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QUOTIENT COMPLETE INTERSECTIONS OF AFFINE SPACES BY FINITE LINEAR GROUPS

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§1. Introduction

Let G be a finite subgroup of $GL_n(C)$ acting naturally on an affine space C^n of dimension n over the complex number field C and denote by C^n/G the quotient variety of C^n under this action of G. The purpose of this paper is to determine G completely such that C^n/G is a complete intersection (abbrev. C.I.) i.e. its coordinate ring is a C.I. when $n > 10$. Our main result is (5.1). Since the subgroup N generated by all pseudo-reflections in G is a normal subgroup of G and C^n/G is obtained as the quotient variety of $C^n/N \cong C^n$ by G/N, without loss of generality, we may assume that G is a subgroup of $SL_n(C)$ (cf. [6, 16, 24, 25]).

Stanley classified G in [21] such that C^n/G is a C.I. under the assumption that $G = G^\# \cap SL_n(C)$ for a finite reflection group $G^\#$ in $GL_n(C)$, and conjectured in [23] that if C^n/G is a C.I., $G^* \supset G \supset [G^*, G^*]$ for a finite reflection group G^* in $GL_n(C)$. In [17, 28], this conjecture was solved negatively. On the other hand, Watanabe ([26]) and Watanabe-Rotillon ([29]) determined G such that C^n/G is a C.I. respectively for abelian G and for any G in $SL_3(C)$. In case of $n = 2$, it is well known and classical that C^2/G is always a hypersurface for every G in $SL_2(C)$.

Recently Goto and Watanabe showed that if C^n/G is a C.I., then its embedding dimension is at most $2n - 1$ i.e. C^n/G can be regarded as a closed subvariety of C^{2n-1} (cf. [27, 31]). This result follows from the main theorem in [11] on rational singularities, because C^n/G is a rational singularity at the induced origin (cf. [10]). Moreover, using Grothendieck's purity theorem, Kac and Watanabe [9] showed that if C^n/G is a C.I., then G is generated by $\{\sigma \in G \mid \dim \operatorname{Im}(\sigma - 1) \leqq 2\}$. Thanks to the last theorem, we can use a classification of some finite linear groups given by Blichfeldt, Huffman and Wales (see the references in [14]), and consequently, for example, have shown

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THEOREM ([13, 14]). *Suppose that $n > 10$, $(C^n)^G = 0$ and G is contained in $SL_n(C)$. Then C^n/G is a hypersurface if and only if $G = G^{\#} \cap SL_n(C)$ for a finite reflection group $G^{\#}$ in $GL_n(C)$ in which all orders of pseudo-reflections are equal to the index $[G^{\#}: G]$.*

The proofs of our theorems, which show that counter-examples for Stanley's conjecture are very few, depend not only on the above results but also on some results on relative invariants of finite groups ([21]) and regular elements of finite reflection groups ([19]). Furthermore the classification of finite reflection groups in [4, 24] plays an essential role in this paper. The manuscript of this paper was completed in 1982. The author was expecting the publication of a part of [27] in English, which has been essentially used in this paper. After this paper was circulated, he learned that Gordeev [32] announced (4.1) and some related partial results. Further classification in small dimensions shall be published elsewhere.

The following notation will be used throughout.

$\boldsymbol{N}$	the additive monoid of all nonnegative integers
$\boldsymbol{Z}$	the ring of all integers
$\det_V$ or $\det$	determinant map on a vector space V
$\operatorname{diag}[a_1, a_2, \cdots, a_n]$	the diagonal matrix whose diagonal entries are $a_1, a_2, \cdots, a_n$
$\sigma[n]$	the permutation matrix associated with σ in the symmetric group S_n of degree n
ζ_m	a primitive m-th root of unity
μ_m	the cyclic group $\langle \zeta_m 1 \rangle$
$\boldsymbol{D}_m$	the binary dihedral group of order $4m$
$\boldsymbol{T}$	the binary tetrahedral group of order 24
$\boldsymbol{O}$	the binary octahedral group of order 48
$\boldsymbol{I}$	the binary icosahedral group of order 120
$(\mu_u \mid \mu_v; H \mid N)$	the subgroup of $GL_2(C)$ defined in [4]
$G(p, q, n)$	the monomial irreducible reflection subgroup in $GL_n(C)$ defined in [4]
$A(p, q, n)$	the diagonal part of $G(p, q, n)$
$\boldsymbol{C}_m$	the group $A(m, m, 2)$
$W(\Gamma)$	the group generated by pseudo-reflections induced from a root graph Γ (cf. [4])
$[\sigma, \tau]$	the commutator $\sigma\tau\sigma^{-1}\tau^{-1}$ for elements σ, τ in a group G
$[G, G]$	the commutator subgroup of a group G

§ 2. Definitions, notations and preliminary results

Throughout this paper all rings are assumed to be commutative with unity. For a ring R, let R^* be the group of all unit elements in R, $\mathrm{ht}(\mathfrak{a})$ the height of an ideal $\mathfrak{a}$ of R and RX the ideal of R generated by a subset X of R.

An algebra A is defined to be *N^m-graded* ($m \in N$) if A is regarded as a graded algebra with a graduation graded by the additive monoid N^m in the natural way, and, for $i=(i_1, \cdots, i_m) \in N^m$, $A_{(i)}$ stands for the i-th graded part of A. If f is an elements of $A_{(i)}$, f is said to be *N^m-graded* and the *N^m-degree* (resp. *j-th degree* $(1 \leqq j \leqq m)$, *total degree*) of f is defined to be $i=(i_1, \cdots, i_m)$ (resp. i_j, $\sum_{j=1}^m i_j$) which is denoted by $\deg^{(m)}(f)$ (resp. $\deg_j(f)$, $\deg(f)$). We say that an N^m-graded algebra A is *defined over* a field K, if $A_{(0)}=K$ and A is finitely generated over K as an algebra, and in this case denote by $\mathrm{emb}(A)$ the embedding dimension of A, i.e., $\dim A_+/A_+^2$, where A_+ is the graded maximal ideal of A. For simplicity, let us use "graded", "degree" and "$\deg(f)$", respectively, instead of "N-graded", "N-degree" and "$\deg^{(1)}(f)$". If A and B are graded algebras defined over a field K, $A \otimes_K B$ is usually regarded as an N^2-graded algebra with the graduation $\{A_{(i)} \otimes_K B_{(j)} \mid (i, j) \in N^2\}$.

By the theorem in [11] on pseudo-rational singularities, the following result is obtained:

THEOREM 2.1 (Goto-Watanabe [27, 31]). *If R is a pseudo-rational local ring and a C.I. whose residue class field is infinite, then* $\mathrm{emb}(R) < 2 \dim R$.

In the case where R is essentially of finite type over a field K of characteristic zero, R is a pseudo-rational singularity if and only if it is a rational singularity.

Remark 2.2. We can determine the relation ideals of graded algebras A such that A_{A_+} are rational singularities. For example, if A are algebras of invariants of reductive algebraic groups over fields of characteristic zero, the minimal generating systems of A are constructive ([15]), and hence their relation ideals are also constructive: In general, let A be an N-graded algebra defined over a field K and $K[X_1, \cdots, X_n]$ an n-dimensional graded polynomial algebra over K. If A_{A_+} is pseudo-rational and φ: $K[X_1, \cdots, X_n] \rightarrow A$ is a graded epimorphism, then $\operatorname{Ker} \varphi \cap K[X_1, \cdots, X_n]_+^{\dim A+1} \subset K[X_1, \cdots, X_n]_+ \operatorname{Ker} \varphi$.

For a finite dimensional vector space V over C, let $\mathrm{Sym}(V)$ be the symmetric algebra of V which is naturally regarded as a graded algebra defined over C. The rank of an element σ in $\mathrm{End}(V)$ (or $\mathrm{M}_n(C)$) is denoted by $\mathrm{rk}(\sigma)$, and, if $\zeta \in C^*$ is a root of 1, the eigenspace of σ corresponds to the eigenvalue ζ is denoted by $V(\sigma, \zeta)$, i.e., $V(\sigma, \zeta) = \{v \in V \mid \sigma(v) = \zeta v\}$ ([19]). An element σ of $GL(V)$ is said to be a *pseudo-reflection* (resp. a *special element*) if $\mathrm{rk}(\sigma - 1) = 1$ (resp. $\mathrm{rk}(\sigma - 1) = 2$), and a finite subgroup of $GL(V)$ is said to be a *reflection group* if it is generated by pseudo-reflections. For a finite group G, a subgroup N of G and a representation ρ: $G \to GL(V)$ of G, we adopt the following notation and terminology: For $x \in V$, G_x stands for the stabilizer of G at x and, for $X \subset V$, put $G_{[X]} = \bigcap_{x \in X} G_x$. G is said to be irreducible (resp. reducible, primitive, imprimitive, monomial) in $GL(V)$, if so is ρ, and moreover G is said to be *irredundant* in $GL(V)$, if there are not nonzero CG-submodules V_i $(i = 1, 2)$ of V such that $V = V_1 \oplus V_2$ and $\rho(G) = \rho(G_{[V_2]}) \times \rho(G_{[V_1]})$. Especially if G is monomial in $GL(V)$, $\{CX_1, \cdots, CX_{\dim V}\}$ is a complete system of imprimitivities of ρ and $X = \{X_1, \cdots, X_{\dim V}\}$ is a C-basis of V, we denote by $\prod_X(G)$ the permutation group of G on $\{CX_1, \cdots, CX_{\dim V}\}$ and by $(CX_{i_1}, \cdots, CX_{i_m})$ the usual cycle on $\{CX_{i_1}, \cdots, CX_{i_m}\}$ in the symmetric group on the letters $\{CX_1, \cdots, CX_{\dim V}\}$. For N such that N is normal in G and $\rho(N)$ is a reflection group, a regular system $\{h_1, \cdots, h_{\dim V}\}$ of graded parameters of $\mathrm{Sym}(V)^N$ is defined to be *G/N-linearlized*, if $\bigoplus_{i=1}^{\dim V} Ch_i$ is a CG-submodule of $\mathrm{Sym}(V)^N$, and it should be noted that such a regular system of parameters of $\mathrm{Sym}(V)^N$ always exists. Let V_N be the CN-submodule $\sum_{\sigma \in N}(\sigma - 1)V$ of V and $\mathscr{R}(V; N)$ the subgroup of $\rho(N)$ generated by all pseudo-reflections in $\rho(N)$. A subspace U of codimension one in V is said to be a *reflecting hyperplane relative to* N if $V^{\langle\sigma\rangle} = U$ for some $\sigma \in N$. Denote by $\mathscr{H}(V, N)$ the set consisting of all reflecting hyperplanes relative to N and by $\mathscr{I}_U(N)$ the subgroup $\{\tau \in \rho(N) \mid V^{\langle\tau\rangle} \supset U\}$ for $U \in \mathscr{H}(V, N)$. An element in N is called a *generic pseudo-reflection* in N if it generates some $\mathscr{I}_U(N)$, and the cardinalities $|\mathscr{I}_U(N)|(U \in H(V, N))$ are called *orders of pseudo-reflections* in N. For each $U \in \mathscr{H}(V, N)$, let $L_U(V, N)$ be a fixed nonzero element in $V_{\mathscr{I}_U(N)}$ and, for a linear character χ of G with $\mathrm{Ker}\,\chi \supset \mathrm{Ker}\,\rho$, put $s_U(V, N, \chi)$ $\min\{a \in N \mid \chi(\tau) = \det_V(\tau)^a$ for all $\tau \in \mathscr{I}_U(N)\}$ and

$$f_\chi(V, N) = \prod_{U \in \mathscr{H}(V, N)} L_U(V, N)^{s_U(V, N, \chi)} .$$

Further $\mathrm{Sym}(V)_\chi^N$ denotes the set $\{f \in \mathrm{Sym}(V) \mid \tau(f) = \chi(\tau)f$ for $\tau \in N\}$, whose elements are known as χ-*invariants* or *invariants of* N *relative to* χ. Since N acts naturally on $\mathscr{H}(V, N)$, $N\backslash\mathscr{H}(V, N)$ stands for a set of all representatives of $\mathscr{H}(V, N)$ modulo N, and, for U, U' in $\mathscr{H}(V, N)$, we say that U is equivalent to U' if U and U' are contained in an N-orbit. The group homomorphisms $\langle \mathscr{I}_{U'}(N) \mid NU \ni U' \rangle \ni \tau \mapsto \det_V(\tau) \in (C^*)_U$ induce the commutative diagram

$$\begin{array}{ccccc} \mathscr{R}(V; N) & \xrightarrow{\Phi_{N,V}} & \bigoplus_{U \in N\backslash\mathscr{H}(V,N)} (C^*)_U & \hookrightarrow & GL_{|N\backslash\mathscr{H}(V,N)|}(C) \\ \cup\uparrow & \nearrow & & & \\ \bigcup_{U \in N\backslash\mathscr{H}(V,N)} \langle \mathscr{I}_{U'}(N) \mid NU \ni U' \rangle & & & & \end{array}$$

where $(C^*)_U = C^*$, $\Phi_{N,V}$ is a group homomorphism and $\oplus_{U \in N\backslash\mathscr{H}(V,N)} (C^*)_U$ is diagonally embedded in $GL_{|N\backslash\mathscr{H}(V,N)|}(C)$ (cf. [12]). For a representation δ: $H \to GL(V)$ of a finite group H, $(\mathscr{R}(V; N), H, V)$ is defined to be a *CI-triplet*, if $\mathscr{R}(V; N) \supset \delta(H) \supset [\mathscr{R}(V; N), \mathscr{R}(V; N)]$ and $\Phi_{N,V}(\delta(H))$ is conjugate to $G_D(C)$ in $GL_{|N\backslash\mathscr{H}(V,N)|}(C)$ for some datum D (see [26], for definition of $G_D(C)$ and D). Moreover H is said to be *extended to a CI-triplet in* $GL(V)$, if (H^*, H, V) is a *CI*-triplet for a finite reflection subgroup H^* in $GL(V)$.

