

RINGS OF MODULAR FORMS ON EICHLER'S PROBLEM

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In his paper [4] or lecture note [3], Eichler asked the problem when the ring of modular forms is Cohen-Macaulay. We shall try to investigate it for the Hilbert or Siegel modular case.

When the dimension n is one, any ring of modular forms for an arithmetic group is Cohen-Macaulay, indeed a normal (graded) ring of Krull dimension two is always Cohen-Macaulay. So we consider the case $n > 1$. Unfortunately rings of modular forms do not always have this nice property. In the case of (symmetric or not) Hilbert modular forms it is essentially Freitag's result (see 7.1 Satz [6] and Proposition A in § 1.1). Let $A(\Gamma) = \bigoplus_k A(\Gamma)_k$ be the ring of Hilbert modular forms for a group Γ . Then the same question for $A(\Gamma)^{(2)} = \bigoplus_{k \equiv 2(0)} A(\Gamma)_k$ with $n = 2$ was raised by Thomas and Vasquez [20], in which it is shown by using the criterion due to Stanley [18], [19] that $A(\Gamma)^{(2)}$ is also Gorenstein if it is Cohen-Macaulay under some condition on Γ . Also Eichler derived some consequence of the 'hypothesis' of $A(\Gamma)^{(2)}$ being Cohen-Macaulay with $n = 2$ in [3].

In this paper we shall show this affirmatively, and moreover get when $A(\Gamma)^{(r)}$ is Cohen-Macaulay for general n and $r \geq 2$, as well as the case of symmetric Hilbert modular forms (Theorem 1). Furthermore if $n = 2$ and if Γ acts freely on H^2 , the necessary and sufficient condition for $A(\Gamma)$ to be Cohen-Macaulay is given as

$$(1) \quad \dim A(\Gamma)_1 = \frac{1}{2}(-\frac{1}{2}\zeta_K(-1) \cdot a + \chi + h)$$

where K is a corresponding real quadratic field, ζ_K is its zeta function, $a = [SL_2(O_K); \Gamma]$, O_K being the ring of integers of K , h is the number of the cusps and χ is the arithmetic genus of the non-singular model of the Hilbert modular surface.

Received January 5, 1984.
Revised May 29, 1984.

Let us refer to the case of Siegel modular forms, and let $A(\Gamma)$ denote the ring of Siegel modular forms for an arithmetic group Γ . When the degree n of the Siegel space is two, if Γ possesses, as its normal subgroup, the principal congruence subgroup $\Gamma_2(2)$ of level two, then $A(\Gamma)$ is Cohen-Macaulay. But the Cohen-Macaulayness does not always hold, indeed $A(\Gamma_2(\ell))$ is such an example if $\ell \geq 6$. When $n = 3$, $A(\Gamma_3(\ell))$, $\ell \geq 3$, is no longer Cohen-Macaulay. We shall show these by disproving the Serre duality theorem for $\text{Proj}(A(\Gamma_n(\ell)))$ which should hold if $A(\Gamma_n(\ell))$ would be Cohen-Macaulay.

This work was done while the author was staying at Harvard University. He wishes to express his hearty thanks to the members of the Department of Mathematics of Harvard University for their hospitality, and to the Educational Project for Japanese Mathematical Scientists for financial support.

§1. Preliminaries

1. Let k be a field, and R be a noetherian k -algebra. We call R Cohen-Macaulay if any ideal generated by a regular sequence has no embedded prime. A k -scheme is called Cohen-Macaulay if all of its local rings are Cohen-Macaulay.

When R is a graded algebra, we have the following (for the detail see Serre [16] Theorem 2 IV-20, Hochster and Robert [13] § 1 (d));

PROPOSITION A. *Let $R = \bigoplus_{m \geq 0} R_m$ ($R_0 = k$) be a normal noetherian graded k -algebra of dimension $N + 1$. Then the following conditions are equivalent:*

- (i) R is Cohen-Macaulay.
- (ii) For some (equivalently any) system of homogeneous elements x_0, \dots, x_N such that R is integral over $k[x_0, \dots, x_N]$, R is free over it.
- (iii) Let $X = \text{Proj}(R)$, and \mathcal{O}_X be its structure sheaf. Then the cohomology group $H^\nu(X, \mathcal{O}_X(m))$ vanishes for $1 \leq \nu \leq N - 1$ and for every $m \in \mathbf{Z}$, where $\mathcal{O}_X(m)$ is Serre's twisting sheaf.

As an easy consequence of this, we get the following;

COROLLARY. *Let R be as in the proposition, and let r be an integer. Then $R^{(r)} = \bigoplus_{m \equiv 0 \pmod{r}} R_m$ is Cohen-Macaulay if and only if $H^\nu(X, \mathcal{O}_X(m)) = 0$ for $1 \leq \nu \leq N - 1$, $m \equiv 0 \pmod{r}$.*

Next one is a part of the famous results in [12].

PROPOSITION B. *Let G be a finite group acting k -linearly on R . Suppose either $\text{char}(k) = 0$, or the order of G is coprime to $\text{char}(k)$. If R is Cohen-Macaulay, then so is the invariant subring R^G .*

If we use the notation of Proposition A, then X is a Cohen-Macaulay scheme if and only if $H^\nu(X, \mathcal{O}_X(m))$ vanishes for $\nu < N$, $m \ll 0$ (see for example, the proof of Theorem 7.6, Chap. III, Hartshorne [9]). So if R is a Cohen-Macaulay algebra, then $X = \text{Proj}(R)$ is a Cohen-Macaulay scheme. The converse is not necessarily true, indeed an N -dimensional projective manifold over C carrying non-trivial holomorphic p -forms ($0 < p < N$) is such an example.

