-1}(Z_{n-g}, R) \longrightarrow \text{Ext}^{g-1}(B_{n-g}, R) \longrightarrow \\ \text{Ext}^g(H_{n-g}, R) &\longrightarrow \text{Ext}^g(Z_{n-g}, R) \longrightarrow \text{Ext}^g(B_{n-g}, R). \end{aligned}$$

Here $\text{Ext}^{g-1}(B_{n-g}, R) = R/I$ from the exactness of the tail of the Koszul complex. On the other hand $\text{Ext}^g(B_{n-g}, R) = \text{Ext}^{g-1}(H_{n-g}, R) = 0$, while $\text{Ext}^g(H_{n-g}, R) = R/I$ since R is a Gorenstein ring. Thus we have the exact sequence

$$0 \longrightarrow \text{Ext}^{g-1}(Z_{n-g}, R) \longrightarrow R/I \xrightarrow{\phi} R/I \longrightarrow \text{Ext}^g(Z_{n-g}, R) \longrightarrow 0.$$

Localizing at primes of height g and $g + 1$, we get that ϕ is an isomorphism since Z_{n-g} is a second syzygy module and the desired assertion follows. \square

COROLLARY 1.2 (see [2]). *Let I be a Cohen-Macaulay ideal of height g that can be generated by $n = g + 2$ elements. Then I is strongly Cohen-Macaulay.*

Remark. If $n = g + 3$ even the condition (SD) may fail to hold; see Section 2.

COROLLARY 1.3. *Let I be an ideal satisfying (SD). If R/I satisfies Serre's condition S_2 , then I is Cohen-Macaulay.*

Proof. (SD) implies that the canonical module of R/I , H_{n-g} , is Cohen-Macaulay. But the argument above shows that $R/I = \text{Ext}^g(H_{n-g}, R)$ given

the condition S_g . □

The main result of this section is the following criterion for (SCM).

THEOREM 1.4. *Let R be a Gorenstein local ring and let I be a Cohen-Macaulay ideal. If I satisfies (SD) and $v(I_p) \leq \max\{\text{ht}(I), \text{ht}(P) - 1\}$ for each prime ideal $P \supset I$, then I is strongly Cohen-Macaulay.*

Proof. Since (SD) and the other conditions localize (cf. [9]), we may assume that I is (SCM) on the punctured spectrum of R . By adding a set of indeterminates to R and to I , we may assume the height g of I is larger than $n - g + 1$, $n =$ minimum number of generators of the new ideal. This clearly leaves the Koszul homology and (SD) unchanged. The net effect however is that we have a Koszul complex K_* whose acyclic tail is longer than the remainder of the complex.

(i) In the conditions above, H_{n-g-i} is the H_{n-g} -dual of H_i [11]; to use the theorem of duality of [6]—see also [11]—one has to verify that the left hand side of the inequality

$$\begin{aligned} \text{depth}(H_i) + \text{depth}(H_{n-g-i}) &\geq (d - n + i) + (d - n + n - g - i) \\ &= (d - g) + (d - n) \end{aligned}$$

exceeds $(d - g) + 1$. If, therefore, $n < d - 1$, it will follow that each H_i is Cohen-Macaulay.

(ii) To set the tone of the argument in case $n = d - 1$, we examine H_1 . Here $\text{depth}(H_{n-g-1}) \geq d - g - 1$ and $\text{depth}(H_1) \geq 2$; we will strengthen the first inequality. Suppose it cannot be done and consider the exact sequence

$$0 \longrightarrow B_{n-g-1} \longrightarrow Z_{n-g-1} \longrightarrow H_{n-g-1} \longrightarrow 0.$$

By (1.1) $\text{depth}(B_{n-g-1}) \geq d - g + 1$ so that if $\text{depth}(H_{n-g-1}) = d - g - 1$ then $\text{depth}(Z_{n-g-1}) = d - g - 1$ as well. It will follow that $\text{depth}(B_{n-g-2}) = d - g - 2$. A similar sequence for $i = n - g - 2$, again by duality, says that $\text{depth}(H_{n-g-2}) = d - g$ or $d - g - 2$. In either case we get that $\text{depth}(Z_{n-g-2}) = d - g - 2$. We repeat this argument until we get

$$\text{depth}(B_i) = \text{depth}(B_{n-g-(n-g-1)}) = d - g - (n - g - 1) = d - n + 1 = 2.$$

Since $\text{depth}(Z_i) = d - g + 2 > 2$, we get a contradiction.