PROPOSITION 2.3 ([12, Sect. 3]). *Let G be a finite subgroup of $GL(V)$ where V is a finite dimensional C-space, and suppose $G^\# \supset G \supset [G^\#, G^\#]$ for some finite reflection subgroup $G^\#$ in $GL(V)$. Then* $\mathrm{Sym}(V)^G$ *is a C.I. if and only if G is extended to a CI-triplet in $GL(V)$.*

LEMMA 2.4. *Let G be a finite group and ρ: $G \to GL(V)$ a representation of G of finite degree over C. Then*:

(1) *If* $\mathrm{Sym}(V)^G$ *is a C.I., then, for any $x \in V$ and any CG-submodule U of V,* $\mathrm{Sym}(V)^{G_x}$ *and* $\mathrm{Sym}(U)^G$ *are C.I.'s.*

(2) *Suppose that* $\rho(G) = \rho(G_{[V_2]}) \times \rho(G_{[V_1]})$ and $V = V_1 \oplus V_2$ *for some nonzero CG-submodules V_i $(i = 1, 2)$ of V. Then* $\mathrm{Sym}(V)^G$ *is a C.I. if and only if* $\mathrm{Sym}(V_i)^G (i = 1, 2)$ *are C.I.'s. Moreover if U is a nontrivial irreducible CG-submodule of V, one of V_i's contains U.*

Proof. (1) and the first assertion of (2) follow from [14, 21]. To show the last assertion, we assume $U \not\subseteq V_i$ $(i = 1, 2)$. Then since U can be embedded in $V_2 \cong V/V_1$ and $V_1 \cong V/V_2$, respectively, as CG-modules, $U \subset V^{G[V_1]} \cap V^{G[V_2]} = (V_1^G \oplus V_2) \cap (V_1 \oplus V_2^G)$, and this shows $U^G = U$, a contradiction.

From now on we will study our subject under the circumstance as follows: Let S be $\mathrm{Sym}(V)$ of an n-dimensional C-space V and G a finite subgroup of $SL(V)$. Let V_i $(1 \leqq i \leqq m)$ be irreducible CG-submodules of V with $\dim V_i = n_i$ which satisfy $V = \oplus_{i=1}^m V_i$, and ρ_i: $G \to GL(V_i)$ the representation of G afforded by the CG-module V_i. Let G_i be $\{\sigma \in GL(V) \mid \sigma(V_j) = V_j \ (1 \leqq j \leqq m),\ \sigma|_{V_j} = 1 \ (i \neq j),\ \sigma|_{V_i} \in \rho_i(G)\}$, and put $\tilde{G} = G_1 \times \cdots \times G_m$, $G^i = \bigcap_{1 \leqq j \leqq m, j \neq i} G_{[V_j]}$ $(1 \leqq i \leqq m)$ and $\mathrm{Spe}(G) = \{\sigma \in G \mid \sigma \notin \bigcup_{1 \leqq i \leqq m} G^i$ and σ is special$\}$ respectively. If G is generated by special elements in $GL(V)$, then $\rho_i(G) = \rho_i(G^i)\rho_i(\langle \mathrm{Spe}(G) \rangle) = \rho_i(G^i)\rho_i(\mathscr{R}(V; \tilde{G}))$ $(1 \leqq i \leqq m)$ and G (resp. $\tilde{G}$) is generated by $\bigcup_{1 \leqq i \leqq m} G^i \cup (\mathscr{R}(V; \tilde{G}) \cap G)$ (resp. $\bigcup_{1 \leqq i \leqq m} G^i \cup \mathscr{R}(V; \tilde{G})$). Since $S \cong \mathrm{Sym}(V_1) \otimes_C \cdots \otimes_C \mathrm{Sym}(V_m)$, we regard S as an N^m-graded C-algebra in the natural way and $\mathrm{Sym}(V)^G$ is an N^m-graded subalgebra of S. Let $\{f_1, \cdots, f_r\}$ be a generating system of S^G as a C-algebra consisting of N^m-graded elements and let $A = C[T_1, \cdots, T_r]$ be an r-dimensional N^m-graded polynomial algebra over C with $\deg^{(m)}(T_i) = \deg^{(m)}(f_i)$. Moreover let Φ: $A \to S^G$ be the N^m-graded C-epimorphism defined by $\Phi(T_i) = f_i$. Then $\mathrm{Ker}\,\Phi$ is minimally generated by N^m-graded elements F_i $(1 \leqq i \leqq s)$.

LEMMA 2.5 (e.g. [14, 27]). *If S^G is a C.I., then*:

(1) $(-n_1, \cdots, -n_m) = \sum_{i=1}^s \deg^{(m)}(F_i) - \sum_{i=1}^r \deg^{(m)}(T_i)$.

(2) $\prod_{i=1}^r \deg(T_i) = |G| \prod_{i=1}^s \deg(F_i)$.

Proof. For the proof of (1), see [14]. If $\{f_1, \cdots, f_r\}$ contains a system $\{f_1, \cdots, f_n\}$ of parameters of S^G, $C[T_{n+1}, \cdots, T_r]$ is a free module over $C[\bar{F}_1, \cdots, \bar{F}_s]$ of rank $\prod_{i=1}^s \deg(F_i) / \prod_{i=n+1}^r \deg(T_i)$ where $\bar{F}_i = F_i(0, \cdots, 0, T_{n+1}, T_{n+2}, \cdots, T_r)$, and hence (2) follows. The general case can easily be reduced to this case.

§ 3. Certain monomial groups of dimension four

In this section, we suppose that $n = 4$ and G is monomial on the C-basis $X = \{X_1, X_2, X_3, X_4\}$ of V such that $\prod_X(G) = \langle (CX_1, CX_2)(CX_3, CX_4), (CX_1, CX_3)(CX_2, CX_4) \rangle$.

PROPOSITION 3.1. *S^G is a C.I. if and only if G is conjugate to one of the groups listed in Table* I.

TABLE I

Groups	Generators	Conditions
G_1	$\gamma_1,\ \gamma_2,\ \sigma_1,\ \sigma_2$	$a\,\vert\, e$
G_2	$\gamma_3,\ \gamma_4,\ \gamma_7,\ \sigma_1,\ \sigma_2$	$a < e/2,\ a\,\vert\, e/2,\ 2\,\vert\, e$
G_3	$\gamma_3^2,\ \gamma_5^2,\ \gamma_7,\ \sigma_1,\ \sigma_2$	$4\,\vert\, e$
G_3'	$\gamma_3^2,\ \gamma_5^2,\ \gamma_7,\ \sigma_1',\ \sigma_2$	$4\,\vert\, e,\ a = 4/e$
G_4	$\gamma_2,\ \gamma_5^2,\ \gamma_6,\ \sigma_1,\ \sigma_2$	$4a\,\vert\, e,\ b - a = e/2,\ a < e/4,\ b/a \equiv 3(4)$
G_4'	$\gamma_2,\ \gamma_5^2,\ \gamma_6,\ \sigma_1',\ \sigma_2$	$4a\,\vert\, e,\ b - a = e/2,\ a < e/4,\ b/a \equiv 3(4)$
G_5	$\gamma_2,\ \gamma_5,\ \gamma_6,\ \sigma_1,\ \sigma_2$	$4a\,\vert\, e,\ b - a = e/2,\ a < e/4,\ b/a \equiv 3(4)$
G_5'	$\gamma_2,\ \gamma_5,\ \gamma_6,\ \sigma_1',\ \sigma_2$	$4a\,\vert\, e,\ b - a = e/2,\ a < e/4,\ b/a \equiv 3(4)$

$\gamma_1 = \mathrm{diag}\,[\zeta_e, 1, 1, \zeta_e^{-1}]$; $\gamma_2 = \mathrm{diag}\,[1, 1, \zeta_a, \zeta_a^{-1}]$; $\gamma_3 = \mathrm{diag}\,[\zeta_{e/2}, \zeta_{e/2}^{-1}, 1, 1]$;
$\gamma_4 = \mathrm{diag}\,[1, \zeta_a, \zeta_a^{-1}, 1]$; $\gamma_5 = \mathrm{diag}\,[1, \zeta_{e/2}, \zeta_{e/2}^{-1}, 1]$; $\gamma_6 = \mathrm{diag}\,[\zeta_e^{-b/a}, \zeta_e^{-1}, \zeta_e^{b/a}, \zeta_e]$;
$\gamma_7 = \mathrm{diag}\,[\zeta_e, \zeta_e^{-1}, \zeta_e^{-1}, \zeta_e]$; $\sigma_1 = (1, 2)(3, 4)[4]$; $\sigma_2 = (1, 3)(2, 4)[4]$;
$\sigma_1' = \mathrm{diag}\,[1, 1, \zeta_{2a}, \zeta_{2a}^{-1}]\sigma_1$; $a, b, e \in \boldsymbol{N}$.

The rest of this section is devoted to the proof of (3.1).

For any element w in S, let $\mathrm{Tr}(w)$ (or $\mathrm{Tr}_G(w)$) denote $\sum_{\sigma\in G/G_w}\sigma(w)$.