2. We shall prepare two lemmas for the later use.

LEMMA 1. *Let D be a domain in C^n and S be a finite group acting on D as holomorphic automorphisms. Let $\pi: D \rightarrow Y = D/S$ be the quotient. Take an automorphy factor $\rho(g, z)$ $g \in S$, $z \in D$ and consider an action on \mathcal{O}_D as*

$$(2) \quad f(z) \longmapsto \rho(g, z)^{-1} f(gz).$$

If \mathcal{F} denotes the invariant subsheaf of $\pi_(\mathcal{O}_D)$ under this action, then we have*

$$i_*(\mathcal{F}|_{Y_0}) = \mathcal{F}$$

where Y_0 is the regular open subset of Y with the inclusion map i .

Proof. Since all non-zero sections f of \mathcal{F} over an open subset V have V as their supports, i.e., $\{P \in V \mid f_P = 0 \text{ in } \mathcal{F} \otimes_{\mathcal{O}_{Y,P}} \mathcal{O}_{Y,P}\} = \emptyset$, \mathcal{F} is a subsheaf of $i_*(\mathcal{F}|_{Y_0})$. $Y' = Y - Y_0$ is of codimension ≥ 2 , since Y is a normal complex space. Hence any holomorphic function g on $\pi^{-1}(V) \cap (D - \pi^{-1}(Y'))$ is extendable to whole $\pi^{-1}(V)$, and moreover if g satisfies (2) on $\pi^{-1}(V) \cap (D - \pi^{-1}(Y'))$, then the extension of g also satisfies (2) on $\pi^{-1}(V)$. This shows that the injection of \mathcal{F} to $i_*(\mathcal{F}|_{Y_0})$ is surjective. q.e.d.

The following is an easy consequence of Corollary to Proposition 5.2.3 Grothendieck [8] (see also Théorème 5.3.1 and its Corollary).

LEMMA 2*. *Let Y be a separated scheme over C , and let \mathcal{F} be a coherent sheaf over Y . Let G be a finite group acting on Y , \mathcal{F} compatibly, and $\pi: Y \rightarrow Y/G$ be the quotient morphism. Then we have*

* The author was informed this by Prof. T. Oda.

$$H^v(Y, \mathcal{F})^G \simeq H^v(Y/G, (\pi_* \mathcal{F})^G).$$

§2. Hilbert modular forms

3. Let K be a totally real algebraic number field of degree $n (> 1)$, and O_K be the ring of integers. $SL_2(O_K)$ acts on the product H^n of n copies of the upper half plane $H = \{z \in \mathbf{C} \mid \text{Im } z > 0\}$ by the modular substitution;

$$z = (z_1, \dots, z_n) \longrightarrow Mz = \left(\frac{\alpha^{(1)}z_1 + \beta^{(1)}}{\gamma^{(1)}z_1 + \delta^{(1)}}, \dots, \frac{\alpha^{(n)}z_n + \beta^{(n)}}{\gamma^{(n)}z_n + \delta^{(n)}} \right)$$

for $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(O_K)$

where $\alpha^{(1)}, \dots, \alpha^{(n)}$ denote the conjugates of $\alpha \in K$ in some fixed order. Let Γ be a subgroup of $SL(O_K)$ of finite index. A holomorphic function f on H^n is called a *Hilbert modular form* for Γ of weight k if it satisfies

$$(3) \quad f(Mz) = \prod_{i=1}^n (\gamma^{(i)}z_i + \delta^{(i)})^k f(z) \quad \text{for any } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma.$$

The symmetric group \mathfrak{S}_n of n letters acts on H^n as permutations of the coordinates

$$z = (z_1, \dots, z_n) \longrightarrow \sigma z = (z_{\sigma(1)}, \dots, z_{\sigma(n)}) \quad \sigma \in \mathfrak{S}_n.$$

The automorphism group $\text{Aut}(K/\mathbf{Q})$ can be regarded as a subgroup of \mathfrak{S}_n because it acts on n -tuples $(\alpha^{(1)}, \dots, \alpha^{(n)})$ as permutations, i.e., for $\sigma \in \text{Aut}(K/\mathbf{Q})$ $((\sigma\alpha)^{(1)}, \dots, (\sigma\alpha)^{(n)})$ is nothing else but the permutation of $(\alpha^{(1)}, \dots, \alpha^{(n)})$. Let us fix some subgroup S of $\text{Aut}(K/\mathbf{Q}) \subset \mathfrak{S}_n$, and let $\hat{\Gamma}$ be the composite of S and Γ as groups acting on H^n . In what follows, we shall always suppose $\Gamma = \hat{\Gamma} \cap SL_2(O_K)$, in other words,

$$(4) \quad \sigma\Gamma\sigma^{-1} = \Gamma \quad \text{for any } \sigma \in S.$$

A holomorphic function f on H^n is called a (symmetric) Hilbert modular form for $\hat{\Gamma}$ if it satisfies both (3) and the identity $f(\sigma z) = f(z)$ for $\sigma \in S$. We shall denote by $A(\hat{\Gamma}) = \bigoplus A(\hat{\Gamma})_k$, the graded \mathbf{C} -algebra of Hilbert modular forms for $\hat{\Gamma}$, $A(\hat{\Gamma})_k$ being the vector space of Hilbert modular forms of weight k , and denote by $A(\hat{\Gamma})^{(r)}$, the subring $\bigoplus_{k \equiv 0(r)} A(\hat{\Gamma})_k$.