(iii) To set up the induction routine, suppose we have shown that H_k and H_{n-g-k} are Cohen-Macaulay; we show that $\text{depth}(Z_{n-g-k}) \geq d - g + 2$.

The argument is similar to (1.1). We have the exact homology sequence

$$0 \longrightarrow \text{Ext}^{g-1}(Z_{n-g-k}, R) \longrightarrow \text{Ext}^{g-1}(B_{n-g-k}, R) \longrightarrow \text{Ext}^g(H_{n-g-k}, R) \\ \longrightarrow \text{Ext}^g(Z_{n-g-k}, R) \longrightarrow 0,$$

since $\text{depth}(B_{n-g-k}) \geq d - g + 1$, by induction. But we also have the isomorphisms $\text{Ext}^{g-1}(B_{n-g-k}, R) = \text{Ext}^{g-2}(Z_{n-g-k+1}, R) = \text{Ext}^{g-2}(B_{n-g-k+1}, R) = \dots = \text{Ext}^{g-k-1}(B_{n-g}, R)$. (This is possible by our ‘increase’ in g .) This last module however, from the self-duality in the Koszul complex, is nothing but H_k . Since $\text{Ext}^g(H_{n-g-k}, R)$ is also a Cohen-Macaulay module, as in (1.1) we conclude that $\text{depth}(Z_{n-g-k}) \geq d - g + 2$. \square

It is clear that one only needs this strengthened (SD) to hold in the lower half range of i . In this regard we have

COROLLARY 1.5. *Let I be a Cohen-Macaulay ideal with (SD). If I is a syzygetic ideal and I/I^2 is a torsion-free R/I -module then H_1 is a Cohen-Macaulay module.*

Proof. The syzygetic condition on I (cf. [15]) simply means that the natural sequence

$$H_1 \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0$$

is exact on the left. In such case H_1 satisfies S_2 , and the argument above goes through. \square

Remark. If R is not a Gorenstein ring (1.5) does not always hold.

§ 2. Codimension three

We exhibit examples of Cohen-Macaulay ideals of height 3 in regular local rings, generated by d -sequences, satisfying (SD) but not (SCM). Since it is known that ideals in the linkage class of a complete intersection are (SCM) [10], we look at non-Gorenstein ideals. For an ideal I with a presentation

$$0 \longrightarrow Z \longrightarrow R^n \longrightarrow I \longrightarrow 0$$

one has the following exact sequences

$$0 \longrightarrow \text{Tor}_1(I, R/I) \longrightarrow Z/IZ \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0$$

and

$$A^2 I \longrightarrow \text{Tor}_1(I, R/I) \longrightarrow \delta(I) \longrightarrow 0$$

where $\delta(I)$ is defined by the associated exact sequence

$$0 \longrightarrow \delta(I) \longrightarrow H_1 \longrightarrow (R/I)^n \longrightarrow I/I^2 \longrightarrow 0,$$

cf. [15]. As remarked, I is called syzygetic if $\delta(I) = 0$. If 2 is invertible in R , we can further add that $\text{Tor}_1(I, R/I) = I^2I \oplus \delta(I)$.

THEOREM 2.1. *Let R be a regular local ring of dimension at least 6 with $2R = R$ and let I be a Cohen-Macaulay ideal of height 3. Denote by W the canonical module of R/I and let $W^* = \text{Hom}_{R/I}(W, R/I)$. Assume that I is syzygetic on the punctured spectrum of R . If W^* has depth at least 3, then I is syzygetic.*

Proof. Let

$$0 \longrightarrow R^p \xrightarrow{\psi} R^m \longrightarrow R^n \longrightarrow I \longrightarrow 0$$

be a minimal resolution of I . By assumption $\delta(I)$ is a module of finite length so that we only have to show that $\text{Tor}_1(I, R/I)$ has depth at least 1. Denote by Z the first-order syzygies of I . We have the exact sequence

$$0 \longrightarrow \text{Tor}_2(I, R/I) \longrightarrow (R/I)^p \xrightarrow{\psi \otimes R/I} (R/I)^m \longrightarrow Z/IZ \longrightarrow 0.$$

On the other hand, $W = \text{coker}(\psi^*) = \text{coker}(\psi^* \otimes (R/I))$, so that $\text{Tor}_2(I, R/I)$ is identified to W^* (see [4, supplement] for general comparisons between these two modules). It follows that Z/IZ —and $\text{Tor}_1(I, R/I)$ along with it—has the required depth. □

For the next two corollaries the hypothesis $2R = R$ is in force.