LEMMA 3.2. *S^{G_i} $(1 \leqq i \leqq 5)$ and $S^{G_i'}$ $(3 \leqq i \leqq 5)$ are C.I.'s.*

Proof. By a direct computation, we easily have $S^{G_1} = \boldsymbol{C}[\mathrm{Tr}_{G_1}(X_1^e)$, $\mathrm{Tr}_{G_1}((X_1X_2)^e)$, $\mathrm{Tr}_{G_1}((X_1X_3)^e)$, $\mathrm{Tr}_{G_1}((X_1X_4)^a)$, $\mathrm{Tr}_{G_1}(X_1^{e+a}X_4^a)$, $X_1X_2X_3X_4]$, $S^{G_2} = \boldsymbol{C}[\mathrm{Tr}_{G_2}(X_1^e)$, $\mathrm{Tr}_{G_2}((X_1X_2)^a)$, $\mathrm{Tr}_{G_2}((X_1X_3)^{e/2})$, $\mathrm{Tr}_{G_2}((X_1X_4)^{e/2})$, $\mathrm{Tr}_{G_2}(X_1^{e+a}X_2^a)$, $X_1X_2X_3X_4]$, $S^{G_3} = \boldsymbol{C}[\mathrm{Tr}_{G_3}(X_1^e)$, $\mathrm{Tr}_{G_3}((X_1X_4)^{e/2})$, $\mathrm{Tr}_{G_3}(X_1^{e/4}X_4^{3e/4})$, $\mathrm{Tr}_{G_3}((X_1X_2)^{e/4})$, $\mathrm{Tr}_{G_3}((X_1X_3)^{e/4})$, $X_1X_2X_3X_4]$, $S^{G_3'} = \boldsymbol{C}[\mathrm{Tr}_{G_3'}(X_1^e)$, $\mathrm{Tr}_{G_3'}((X_1X_4)^{e/2})$, $\mathrm{Tr}_{G_3'}(X_1^{e/4}X_4^{3e/4})$, $\mathrm{Tr}_{G_3'}((X_1X_2)^{e/4})$, $\mathrm{Tr}_{G_3'}((X_1X_3)^{e/4})$, $X_1X_2X_3X_4]$, $S^{G_4} = \boldsymbol{C}[\mathrm{Tr}_{G_4}(X_1^e)$, $\mathrm{Tr}_{G_4}((X_1X_2)^{e/4})$, $\mathrm{Tr}_{G_4}((X_1X_3)^{e/4})$, $\mathrm{Tr}_{G_4}(X_1^aX_4^b)$, $\mathrm{Tr}_{G_4}((X_1X_4)^{2a})$, $\mathrm{Tr}_{G_4}((X_1X_4)^aX_2^{e/2})$, $X_1X_2X_3X_4]$, $S^{G_4'} = \boldsymbol{C}[\mathrm{Tr}_{G_4'}(X_1^e)$, $\mathrm{Tr}_{G_4'}((X_1X_2)^{e/4})$, $\mathrm{Tr}_{G_4'}((X_1X_3)^{e/4})$, $\mathrm{Tr}_{G_4'}(X_1^aX_4^b)$, $\mathrm{Tr}_{G_4'}((X_1X_4)^{2a})$, $\mathrm{Tr}_{G_4'}((X_1X_4)^aX_2^{e/2})$, $X_1X_2X_3X_4]$, $S^{G_5} = \boldsymbol{C}[\mathrm{Tr}_{G_5}(X_1^e)$, $\mathrm{Tr}_{G_5}((X_1X_2)^{e/2})$, $\mathrm{Tr}_{G_5}((X_1X_4)^{2a})$, $\mathrm{Tr}_{G_5}((X_1X_3)^{e/2}$, $\mathrm{Tr}_{G_5}(X_1^aX_4^b)$, $\mathrm{Tr}_{G_5}((X_1X_4)^aX_2^{e/2})$, $X_1X_2X_3X_4]$ and $S^{G_5'} = \boldsymbol{C}[\mathrm{Tr}_{G_5'}(X_1^e)$, $\mathrm{Tr}_{G_5'}((X_1X_2)^{e/2})$, $\mathrm{Tr}_{G_5'}((X_1X_3)^{e/2})$, $\mathrm{Tr}_{G_5'}(X_1^aX_4^b)$, $\mathrm{Tr}_{G_5'}((X_1X_4)^{2a})$, $\mathrm{Tr}_{G_5'}((X_1X_4)^aX_2^{e/2})$, $X_1X_2X_3X_4]$. Then S^{G_i} $(1 \leqq i \leqq 3)$ and $S^{G_3'}$ are C.I.'s (cf. [25, 18]). Suppose $G = G_4$ or G_4' and put $u = b/a$, $f_1 = \mathrm{Tr}(X_1^e)$, $f_2 = \mathrm{Tr}((X_1X_2)^{e/4})$, $f_3 = \mathrm{Tr}((X_1X_3)^{e/4})$, $f_4 = \mathrm{Tr}(X_1^aX_4^b)$, $f_5 = \mathrm{Tr}((X_1X_4)^{2a})$, $f_6 = \mathrm{Tr}((X_1X_4)^aX_2^{e/2})$, $f_7 = X_1X_2X_3X_4$. We effectively find all relations of degree $\leqq 2(a+b)$: $\deg(F_1) = e$, $\deg(F_2) = \deg(F_3) = 2(a+b)$, and $2(a+b) < \deg(F_4) \leqq \deg(F_5) \leqq \cdots$ if $s > 3$. (For our purpose, it suffices to show $(F_1, F_2, F_3)A = \mathrm{Ker}\,\Phi$, but this is not easy).

Assume that S^G is not a C.I. and let

$$0 \longrightarrow L_3 \xrightarrow[\Phi_3]{} L_2 \xrightarrow[\Phi_2]{} L_1 \xrightarrow[\Phi_1]{} L_0(=A) \xrightarrow[\Phi]{} S^G \longrightarrow 0$$

be a minimal free resolution of S^G, where each L_i is a graded free A-module $\oplus_j AY_{ij}$ with graded elements Y_{ij} $(1 \leqq j \leqq \operatorname{rank} L_i)$ and Φ_i is a graded homomorphism. Since S^G is a Gorenstein ring, $L_3 \cong A$ and there is a pairing $\langle\ ,\ \rangle\colon L_2 \otimes_A L_1 \to L_3 = AY_{31}$ which preserves the graduation and induces an isomorphism $L_1 \cong L_2^* = \operatorname{Hom}_A(L_2, A)$ (cf. [3, 22]). Thus we may suppose $\deg(Y_{1j}) + \deg(Y_{2j}) = \deg(Y_{31})$, $\deg(Y_{11}) = 2(u-1)a$, $\deg(Y_{12}) = \deg(Y_{13}) = 2(u+1)a$. On the other hand $\deg(Y_{31}) = \sum_{i=1}^{7} \deg(f_i) - 4$ (cf. [22] and the proof of [14, (2.8)]). Moreover, because $F_1 = T_2T_3 + w$ for some graded element w in $C[T_1, T_4, T_5, T_6, T_7]$, $s = 5$ and there is a 5×5 alternating matrix $\Theta = [v_{ij}]$ whose entries are graded elements of positive degree in A such that $\operatorname{Pf}(\Theta_i)$ $(1 \leqq i \leqq 5)$ generate $\operatorname{Ker}\Phi$ (cf. [3]). Here Θ_i is the 4×4 submatrix of Θ deleted the i-th column and i-th row from Θ and $\operatorname{Pf}(\Theta_i)$ is the Paffian of Θ_i. We may suppose that $v_{ij} = \langle Y_{2i}, \Phi_2(Y_{2j})\rangle Y_{31}^{-1}$ (cf. [3]), and $\deg(v_{ij}) = \deg(Y_{2j}) + \deg(Y_{2i}) - \deg(Y_{31})$, which implies $\deg(\operatorname{Pf}(\Theta_i)) = \sum_{j \neq i} \deg(Y_{2j}) - 2\deg(Y_{31}) = 2\deg(Y_{31}) - \sum_{j \neq i} \deg(Y_{1j})$; $\deg(\operatorname{Pf}(\Theta_1)) = 8ua - \deg(Y_{14}) - \deg(Y_{15})$, $\deg(\operatorname{Pf}(\Theta_2)) = (8u+4)a - \deg(Y_{14}) - \deg(Y_{15})$, $\deg(\operatorname{Pf}(\Theta_3)) = (8u+4)a - \deg(Y_{14}) - \deg(Y_{15})$, $\deg(\operatorname{Pf}(\Theta_4)) = (6u+2)a - \deg(Y_{15})$, $\deg(\operatorname{Pf}(\Theta_5)) = (6u+2)a - \deg(Y_{14})$. As $\deg(Y_{14}) = \deg(F_4) > \deg(Y_{13})$, $\deg(\operatorname{Pf}(\Theta_5)) \geqq \deg(\operatorname{Pf}(\Theta_4)) > \deg(\operatorname{Pf}(\Theta_3)) = \deg(\operatorname{Pf}(\Theta_2)) > \deg(\operatorname{Pf}(\Theta_1))$ and hence $\deg(\operatorname{Pf}(\Theta_i)) = \deg(Y_{1i})$ and $\deg(Y_{14}) + \deg(Y_{15}) = \deg(Y_{31})$. Then $\deg(v_{45}) = \deg(Y_{24}) + \deg(Y_{25}) - \deg(Y_{31}) = 0$, which requires $v_{45} = 0$ and $\operatorname{Pf}(\Theta_1) = v_{24}v_{35} - v_{34}v_{25}$. Obviously $\deg(v_{ij}) > 0$ $(i = 2, 3;\ j = 4, 5)$ Substituting 0 for T_i $(i \neq 2, 3)$, one sees $\deg(v_{24}) = \deg(T_2) = \deg(T_3) = \deg(v_{35})$ or $\deg(v_{34}) = \deg(T_4) = \deg(T_5) = \deg(v_{25})$, which shows $\deg(Y_{14}) = \deg(Y_{15})$. Therefore $\deg(v_{24}) = \deg(v_{34}) = \deg(v_{35}) = \deg(v_{25})$, and $v_{ij} \in C[T_5, T_7] \oplus CT_2 \oplus CT_3$. This conflicts with the expression of F_1, and consequently S^G is a C.I.. Similarly we can prove that S^{G_5} and $S^{G'_5}$ are C.I.'s (in this case, $\deg(F_1) = \deg(F_2) = 2(a+b)$ and $\deg(F_3) = 2e$).

In order to show the "only if" part of (3.1), we suppose that S^G is a C.I. and may assume that the subgroup D consisting of all diagonal matrices in G is nontrivial. Clearly G is generated by D and the elements $\sigma = \operatorname{diag}[1, u, v, w](1, 2)(3, 4)[4]$, $\tau = (1, 3)(2, 4)[4] = \sigma_2$, since G is generated by special elements. Here $u, v, w \in C^*$ with $uvw = 1$. Moreover we may suppose $u = 1$ and $v = w^{-1}$. Let us assume $r = \mathrm{emb}\,(S^G)$. Because G is transitively monomial, f_i may be identified with $\mathrm{Tr}\,(M_i)$ for some monomial M_i of variables X_j ($1 \leqq j \leqq 4$) such that M_i is divisible by X_1 in S and moreover G_{M_i} is equal to the stabilizer of G at the line CM_i. For each $2 \leqq j \leqq 4$, let ψ_j: $S \to C[X_1, X_j]$ be a C-algebra map defined by $\psi_j(X_1) = X_1$, $\psi_j(X_j) = X_j$, $\psi_j(X_i) = 0$ ($i \neq 1, j$) and let S' be a C-subalgebra of S generated by $\bigcup_{i \neq j} C[X_i, X_j]^D$. Clearly $\psi_j(S^G) = C[\psi_j(f_i) \mid M_i \in C[X_1, X_j]]$, $\psi_2(S^G) = C[X_1, X_2]^{\langle D, \sigma \rangle}$, $\psi_3(S^G) = C[X_1, X_3]^{\langle D, \tau \rangle}$, $\psi_4(S^G) = C[X_1, X_4]^{\langle D, \sigma\tau \rangle}$ and $C[X_i]^D = C[X_i^e]$ ($1 \leqq i \leqq 4$) for some $e \in N$. Put $r_j = \mathrm{emb}\,(\psi_j(S^G))$ and $d_j = \mathrm{emb}\, C([X_1, X_j]^D)$, $2 \leqq j \leqq 4$. Exchanging the indices of f_i, we assume $\psi_2(S^G) = C[\psi_2(f_1), \psi_2(f_2), \cdots, \psi_2(f_{r_2})]$, $\psi_3(S^G) = C[\psi_3(f_1), \psi_3(f_{r_2+1}), \psi_3(f_{r_2}), \cdots, \psi_3(f_{r_2+r_3-1})]$ and $\psi_4(S^G) = C[\psi_4(f_1), \psi_4(f_{r_2+r_3}), \psi_4(f_{r_2+r_3+1}), \cdots, \psi_4(f_{r_2+r_3+r_4-2})]$.

LEMMA 3.3. $2 + \sum_{j=2}^{4} (r_j - 1) \leqq 7$.

Proof. As D is nontrivial, $X_1X_2X_3X_4$ is not contained in $((SV)^G)^2$. Thus this lemma follows from the above observation and (2.1).

