T. Hida, K.-S. Lee and S.-S. Lee
Nagoya Math. J.
Vol. 98 (1985), 87–98

CONFORMAL INVARIANCE OF WHITE NOISE

TAKEYUKI HIDA, KE-SEUNG LEE AND SHEU-SAN LEE

§ 0. Introduction

The remarkable link between the structure of the white noise and that of the infinite dimensional rotation group has been exemplified by various approaches in probability theory and harmonic analysis. Such a link naturally becomes more intricate as the dimension of the time-parameter space of the white noise increases. One of the powerful method to illustrate this situation is to observe the structure of certain subgroups of the infinite dimensional rotation group that come from the diffeomorphisms of the time-parameter space, that is the time change. Indeed, those subgroups would shed light on the probabilistic meanings hidden behind the usual formal observations. Moreover, the subgroups often describe the way of dependency for Gaussian random fields formed from the white noise as the time-parameter runs over the basic parameter space.

The main purpose of this note is to introduce finite dimensional subgroups of the infinite dimensional rotation group that have important probabilistic meanings and to discuss their roles in probability theory. In particular, we shall see that the conformal invariance of white noise can be described in terms of the conformal group which is a finite dimensional Lie subgroup of the infinite dimensional rotation group.

As is well known, *the projective invariance* of the ordinary Brownian motion with one-dimensional parameter was discovered by P. Lévy [7], and a group theoretic as well as probabilistic interpretation was given in [6]. One may naturally ask "what is the higher dimensional parameter analogue of this property?" (See also [9]). This was the motivation of our present work. Our approach is rather group theoretic in technique, although it is probabilistic in spirit. For one thing, it is not so obvious to introduce a d-dimensional ($d \geq 2$) parameter analogue of a Brownian bridge, from which the discussion in [6] was originated. We shall there-

Received February 20, 1984.

fore give a plausible explanation to the reason why we take the conformal group, in Section 2. Then, we propose a d-dimensional version of the projective invariance of white noise (or of Brownian motion), namely *the conformal invariance of white noise.*

It is our hope the present work would develop to serve in the study of dependency for various Gaussian random fields that are formed from white noise.

§1. Background

This section is devoted to a summary of basic notions and a short review of some known results.

We start with a Gel'fand triple:

$$E \subset L^2(R^d) \subset E^* , \qquad d \geq 2 ,$$

where E is a nuclear space and E^* is the dual space of E. Given a characteristic functional

$$C(\xi) = \exp\left[-\tfrac{1}{2}\|\xi\|^2\right] , \quad \xi \in E , \quad \|\ \|: L^2(R^d)\text{-norm} , \tag{1.1}$$

a probability measure μ is introduced in the space E^* in such a way that

$$C(\xi) = \int_{E^*} \exp\left[i\langle x, \xi\rangle\right] d\mu(x) . \tag{1.2}$$

In fact, the measure μ is nothing but the probability distribution of white noise with time-parameter space R^d. Thus, each x in the space E^* with μ may be thought of as a sample function of a white noise. The Hilbert space $(L^2) = L^2(E^*, \mu)$ is therefore the collection of all functionals of a white noise with finite variance. A member of (L^2) is often called a Brownian functional.

A rotation g of E is a linear isomorphism of E such that $\|g\xi\| = \|\xi\|$ for any $\xi \in E$. The collection of all rotations of E, denote it by $O(E)$, forms a group under the usual product. This group $O(E)$ is called *the rotation group of* E or *the infinite dimensional rotation group* when the basic nuclear space is not necessarily mentioned.

Associated with g in $O(E)$ is the adjoint g^* determined by the relation

$$\langle x, g\xi\rangle = \langle g^*x, \xi\rangle , \quad \xi \in E , \quad x \in E^* . \tag{1.3}$$

Set

$$O^*(E^*) = \{g^*, g \in O(E)\} .$$

Then $O^*(E^*)$ is a group isomorphic to $O(E)$ under the correspondence:

$$(1.4) \qquad g \longleftrightarrow g^{*-1} \in O(E^*)\,, \qquad g \in O(E)\,.$$

PROPOSITION 1. *For any g^* in $O^*(E^*)$ the relation*

$$g^*\mu = \mu$$

holds.