4. Let h denote the number of the cusps for $\hat{\Gamma}$. $X = H^n/\hat{\Gamma}$ is compactified by adding h points, and we get a normal projective variety X^* ,

which is isomorphic to $\text{Proj}(A(\hat{\Gamma}))$. We shall denote by X_0 , the regular open subset of X , hence of X^* .

Let $\mathcal{L}(i)$ denote the coherent sheaf on X^* corresponding to modular forms of weight $i \in \mathbf{Z}$, and let $\mathcal{L} = \mathcal{L}(1)$. Obviously we have $\mathcal{L}(i) \otimes \mathcal{L}(j) \subset \mathcal{L}(i+j)$ for $i, j \geq 0$. Let $\pi: \tilde{X} \rightarrow X^*$ be a desingularization. The canonical coherent sheaf K_{X^*} on X^* is given by $K_{X^*} = \pi_* K_{\tilde{X}}$, $K_{\tilde{X}}$ being the canonical invertible sheaf on \tilde{X} . K_{X^*} is determined up to desingularizations (Grauert-Riemenschneider [7]). We shall need also the dualizing sheaf ω_{X^*} which gives rise to the functorial isomorphism $\text{Hom}(\mathcal{F}, \omega_{X^*}) \simeq H^n(X^*, \mathcal{F})^\vee$ for coherent sheaves \mathcal{F} . Again by [7], ω_{X^*} equals $i_* K_{X_0}$ where i denotes the inclusion of X_0 to X^* . If X^* is Cohen-Macaulay, then there are natural isomorphisms $H^\nu(X^*, \mathcal{F}) \simeq H^{n-\nu}(X^*, \mathcal{F}^\vee \otimes \omega_{X^*})^\vee$ for any locally free sheaf \mathcal{F} and for its dual \mathcal{F}^\vee . We have the canonical inclusion $K_{X^*} \subset \omega_{X^*}$ (loc. cit). Moreover by Freitag [5] Satz 1 we have an equality $K_{X^*}|_X = \omega_{X^*}|_X$.

If S is a subgroup of the alternating group, then $\omega_{X^*}, \mathcal{L}(2)$ are isomorphic. Let us show this. Let X^0 be the open subset of X which is the complement of the fixed points set. Obviously $X^0 \subset X_0$. For an open subset U of X^0 if f is a section of $\Gamma(U, \mathcal{L}(2))$, then $f dz_1 \wedge \cdots \wedge dz_n$ gives a section of $\Gamma(U, K_{X_0})$ and vice versa. So K_{X_0} and $\mathcal{L}(2)$ are isomorphic on X^0 . Since the codimension of X^0 in X is larger than or equal to two as one can easily see, $K_{X_0}, \mathcal{L}(2)$ are isomorphic on X_0 by the extendability of holomorphic functions. So by Lemma 1 we get $\omega_{X^*}|_X \simeq \mathcal{L}(2)|_X$. Let ∞ be any cusp, and let U be an open neighborhood at ∞ . Then a section f of $\Gamma(U - \{\infty\}, \mathcal{L}(j))$, $j \in \mathbf{Z}$, admits the Fourier expansion

$$f(z) = c_0 + \sum_{\lambda} c_{\lambda} \exp(2\pi\sqrt{-1}(\lambda^{(1)}z_1 + \cdots + \lambda^{(n)}z_n))$$

where λ varies over some lattice of totally positive numbers in K . So f is holomorphic at ∞ , and hence we get $i_*(\mathcal{L}(j)|_{U-\{\infty\}}) = \mathcal{L}(j)$, i being the inclusion $U - \{\infty\} \rightarrow U$. Thus

$$\mathcal{L}(2) \simeq \omega_{X^*}.$$

Suppose that S is a group not contained in the alternating group. Let S' be the normal subgroup of S of index two which is a subgroup of the alternating group. Let $\psi: (H^n/\hat{\Gamma}')^* \rightarrow X^* = (H^n/\hat{\Gamma})^*$ be the canonical projection where $\hat{\Gamma}' = S' \cdot \Gamma$. If $\mathcal{L}'(2)$ is the coherent sheaf on $(H^n/\hat{\Gamma}')^*$ corresponding to modular forms of weight two, then we define a coherent

sheaf $\mathcal{L}(2)_-$ on X^* by

$$(5) \quad \Gamma(U, \mathcal{L}(2)_-) = \{f \in \Gamma(\psi^{-1}(U), \mathcal{L}'(2)) \mid f(\sigma z) = \text{sgn}(\sigma)f(z), \sigma \in S\},$$

U being an open subset of X^* . Any section of $\mathcal{L}(2)_-$ vanishes along the fixed points set under the action of the group S/S' . Then $\mathcal{L}(2)_-|_{x_0}$ is isomorphic to K_{x_0} by the similar argument as above and by [5] the proof of Hilfssatz 4, and moreover

$$\mathcal{L}(2)_- \simeq \omega_{x^*}.$$

5. Let $X^* = (H^n/\hat{\Gamma})^*$, $\mathcal{L}(i)$, $\mathcal{L} = \mathcal{L}(1)$ be as above.

LEMMA 3. Assume that $\Gamma = \hat{\Gamma} \cap SL_2(O_K)$ acts freely on H^n . Then

- i) $\mathcal{L}(i)$ is invertible, and $\mathcal{L}^i = \mathcal{L}(i)$,
- ii) $H^\nu(X^*, \mathcal{L}(i)) = 0$ for $i \geq 2$, $\nu > 0$, $(i, \nu) \neq (2, n)$.