We may suppose $f_r = X_1X_2X_3X_4$. Let δ_j: $D \to GL\,(CX_1 \oplus CX_j)$ ($2 \leqq j \leqq 4$) be the natural representation of D whose matrix representation is afforded by $\{X_1, X_j\}$, and c_j the order of pseudo-reflections in $\delta_j(D)$, which equals to $|\delta_j(D_{x_1})|$ (note that $\mathscr{R}(CX_1 \oplus CX_j;\, D) = \langle \mathrm{diag}\,[\zeta_{c_j}, 1], \mathrm{diag}\,[1, \zeta_{c_j}] \rangle$). Since $C[X_2, X_3, X_4]^{G_{X_1}}$ is a C.I. (cf. (2.4)), D_{X_1} is equal to one of $\langle \mathrm{diag}\,[\zeta_{c_2}, \zeta_{c_2}^{-1}, 1], \mathrm{diag}\,[1, \zeta_{c_3}, \zeta_{c_3}^{-1}] \rangle$ ($c_2 | c_3$, $c_3 = c_4$), $\langle \mathrm{diag}\,[\zeta_{c_3}, \zeta_{c_3}^{-1}, 1], \mathrm{diag}\,[\zeta_{c_2}, 1, \zeta_{c_2}^{-1}] \rangle$ ($c_3 | c_2$, $c_2 = c_4$), $\langle \mathrm{diag}\,[\zeta_{c_2}, \zeta_{c_2}^{-1}, 1], \mathrm{diag}\,[\zeta_{c_4}, 1, \zeta_{c_4}^{-1}] \rangle$ ($c_4 | c_2$, $c_2 = c_3$) on the C-basis $\{X_2, X_3, X_3\}$ (cf. [26]). Obviously D/D_{X_1} is a cyclic group of order e, and $\delta_j(D)/\mathscr{R}(CX_1 \oplus CX_j;\, D)$ is also cyclic. Let $N_{ji} = X_1^{a_{ji}} X_j^{b_{ji}}$ ($2 \leqq j \leqq 4$; $1 \leqq i \leqq d_j$) be defined to satisfy that $\{N_{ji} \mid 1 \leqq i \leqq d_j\}$ is a minimal generating set of $C[X_1, X_j]^D$ and $a_{j1} \leqq a_{j2} \leqq \cdots \leqq a_{jd_j}$.

LEMMA 3.4. *For any* $2 \leqq j \leqq 4$;

(1) $0 = a_{j1} < a_{j2} < \cdots < a_{jd_j}$.

(2) $a_{ji} = b_{jd_j-i+1}$,

(3) $a_{j2} = c_j$ *divides* a_{ji},

(4) $a_{j2} + b_{j2} \leqq e$, *and especially if* $a_{j_2} + b_{j_2} = e$, *then* $a_{ji} = (i-1)c_j$,

(5) $r_j \geqq [(d_j + 1)/2]$ ([ ] *is Gaussian symbol*).

Proof. (1) and (2) are known ([30]), and (5) follows easily from (2). To show (3) and (4), we may assume that $c_j = 1$. Then $\delta_j(D) = \langle \mathrm{diag}\,[\zeta_e, \zeta_e^k] \rangle$

for some $1 \leqq k < e$ with $(k, e) = 1$. Thus the assertions are evident (cf. [30]).

LEMMA 3.5. *For some* $1 \leqq j \leqq 4$, *if* $d_j \geqq 6$, *then* $r_j \geqq 4$.

Proof. If $d_j \geqq 7$, this assertion follows from (3.4), so we suppose $d_j = 6$ and $r_j = 3$. Say $j < 4$. Since $\psi_j(S^G)$ is obtained as the ring of invariants of some monomial subgroup L of $GL(CX_1 \oplus CX_j)$ in $B = C[X_1, X_j]$, $B^{\mathscr{R}(CX_1 \oplus CX_j; L)}$ is equal to $C[X_1^p, X_j^p]$ $(p \in N)$ or $C[X_1^p + X_j^p, (X_1X_j)^q]$ $(p, q \in N, q | p)$. If the former case occurs, B^D is a hypersurface ([25]). Therefore $B^{\mathscr{R}(CX_1 \oplus CX_j; L)} = C[X_1^p + X_j^p, (X_1X_j)^q]$. Since $(X_1X_j)^{q-1}(X_1^p - X_j^p) = f_{\det^{-1}}(CX_1 \oplus CX_j, L)$ is a $\det^{-1}$-invariant of L (cf. [25, 21]), $X_1^p - X_j^p$ is a relative invariant of L, and hence both $X_1^p + X_j^p$ and $(X_1X_j)^q$ are relative invariants of L. Clearly $L/\mathscr{R}(CX_1 \oplus CX_j; L)$ is cyclic, and we must have $S^L = [C(X_1^p + X_j^p)^u, (X_1X_j)^{qu}, (X_1^p + X_j^p)(X_1X_2)^q]$ for $u \in N$. On the other hand, by our assumption, $\psi_j(S^G)$ must be written as $C[N_{j1} + N_{j6}, N_{j2} + N_{j5}, N_{j3} + N_{j4}]$, which conflicts with the above computation (cf. (3.4)).

LEMMA 3.6. *If* $r_{j'} = 4$ *for some* j', *then*;
(1) $S^D = S'[X_1X_2X_3X_4]$,
(2) $C[X_1, X_j]^D = C[X_1^e, X_j^e, (X_1X_j)^{c_j}]$ $(j \neq j')$,
(3) $a_{j'i} + b_{j'i} = e$.

Proof. For simplicity, we assume $j' = 2$. Since $\psi_j(S^G)$ is generated by $\psi_j(f_i)$ such that $M_i \in C[X_1, X_j]$, $r = 7$ and $f_r = X_1X_2X_3X_4$, we see that $S^D = S'[f_r]$ and, for $j \neq 2$, $\psi_j(S^G)$ are polynomial rings over C, which implies (2). As $N_{23}X_3^{a_{22}}X_4^{b_{22}}$ is an invariant of D,

$$\begin{aligned} X_1^{a_{23}-a_{22}}X_2^{b_{23}-a_{22}}X_4^{b_{22}-a_{22}} &\in C[X_1, X_2, X_4]^D \\ &= C[N_{21}, \cdots, N_{2d_2}, (X_1X_4)^{c_4}, X_4^e, (X_2X_4)^{c_3}] \end{aligned}$$

(cf. (1)), and hence $X_1^{a_{23}-a_{22}}X_2^{b_{23}-a_{22}}X_4^{b_{22}-a_{22}} \in C[(X_1X_4)^{c_4}, (X_2X_4)^{c_3}]$ (cf. (3.4)). From this it follows that $a_{23} + b_{23} = a_{22} + b_{22}$, which proves (3) (cf. (2)).

Suppose one of d_j's is $\geqq 6$, say $d_2 \geqq 6$. Then $r_2 = 4$ and $r = 7$. Clearly $\deg(f_1) = \deg(f_2) = \deg(f_3) = e$, $\deg(f_5) = 2c_3$, $\deg(f_6) = 2c_4$, $\deg(f_7) = 4$ and

$$\deg(f_4) = \begin{cases} 2e & \text{if } d_2 = 6 \\ e & \text{otherwise}. \end{cases}$$

By (3.5)

$$\sum_{i=1}^{3} \deg(F_i) = \begin{cases} 5e + 2c_3 + 2c_4 = 30c_2 + 2c_3 + 2c_4 & \text{if } d_2 = 6 \\ 4e + 2c_3 + 2c_4 = 4d_2c_2 + 2c_3 + 2c_4 & \text{otherwise}, \end{cases}$$

and

$$\prod_{i=1}^{3} \deg(F_i) = \begin{cases} 8e^3c_3c_4/|D_{X_1}| & \text{if } d_2 = 6 \\ 4e^3c_3c_4/|D_{X_1}| & \text{otherwise} \end{cases}$$

where $|D_{X_1}| = \min\{c_2, c_3, c_4\} \cdot \max\{c_2, c_3, c_4\}$. From these equalities and $\prod_{i=1}^{3} \deg(F_i) \leqq (\sum_{i=1}^{3} \deg(F_i)/3)^3$, we easily deduce a contradiction. (For example, suppose $c_3 = c_4$ (and so $c_2 | c_3$) and $d_2 = 6$. As $\sum_{i=1}^{3} \deg(F_i) \leqq 9e$, $\prod_{i=1}^{3} \deg(F_i) = 8e^3c_3/c_2 \leqq 27e^3$. Thus $c_3/c_2 = 3$, and $\sum_{i=1}^{3} \deg(F_i) \leqq 6e$, which implies $8e^3c_3/c_2 \leqq 8e^3$. Consequently $c_2 = c_3 = c_4$, and $\sum_{i=1}^{3} \deg(F_i) = 34c_2$. However $\prod_{i=1}^{3} \deg(F_i) = 8e^3 > (\sum_{i=1}^{3} \deg(F_i)/3)^3$, a contradiction.) Hence $d_i \leqq 5$, $2 \leqq j \leqq 4$.

Since $C[X_1, X_j]^D$ is normal and $r_j \leqq 4$, $a_{j3} = 2a_{j2}$, $2(a_{j4} - a_{j2}) = e$, $4a_{j2} | e$ and a_{j4}/a_{j2} is odd, in case of $d_j = 5$.

LEMMA 3.7. $|\{j | r_j = 3\}| = 1$.

Proof. We assume that this lemma is false, and may suppose $\{j | r_j = 3\} = \{3, 4\}$. Then $r_2 = 2$ and $d_2 \leqq 3$. We need only to consider this in the following cases; Case 1 "$d_3 = 4$, $d_4 = 5$"; Case 2 "$d_3 = 5$, $d_4 = 3$"; Case 3 "$d_3 = 4$, $d_4 = 4$"; Case 4 "$d_3 = 4$, $d_4 = 3$"; Case 5 "$d_3 = 5$, $d_4 = 5$".

Case 1: $N_{44}X_2^{a_{42}}X_3^{a_{44}}$ is an invariant of D, and this implies $(X_1X_2)^{e/2} = (X_1X_3)^{a_{44}-a_{42}} \in C[X_1, X_3]^D$. On the other hand, as $r_3 = 3$, $C[X_1, X_3]^D = C[X_1^e, X_1^{e/3}X_3^{2e/3}, X_1^{2e/3}X_2^e, X_3^e]$, which conflicts with the above argument.

Case 3: $a_{34} - a_{32}$ $(=e/2)$ is divisible by c_2 and c_4, respectively, in N. On the other hand $\psi_4(S^G) = C[X_1^e - X_4^e, (X_1^e + X_4^e)(X_1X_4)^{c_4}, (X_1X_4)^{2c_4}]$. Since $\mathrm{Tr}((X_1X_4)^{c_4}(X_1X_3)^{2c_3}) \in ((SV)^G)^2$, substituting 0 for X_2, we see that $(X_1X_4)^{c_4}(X_1X_3)^{2c_3}$ is a product of monomial in $C[X_1, X_3]^D$ and a monomial in $C[X_3, X_4]^D$. Therefore $X_1^e(X_3X_4)^{c_4} = (X_1X_4)^{c_4}(X_1X_3)^{2c_3}$, which implies $c_4 = 2c_3$ and $c_4 + 2c_3 = e$, i.e., $e = 4c_3 = 2c_4$. As some two elements of c_2, c_3, c_4 agree each other, the degrees of $\{f_i\}$ can be calculated. Then, by (2.5), $\prod_{e=1}^{3} \deg(F_i) = 2048c_3^3 \leqq (\sum_{i=1}^{3} \deg(F_i)/3)^3 = (32c_3/3)^3 < 1331c_3^3$, which is a contradiction.

In Cases 2, 4 and 5, we can similarly deduce a contradiction.

In case of $d_j = 4$, $r_j = 3$ if and only if $a_{j2} + a_{j3} = e$. Thus, by (3.6), we have:

LEMMA 3.8. *If* $r_j = 4$, *then* $d_j = 5$.

LEMMA 3.9. *If, for some* $2 \leqq j' \leqq 4$, $d_{j'} \leqq 3$ *and* $c_{j'} = \min\{c_2, c_3, c_4\}$, *then* $S^D = S[f_r]$.