This property is the first bridge that connects the measure μ of white noise and the infinite dimensional rotation group.

Coming back to the Hilbert space (L^2), we take a particular member $\langle x, \xi \rangle$, ξ being fixed. It is a random variable on the probability space (E^*, μ) and is Gaussian $N(0, \|\xi\|^2)$ in distribution. Suppose a sequence ξ_n converges to f in $L^2(R^d)$. Then $\{\langle x, \xi_n \rangle\}$ forms a Cauchy sequence in (L^2), so that there exists the limit of the $\langle x, \xi_n \rangle$ in the mean square sense. We denote this limit by $\langle x, f \rangle$, although it is no more continuous bilinear functional, but additive in $f \in L^2(R^d)$. Such a functional is often called *a stochastic bilinear form.* It is still Gaussian in distribution.

§ 2. Conformal group

In this section we focus our attention to one-parameter subgroups of $O(E)$ that come from the change of time variables. Such a one-parameter subgroup is often called a "whisker", and it is known that within the group $O(E)$ the family of whiskers is sitting entirely outside of the class of subgroups isomorphic to finite dimensional rotation groups or even outside of their inductive limit.

In what follows the basic nuclear space E is taken to be the space $D_0(R^d)$ defined by

$$(2.1) \qquad D_0(R^d) = \{\xi;\ \xi \text{ and } w\xi \text{ are } C\text{-functions on } R^d\}$$

where w denotes the inversion:

$$(w\xi)(u) = \xi(u/|u|^2)\,|u|^{-d}\,, \qquad u \in R^d\,.$$

We shall see later that such a choice of E is fitting for our purpose.

Now start with the most important, and in fact very simple example of a whisker; namely it is the shifts $\{S_t^j,\ t \in R^1\}$, $j = 1, 2, \cdots, d$, given by

$$(2.2) \qquad (S_t^j\xi)(u) = \xi(u - te_j)\,, \qquad e_j = (0, \cdots, 0, \overset{j}{\check{1}}, 0, \cdots, 0) \in R^d\,.$$

Such transformations certainly come from the transformation of u, indeed translations on R^d, and obviously each family $\{S_t^j\} \equiv \{S_t^j, t \in R^1\}$ forms a one-parameter group:

$$S_t^j S_s^j = S_{t+s}^j, \qquad t, s \in R^1 .$$

Take the adjoint $(S_t^j)^* = T_t^j$ to see that, by Proposition 1, $\{T_t^j, t \in R^1\}$ is a flow on (E^*, μ) i.e. that it is a one-parameter group of μ-measure-preserving transformations on E^*; in addition, it is continuous in t. By using these flows $\{T_t^j\}$, $j = 1, \cdots, d$, or alternatively the shifts $\{S_t^j\}$'s, we are able to describe random phenomena that are realized in (L^2) and that change as the time u goes by.

Another example, which is also interesting from a probabilistic viewpoint, is the isotropic dilation $\{\tau_t, t \in R^1\}$ given by

$$(\tau_t \xi)(u) = \xi(e^t u) e^{td/2}, \qquad t \in R^1 . \tag{2.3}$$

Obviously, it is a whisker. As for the probabilistic role of $\{\tau_t\}$ we may say, for instance, that the flow $\{\tau_t^*, t \in R^1\}$ gives an Ornstein-Uhlenbeck process $U(t)$ in such a manner that for any balanced set $A \subset R^d$

$$U(t) \equiv U(t, x) \equiv \langle \tau_t^* x, \chi_A \rangle, \ \chi_A\text{: indicator function of } A , \tag{2.4}$$

is a stationary Gaussian process with mean 0 and covariance function $|A| e^{-|t| d/2}$, $|A|$ being the volume of the set A.