Proof. Let $\pi: \tilde{X} \rightarrow X^*$ be the desingularization. Then the cohomology group $H^\nu(\tilde{X}, K_{\tilde{X}})$ ($\simeq H^{n-\nu}(\tilde{X}, \mathcal{O}_{\tilde{X}})^\vee$) vanishes for $0 < \nu < n$ (cf. [6]). Since all the higher direct image sheaves $R^i\pi_*K_{\tilde{X}}$ ($i \geq 1$) vanish by [7], also $H^\nu(X^*, K_{X^*})$, $0 < \nu < n$, vanish by using the Leray spectral sequence.

At first let us suppose $\hat{\Gamma} = \Gamma$. Then i) is obvious, and this implies that \mathcal{L} is ample. We have an exact sequence

$$0 \longrightarrow K_{X^*} \longrightarrow \mathcal{L}^2 \longrightarrow \mathcal{L}^2/K_{X^*} \longrightarrow 0,$$

where \mathcal{L}^2/K_{X^*} is supported only at cusps by the observation of Section 2.4. Tensoring \mathcal{L}^i and taking the long exact sequence

$$\longrightarrow H^\nu(X^*, \mathcal{L}^i \otimes K_{X^*}) \longrightarrow H^\nu(X^*, \mathcal{L}^{2+i}) \longrightarrow H^\nu(X^*, \mathcal{L}^i \otimes (\mathcal{L}^2/K_{X^*})) \longrightarrow,$$

we get the desired result since $H^\nu(X^*, \mathcal{L}^i \otimes (\mathcal{L}^2/K_{X^*}))$, $\nu > 0$, vanishes, and since also $H^\nu(X^*, \mathcal{L}^i \otimes K_{X^*})$, $i \geq 0$, $\nu > 0$, $(i, \nu) \neq (0, n)$, vanishes by the generalized Kodaira vanishing theorem (cf. [7] Satz 2.1) and by the above observation.

Let us consider the general case. Let us put $Y = H^n/\Gamma$, and let \mathcal{M} be the invertible sheaf on Y^* corresponding to modular forms of weight one. We have shown above that i), ii) hold for Y^* , \mathcal{M} . To prove i) for X^* , \mathcal{L} , it is enough to show that for any point x of X^* , there is a neighborhood V at x such that $\Gamma(V, \mathcal{L})$ has a section not vanishing at x as a function. If x is not a ramification point of the canonical projection $p: Y^* \rightarrow Y^*$, then nothing is a problem. If x is such a point, then we can take a point $y \in Y$ with $x = p(y)$, and its neighborhood W

such that $\Gamma(W, \mathcal{M})$ has a section f not vanishing at y . Then $g = \sum \sigma f$, σ running over the stabilizer subgroup at y of $S \simeq \hat{\Gamma}/\Gamma$, is a desired element, indeed if we take a sufficiently small neighborhood V at x , then $g|_V$ is a section of $\Gamma(V, \mathcal{L})$ whose value $g(x)$ at x is not zero. This shows i). ii) is a direct consequence of Lemma 2, noticing $\mathcal{L} = (p_*\mathcal{M})^S$. q.e.d.

For any $\hat{\Gamma}$, there is a normal subgroup $\hat{\Gamma}'$ of finite index such that $\hat{\Gamma}' \cap SL_2(O_K)$ acts freely on H^n . Then $\hat{\Gamma}/\hat{\Gamma}'$ acts on $(H^n/\hat{\Gamma}')^*$ as a finite automorphism group, and the quotient morphism $p: (H^n/\hat{\Gamma}')^* \rightarrow (H^n/\hat{\Gamma})^* = X^*$ is induced. If $\mathcal{L}'(i)$ is the invertible sheaf on $(H^n/\hat{\Gamma}')^*$ corresponding to modular forms of weight i , then $\mathcal{L}(i)$ equals to $(p_*\mathcal{L}'(i))^{\hat{\Gamma}/\hat{\Gamma}'}$. So by Lemma 3 and Lemma 2, we have the following;

PROPOSITION 1.

$$H^\nu(X^*, \mathcal{L}(i)) = 0 \quad \text{for } i \geq 2, \quad \nu > 0, \quad (i, \nu) \neq (2, n).$$

6. Our main theorem is as follows;

THEOREM 1. *Let $n, \Gamma, \hat{\Gamma}, A(\hat{\Gamma})$ be as in Section 2.3, and let Γ be under the condition (4) Then $A(\hat{\Gamma})^{(r)}$ is Cohen-Macaulay for any (equivalently some) $r \geq 2$ if and only if*

$$(6) \quad \begin{aligned} & n \leq 2, \quad \text{or} \\ & n = 3, \quad \hat{\Gamma}_\infty/\Gamma_\infty \simeq \mathbf{Z}/3\mathbf{Z} \quad \text{at each cusp } \infty, \quad \text{or} \\ & n = 4, \quad \hat{\Gamma}_\infty/\Gamma_\infty \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} \quad \text{at each cusp } \infty, \end{aligned}$$

where $\hat{\Gamma}_\infty$ (resp. Γ_∞) is denoting the stabilizer subgroup of $\hat{\Gamma}$ (resp. Γ) at ∞ .

By Thomas and Vasquez [20], Theorem 2, we also get the following corollary;

COROLLARY. *Let $n = 2$, and let Γ be $SL_2(O_K)$ or its torsion free subgroup. Then $A(\Gamma)^{(2)}$ is Gorenstein.*

To prove Theorem 1 the following is a key proposition.