Proof. Let M be a monomial in S^D such that, for $0 \leqq i < \deg(M)$, the i-th graded part of S^D is contained in $S'[f_r]$. We may suppose $j' = 2$ and $M = X_1^x X_3^y X_4^z$ for $x, y, z \in N$. Since $X_3^y X_4^z$ is contained in $C[X_2, X_3, X_4]^{D_{X_1}}$ (which equals to $C[X_2^{c_2}, X_3^{c_3}, X_4^{c_4}, X_2X_3X_4, (X_3X_4)^{c_2}]$) and $c_2 | c_3$ $(= c_4)$, M is divisible by $(X_3X_4)^{c_2}$. On the other hand, by our assumption, $C[X_3, X_4]^D = C[X_3^e, X_4^e, (X_3X_4)^{c_2}]$, which shows $M/(X_3X_4)^{c_2} \in S^D$. Thus the assertion follows.

LEMMA 3.10. $d_j \neq 4$ *for* $2 \leqq j \leqq 4$.

Proof. Suppose, for example, $d_4 = 4$. Then $r_4 = 3$, $c_4 = a_{42} = e/3$, $a_{43} = 2e/3$, $r_j = 2$ and $d_j = 3$ $(j \neq 4)$. By (3.7), we may assume that $f_1 = \mathrm{Tr}(X_1^e)$, $f_2 = \mathrm{Tr}((X_1X_2)^{c_2})$, $f_3 = \mathrm{Tr}((X_1X_3)^{c_3})$, $f_4 = \mathrm{Tr}(X_1^{e/3}X_4^{2e/3})$, $f_5 = \mathrm{Tr}((X_1X_4)^e)$. $e/3$ is divisible by c_2 and c_3, respectively, in N. Suppose $c_2 \leqq c_3$ (this implies $c_3 = c_4 = e/3$). Clearly $\mathrm{Tr}(X_1^e(X_1X_2)^{c_2})$ is not contained in $C[f_1, \cdots, f_5, f_r]$. Since $\mathrm{Tr}((X_1X_2)^{c_2}(X_1X_3)^{c_3}) \in C[f_1, \cdots, f_5, f_r]$ and $S^D = S'[f_r]$ (cf. (3.5)), we must have $f_6 = \mathrm{Tr}(X_1^e(X_1X_2)^{c_2})$. Put $u = 2e/3c_2 \in N$. Then, by (2.5), $\sum_{i=1}^3 \deg(F_i) = (17u + 4)c_2$ and $\prod_{i=1}^3 \deg(F_i) = 72u^2(3u + 2)c_2^3$. Thus $\prod_{i=1}^3 \deg(F_i) = 72u^2(3u + 2)c_2^3 \leqq (\sum_{i=1}^3 \deg(F_i)/3)^3 \leqq (6u + 1)^3 c_2^3$, which is a contradiction.

LEMMA 3.11. *If* $d_j = 5$ *for some* $2 \leqq j \leqq 4$, *then* G *is conjugate to one of* $G_3, G_3', G_4, G_4', G_5, G_5'$.

Proof. We may suppose that $d_4 = 5$ (and have already known that $r_i = 2$ for $i \neq 4$) and $c_2 \leqq c_3$. Since $a_{44} - a_{42}$ is divisible by c_2 (and c_3), the fact "$N_{42}X_3^e \in S^D$" implies $(X_1X_4)^{c_4}X_3^{e/2} \in S^D$. Thus, under the assumption that "$S^D = S'[f_r]$", $e = 4c_4$, $a_{44} = 3c_4$ and $c_2 \leqq c_3 = c_4$. Clearly

$$\psi_4(S^G) = \begin{cases} C[X_1^e + X_4^e, N_{42} + w^{a_{42}}N_{44}, N_{43}] & \text{if } w^{2a_{42}} = 1 \\ C[X_1^e + X_4^e, N_{42} + w^{a_{42}}N_{44}, N_{43}^2, N_{43}(N_{42} - w^{a_{42}}N_{44})] & \text{otherwise} \end{cases}$$

(note that $(\sigma\tau)^2 \in D$). Assume that $r_4 = 4$. Then $r = 7$ and $S^D = S'[f_7]$. Put $u = c_3/c_2 \in N$. Since each f_i satisfies $\psi_j(f_i) \neq 0$ for some j, we can easily compute $\deg(f_i)$ and, by (2.5), $\sum_{i=1}^3 \deg(F_i) = (26u + 2)c_2$ and $\prod_{i=1}^3 \deg(F_i) - (32)^2u^3c_2^3$, which is a contradiction. Hence $r_4 = 3$.

Case 1 "$c_4 < c_3$ $(= c_2)$": Obviously $(X_1X_4)^{c_4}X_3^{e/2} \notin S'$, and because $X_3^{e/2}X_4^{c_4}$

$\in C[X_2, X_3, X_4]^{G_{X_1}} = C[X_2^{c_2}, X_3^{c_3}, X_4^{c_4}, (X_2X_3)^{c_4}, X_2X_3X_4]$, we easily see that $(X_1X_4)^{c_4}X_3^{e/2} \notin ((SV)^D)^2$ and may identify f_6 with $\mathrm{Tr}((X_1X_4)^{c_4}X_3^{e/2})$. $X_1^{e-c_2}X_4^{c_2} \in C[X_1, X_4]^D$, and so if $c_2 \neq e/2$, $e/4 = c_2$ $(= c_3)$ and $c_2 \equiv c_4 \bmod 2c_4$. Consequently the minimal system of generators of S^D can be obtained, and G is conjugate to $\boldsymbol{G}_4$, $\boldsymbol{G}_4'$, $\boldsymbol{G}_5$ or $\boldsymbol{G}_5'$.

Case 2 "$c_4 = c_3 \geqq c_2$"; Clearly $S^D = S'[f_r]$ (cf. (3.9)) and $4c_4 = e$. If $c_3 > c_2$, as in the proof of (3.10), we can similarly identify f_6 with $\mathrm{Tr}(X_1^e(X_1X_2)^{c_2})$, and, by (2.1), get a contradiction. Thus $c_2 = e/4$. D is effectively determined by S^D, which implies that G is conjugate to $\boldsymbol{G}_3$ or $\boldsymbol{G}_3'$.

Finally let us assume $d_j \leqq 3$ for all $2 \leqq j \leqq 4$, which implies $S^D = S'[f_r]$. Obviously $r_j = 2$ $(j = 2, 3)$. If $d_j = 2$, $c_j = e$, and especially if $d_4 = 2$, $r_4 = 2$. We easily see that $\max\{c_2, c_3, c_4\} = e/2$, if $\max\{c_2, c_3, c_4\} < e$ (if $\max\{c_2, c_3, c_4\} = c_3$, $(X_1X_3)^{c_3}X_2^e \in S^D$, which shows $c_3 = e/2$).

LEMMA 3.12. *$r_j = 2$ for $2 \leqq j \leqq r$.*

Proof. Suppose that the assertion is false. Then $r = 7$ and $2c_4 | e$ in $\boldsymbol{N}$. As in the proof of (3.10), we can similarly identify f_6 with $\mathrm{Tr}((X_1X_2)^{c_2}(X_1X_4)^{c_4})$ (resp. $\mathrm{Tr}((X_1X_4)^{c_4}X_2^e)$) if $\max\{c_2, c_3, c_4\} < e$ (resp. if $\max\{c_2, c_3, c_4\} = e$). One can easily compute the degrees of f_i's, and, by (2.1), get a contradiction.

We now can determine S^D and see that G is conjugate to $\boldsymbol{G}_1$ or $\boldsymbol{G}_2$. Thus the proof of (3.1) is completed.

§ 4. Reducible groups

The purpose of this section is to prove

PROPOSITION 4.1. *If S^G is a C.I., then $G \supset [\tilde{G}, \tilde{G}]$.*

Let us assume that (4.1) is false, and let G be a minimal counterexample with $V^G = 0$, i.e., let G be a minimal subgroup such that $V^G = 0$, S^G is a C.I. and $G \not\supset [\tilde{G}, \tilde{G}]$. Since G is generated by special elements, by (2.4) and the minimality of G, we see that $m = 2$, $n_i = 2$ $(i = 1, 2)$ and both V_i's are $C\langle \mathrm{Spe}(G)\rangle$-irreducible (cf. [14, Sect. 3]).

LEMMA 4.2. *Each $\rho_i(\langle \mathrm{Spe}(G)\rangle)$ agrees with $\rho_i(G)$. Moreover, for $i = 1$ or 2, if G is primitive in $GL(V_i)$, $\rho_i(G^i)$ can be identified with $\boldsymbol{D}_2$, $\langle -1\rangle$, 1 in $GL(V_i)$, and otherwise G^i is cyclic.*

Proof. It suffices to treat the case where $i = 1$. Let us identify $\rho_1(G^1)$ with one of $\boldsymbol{C}_u$, $\boldsymbol{D}_u$ $(u \geqq 2)$, $\boldsymbol{T}$, $\boldsymbol{O}$, $\boldsymbol{I}$ in $SL(V_i)$. If $\rho_1(G^1)$ equals $\boldsymbol{D}_u$ $(u > 2)$,

$\boldsymbol{T}$, $\boldsymbol{O}$ or $\boldsymbol{I}$, then $S^{G^1} = \boldsymbol{C}[g_1, g_2, g_3] \otimes_{\boldsymbol{C}} \mathrm{Sym}(V_2)$ for some graded elements g_i $(1 \leqq i \leqq 3)$ in $\mathrm{Sym}(V_1)$ with $\deg(g_1) < \deg(g_2) < \deg(g_3)$ and $\rho_1(G)/\rho_1(G^1)$ acts faithfully on $\boldsymbol{C}[g_1, g_2, g_3]$, which shows $[\rho_1(G), \rho_1(G)] \subset \rho_1(G^1)$. Thus $G^1 = \boldsymbol{C}_u$ or $\boldsymbol{D}_2$. Suppose that G is primitive in $GL(V_1)$. By Clifford's theorem, $\rho_1(G^1) = \langle -1 \rangle$ or 1 in the case where G^1 is cyclic. If $\langle \mathrm{Spe}(G) \rangle$ is imprimitive in $GL(V_1)$, $\rho_1(\langle \mathrm{Spe}(G) \rangle)$ is equivalent to $G(4, 2, 2)$ (cf. [4, (2. 13)]), and we have $G^1 = 1$. Thus $\rho_1(\langle \mathrm{Spe}(G) \rangle)$ is a primitive reflection group. Then $\rho_1(\langle \mathrm{Spe}(G) \rangle) \supset \rho_1(G^1)$, which implies $\rho_1(G) = \rho_1(\langle \mathrm{Spe}(G) \rangle)$. Suppose that G is imprimitive in $GL(V_1)$, i.e., G is monomial on a $\boldsymbol{C}$-basis $\{X_1, X_2\}$ of V_1. We may assume that $\rho_1(\langle \mathrm{Spe}(G) \rangle)$ is expressed as $G(p, q, 2)$ on this basis. If $\rho_1(G^1)$ contains a non-diagonal matrix, $\rho_1(\sigma G^1)$ contains a diagonal matrix for each $\sigma \in \mathrm{Spe}(G)$, and $\rho_2(G^2) \supset \rho_2([\mathrm{Spe}(G), \mathrm{Spe}(G)])$, which is a contradiction. Thus $\rho_1(G^1)$ is diagonal on $\{X_1, X_2\}$. Let τ be an element of $\mathrm{Spe}(G)$ whose restriction to V_1 is not diagonal. Then $G^1\tau \subset \mathrm{Spe}(G)$, which shows $\rho_1(\langle \mathrm{Spe}(G) \rangle) \supset \rho_1(G^1)$.

Now, we assume $r = \mathrm{emb}(S^G)$, $\mathrm{Sym}(V_1)^G = \boldsymbol{C}[f_1, f_2]$, $\mathrm{Sym}(V_2)^G = \boldsymbol{C}[f_3, f_4]$, $V_1 = \boldsymbol{C}X_1 \oplus \boldsymbol{C}X_2$ and $V_2 = \boldsymbol{C}X_3 \oplus \boldsymbol{C}X_4$.