We have so far obtained two kinds of whiskers, and now one may ask the relationship between them. The answer is that the shift is transversal to the dilation, in terms of dynamical system. More explicitly, we have

$$S_t^j \tau_s = \tau_s S_{te^s}^j , \qquad \text{for every } j . \tag{2.5}$$

In this sense these two whiskers are in a good relation.

We are now in search of other whiskers which are mutually in good relations together with the shifts and the dilation. The idea of our approach to this problem is the same as in [6], but of course much complicated. For our purpose we first establish the general form of a whisker $\{g_t\}$. By assumption g_t has to be of the form

$$(g_t \xi)(u) = \xi(\psi_t(u)) \sqrt{\left| \frac{\partial}{\partial u} \psi_t(u) \right|} , \tag{2.6}$$

where ψ_t is an automorphism of $\overline{R^1}$, the one-point compactification of R^1, satisfying

$$(\psi_t \circ \psi_s)(u) = \psi_{t+s}(u) , \tag{2.7}$$

which comes from the group property $g_t g_s = g_{t+s}$. The equation (2.7) above is the well-known translation equation, so that, by noting the inequality $1 = \dim t < \dim u = d$, it can be solved as follows (see J. Aczél [1]):

$$\psi_t(u) = f^{-1}(f(u) + tc) , \tag{2.8}$$

where f is an automorphism of R^d and c is a constant d-vector. Simple computations give us an explicit expression of the infinitesimal generator of the $\{g_t\}$ as is prescribed below.

THEOREM 1. *The infinitesimal generator α of the one-parameter subgroup of $O(E)$ defined by* (2.6) *with ψ_t given by* (2.8) *is expressed in the form*

$$\alpha = (a, \nabla) + \frac{1}{2}(\nabla, a) , \qquad \nabla = \left(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_d}\right), \tag{2.9}$$

where $a = c^t(\partial f^{-1}/\partial u)$ evaluated at $f(u)$, $(\partial f^{-1}/\partial u)$ being the Jacobian of the transformation f^{-1} and c being the constant appeared in (2.8).

Good relations among the whiskers have so far had only vague meaning, but we now understand rigorously in such a way that they generate a finite dimensional Lie subgroup of $O(E)$. A powerful technique for the investigation of this concept is the use of the Lie algebra generated by those infinitesimal generators of the form (2.9). The algebra has to be finite dimensional.

Associated with the shifts and the dilation are generators expressed in the forms

$$s_j = \frac{d}{dt} S_t^j |_{t=0} = -D_j , \quad D_j = \frac{\partial}{\partial u_j}, \quad j = 1, \cdots, d ,$$

$$\tau = \frac{d}{dt} \tau_t |_{t=0} = (u, \nabla) + \frac{d}{2} .$$

The commutation relation of them is

$$[\tau, s_j] \equiv \tau s_j - s_j \tau = - s_j , \tag{2.10}$$

which comes also from (2.5).

With this information we now find a possible class of infinitesimal generators, involving s_j's and τ, which generate a finite dimensional Lie algebra under the Lie product $[\alpha, \beta] = \alpha\beta - \beta\alpha$. First note that for α of the form (2.9)

$$(2.11)\qquad [s_j, \alpha] = -(D_j a, \nabla) = \tfrac{1}{2}(\nabla, D_j a)\,, \quad D_j a = (D_j a_1, \cdots, D_j a_d)\,,$$

$$(2.12)\qquad [\tau, \alpha] = (a', \nabla) + \tfrac{1}{2}(\nabla, a')\,, \quad a' = \sum_j u_j D_j a - a\,.$$

As in the case of [6] or of [3, § 5.3] we consider such α's as

$$[s_j, \alpha] = \sum_k \lambda_{j,k} s_k\,.$$

Then the coefficient vector a of α is an affine function of u. These α's and the s_j's form a finite dimensional (in fact, it is $(d + d^2)$-dimensional) Lie algebra, but the whole algebra is not interested probabilistically. We take only the following generators of rotations out of them:

$$(2.13)\qquad \gamma_{j,k} \equiv u_j \frac{\partial}{\partial u_k} - u_k \frac{\partial}{\partial u_j}\,, \qquad j \neq k,\ 1 \leq j,\ k \leq d\,.$$

The rotations $\{\gamma_\theta^{j,k}\}$ of the parameter space R^d with generator $\gamma_{j,k}$ also can define whiskers in an obvious way.