PROPOSITION C (Freitag [6]). *The condition (6) and the following two conditions (a), (b) are all equivalent to each other.*

- (a) X^* is Cohen-Macaulay,
- (b) $H^\nu(X^*, \mathcal{O}_{X^*}) = 0$ for $0 < \nu < n$.

By the fact we saw in Section 1.1, $A(\hat{\Gamma})^{(r)}$ cannot be Cohen-Macaulay for any r unless (6) is the case. In [20] it is shown that $A(\Gamma)^{(2)}$ is never

Gorenstein under some condition on Γ with $n = 3$. But it is not even Cohen-Macaulay.

7. *Proof of Theorem 1.* By Propositions A and C it is enough to show merely 'if' part. We shall give the proofs of two kinds, however one is available only for $n = 2$. At first we assume $n = 2$. Then X^* is a normal surface and hence it is Cohen-Macaulay. We may assume $\Gamma (= \hat{\Gamma} \cap SL_2(O_K))$ acts freely on H^2 by replacing $\hat{\Gamma}$ by a normal subgroup $\hat{\Gamma}'$ of finite index if necessary. Indeed if $A(\hat{\Gamma}')^{(r)}$ is Cohen-Macaulay, then so is $A(\hat{\Gamma})^{(r)}$ by Proposition B because $A(\hat{\Gamma})^{(r)}$ is the invariant subring of $A(\hat{\Gamma}')^{(r)}$ under the action of $\hat{\Gamma}/\hat{\Gamma}'$. So we may assume that \mathcal{L} is an ample invertible sheaf by Lemma 3. Since X^* is Cohen-Macaulay, we have an isomorphism between the cohomology groups

$$H^\nu(X^*, \mathcal{L}^{-i}) \simeq H^{n-\nu}(X^*, \mathcal{L}^i \otimes \omega_{X^*})^\vee$$

by Serre's duality theorem. As we have seen in Section 2.4, K_{X^*} is a subsheaf of ω_{X^*} and ω_{X^*}/K_{X^*} is supported only at cusps. Now the similar argument as in the proof of Lemma 3 will derive the vanishing of the cohomology groups $H^{n-\nu}(X^*, \mathcal{L}^i \otimes \omega_{X^*})$ for $n - \nu > 0$, $i > 0$, $(i, n - \nu) \neq (0, n)$ (use ω_{X^*} instead of \mathcal{L}^2). So $H^\nu(X^*, \mathcal{L}^{-i})$ vanishes for $0 < \nu < n$, $i > 0$. Together with Proposition 1 and Proposition C (b) we get

$$\begin{aligned} H^\nu(X^*, \mathcal{L}^i) &= 0 \\ \text{for } 0 < \nu < n, \quad i &\equiv 0 \pmod{r}, \end{aligned}$$

where r is any integer greater than one. By Corollary to Proposition A this implies that $A(\hat{\Gamma})^{(r)}$ is Cohen-Macaulay, and our assertion is proved when $n = 2$.

In the case $n = 3, 4$ the above argument does not work since $(H^n/\hat{\Gamma}')^*$ may not be Cohen-Macaulay even if so is $(H^n/\hat{\Gamma})^*$, where $\hat{\Gamma}'$ is a subgroup of $\hat{\Gamma}$. Let us take a normal subgroup Γ' of $\hat{\Gamma}$ which acts freely on H^n . Then by the virtue of [1] we have a smooth toroidal compactification \bar{X}' of $X' = H^n/\Gamma'$, on which, we may assume, the finite quotient group $\hat{\Gamma}'/\Gamma'$ acts in the natural way (cf. [1], [22]). Let us put $\bar{X} = \bar{X}'/(\hat{\Gamma}'/\Gamma')$ that has only quotient singularities. \bar{X} (resp. \bar{X}') has $X = H^n/\hat{\Gamma}$ (resp. X') as its Zariski open subset. Let π (resp. π') be the morphism of the blowing up $\bar{X} \rightarrow X^*$ (resp. $\bar{X}' \rightarrow X'^*$), and let ψ (resp. $\bar{\psi}$) be the quotient map of X'^* to X^* (resp. \bar{X}' to \bar{X}). We have a commutative diagram

$$\begin{array}{ccc}
 \bar{X}' & \xrightarrow{\bar{\psi}} & \bar{X} \\
 \pi' \downarrow & & \downarrow \pi \\
 X'^* & \xrightarrow{\psi} & X^* .
 \end{array}$$

We shall show that the morphism π enjoys

$$R^\nu \pi_* \mathcal{O}_X = 0, \quad 0 < \nu < n - 1,$$

by using our assumption of X^* being Cohen-Macaulay. Let $\tilde{\pi}: \tilde{X} \rightarrow \bar{X}$ be the desingularization. Since \bar{X} has only rational singularities, the higher direct image sheaves $R^\nu \pi_* \mathcal{O}_{\tilde{X}}$, $\nu > 0$, vanish. $\pi \circ \tilde{\pi}: \tilde{X} \rightarrow X^*$ is the desingularization of X^* , and by the same reason as above $R^\nu(\pi \circ \tilde{\pi})_* \mathcal{O}_{\tilde{X}|X}$ vanish for $\nu > 0$. Since $(R^\nu(\pi \circ \tilde{\pi})_* \mathcal{O}_{\tilde{X}})_\infty = 0$ for $0 < \nu < n - 1$ if the local ring at a cusp ∞ is Cohen-Macaulay as Freitag showed [6] 4.5, $R^\nu(\pi \circ \tilde{\pi})_* \mathcal{O}_{\tilde{X}}$ vanish for $0 < \nu < n - 1$. Considering the Leray spectral sequence $E_2^{p,q} = R^p \pi_* (R^q \tilde{\pi}_* \mathcal{O}_{\tilde{X}}) \Rightarrow R^{p+q}(\pi \circ \tilde{\pi})_* \mathcal{O}_{\tilde{X}}$, we get the vanishing of $R^\nu \pi_* \mathcal{O}_X$ for $0 < \nu < n - 1$.