LEMMA 4.3. *One of ρ_i's is primitive.*

Proof. Let $\rho_i(G) = G(p_i, q_i, 2)$, $i = 1, 2$. Put $\mathrm{Spe}_1(G) = \{\sigma \in \mathrm{Spe}(G) \mid \rho_1(\sigma)$ is non-diagonal and $\rho_2(\sigma)$ is diagonal$\}$, $\mathrm{Spe}_2(G) = \{\sigma \in \mathrm{Spe}(G) \mid \rho_2(\sigma)$ is non-diagonal and $\rho_1(\sigma)$ is diagonal$\}$, $\mathrm{Spe}_d(G) = \{\sigma \in \mathrm{Spe}(G) \mid \rho_i(\sigma)$ $(i = 1, 2)$ are diagonal$\}$ and suppose $\mathrm{Spe}_1(G) \cup \mathrm{Spe}_2(G)$ is non-empty. Exchanging the indices of V_i, we can choose elements $\sigma = \mathrm{diag}[a, a^{-1}, -1, 1]\cdot(1, 2)[4]$, $\tau = \mathrm{diag}[b, b^{-1}, 1, -1]\cdot(1, 2)[4]$ $(a, b \in \boldsymbol{C}^*)$ from $\mathrm{Spe}_1(G)$. Obviously every element in $\mathrm{Spe}_d(G)$ is of odd order (in fact, if $\mathrm{Spe}_d(G)$ contains an element of even order, $\rho_1(G^1)$ have a non-diagonal element). As $\mathrm{Spe}_1(G) \neq \phi$, $\mathrm{diag}[c, c^{-1}] \in \rho_1(G^1)$ and $\mathrm{diag}[c, c^{-1}] \in \rho_2(G^2)$ if $\mathrm{diag}[c, 1]$ or $\mathrm{diag}[1, c]$ $(c \in \boldsymbol{C}^*)$ belongs to $\rho_1(\mathrm{Spe}_d(G))$. Therefore we easily see $S^{\langle \mathrm{Spe}_d(G) \rangle} = \boldsymbol{C}[X_1^e, X_2^e, X_3^e, X_4^e, X_1X_2X_3X_4]$ for some $e \in \boldsymbol{N}$ and $S^N = \boldsymbol{C}[X_1^{ew}, X_2^{ew}, X_3^{et}, X_4^{et}, (X_1X_2)^e, (X_3X_4)^e, X_1X_2X_3X_4]$, where $N = \langle G^1 \cup G^2 \cup \mathrm{Spe}_d(G) \rangle$ and w, $t \in \boldsymbol{N}$ with $we = |G^1|$, $te = |G^2|$. Recalling the definition of $G(p_i, q_i, 2)$ and $G = \langle \mathrm{Spe}(G) \rangle$, one has $p_1/q_1 = 2e$ if $\mathrm{Spe}_2(G) \neq \phi$, $p_1/q_1 = e$ if $\mathrm{Spe}_2(G) = \phi$, and $p_2/q_2 = 2e$ (observe $p_i/q_i = |\det(A(p_i, q_i, 2))|$; for definition of $A(p, q, n)$, see [4]). Let $\lambda = \mathrm{diag}[x, y, z, w]$ be an element of G which acts trivially on $\boldsymbol{C}[X_1^{ew}, X_2^{ew}, X_3^{et}, X_4^{et}]$ and non-trivially on S^N ($\lambda((X_1X_2)^e) = -(X_1X_2)^e$, $\lambda((X_3X_4)^e) = -(X_3X_4)^e$ as $p_2/q_2 = 2e$ and $\lambda \in SL(V)$). Because $\mathrm{diag}[x^{-e}, x^e, 1, 1] \in G^1$ and $\mathrm{diag}[1, 1,$

$z^{-e}, z^{e}] \in G^2$, $\operatorname{diag}[-1, 1, -1, 1] = \lambda^e \operatorname{diag}[x^{-e}, x^e, 1, 1] \operatorname{diag}[1, 1, z^{-e}, z^e]$ and consequently this element belongs to $\mathrm{Spe}_d(G)$, which is a contradiction. Therefore G/N acts faithfully on $C[X_1^{ew}, X_2^{ew}, X_3^{et}, X_4^{et}]$. For any element $\gamma = \operatorname{diag}[c, c^{-1}, d, d^{-1}] \cdot (1, 2)(3, 4)[4] \in \mathrm{Spe}(G)$ $(c, d \in C^*)$, $[\sigma, \gamma] = \operatorname{diag}[a^2c^{-2}, a^{-2}c^2, -1, -1]$ and hence $\operatorname{diag}[a^2c^{-2}, a^{-2}c^2] \in \rho_1(G^1)$ if and only if t is even. If $\mathrm{Spe}_2(G) \neq \phi$ (we have already assumed $\mathrm{Spe}_1(G) \neq \phi$), $[\mathrm{Spe}_1(G), \mathrm{Spe}_1(G)] \ni -1$. Thus $\mathrm{Spe}_2(G) = \phi$ in the case where only one of w and t is even. If t is even, by these observations, we easily see $[\mathrm{Spe}(G), \mathrm{Spe}(G)] \subseteq G^1 \times G^2$, which conflicts with our circumstances. Let δ be any element of G which acts trivially on $CX_1^{ew} \oplus CX_2^{ew}$. If $\delta((X_1X_2)^e) = (X_1X_2)^e$, exchanging δ by some element in δN, we may assume $\delta(X_1X_2) = X_1X_2$. If $\delta((X_1X_2)^e) \neq (X_1X_2)^e$, $\delta((X_1X_2)^e) = -(X_1X_2)^e$ and hence $\mathrm{Spe}_2(G) \neq \phi$, which implies w is odd. But in this case, $(X_1X_2)^{ew} = \delta(X_1X_2)^{ew} = (\delta((X_1X_2)^e))^w = (-(X_1X_2)^e)^w = -(X_1X_2)^{ew}$, and consequently $\delta((X_1X_2)^e) = (X_1X_2)^e$. Since $C[X_1, X_2]^N = C[X_1^{ew}, X_2^{ew}, (X_1X_2)^e]$, by the Galois theory and the definition of N, we have $\delta \in N$. Therefore the natural representation $\bar{\rho}_1$: $G/N \to GL(CX_1^{ew} \oplus CX_2^{ew})$ of G/N is faithful and, because $\rho_1(N) \cap SL(V_1) = \rho_1(G^1)$ and $\rho_1([G, G]) \subseteq SL(V_1)$, $\bar{\rho}_1(G/N)$ is a nonabelian reflection group i.e. it can be identified with the irreducible reflection group $G(\bar{p}_1, \bar{q}_1, 2)$ $(\bar{p}_1, \bar{q}_1 \in N, \bar{q}_1 | \bar{p}_1)$ on the C-basis $\{X_1^{ew}, X_2^{ew}\}$. Obviously $\langle \bar{\rho}_1(\sigma), \bar{\rho}_1(\tau) \rangle$ is abelian, and recalling that et is odd, one sees that it is Klein's four group. Let $\{Y_1, Y_2\}$ be a C-basis of $CX_1^{ew} \oplus CX_2^{ew}$ on which $\bar{\rho}_1(\sigma)$ and $\bar{\rho}_1(\tau)$ are diagonal. $\langle \bar{\rho}_1(\sigma), \bar{\rho}_1(\tau) \rangle = \bar{\rho}_1(\langle N, \mathrm{Spe}_1(G) \rangle / N)$ is normal in $\bar{\rho}_1(G/N)$, and therefore $\{CY_1, CY_2\}$ is a complete system of imprimitivities of $\bar{\rho}_1$. Then it follows from [4, (2.13)] that $(\bar{p}_1, \bar{q}_1) = (2, 1)$, $(4, 4)$ or $(4, 2)$. If s is even, recalling that ($\mathrm{Spe}_2(G) = \phi$ and) p_1/q_1 is odd, we see $(\bar{p}_1, \bar{q}_1) = (4, 4)$ and if w is odd, $\mathrm{Spe}_2(G) \neq \phi$ and $(\bar{p}_1, \bar{q}_1) = (2, 1)$ or $(4, 2)$. Consequently the action of G/N on S^N may be given by one of the following rules; Case 1: $G/N = \langle \sigma N, \tau N, \varphi N \rangle$, $\bar{\rho}_1(G) = G(4, 4, 2)$, $\bar{\rho}(\sigma N) = \operatorname{diag}[1, 1, -1, 1] \cdot (1, 2)[4]$, $\bar{\rho}(\tau N) = \operatorname{diag}[-1, -1, 1, -1] \cdot (1, 2)[4]$, $\bar{\rho}(\varphi N) = \operatorname{diag}[\zeta_4^{-1}, \zeta_4, 1, 1] \cdot (1, 2)(3, 4)[4]$, $\sigma((X_1X_2)^e) = \tau((X_1X_2)^e) = \varphi((X_1X_2)^e) = (X_1X_2)^e$, $\sigma((X_3X_4)^e) = \tau((X_3X_4)^e) = -(X_3X_4)^e$, $\varphi((X_3X_3)^e) = (X_3X_4)^e$, $\sigma(X_1X_2X_3X_4) = \tau(X_1X_2X_3X_4) = -X_1X_2X_3X_4$, $\varphi(X_1X_2X_3X_4) = X_1X_2X_3X_4$; Case 2: $G/N = \langle \sigma N, \tau N, \varphi N, \psi N \rangle$, $\bar{\rho}_1(G) = G(4, 2, 2)$, the action of σ, τ, φ is the same one as in Case 1, $\bar{\rho}(\psi) = \operatorname{diag}[-1, 1, \zeta_4, \zeta_4^{-1}] \cdot (3, 4)[4]$, $\psi((X_1X_2)^e) = -(X_1X_2)^e$, $\psi((X_3X_4)^e) = (X_3X_4)^e$, $\psi(X_1X_2X_3X_4) = -(X_1X_2X_3X_4)$; Case 3: $G/N = \langle \sigma N, \tau N, \varphi' N \rangle$, $\bar{\rho}_1(G) = G(2, 1, 2)$, the action of σ, τ is the same one as in Case 1, $\bar{\rho}(\varphi') = \operatorname{diag}[-1, 1, 1, 1] \cdot (3, 4)[4]$, $\varphi'((X_1X_2)^e) = (X_1X_2)^e$, $\varphi'((X_3X_4)^e) = (X_3X_4)^e$, $\varphi'(X_1X_2X_3X_4) =$

$X_1X_2X_3X_4$; where $\bar{\rho}$: $G/N \to GL(CX_1^{ew} \oplus CX_2^{ew} \oplus CX_3^{et} \oplus CX_4^{et})$ is the natural representation of G/N and its matrix representation stated above is afforded by the basis $\{X_1^{ew}, X_2^{ew}, X_3^{et}, X_4^{et}\}$. Let χ be a linear character of $\langle \sigma N, \tau N, \varphi N\rangle/N$ such that $(X_3X_4)^e$ is a χ-invariant of $\langle \sigma N, \tau N, \varphi N\rangle/N$ and put $y_1 = X_1^{ew} - X_2^{ew}$, $y_2 = \zeta_4(X_1^{ew} + X_2^{ew})$, $y_3 = X_3^{et}$, $y_4 = X_4^{et}$, $y_5 = (X_1X_2)^e$, $y_6 = (X_3X_4)^e$, $y_7 = X_1X_2X_3X_4$. Clearly $(S^N)^{\mathrm{Ker}\,\chi} = C[y_1^2 + y_2^2, y_1y_2, y_3^2 + y_4^2, y_3y_4, (y_1 + y_2)(y_3 + y_4), (y_1 - y_2)(y_3 - y_4), y_5, y_6, y_7]$ (since $\mathrm{Ker}\,\chi$ is an abelian group, a set of generators of the ring of invariants can easily be obtained). The element $\sigma N(\mathrm{Ker}\,\chi)$ acts on $(S^N)^{\mathrm{Ker}\,\chi}$ as follows; $\sigma(y_1y_2) = -y_1y_2$, $\sigma(y_5) = y_5$, $\sigma(y_3^2 + y_4^2) = y_3^2 + y_4^2$, $\sigma(y_6) = -y_6$, $\sigma(y_1y_3 + y_2y_4) = y_1y_3 + y_2y_4$, $\sigma(y_2y_3 + y_1y_4) = -y_2y_3 - y_1y_4$, $\sigma(y_7) = -y_7$. Thus $(S^N)^{\langle \sigma N, \tau N, \varphi N\rangle} = C[y_1^2y_2^2, y_5, y_3^2 + y_4^2, y_6^2, y_1y_2y_6, (y_2y_3 + y_1y_4)^2, y_7^2, y_1y_2(y_2y_3 + y_1y_4), y_1y_2y_7, y_6(y_2y_3 + y_1y_4), y_6y_7, y_7(y_2y_3 + y_1y_4), y_1y_3 + y_2y_4]$ and we denote by Ω' this generating system of the algebra. Let Ω be a minimal system of generators of $(S^N)^{\langle \sigma N, \tau N, \varphi N\rangle}$ contained in Ω'.