COROLLARY 2.2. *Let I be the ideal generated by the $(n-1)$ -sized $(n > 1)$ minors of a generic, symmetric $n \times n$ matrix. Then I is syzygetic.*

Proof. The assumption is that $R = k[[x_{ij}]]$, where $k = \text{field}$ and x_{ij} , $1 \leq i, j \leq n$, are indeterminates and the entries of a symmetric matrix $= \phi$. The hypothesis on the punctured spectrum follows by induction and the discussion in [12] of such ideals. On the other hand, Goto [3] proved that R/I is integrally closed with divisor class group $Z/(2)$, generated by the class of W . □

Remark. Let I be the ideal generated by the 2×2 minors of a generic 2×4 matrix. In view of the Plücker relations, I is not syzygetic. Since I is a complete intersection on the punctured spectrum of the corresponding ring, W^* must have depth 2.

COROLLARY 2.3. *Let I be the ideal generated by the 2×2 minors of a generic, symmetric 3×3 matrix ϕ . Then:*

- (a) *I is generated by a d -sequence, satisfies (SD) but not (SCM).*
- (b) *The Rees algebra of I , $\mathcal{R}(I)$, is an integrally closed, Cohen-Macaulay domain.*
- (c) *The associated graded ring of I , $\text{gr}_I(R)$, is a non-reduced, non-Gorenstein, Cohen-Macaulay ring.*

Proof. Let d be the determinant of the matrix ϕ . It is easily verified that $dx_{ij} \in I^2$ for each entry of ϕ ; since $d \notin I^2$, the class of d in I/I^2 is annihilated by the maximal ideal of R . Since I is syzygetic by (2.2), $\text{depth}(H_i) = 1$. Furthermore, as $d^2 \in I^3$, $\text{gr}_I(R)$ is non-reduced.

(a) We compute the depths of the modules Z_i , $i = 1, 2$ and 3 , of the Koszul complex on the canonical 6 generators of I . Since $\text{depth}(H_i) = 1$, $\text{depth}(Z_2) = 1 + \text{depth}(B_1) = 3$. On the other hand, $\text{depth}(Z_3) = 5$ by (1.1), so that I satisfies (SD) but not (SCM). Moreover, since I is also a complete intersection on the punctured spectrum of R , the approximation complex of I is acyclic and thus I is generated by a d -sequence (cf. [8]).

(b) and (c) follow now from [9, (6.5)], for the Cohen-Macaulay assertions. That $\mathcal{R}(I)$ is integrally closed can be verified either by a direct application of the Jacobian criterion— $\mathcal{R}(I)$ can be presented as a quotient $R[T_{ij}]/J$, with J derived from the explicit resolution of I —or more rapidly in the following manner. Since $\mathcal{R}(I)$ is Cohen-Macaulay, by Serre's normality criterion it suffices to check the localizations at its height 1 primes. Let P be such a prime and $\mathfrak{p} = P \cap R$. If $\mathfrak{p} \neq \mathfrak{m} = \text{maximal ideal of } R$ there is no difficulty since $I_{\mathfrak{p}}$ is a complete intersection. If $\mathfrak{p} = \mathfrak{m}$, $P = \mathfrak{m}R(I)$. Let Q be the corresponding prime of $R[T_{ij}]$ —i.e. $Q = \mathfrak{m}R[T_{ij}]$. Looking at the image of J in the vector space $(Q/Q^2)_{\mathfrak{p}}$ one easily gets that it has the desired rank 5. \square

The crucial hypothesis of (2.2) never occurs in codimension two.