If $\mathcal{L}'(i)$ is the invertible sheaf on X'^* corresponding to modular forms of weight i , then $(\bar{\psi}_* \pi'^* \mathcal{L}'(i))^{\hat{\Gamma}'}$ is equal to $\pi^* \mathcal{L}(i)$ by using the facts that (i) π, π' are birational, (ii) X^*, X'^* are normal, and (iii) $\psi_* \mathcal{L}'(i)^{\hat{\Gamma}'} = \mathcal{L}(i)$. $H^\nu(\bar{X}', \pi'^* \mathcal{L}'(i))$ vanishes for $\nu < n$, $i < 0$ by the generalized Kodaira vanishing theorem [7]. So applying Lemma 2, we have $H^\nu(\bar{X}, \pi^* \mathcal{L}(i)) = 0$ for $\nu < n$, $i < 0$. Since $\mathcal{L}(i)$ is locally free near at each cusp, the projection formula $R^\nu \pi_* \mathcal{L}(i) = \mathcal{L}(i) \otimes R^\nu \pi_* \mathcal{O}_X$ holds and hence there exists the Leray spectral sequence $E_2^{p,q} = H^p(X^*, \mathcal{L}(i) \otimes R^q \pi_* \mathcal{O}_X) \Rightarrow H^{p+q}(\bar{X}, \pi^* \mathcal{L}(i))$. It follows from this that $H^\nu(X^*, \mathcal{L}(i)) = 0$ for $\nu < n$, $i < 0$. Now the same argument as above shows our assertion. q.e.d.

In the above proof, we have shown under the condition (6) that $H^\nu(X^*, \mathcal{L}(i))$ vanishes for $0 < \nu < n$ if $i \neq 1$. As a consequence of this we get the following;

PROPOSITION 2. $A(\hat{\Gamma})$ is Cohen-Macaulay if and only if

$$H^\nu(X^*, \mathcal{L}) = 0 \quad \text{for } 0 < \nu < n$$

together with the condition (6).

8. In what follows, we always assume $n = 2$, and that Γ acts freely on H^2 . Let a be the index $a = [SL_2(O_K); \Gamma]$, and let χ be the arithmetic

genus $\sum_{\nu=0}^2 (-1)^\nu \dim H^\nu(\tilde{X}, \mathcal{O}_{\tilde{X}})$ where \tilde{X} is the nonsingular model of $X^* = (H^2/\Gamma)^*$. χ is equal to $1 + \dim$. of the space of cusp forms of weight two. By Shimizu [17] (see also Hirzebruch [10] § 2 Theorem, Freitag [6] 7.2 Satz) we have a Hilbert polynomial $P(k)$ of $A(\Gamma)$:

$$(7) \quad P(k) = \frac{1}{2} \cdot \zeta_K(-1) \cdot ak(k-2) + \chi + h,$$

where ζ_K is the zeta function of K , and h is the number of cusps. $P(k)$ gives the dimension of $A(\Gamma)_k$ for $k \geq 3$, and $P(2)$ equals $\dim A(\Gamma)_2 + 1$. $P(k)$ must be equal to the Euler-Poincaré characteristic $\chi(\mathcal{L}^k) = \sum_{\nu=0}^2 (-1)^\nu \dim H^\nu(X^*, \mathcal{L}^k)$, which is known to be a polynomial of k (cf. [15]). Hence we have

$$\begin{aligned} & -\frac{1}{2} \zeta_K(-1) \cdot a + \chi + h \\ & = \dim H^0(X^*, \mathcal{L}) - \dim H^1(X^*, \mathcal{L}) + \dim H^2(X^*, \mathcal{L}). \end{aligned}$$

Since \mathcal{L}^2 is now Serre's dualizing sheaf (§ 2.4), $H^2(X^*, \mathcal{L})$ is just dual to $H^0(X^*, \mathcal{L})$. So we obtain

$$\dim A(\Gamma)_1 = \frac{1}{2}(-\frac{1}{2} \zeta_K(-1)a + \chi + h) + \frac{1}{2} \dim H^1(X^*, \mathcal{L}).$$

Especially the inequality

$$\dim A(\Gamma)_1 \geq \frac{1}{2}(-\frac{1}{2} \zeta_K(-1)a + \chi + h)$$

always holds, and $A(\Gamma)$ is Cohen-Macaulay if and only if the equality (1) holds by Proposition 2.

(1) is a nice equality in the following sence. If (1) is the case, then we can compute the generating function $Q(t) = \sum \dim A(\Gamma)_k t^k$ together with (7) and with $\dim A(\Gamma)_2 = P(2) - 1$ as

$$\begin{aligned} Q(t) = \frac{1}{(1-t)^3} \{ & 1 + t^5 + (t+t^4) \{ \frac{1}{2}(-\frac{1}{2} \zeta_K(-1)a + \chi + h) - 3 \} \\ & + (t^2 + t^3) \{ \frac{1}{2}(\frac{3}{2} \zeta_K(-1)a - \chi - h) + 2 \} \}. \end{aligned}$$

It is easy to see $Q(t)$ satisfies $-t^2 Q(t^{-1}) = Q(t)$. By Stanley [18] this implies that $A(\Gamma)$ is Gorenstein.