We then consider the relationship between τ and α of the form (2.9) with a polynomial coefficient of degree 2; in particular, we consider the case where

$$[\tau, \alpha] = \lambda\alpha$$

holds for some constant λ. For simplicity we may take $\lambda = 1$. Then we can propose differential operators of the form

$$(2.14)\qquad \kappa_j = 2u_j(u, \nabla) - |u|^2 \frac{\partial}{\partial u_j} + d \cdot u_j\,, \qquad j = 1, \cdots, d\,, \quad |u|^2 = \sum_j u_j^2$$

which are infinitesimal generators of the special conformal transformations

$$(2.15)\qquad \kappa_t^j = w S_t^j w\,, \quad t \in R^1\,, \quad j = 1, \cdots, d\,.$$

The action of κ_t^j can be expressed in the form

$$(2.16)\qquad (\kappa_t^j \xi)(u) = \xi\Big(\frac{u_1}{1 - 2tu_j + t^2|u|^2}, \cdots, \frac{u_j - t|u|^2}{1 - 2tu_j + t^2|u|^2}, \cdots, \frac{u_d}{1 - 2tu_j + t^2|u|^2}\Big)|1 - 2tu_j + t^2|u|^2|^{-d/2}\,.$$

Collecting all the infinitesimal generators obtained so far, the list of commutation relations is given as follows:

$$
\begin{aligned}
&[s_1, s_j] = 0\,, \quad [\kappa_i, \kappa_j] = 0\,, \quad [\tau, \gamma_{i,j}] = 0 \\
&[s_i, \kappa_i] = -2\tau\,, \quad [s_i, \kappa_j] = 2\gamma_{ij} \quad (i \neq j) \\
&[s_i, \tau] = s_i\,, \qquad [\kappa_i, \tau] = -\kappa_i \\
&\begin{cases} [\gamma_{i,j}, \gamma_{j,l}] = \gamma_{i,l}, \\ [\gamma_{i,j}, \gamma_{k,l}] = 0\,, \qquad (i, j, k, l \text{ different}) \end{cases} \\
&\begin{cases} [\gamma_{i,j}, \kappa_j] = \kappa_i \\ [\gamma_{i,j}, \kappa_k] = 0\,, \qquad (i, j, k \text{ different}) \end{cases} \\
&\begin{cases} [\gamma_{i,j}, s_j] = s_i\,, \\ [\gamma_{i,j}, s_k] = 0\,, \qquad (i, j, k \text{ different}) \end{cases}
\end{aligned}
$$

It can easily be seen that a $\{d(d+3)/2+1\}$-dimensional real simple Lie algebra, call it $\mathfrak{c}(d)$, is generated by all the members appeared in the above list, and be seen that the algebra is associated with the *conformal group* $C(d)$ which is generated by

$$
\begin{aligned}
&d \ \text{ shifts} && \{S_t^j\}\,, \quad j = 1, \cdots, d\,, \\
&1 \ \text{ dilation} && \{\tau_t\} \\
&\binom{d}{2} \text{ rotations} && \{\gamma_\theta^{i,j}\} \ i \neq j\,, \quad 1 \leq i, \quad j \leq d\,, \\
&d \ \text{ special conformal transformations} && \{\kappa_t^j\},\ j = 1, \cdots, d.
\end{aligned}
$$

The group $C(d)$ is certainly a subgroup of $O(E)$ involving whiskers. Among them the rotation has obvious probabilistic role as a particular transformation of the time variable u, while that of $\{\kappa_t^j\}$'s combining some others will be elucidated in the following section. (See also [8].)