THEOREM 2.4. *Let R be a regular local ring and let I be a Cohen-Macaulay ideal of height 2 which is generically a complete intersection. If the class of W in the divisor class monoid of R/I is 2-torsion, then I is a complete intersection.*

Proof. Let

$$0 \longrightarrow R^{n-1} \longrightarrow R^n \longrightarrow I \longrightarrow 0$$

be a resolution of I . Tensoring over with R/I we obtain the exact sequence

$$0 \longrightarrow \text{Tor}_1(I, R/I) \longrightarrow (R/I)^{n-1} \longrightarrow H_1 \longrightarrow 0,$$

since I is syzygetic (cf. [15]). As in the proof of (2.1), $\text{Tor}_1(I, R/I) = W^*$; if the class of W is 2-torsion, we have the exact sequence

$$0 \longrightarrow W \longrightarrow (R/I)^{n-1} \longrightarrow H_1 \longrightarrow 0.$$

Since H_1 is Cohen-Macaulay ([2]) and W is the canonical module of R/I , this sequence will split—as it does so after reduction modulo a maximal regular sequence of R/I . Therefore R/I will be a Gorenstein ring, and hence a complete intersection by Serre’s criterion ([14]). \square

§ 3. Residually Cohen-Macaulay ideals

We prove here the naturality of sliding depth in a theorem of Huneke ([11]) on residual intersections. We also relate (SD) to various notions of syzygetic sequences (cf. [7]).

In this section (R, \mathfrak{m}) is a Cohen-Macaulay local ring of dimension d with infinite residue field.

DEFINITION 3.1. Let I be an ideal of R and let $\mathbf{x} = \{x_1, \dots, x_s\}$ be a sequence of elements of I satisfying:

- (1) $\text{ht}((\mathbf{x}): I) \geq s \geq g = \text{ht}(I)$.
- (2) For all primes $P \supset I$ will $\text{ht}(P) \leq s$, one has
 - (i) $(\mathbf{x})_P = I_P$;
 - (ii) $v((\mathbf{x})_P) \leq \text{ht}(P)$.

I is said to be *residually Cohen-Macaulay* if for any such sequence, one has:

- (a) $R/(\mathbf{x}): I$ is Cohen-Macaulay of dimension $d - s$;
- (b) $((\mathbf{x}): I) \cap I = (\mathbf{x})$;
- (c) $\text{ht}((\mathbf{x}): I) > \text{ht}((\mathbf{x}): I)$.

Remark 3.2. Let $\mathbf{x} = \{x_1, \dots, x_s\}$ I be a sequence satisfying (1) and (2) above. Then:

- (a) $\text{ht}(\mathbf{x}) = \text{ht}(I)$;
- (b) $v((\mathbf{x})_P) \leq \text{ht}(P)$ for all primes $P \supset (\mathbf{x})$.

Proof. (a): Let P be a minimal prime of (\mathbf{x}) . Suppose $I \not\subset P$; then $((\mathbf{x}): I)_P = (\mathbf{x})_P$. It will follow from (1) that $\text{ht}(P) \geq s \geq \text{ht}(I)$.

(b): If $\text{ht}(P) \geq s$, the assertion is trivial; if $\text{ht}(P) < s$, the proof of (a) shows that $P \supset I$ and (2) applies.

THEOREM 3.3. *If I satisfies the sliding depth condition, then I is residually Cohen-Macaulay.*

THEOREM 3.4. *Suppose $v(I) \leq \text{ht}(P)$ for all primes $P \supset I$. The following conditions are equivalent:*

- (a) *I satisfies the sliding depth condition.*
- (b) *I is residually Cohen-Macaulay.*
- (c) *I can be generated by a d -sequence $\{x_1, \dots, x_n\}$ satisfying: $(x_1, \dots, x_{i+1})/(x_1, \dots, x_i)$ is a Cohen-Macaulay module of dimension $d - i$, for $i = 0, \dots, n - 1$.*

Remark. The ideals occurring in the filtration of (3.4c) have the following homological properties. Assume that R is a regular local ring and that I is a Cohen-Macaulay ideal of height g . Consider the sequences

$$0 \longrightarrow I_i \longrightarrow I_{i+1} \longrightarrow Q_i \longrightarrow 0$$

where $I_i = (x_1, \dots, x_i)$. We claim that the projective dimension of $I_i = i - 1$ for each $i < n$. Suppose one inequality holds; pick j largest with $\text{pd}(I_j) < j - 1$. Note that $j < n - 1$ since $I = I_n$ is assumed Cohen-Macaulay and Q_{n-1} has projective dimension $n - 1$. Localize R at an associated prime of Q_j ; this implies that each $Q_{j+k} = 0$ for $k > 0$, and thus $I_{j+1} = \dots = I_n$. Consider the (localized) sequence

$$0 \longrightarrow I_j \longrightarrow I_{j+1} \longrightarrow Q_j \longrightarrow 0;$$

since $\text{pd}(Q_j) = j$ and—now— $\text{pd}(I_{j+1}) = 0$ or $g - 1$, we conclude $\text{pd}(I_j) = j - 1$, which is a contradiction. \square

The proofs of (3.3) and (3.4) require some technical lemmata on sliding depth.