9. Let X^* , Γ be as above. Let \hat{X}^* denote $(H^2/\hat{\Gamma})^*$ where $\hat{\Gamma} = \mathfrak{S}_2 \cdot \Gamma$, and let $\hat{\mathcal{L}}$ be the invertible sheaf on \hat{X} corresponding to symmetric Hilbert modular forms of weight one. Let us suppose (4). Then if $p: X^* \rightarrow \hat{X}^*$ is the canonical projection, we have the direct decomposition

$$p_* \mathcal{L} = \hat{\mathcal{L}} \oplus \mathcal{L}_-$$

where \mathcal{L}_- is the coherent sheaf given in the similar way as (5). Since $\hat{\mathcal{L}} \otimes \mathcal{L}_- = \mathcal{L}(2)_-$, it gives Serre's dualizing sheaf on \hat{X}^* (§ 2.4). Thus we have

$$H^1(\hat{X}^*, \mathcal{L}_-) = \text{Ext}^1(\mathcal{O}_{\hat{X}^*}, \mathcal{L}_-) \simeq \text{Ext}^1(\hat{\mathcal{L}}, \hat{\mathcal{L}} \otimes \mathcal{L}_-) \simeq H^1(\hat{X}^*, \hat{\mathcal{L}})^\vee$$

and hence

$$\begin{aligned} H^1(X^*, \mathcal{L}) &\simeq H^1(\hat{X}^*, p_*\mathcal{L}) \simeq H^1(\hat{X}^*, \hat{\mathcal{L}}) \oplus H^1(\hat{X}^*, \mathcal{L}_-) \\ &\simeq H^1(\hat{X}^*, \hat{\mathcal{L}}) \oplus H^1(\hat{X}^*, \hat{\mathcal{L}})^\vee, \end{aligned}$$

and hence $\dim H^1(X^*, \mathcal{L}) = 2 \dim H^1(\hat{X}^*, \hat{\mathcal{L}})$. Thus $A(\Gamma)$ and $A(\hat{\Gamma})$ are Cohen-Macaulay or not alike by Proposition 2. Summing up the above, we shall state it as the proposition.

PROPOSITION 3. *Let K be a real quadratic field, and Γ be a subgroup of $SL_2(O_K)$ of finite index acting freely on H^2 . Then the following are equivalent;*

- (a) $A(\Gamma)$ is Gorenstein.
- (b) $A(\Gamma)$ is Cohen-Macaulay.
- (c) The equality (1) $\dim A(\Gamma)_1 = \frac{1}{2}(-\frac{1}{2}\zeta_K(-1)a + \chi + h)$ holds.

Assuming (4) for $S = \mathfrak{S}_2$,

- (d) $A(\hat{\Gamma})$ is Cohen-Macaulay.

The known examples of full rings $A(\Gamma)$ for above Γ are quite a few yet. At any rate such examples in Hirzebruch [11], which are

$$K = \mathbb{Q}(\sqrt{5}), \quad \Gamma = \Gamma(\sqrt{5}) = \{M \in SL_2(O_K) \mid M \equiv 1_2 \pmod{\sqrt{5}}\},$$

and

$$\begin{aligned} K &= \mathbb{Q}(\sqrt{2}), \quad \Gamma = \Gamma(2) \cdot \left\langle \begin{pmatrix} 1 + \sqrt{2} & \\ & 1 - \sqrt{2} \end{pmatrix} \right\rangle, \\ \Gamma(2) &= \{M \in SL_2(O_K) \mid M \equiv 1_2 \pmod{2}\}, \end{aligned}$$

are satisfying the conditions in Proposition 3. It may not be unreasonable to expect it in more general case.

10. Let Γ be as above. By the method of [20], we can show that $A(\Gamma)$ may possibly be a complete intersection ring only in a finite number of cases as following. The index $a = [SL_2(O_K) : \Gamma]$ is divisible by 6 because $SL_2(O_K)$ has torsion points of order 2, 3, on the other hand Γ not (cf. Hirzebruch [10] § 1.7). Let us put $a = 6f$. Then $A(\Gamma)$ may possibly be a complete intersection ring only if $(2\zeta_K(-1)f, h + \chi)$ is one of

$$(32/3, 32), (8, 26), (16/3, 20), (8/3, 14), (4, 16), (2, 11), \\ (4/3, 8), (10/3, 15), (4/3, 10), (2/3, 7), (2/3, 9).$$

Considering the values of the zeta functions at -1 , this cannot happen if the discriminant of K is larger than 105. We skip the proof, which will be almost the same as in [20].

§3. Single modular forms

11. Let H_n be the Siegel space of degree n , i.e., $\{Z \in M_n(\mathbb{C}) \mid Z = Z, \operatorname{Im} Z > 0\}$. The symplectic group $Sp(\mathbb{R})$ acts on H_n by the usual modular substitution

$$Z \longmapsto MZ = (AZ + B)(CZ + D)^{-1} \quad M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in Sp_n(\mathbb{R}).$$

We shall denote by $\Gamma_n(\ell)$, the principal congruence subgroup of level ℓ ; $\{M \in Sp_n(\mathbb{Z}) \mid M \equiv 1_{2n} \pmod{\ell}\}$.