First we will consider the case where $e \neq 1$. By the computation of degrees of elements in Ω', $y_7^2 \in \Omega$. Assume $\Omega \not\ni y_1y_2y_6$. Then $y_1y_2y_6 \in C[(y_1y_2)^2, y_5, y_3^2 + y_4^2, y_6^2, y_6y_7, y_1y_2y_7, y_1y_3 + y_2y_4, y_7^2]$, which implies $t \leqq 2$. If $t = 2$, $y_1y_2y_6 \in C[y_1y_3 + y_2y_4, y_7^2, y_5]$, and substituting 0 for X_4, we see $y_1y_2y_6 \in C[y_7^2, y_5]$, which conflicts with $y_1y_2y_6 = \zeta_4(X_1^{2ew}X_3^eX_4^e - X_2^{2ew}X_3^eX_4^e)$. When $t = 1$, we similarly get a contradiction. Hence $\{y_7^2, y_1y_2y_6\} \subseteqq \Omega$. Next, suppose $e = 1$. Clearly $y_1y_2y_6 \in \Omega$. If $y_6(y_2y_3 + y_1y_4) \notin \Omega$, for some u, $v_{ij} \in C$,

$$\begin{aligned} y_6(y_2y_3 + y_1y_4) &= u(y_1y_3 + y_2y_4)(y_3^2 + y_4^2) + y_5^{w/2}\Big(\sum_{2it+4j=t+2} v_{ij}(y_3^2 + y_4^2)^i y_6^{2j}\Big) \\ &= u(y_1y_3 + y_2y_4)(y_3^2 + y_4^2) + v_{0(t+2)/4}\, y_5^{w/2} y_6^{(t+2)/2}\,, \end{aligned}$$

and we obtain $u = 0$ (, substituting 0 for X_4). Then $y_2y_3 + y_1y_4 = v_{0(t+2)/4}\, y_5^{w/2} y_6^{t/2}$, which is a contradiction. We see $\{y_1y_2y_6, y_6(y_2y_3 + y_1y_4)\} \subseteqq \Omega$, and consequently, Ω always contains invariants h_1, h_2 such that $\nu_4(h_1) = \nu_4(h_2) = 0$ where ν_4: $S \to S$ is the C-algebra map defined by $\nu_4(X_i) = X_i$ $(1 \leqq i \leqq 3)$, $\nu_4(X_4) = 0$. We may suppose that $f_1 = X_1^{4ew} + X_2^{4ew}$, $f_2 = (X_1X_2)^e$, $\nu_4(f_3) = X_3^{2et}$ and $\nu_4(f_4) = 0$. Clearly $C[X_1, X_2, X_3]^{\langle D, \sigma\rangle}$ is minimally generated by $X_1^{2ew} + X_2^{2ew}$, $(X_1X_2)^e$, X_3^{2et}, $X_1^{ew}X_3^{et} - X_2^{ew}X_3^{et}$.

Case 1. As $\mathrm{emb}(S^G) \leqq 7$, $\nu_4(S^G) = C[\nu_4(f_1), \nu_4(f_2), \nu_4(f_3), \nu_4(h_3)]$ for some N^2-graded element h_3 in S^G. On the other hand $\sum_{\theta \in G/D} \theta(X_1^{ew}X_3^{et})$ is a nonzero invariant of G, and so $\deg^{(2)}(h_3) = (ew, et)$. $\nu_4(\sum_{\theta \in G/D} \theta(X_1^{3ew}X_3^{et})) = (X_1^{3ew} - X_2^{3ew})X_3^{et}$ belongs to $\nu_4(S^G)$, which implies that it is an element of $C[f_2, \nu_4(h_3)]$ (compare degrees of the invariants). Substituting 0 for X_2, we see $X_1^{3ew}X_3^{et} \in C[\nu_4(h_3)]$, a contradiction.

Case 2. Let us choose an N^2-graded element h_3 from S which satisfies $S^G = C[f_1, f_2, f_3, f_4, h_1, h_2, h_3]$. Then $\nu_4(S^G) = C[y_1^2y_2^2, y_5^2, y_3^4, (y_2y_3)^2, y_1y_2^2y_3, y_1^2y_3^2, y_5y_3^2, y_5y_1y_3, y_1y_3^2] = \nu_4(S^{\tilde{G}})[\nu_4(h_3)] = C[y_1^2y_2^2, y_5^2, y_3^4, \nu_4(h_3)]$. Since $y_5y_1y_3 \in C[y_5^2, \nu_4(h_3)]$, $\deg_2(y_5y_1y_3) = et$ and $\deg_2(\nu_4(h_3)) = et$, and hence $\nu_4(h_3)$ may be identified with one of $y_1y_2^2y_3$, $y_5y_1y_3$. On the other hand, computing degrees, we see $y_5y_3^2 \in C[\nu_4(h_3)]$ and choose elements $u' \in C$, $r' \in N$ such that $y_5y_3^2 = u'\nu_4(h_3)^{r'}$. Therefore $r' = 2$ and $\deg^{(2)}(\nu_4(h_3)) = (e, et)$, which conflicts with $\deg_1(y_1y_2^2y_3) \neq e \neq \deg_1(y_5y_1y_3)$.

In Case 3, we can obtain a generating set of S^G and similarly get a contradiction as in Case 1. (Let Γ be the set consisting of nonzero N^2-graded elements in S^G which do not belong to $S^{\tilde{G}}$. Let h_1' be an element of Γ whose $\deg_2$ is minimal in Γ and let h_2' be an element of $\Gamma - (Ch_1' + S^{\tilde{G}})$ whose $\deg_2$ is minimal in this set. Then S^G must be generated by f_i $(1 \leqq i \leqq 4)$, h_1', h_2', h_3' for some N^2-graded element h_3' in S and $\nu_4(h_1') = \nu_4(h_2') = 0$. From this we deduce a contradiction.) Consequently $\mathrm{Spe}_1(G) \cup \mathrm{Spe}_2(G) = \phi$. G can be identified with $\langle D, \xi = (1, 2)(3, 4)[4]\rangle$ where D is a diagonal group, and D is generated by $\mathrm{Spe}_d(G) \cup \{\xi\beta \mid \beta \in \mathrm{Spe}(G) - \mathrm{Spe}_d(G)\} \cup G^1 \cup G^2$.

Suppose $\mathrm{Spe}_d(G) = \phi$. Since S^G is free over $C[X_1^{|G^1|}, X_2^{|G^1|}, X_3^{|G^2|}, X_4^{|G^2|}]^G$ (note X_1X_2, $X_3X_4 \in S^G$), we may assume $G^1 = G^2 = 1$. Then D is a cyclic group. If $|D| = 2$, $\rho_1(G)$ is abelian, and if $|D| = 3$, each $\rho_i(G)$ is conjugate to $W(A_2)$, which conflicts with [14, (4.1)]. Moreover, recalling that G is generated by $\mathrm{Spe}(G)$, we may suppose $D = \langle \mathrm{diag}[\zeta_d, \zeta_d^{-1}, \zeta_d^c, \zeta_d^{-c}]\rangle$ where $d = |D|$ and $c \in N$ such that $(c, d) = 1$. As $\mathrm{emb}(C[X_1, X_3]^D) = 5$ and $\mathrm{emb}(C[X_1, X_4]^D) = 3$, $\mathrm{emb}(S^G) = 4 + \mathrm{emb}(C[X_1, X_3]^D) - 2 + \mathrm{emb}(C[X_1, X_4]^D) - 2 = 8$, and therefore $\mathrm{Spe}_d(G) \neq \phi$.

Suppose $M_\infty = X_1X_2X_3X_4$ belongs to a minimal system of generators of S^D consisting of monomial matrices. Put $e = |\{\beta|_{CX_1} \mid \beta \in \mathrm{Spe}_d(G)\}|$, $u = |\{\beta|_{CX_1} \mid \beta \in D\}|$, $v = |\{\beta|_{CX_2} \mid \beta \in D\}|$, $N_1 = (X_1X_2)^e$, $N_2 = (X_3X_4)^e$, respectively. There are monomials M_i $(1 \leqq i \leqq q$; q may be zero) such that $\{X_1^u, X_2^u, X_3^v, X_4^v, N_1, N_2, M_i\ (1 \leqq i \leqq q), M_\infty\}$ is a minimal system of generators of the C-algebra S^D. Then $q \leqq 4$, since $\mathrm{emb}(S^G) = r \leqq 7$ and M_∞ is an invariant of G. Obviously $q = 0$, 2 or 4. If $q = 0$, $S^G = S^{\tilde{G}}[(X_1^u - X_2^u)\cdot(X_1^v - X_4^v), M_\infty]$, which implies $G \supseteqq [\tilde{G}, \tilde{G}]$ (observe that $(X_1^u - X_2^u)(X_3^v - X_4^v)$ and M are relative invariants of $\tilde{G}$). Suppose $q = 4$. Exchanging indices of M_i and X_j, we have $\nu_4(M_1) = M_1$, $\deg^{(2)}(M_1) = \deg^{(2)}(M_2)$, $\deg^{(2)}(M_3) = \deg^{(2)}(M_4)$ and $S^G = C[X_1^u + X_2^u, N_1, X_3^v + X_4^v, N_2, M_1 + M_2, M_3 + M_4, M_\infty]$.

If $\nu_4(M_3)=\nu_4(M_4)=0$, $\nu_4((X_1^u-X_2^u)(M_1-M_2))=(X_1^u-X_2^u)M_1\in C[X_1^u+X_2^u, N_1, M_1]$, as $(X_1^u-X_2^u)(M_1-M_2)\in S^G$ and $\deg_2(M_1)<v$, and this implies $X_1^u-X_2^u\in C[X_1^u+X_2^u, N_1]$. So we may assume $\nu_4(M_3)=M_3$ and $\deg_2(M_3)=\deg_2(M_1)$. Observing that $(X_1^u-X_2^u)(M_3-M_4)$, $(X_3^v-X_4^v)(M_1-M_2)$ and $(X_3^v-X_4^v)(M_3-M_4)$ are invariants of G, by a similar reason, moreover we may assume that $M_1=X_1^aX_3^b$, $M_2=X_2^aX_4^b$, $M_3=X_2^aX_3^b$ and $M_4=X_1^aX_4^b$ for some a, $b\in N$. Clearly S^D is contained in the normal ring $C[X_1^a, X_2^a, X_1X_2, X_3, X_4]$ and this implies $G^1\ni\operatorname{diag}[\zeta_a, \zeta_a^{-1}, 1, 1]$. On the other hand $X_1^uM_1+X_2^uM_2\in S^G$ and $\nu_4(X_1^uM_1+X_2^uM_2)=X_1^uM_1\in C[X_1^u+X_2^u, N_1, M_1, M_3]$, which shows that $X_2^uM_1=N_1^{u'}M_3$ for some $u'\in N$. Hence $e|a$ in N and $2a=u$, It follows easily from these facts that $\rho_1(G)/\rho_1(G^1)$ is abelian, which is a contradiction. Let us treat the case that $q=2$. As $\xi(M_1)=M_2$ and $\operatorname{emb}(S^G)\leqq 7$, $S^G=B[h_3]$, where $B=C[X_1^u+X_2^u\ N_1, X_3^v+X_4^v, N_2, M_1+M_2, M_\infty]$ and h_3 is one of the polynomials $(X_1^u-X_2^u)(M_1-M_2)$, $(X_3^v-X_4^v)\cdot(M_1-M_2)$ and $(X_1^u-X_2^u)(X_3^v-X_4^v)$. As in case of $q=4$, we can similarly show that, for each $1\leqq j\leqq 4$, $\{i\,|\,\nu_j(M_i)\neq 0\}\neq\phi$ where ν_j defined by $\nu_j(X_i)$ $(1-\delta_{ij})X_i$ (δ_{ij} is Kronecker's δ), and using ν_j, easily see that $(X_3^v-X_4^v)(M_1-M_2)\notin B[(X_1^u-X_2^u)(M_1-M_2)]$, $(X_1^u-X_2^u)(M_1-M_2)\notin B[(X_3^v-X_4^v)(M_1-M_2)]$ and $(X_1^u-X_2^u)(M_1-M_2)\notin B[(X_1^u-X_2^u)(X_3^v-X_4^v)]$. This is a contradiction.