LEMMA 3.5. *Let $\{x_1, \dots, x_k\}$ be a regular sequence in I . Let “ $'$ ” denote the canonical epimorphism $R \rightarrow R/(x_1, \dots, x_k)$. I satisfies (SD) if and only if I' satisfies (SD) (in R').*

Proof. Complete the sequence to a generating set $x = \{x_1, \dots, x_n\}$ of I . The condition follows from the fact that $\dim(R') = d - k$, and the isomorphism (see [13]):

$$H_i(x_1, \dots, x_n; R) = H_i(x'_1, \dots, x'_n; R'). \quad \square$$

LEMMA 3.6. *Suppose $I \neq 0$, and $I_p = 0$ for all minimal primes $P \supset I$. Then*

- (a) $(0: I) \cap I = 0$;
- (b) $\text{ht}((0: I) + I) = 1$.

Moreover, if I satisfies (SD), then so does I^ , and $R/0: I$ is Cohen-Macaulay. (Here “*” denotes the canonical epimorphism $R \rightarrow R/(0: I)$.)*

Proof. (a) and (b) follow directly from the Abhyankar-Hartshorne lemma ([5]).

To prove the second assertion of the lemma, we use the exact sequences

$$0 \longrightarrow L_i \longrightarrow H_i(x_1, \dots, x_n; R) \longrightarrow H_i(x_1^*, \dots, x_n^*; R^*) \longrightarrow 0$$

of [11], where L_i is a direct sum of copies of $0: I$.

If I satisfies (SD), then $\text{depth}(0: I = Z_n) = d$. From the sequences we have

$$\text{depth } H_i(x_1^*, \dots, x_n^*; R) \geq d - n + i \quad \text{for } i < n,$$

while by (b) $\text{ht}(I^*) = 1$, and hence $H_n(x_1^*, \dots, x_n^*; R^*) = 0$.

To see that $R/0: I$ is Cohen-Macaulay, note that $R/0: I = B_{n-1}$, where $n = v(I)$. The assertion then follows from the exact sequence

$$0 \longrightarrow B_{n-1} \longrightarrow Z_{n-1} \longrightarrow H_{n-1} \longrightarrow 0$$

and the fact that Z_{n-1} is Cohen-Macaulay, cf. Section 1. □

LEMMA 3.7. *Suppose I is a generated by a proper sequence $\mathbf{x} = \{x_1, \dots, x_n\}$ (cf. [7]). The following conditions are equivalent:*

- (a) I satisfies (SD).
- (b) $\text{depth } R/(x_1, \dots, x_i) \geq d - i$, for $i = 0, \dots, n$.
- (c) $\text{depth } (x_1, \dots, x_{i+1})/(x_1, \dots, x_i) \geq d - i$, for $i = 0, \dots, n - 1$.

Proof. Since \mathbf{x} is a proper sequence, we have exact sequences

$$0 \longrightarrow H_i(x_1, \dots, x_j) \longrightarrow H_i(x_1, \dots, x_{j+1}) \longrightarrow H_{i-1}(x_1, \dots, x_j) \longrightarrow 0$$

for all $i > 1$. It follows by descending induction that if \mathbf{x} satisfies (SD), then $\text{depth } H_1(x_1, \dots, x_i) \geq d - i + 1$ for $i = 1, \dots, n$. It is also clear that, conversely, this diagonal condition will imply that $\text{depth } H_i(x_1, \dots, x_n) \geq d - i + 1$ for $i \geq 1$. We shall use this remark further in the proof.

Denote $M_i = ((x_1, \dots, x_i): x_{i+1})/(x_1, \dots, x_i)$ and $Q_i = (x_1, \dots, x_{i-1})/(x_1, \dots, x_i)$. We have exact sequences:

$$(1) \quad 0 \longrightarrow H_1(x_1, \dots, x_i) \longrightarrow H_1(x_1, \dots, x_{i+1}) \longrightarrow M_i \longrightarrow 0$$

$$(2) \quad 0 \longrightarrow M_i \longrightarrow R/(x_1, \dots, x_i) \longrightarrow Q_i \longrightarrow 0$$

and

$$(3) \quad 0 \longrightarrow Q_i \longrightarrow R/(x_1, \dots, x_i) \longrightarrow R/(x_1, \dots, x_{i+1}) \longrightarrow 0.$$

(b) \Rightarrow (c): Follows from the exact sequence (3).