Next comes another important observation: If we add other differential operators of the form (2.9) with a coefficient polynomial of degree ≥ 2 to the algebra $\mathfrak{c}(d)$, then the generated algebra is to be infinite dimensional. We can therefore prove, in line with our approach to discover whiskers, the following

PROPOSITION 2 ([4]). *The conformal group $C(d)$ in our expression is a maximal finite dimensional Lie subgroup of $O(E)$.*

Remark. i) It is known that $C(d)$ is isomorphic to the Lie group $SO_0(d+1, 1)$. In fact, $C(d)$ may be viewed as a unitary representation of $SO(d+1, 1)$.

ii) If we introduce the Iwasawa decomposition of $C(d)$

$$C(d) = KAN\,,$$

then A is taken to be the isotropic dilation which is abelian and one-dimensional regardless the number of the dimension of the parameter space. We can herewith see a position of the dilation or an Ornstein-Uhlenbeck process in the conformal invariance of white noise that will be the topic of the next section.

§ 3. Conformal invariance

This section is devoted to the investigation of a particular class of Gaussian random fields expressed as stochastic bilinear form in terms of white noise by using the conformal group established in the last section.

Let $S(p)$ denote a ball in R^d with diameter $\overline{Op}$, O being the origin of R^d, i.e. the point p is antipodal to the origin with respect to the ball $S(p)$. It is noted that the class $\boldsymbol{S} = \{S(p), p \in R^d\}$ is invariant under such transformation acting on R^d as i) the isotropic dilation, ii) the rotations and iii) the special conformal transformations like

$$u \longrightarrow \frac{u - t|u|^2}{1 - 2(t, u) + |t|^2|u|^2}\,, \qquad t \in R^d\,.$$

(Here one should not have confusion with the transformations acting on E with the same name introduced in the last section.) Now remained that any g^* in $O^*(E^*)$ is a μ-measure preserving transformation on E^*. Given a Gaussian random field $\{X(p)\} \equiv \{X(p, x)\} \equiv \langle x, f(p, \cdot)\rangle$, $p \in R^d$ with a suitable choice of a family $\{f(p, \cdot)\}$ of $L^2(R^d)$-functions indexed by $p \in R^d$, we immediately see that $\{X(p, x), p \in R^d\}$ and $\{X(p, g^*x), p \in R^d\}$ have the same probability distribution, namely they are the same Gaussian random field.

We are now in a position to introduce a special class of functions on R^d indexed by R^d as well:

$$f(p, u) = \alpha(|p|)\cdot\chi_{S(p)}(u)\{(p, u) - |p||u|^2\}^{-1}|u|^{-d+2} \tag{3.1}$$

and set

$$Y(p, x) = \langle x, f(p, \cdot)\rangle\,, \tag{3.2}$$

where $\alpha(\lambda)$ is a real valued function on $(0, 1)$ determined later. Because of the singularity at the origin the above expression can not be an ordinary stochastic bilinear form, however it does have meaning as a generalized Brownian functional if the regularization technique due to Gel'fand

and Shilov [2] is applied. For this topic we refer to the paper [5], and here we only note that the kernel $f(p, \cdot)$ has singularity of polynomial order at $u = O$ and that for ξ with support apart from O, $Y(p, x)$ can be evaluated at $\langle x, \xi\rangle$ to take the value $\int f(p, u)\xi(u)du$. The situation never changes even if x is replaced by g^*x with $g \in O(E)$. Such a family $\{Y(p, x)\}$ of generalized Brownian functionals is called *a generalized Gaussian random field.*

The interesting part of its property is that if g is restricted to $C(d)$ and does not involve the shift, then the action g^* turns out eventually to be the transformation of the parameter p. Let us observe in what follows the change of $Y(p, x)$ under g^* explicitly.

We restrict our attention to the case where the parameter p runs over the unit ball S with center at origin and where we take the g's in $C(d)$ that carry S onto itself. As a result the boundary ∂S is kept invariant under such g's.