Let f be a holomorphic function on H_n . f is called a *Siegel modular form* of weight k for a congruence subgroup Γ , if it satisfies

$$f(MZ) = |CZ + D|^k f(Z) \quad \text{for } M = \begin{pmatrix} AB \\ CD \end{pmatrix} \in \Gamma.$$

When $n = 1$, we need an additional condition that f is holomorphic also at cusps, which is automatic if $n > 1$. We denote by $A(\Gamma) = \bigoplus_{k \geq 0} A(\Gamma)_k$ (resp. $S(\Gamma) = \bigoplus_{k \geq 0} S(\Gamma)_k$), the graded ring of modular forms (resp. the graded ideal of cusp forms).

Let $X = H_n/\Gamma$, and let X^* be its Satake compactification, which is a normal projective variety isomorphic to $\operatorname{Proj}(A(\Gamma))$.

12. Let Γ be neat, and let \mathcal{L} be an invertible sheaf on X^* corresponding to modular forms of weight one. The regular open subset of X^* coincides with X , and $\mathcal{L}|_X$ is isomorphic to the canonical invertible sheaf K_X on X . Then by [7], the dualizing sheaf ω_{X^*} on X^* is given by $i_* \mathcal{L}|_X^{n+1}$, i being the inclusion map of X into X^* , where ω_{X^*} gives rise to the functorial isomorphism $\operatorname{Hom}(\mathcal{F}, \omega_{X^*}) \simeq H^{n(n+1)/2}(X^*, \mathcal{F})^\vee$ for coherent sheaves \mathcal{F} on X^* . Here we note

$$\omega_{X^*} \simeq \mathcal{L}^{n+1}$$

by Koecher's principle. So if X^* is Cohen-Macaulay, then $H^v(X^*, \mathcal{L}^k)$ is isomorphic to the dual of $H^{n(n+1)/2-v}(X^*, \mathcal{L}^{n+1-k})$ and hence $P(k) =$

$(-1)^{n(n+1)/2} P(n+1-k)$, $P(k)$ denoting the Hilbert polynomial of the graded ring $A(\Gamma)$ or equivalently $\chi(\mathcal{L}^k)$.

On the other hand it is shown in [2] Vol. 2-16 that

$$P(k) = \dim A(\Gamma)_k = \dim S(\Gamma)_k + \sum_{\Gamma' \subset S_{2n-1}(\mathbf{R})} \dim S(\Gamma')_k + \cdots + \sum_{\Gamma' \subset S_{21}(\mathbf{R})} \dim S(\Gamma')_k + \#(0\text{-dimensional cusps}).$$

for $k \gg 0$ where Γ' varies over the set of all the subgroups attached to cusps of X^* . (The above is shown in [2] for $k \gg 0$, $k \equiv 0 \pmod{2}$. However both sides must be numerical polynomials of k for $k \gg 0$, so we get the above formula.)

13. Let us consider the case $n = 2$. $A(\Gamma_2(2))$ was shown to be Cohen-Macaulay in Igusa [14]. So for any arithmetic group Γ containing $\Gamma_2(2)$ as a normal subgroup, $A(\Gamma)$ is Cohen-Macaulay by Proposition B. However the Cohen-Macaulayness fails for $\Gamma_2(\ell)$ $\ell \geq 6$. We shall show it.

Let X^* be the Satake compactification of $H^2/\Gamma_2(\ell)$ for some $\ell \geq 3$, and let $P(k)$ be the Hilbert polynomial for $A(\Gamma_2(\ell))$. Then if X^* is Cohen-Macaulay, we would have $P(3/2) = 0$ since $P(k) = -P(3-k)$ by the observation in Section 3.12. By Yamazaki [23] we can actually calculate $P(k)$ and hence $P(3/2)$;

$$P(3/2) = 2^{-4} 3^{-1} \ell^4 \prod_{p|\ell} (1-p^4) \{(\ell^3 - 6\ell^2) \prod_{p|\ell} (1-p^{-2}) + 2^3 3\}.$$

This is not zero if $\ell \geq 6$, so in this case X cannot be Cohen-Macaulay and hence $A(\Gamma_2(\ell))$ $\ell \geq 6$ are not Cohen-Macaulay algebras.

The similar argument works also for $\Gamma = \Gamma_3(\ell)$ $\ell \geq 3$ by using the formula by Tsushima [21]. Indeed if $(H_3/\Gamma_3(\ell))^*$ were Cohen-Macaulay, then the Hilbert polynomial $P(k)$ of the graded ring $A(\Gamma_3(\ell))$ would satisfy $P(k-2) - P(2-k) = 0$ by the observation in Section 3.12. However actually we have

$$\begin{aligned} P(k-2) - P(2-k) &= 2^{-7} 3^{-3} 5^{-1} \ell^{16} \prod_{p|\ell} (1-p^{-2})(1-p^{-4})(1-p^{-6})k^3 + O(k^2). \end{aligned}$$

So $(H_3/\Gamma_3(\ell))^*$ is not Cohen-Macaulay. We obtain the following;

PROPOSITION 4. *Let $\Gamma = \Gamma_n(\ell)$ with $n = 2$, $\ell \geq 6$ or $n = 3$, $\ell \geq 3$. Then the Satake compactification of $H_n/\Gamma_n(\ell)$ is not a Cohen-Macaulay variety. Especially if $A(\Gamma)$ denotes the ring of Siegel modular forms for Γ , then $A(\Gamma)^{(r)}$ is not Cohen-Macaulay for any integer r .*

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