(c) \Rightarrow (a): Using the exact sequences (1), (2), (3) and the earlier remark the assertion follows by induction on i .

(a) \Rightarrow (b): We show by induction on i that $\text{depth } R/(x_1, \dots, x_{n-i}) \geq d - n + i$. For $i = 0$ this is our assumption. Suppose the assertion has been proved for $j = n - i \leq n$, and assume that

$$\text{depth } R/(x_1, \dots, x_{j-1}) = k < d - j + 1.$$

Now by (1) we have $\text{depth } M_{j-1} \geq d - j + 1$; hence the map

$$\alpha: \text{Ext}^k(R/\mathfrak{m}, R/(x_1, \dots, x_{j-1})) \longrightarrow \text{Ext}^t(R/\mathfrak{m}, Q_{j-1})$$

induced by (2) is injective. On the other hand (3) gives rise to the mapping

$$\beta: \text{Ext}^k(R/\mathfrak{m}, Q_{j-1}) \longrightarrow \text{Ext}^k(R/\mathfrak{m}, R/(x_1, \dots, x_{j-1}))$$

that is injective as well. It follows that the composite $\beta\alpha$ is injective. But this is a contradiction since $\beta\alpha$ is induced by multiplication by x_j , and is thus the null mapping. \square

Proof of (3.3): Suppose I satisfies (SD), $\text{ht}(I) = g$ and $\{x_1, \dots, x_s\}$, $s \geq g$, is a sequence satisfying (1) and (2) of (3.1). All assertions depend solely on the ideal (x_1, \dots, x_s) ; we may therefore switch to a different set of generators. We use the general position argument of [1] (see [11]) to obtain a system of generators $\{x_1, \dots, x_s\}$ such that for all primes $P \supset I$ with $g \leq \text{ht}(P) = k \leq s$ we have

$$(*) \quad (x_1, \dots, x_s)_P = (x_1, \dots, x_k)_P$$

(see Remark (3.2b)).

We now proceed by induction on s . Let $s = g$. Since by (3.2a)

$\text{ht}(x_1, \dots, x_g) = \text{ht}(I) = g$, it follows that $\{x_1, \dots, x_g\}$ is a regular sequence. Denote by “'” the epimorphism $R \rightarrow R/(x_1, \dots, x_g)$. According to (3.5), I' satisfies (SD) and therefore $R'/(0: I')$ is Cohen-Macaulay of dimension $d - g$ (cf. 3.6). But $R/(x_1, \dots, x_g): I = R'/(0: I')$, and hence condition (a) in (3.1) is realized. For the conditions (b) and (c), we have by (3.6) that $(0: I') \cap I' = 0$ and $\text{ht}((0: I') + I') > 0$, which translate as desired.

We now assume that $s > g$.

1. Case $g > 0$: This is immediate from (*) and the reduction to the ring R' . I' and $\{x'_1, \dots, x'_s\}$ satisfy all the hypotheses of the theorem. By induction the statements (a), (b) and (c) of (3.1) hold then and it is easily lifted to R .

2. Case $g = 0$: Let “*” denote the canonical epimorphism $R \rightarrow R/0: I$. By (3.6) R^* is Cohen-Macaulay of dimension d , I^* and $\{x_1^*, \dots, x_s^*\}$ satisfy (1) of (3.1). As for (2), we only have to check that $((x_1^*, \dots, x_s^*): I^*) = ((x_1, \dots, x_s): I)^*$. The inclusion \supset is obvious. Let a^* be an element of $(x_1^*, \dots, x_s^*): I^*$; then $aI \subset (x_1, \dots, x_s) + 0: I$. For x in I we can therefore write $ax = y + z$, $y \in (x_1, \dots, x_s)$, $z \in 0: I$. It follows that $z = ax - y$ lies in $I \cap 0: I = 0$, by (3.6). Furthermore we now have $\text{ht}(x_1^*, \dots, x_s^*) = \text{ht}(I^*) > 0$ and I^* satisfies (SD); we are then back in case 1. Therefore $\{x_1^*, \dots, x_s^*\}$ and I^* satisfy (a), (b) and (c) of (3.1); again it is easy to lift back to R . □

Proof of (3.4): (a) \Rightarrow (b) is already proved more generally in (3.3).