First apply $(\kappa_{t_1}^1)^*$ to x to obtain

$$Y(p, (\kappa_{t_1}^1)^*x) = \langle x, \kappa_{t_1}^1 f(p, \cdot)\rangle ,$$

where

$$\kappa_{t_1}^1 f(p, u) = \frac{\alpha(|p|)\chi_{S(p^{(1)})}(u)}{(p, u) - (p_1 t_1 + |p|)|u|^2}|u|^{-d+2}$$

with $p = (p_1, p_2, \cdots, p_d)$, $p^{(1)} = (1 + p_1 t_1)^{-1}p$. Then apply $(\kappa_{t_j}^j)$, $j = 2, 3, \cdots, d$, to x successively, we have

$$Y\langle p, (\kappa_{t_1}^1)^* \cdots (\kappa_{t_d}^d)^*x\rangle = \langle x, \kappa_{t_d}^d \cdots \kappa_{t_1}^1 f(p, \cdot)\rangle , \tag{3.3}$$

where

$$\kappa_{t_d}^d \cdots \kappa_{t_1}^1 f(p, u) = \frac{\alpha(|p|)\chi_{S(p^{(1)})}(u)}{(p, u) - \{(p, t) + |p|\}|u|^2}|u|^{-d+2}$$

with $p^{(d)} = (1 + (p, t))^{-1}p$ and $(p, t) = \sum_j p_j t_j$. It is noted that we should exclude such t as $(p, t) = -1$.

Observe now that the image of the mapping

$$p \longrightarrow p^{(d)} , \qquad p \in S ,$$

does not agree with S in general. We must therefore apply a dilation so that the ball S is carried onto itself. Unfortunately, the magnification rate depends on p, but it is constant if p is restricted to a radius of S.

In view of this, we fix p° on ∂S, i.e. $|p^\circ| = 1$, and let p run over the radius $\overline{Op^\circ}$. Set $p = \lambda p^\circ, 0 \leq \lambda < 1$. Apply τ_s, to $f(p, \cdot)$, where s is determined by

$$e^s(1 + (p^\circ, t)) = 1 . \tag{3.4}$$

Then we have

$$\tau_s \kappa_t f(p, u) = \frac{\alpha(\lambda)}{1 + \lambda(p^\circ, t)} \frac{\chi_{S(p')}(u) e^{-sd/2}}{(p', u) - |p'||u|^2} |u|^{-d+2} , \tag{3.5}$$

where $\kappa_t = \kappa^d_{t_d} \cdots \kappa^d_{t_1}$, $t = (t_1, \cdots, t_d)$, and $p' = e^s(1 + (p, t))^{-1}p$. Set

$$\alpha(\lambda) = \alpha \lambda^{d/2}(1 - \lambda)^{-d/2+1} , \quad 0 \leq \lambda < 1 , \quad \alpha \text{ constant} . \tag{3.6}$$

Then, the expression (3.5) can be rephrased in the form

$$\tau_s \kappa_t f(p, u) = f(p', u) , \qquad p' = e^s(1 + (p, t))^{-1} p . \tag{3.7}$$

We are now almost ready to state our main theorem, but before doing so, some notations are introduced. Let A be the same as in ii) of the Remark at the end of Section 2, and let K_0 and N be the subgroup ($\subset C(d)$) of rotations and that of the special conformal transformations, respectively. Note that for any $k \in K_0$, there exists a rotation $\tilde{k} \in SO(d)$ such that

$$(kf)(u) = f(\tilde{k}u) .$$

Let $\tilde{\kappa}_t$ and $\tilde{\tau}_s$ be the transformations such that

$$(\tilde{\kappa}_t f)(u) = f\left(\frac{u}{-tu + 1}\right) ,$$
$$(\tilde{\tau}_t f)(u) = f(e^t u) , \qquad t, \ u \in R^1 .$$