(b) \Rightarrow (c): Since $v(I_P) \leq \text{ht}(P)$ for all primes $P \supset I$, we may choose generators $\{x_1, \dots, x_n\}$ of I such that

- (i) $(x_1, \dots, x_s)_P = I_P$, for all $P \supset I$, $\text{ht}(P) \leq s$, and
- (ii) $\text{ht}((x_1, \dots, x_s): I) \geq s$.

Since I is residually Cohen-Macaulay, we then have that for $s \geq g = \text{ht}(I)$, (a), (b) and (c) of (3.1) hold.

It is clear that $\{x_1, \dots, x_g\}$ is a regular sequence. Next we show that x_{s+1} is not a zero-divisor on $R/(x_1, \dots, x_s): I$ for $g \leq s < n$. It will then follow that $(x_1, \dots, x_s): I = (x_1, \dots, x_s): x_{s+1}$. Together with condition (b) this will imply that $\{x_1, \dots, x_n\}$ is a d -sequence.

Denote by “'” the canonical epimorphism $R \rightarrow R/(x_1, \dots, x_s): I$. (a) and (c) imply the I' contains a non-zero divisor z . Suppose x'_{s+1} is a zero divisor. Let $y \in (x'_{s+1}): I'$; then $zy \in (x'_{s+1})$. This shows that $(x'_{s+1}): I'$ con-

sists of zero-divisors. Since R' is Cohen-Macaulay, this implies that $\text{ht}((x'_{s+1}): I') = 0$, contradicting (a). Since $(x_1, \dots, x_{s+1})/(x_1, \dots, x_s) = R/(x_1, \dots, x_s)$; $x_{s+1} = R/(x_1, \dots, x_s): I$, the implication is proved.

(c) \Rightarrow (a): Apply (3.7). \square

REFERENCES

- [1] M. Artin and M. Nagata, Residual intersection in Cohen-Macaulay rings, *J. Math. Kyoto Univ.*, **12** (1972), 307–323.
- [2] L. Avramov and J. Herzog, The Koszul algebra of a codimension 2 embedding, *Math. Z.*, **175** (1980), 249–280.
- [3] S. Goto, The divisor class group of a certain Krull domain, *J. Math. Kyoto Univ.*, **17** (1977), 47–50.
- [4] A. Grothendieck, Théorèmes de dualité pour les faisceaux algébriques cohérents, *Sem. Bourbaki*, t. **9** (1956/57).
- [5] R. Hartshorne, Complete intersections and connectedness, *Amer. J. Math.*, **84** (1962), 497–508.
- [6] R. Hartshorne and A. Ogus, On the factoriality of local rings of small embedding codimension, *Comm. Algebra*, **1** (1974), 415–437.
- [7] J. Herzog, A. Simis and W. V. Vasconcelos, Approximation complexes of blowing-up rings. I, *J. Algebra*, **74** (1982), 466–493.
- [8] J. Herzog, A. Simis and W. V. Vasconcelos, Approximation complexes of blowing-up rings. II, *J. Algebra*, **82** (1983), 53–83.
- [9] J. Herzog, A. Simis and W. V. Vasconcelos, On the arithmetic and homology of algebras of linear type, *Trans. Amer. Math. Soc.*, **283** (1984), 661–683.
- [10] C. Huneke, Linkage and the Koszul homology of ideals, *Amer. J. Math.*, **104** (1982), 1043–1062.
- [11] C. Huneke, Strongly Cohen-Macaulay schemes and residual intersections, *Trans. Amer. Math. Soc.*, **277** (1983), 739–763.
- [12] T. Józefiak, Ideals generated by minors of a symmetric matrix, *Comment. Math. Helv.*, **53** (1978), 595–607.
- [13] H. Matsumura, *Commutative Algebra*, Benjamin-Cummings, New York, 1980.
- [14] J.-P. Serre, Sur les modules projectifs, *Sem. Dubreil-Pisot*, 1960/61, exposé 2.
- [15] A. Simis and W. V. Vasconcelos, The syzygies of the conormal module, *Amer. J. Math.*, **103** (1981), 203–224.
- [16] A. Simis and W. V. Vasconcelos, On the dimension and integrality of symmetric algebras, *Math. Z.*, **177** (1981), 341–358.

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