THEOREM 2. *Let $Y = \{Y(p, x);\ p \in S\}$ be the generalized Gaussian random field given by* (3.1) *and* (3.2) *with $\alpha(|p|)$ as in* (3.6). *Let g be a member of $C(d)$ expressed in the form*

$$g = kan , \quad k \in K_0 , \quad a \in A , \quad n \in N$$

with $n = \kappa_t = \kappa^d_{t_d} \cdots \kappa^1_{t_1}$ and $a = \tau_s$. Fix a point p° on ∂S. If n and a are taken so as the requirement (3.4) *to be fulfilled, then*

$$Y(p, n^* \tau^*_s k^* x) = \tilde{\kappa}^{-1}_{(p^\circ, t)} \tilde{\tau}^{-1}_s \tilde{k}^{-1} Y(p, x) , \qquad p \in \overline{Op^\circ} , \tag{3.8}$$

where $Y(p, \cdot)$ with $p = \lambda p^\circ$ is viewed as a function of λ so that $\tilde{\kappa}$ and $\tilde{\tau}$ can act on it.

Proof. The role of the rotation in the expression (3.8) is obvious since the kernel $f(p, u)$ involves only the inner product and the norm in R^d, so that we may ignore k. The rest of the proof follows immediately from the result (3.7).

Set $Y^*(p, x) = Y(p, n^*\tau_s^*k^*x)$. Then $\{Y^*(p, x); p \in S\}$ is of course the same random field (in distribution) as $\{Y(p, x); p \in S\}$, since n^*, τ_s^* and k^* are μ-measure preserving transformations. The theorem above claims that $Y^*(p, x)$ comes from $Y(p, x)$ by applying suitable conformal transformations of the variable p. In view of this, we say that $\{Y(p, x)\}$ describes the so-to-speak "Conformal Invariance of white noise".

To close this section two remarks are now in order.

Remark. i) Our theorem above may be thought of as a multidimensional parameter analogue of the projective invariance of Brownian motion discussed in [6], if we observe the formula (3.8). In particular, if p° is taken to be $(1, 0, \cdots, 0)$, and if p is restricted to the one-dimensional subspace $(p_1, 0, \cdots, 0)$, $0 \leq p_1 \leq 1$, then we can easily compare with the result in [6].

ii) The unit ball can be changed to any ball with center at O, and the same result follows with slight modification.

ACKNOWLEDGEMENT. The authors are grateful to Professors K. Aomoto, F. Jegerlehner, P. Winternitz and H. Yoshizawa for their comments and discussions.

REFERENCES

[1] J. Aczél, Functional equations and their applications, Academic Press, 1966.
[2] I. M. Gel'fand and G. E. Shilov, Generalized functions, vol. 1, English trans. Academic Press, 1964.
[3] T. Hida, Brownian motion, Springer-Verlag, 1980, Applications of Math. vol. 11.
[4] ——, White noise analysis and its application of Quantum Dynamics, Proc. 7th International Congress on Mathematical Physics, 1983, Boulder, Physica, **124A** (1984), 399–412.
[5] ——, Generalized Brownian functionals and stochastic integrals, Appl. Math. Optim., **12** (1984), 115–123.
This article is based on the lecture delivered at AMS Conference at Baton Rouge, May–June 1983.
[6] T. Hida, I. Kubo, H. Nomoto and H. Yoshizawa, On projective invariance of Brownian motion, Publ. RIMS Kyoto Univ. A, **4** (1968), 595–609.
[7] P. Lévy, Processus stochastiques et mouvement brownien, Gauthier-Villars, 1948.
[8] I. T. Todorov et al, Conformal invariance in Quantum field theory, Scuola Normale Superiore, Pisa 1978.
[9] H. Yoshizawa, Rotation group of Hilbert space and its application to Brownian

motion, Proc. International Conference on Functional Analysis and Related Topics, 1969, Tokyo, 414–423.

T. Hida
Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya 464
Japan

Ke-Seung Lee
College of Leberal arts and Sciences
Department of Mathematics
Korea University
Korea

Sheu-San Lee
Shenyang Chemical Engineering Institute
Shenyang
China