Therefore both X_1X_3 and X_2X_4 are contained in the minimal system of generators of S^D consisting of monomials, and we conclude that $G^1=G^2=1$. Then $S^D=C[X_1^{ew'}, X_2^{ew'}, X_3^{ew'}, X_4^{ew'}, (X_1X_2)^e, (X_3X_4)^e, X_1X_3, X_2X_4, (X_1^{w'-1}X_4)^e, (X_1^{w'-2}X_4^2)^e, \cdots, (X_1X_4^{w'-1})^e, (X_2^{w'-1}X_3)^e, (X_2^{w'-2}X_3^2)^e, \cdots, (X_2X_3^{w'-1})^e]$. From the above equality, as $e\geqq 2$, we can easily infer $\operatorname{emb}(S^G)\geqq 8$ (in fact, the polynomials $X_1^{ew'}+X_2^{ew'}$, $(X_1X_2)^e$,$(X_3X_4)^e$, $X_3^{ew'}+X_4^{ew'}$, $X_1X_3+X_2X_4$, $X_1^{ew'+1}X_3+X_2^{ew'+1}X_4$, $X_1X_2X_3X_4$ and $X_1X_3^{ew'+1}+X_2X_4^{ew'+1}$ are contained in a minimal system of graded generators of S^G), which is a contradiction.

EXAMPLE 4.4. Suppose that $\rho_i(G)=W(L_2)$ in $GL(V_i)$, $i=1, 2$. Since S^G is not a hypersurface (cf. [14]), r is equal to 6 or 7. Exchanging indices of T_i and F_i, we may suppose that $\deg(T_{4+1})\leqq\deg(T_{4+2})\leqq\cdots$, $\deg(F_1)\leqq\deg(F_2)\leqq\cdots$ and $\deg(F_i)>\deg(T_{4+i})$, because $\operatorname{Ker}\Phi$ is contained in the square of the graded maximal ideal of A. Degrees of $W(L_2)$ are known and thus, by (2.5), $\sum_{i=1}^{r-4}(\deg^{(2)}(F_i)-\deg^{(2)}(T_{4+i}))=(8.8)$. Since $f_{4+i}\notin\operatorname{Sym}(V_1)\cup\operatorname{Sym}(V_2)$, $2\leqq\deg(T_{4+1})$, and if $\deg(T_{4+i})=2$, $\deg^{(2)}(T_{4+i})=(1, 1)$. Let σ be an element of $\operatorname{Spe}(G)$ and let $\{X_{i1}, X_{i2}\}$ be a C-basis of V_i on which $\rho_i(\sigma)$ is represented as

$$\begin{bmatrix} * & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $S^{\langle\sigma\rangle}_{(1,1)} = CX_{11}X_{21} \oplus CX_{12}X_{22}$, and hence, if $\dim S^G_{(1,1)} = 2$, $X_{11}X_{21} \in S^G$, which conflicts with the irreducibility of ρ_1. If $G^1 = G^2 = 1$, because both ρ_i are faithful and $Z(W(L_2))$ (the centre of $W(L_2)) = \langle -1 \rangle$, G contains -1. Thus $S^G_{(2,1)} = S^G_{(1,2)} = 0$, and we always have $\deg(T_{4+1}) \geqq 2$, $\deg(T_{4+i}) \geqq 4$ $(i > 1)$. Obviously $S^G_{(1,1)} = 0$ in case of $G^1 \cong G^2 \cong \langle -1 \rangle$. By (2.5),

$$\prod_{i=2}^{r-4} \{1 + (\deg(F_i) - \deg(T_{4+i}))/4\} \geqq \begin{cases} 24\{1 + (\deg(F_1) - \deg(T_5))/2\}^{-1} & \text{if } G^1 = G^2 = 1\,, \\ 12\{1 + (\deg(F_1) - \deg(T_5))/4\}^{-1} & \text{otherwise}\,. \end{cases}$$

We examine this in all possible cases, and easily deduce a contradiction.

Remark 4.5. Using Stanley's theorem (cf. [22]), as in [15, p. 364], we similarly see that $\deg(f_i) \leqq \sum_{j=1}^n \deg(f_j) - 4$ and moreover, by [3, 22], have $\deg(F_i) \leqq \sum_{j=1}^r \deg(f_j) - 4$.

LEMMA 4.6. *Suppose that both ρ_i's are primitive and G^1 is not isomorphic to $\boldsymbol{D}_2$. Then:*

(1) *G^2 is isomorphic to G^1.*

(2) *$\rho_1(G)$ is conjugate to $\rho_2(G)$ in $GL_2(C)$ (where we identify $GL(V_i)$ with $GL_2(C)$).*

(3) *If $G^1 \neq 1$, then* $\mathrm{Sym}^2(\rho_1)$ *is equivalent to* $\mathrm{Sym}^2(\rho_2)$ *modulo a tensor product of a linear character of G.*

(4) *Suppose that $G = \langle \Delta, G^1 \rangle$ for a normal subgroup Δ such that $\Delta \cap G^1 = 1$. Unless, on Δ, ρ_2 is equivalent to a tensor product of ρ_1 and a linear character of Δ, then $\rho_1(G) = \mu_{2u}\boldsymbol{I}$ and u is not divisible by* 5.

(5) *If the Shephard-Todd number of $\rho_1(G)$ is none of* 8, 9, 10, 11, 12, 14 *then ρ_1 is split.*

Proof. (1) and (2) are easy. (3) and (4) follow from the character theory of $\boldsymbol{D}_2$, $\boldsymbol{T}$ and $\boldsymbol{I}$. For the proof of (4), observe that the stabilizer of G at any point of V is generated by special elements. To check (5), we need only to consider a Sylow 2-group of G and use the above fact on stabilizers.

LEMMA 4.7. *One of ρ_i's is imprimitive.*

Proof. We assume that both ρ_i's are primitive and shall give a contradiction. Suppose $G^1 \neq \boldsymbol{D}_2$. Since the proofs are similar (cf. (3) of (4.6)), we may treat only the case where ρ_1 is split. Let Δ be the subgroup defined in (4) of (4.6). Assume that, on Δ, ρ_2 is never equivalent to a product of ρ_1 and a linear character of Δ. Then $\rho_i(G) = \mu_{2u}\boldsymbol{I}$ and $u = 2$, 3 or 6. Because $\mathrm{Sym}^i(V_1)\otimes_{\boldsymbol{C}}\mathrm{Sym}^j(V_2) \simeq \mathrm{Sym}^i(V_1)\otimes_{\boldsymbol{C}}\mathrm{Sym}^j(V_1)$ ($j \equiv 3$, 4, 5 (5)) as $\boldsymbol{C}\rho_1^{-1}(\boldsymbol{I})\cap\Delta$-modules. By this we can estimate (calculate) the lower terms of the Taylor expansion of the Poincare series of S^G and get a contradiction; say $u = 3$. There are nonzero $\boldsymbol{N}^2$-graded elements g_i $(1 \leqq i \leqq 3)$ in S^Δ with $\deg^{(2)}(g_1) = (9, 3)$, $\deg^{(2)}(g_2) = (27, 3)$ and $\deg^{(2)}(g_3) = (3, 9)$, which requires $\mathrm{emb}\, S^G > 7$. Thus, on Δ, ρ_2 is equivalent to $\chi\rho_1$ for a linear character χ of Δ such that $\chi^2 = \det_{V_1}^{-2}$. For a simplicity, let us treat only the case where $\chi = \det_{V_1}^{-1}$. Let $W_1 = \boldsymbol{C}Y_1 \oplus \boldsymbol{C}Y_2$ and $W_2 = \boldsymbol{C}Y_3 \oplus \boldsymbol{C}Y_4$ be $\boldsymbol{C}\Delta$-modules such that $W_1 \cong V_1$ as $\boldsymbol{C}\Delta$-modules, $\boldsymbol{C}Y_3$ is a trivial $\boldsymbol{C}\Delta$-module and $\sigma(Y_4)/Y_4 = \det_{V_1}(\sigma)^{-1}$, $\sigma \in \Delta$. Putting $W = W_1 \oplus W_2$ and $B = \mathrm{Sym}(W_1)\#\mathrm{Sym}(W_2)$ (the Segre product of graded algebras), we naturally regard $\mathrm{Sym}(W)$ and B as $\boldsymbol{N}^2$-graded $\boldsymbol{C}$-algebras. There is a Δ-equivariant $\boldsymbol{C}$-algebra epimorphism $\varphi\colon S \to B$ whose kernel is generated by a graded element w of degree 2. Clearly w is an invariant of Δ, and it is a relative invariant of G satisfying $w^2 \in S^G$ if $G \neq \Delta$. So, G always acts on B and one has the natural epimorphism $S^G \to B^G$. Let d_1, d_2 be the degrees of the reflection group $\rho_1(G)$, c the least common multiplier of the orders of pseudo-reflections in $\rho_1(G)$ and put $d_3 = \deg(f_{\det}(V_1, G))$. Let g_i $(1 \leqq i \leqq 3)$ be graded elements in $\mathrm{Sym}(W_1)$ of $\deg(g_i) = d_i$ such that $\mathrm{Sym}(W_1)^\Delta = \boldsymbol{C}[g_1, g_2]$ and $\mathrm{Sym}(W_1)^{SL(W_1)\cap\Delta|W_1} = \boldsymbol{C}[g_1, g_2, g_3]$. Then $B^\Delta = \mathrm{Sym}(W)^\Delta \cap B = B \cap \boldsymbol{C}[g_1, g_2, g_3Y_4, Y_3, Y_4^c]$. Because $d_1, d_2 \geqq 4$, w or w^2 belongs to a minimal system of graded generators of S^G (, and $\mathrm{emb}(B^G) \leqq 6$ (cf. (2.1))). By the above observations, one can easily give a contradiction as follows: As the proofs are similar, for example, let $\rho_1(G) = (\mu_8|\mu_4;\, \boldsymbol{O}|\boldsymbol{T})$. Then $d_1 = 8$, $d_2 = 12$, $d_3 = 6$ and $c = 4$. The polynomials $g_1Y_3^8$, $g_1Y_4^8$, $g_1(Y_3Y_4)^4$, $g_2Y_3^{12}$, $g_2Y_3^8Y_4^4$, $g_2Y_3^4Y_4^8$ and $g_2Y_4^{12}$ are members of a minimal system of graded generators of B^G, which conflicts with $\mathrm{emb}(B^G) \leqq 6$.

We see $G^1 = G^2 = \boldsymbol{D}_2$ and $\rho_i(G) = (\mu_{2u}|\mu_u;\, \boldsymbol{O}|\boldsymbol{T})$ or $\mu_{2u}\boldsymbol{O}$. Let g_{ij} $(1 \leqq j \leqq 3)$ be graded elements in $\mathrm{Sym}(V_i)$ such that $\mathrm{Sym}(V_i)^{G^i} = \boldsymbol{C}[g_{i1}, g_{i2}, g_{i3}]$, $\deg(g_{i1}) = \deg(g_{i2}) = 4$, $\deg(g_{i3}) = 6$. g_{i3}'s are relative invariants of G: Since $g_{i3}^2 \in \boldsymbol{C}[g_{i1}, g_{i2}]$, $S^{G^1\times G^2} = \tilde{S} \oplus \tilde{S}g_{13} \oplus \tilde{S}g_{23} \oplus \tilde{S}g_{13}g_{23}$ where $\tilde{S} = \boldsymbol{C}[g_{11}, g_{12}, g_{21}, g_{22}]$. We may suppose that $\{f_1, \cdots, f_d\} \subset \tilde{S}^G$ and $\{f_{d+1}, f_{d+2}, \cdots, f_r\} \subset$

$(\tilde{S}g_{13})^G \cup (\tilde{S}g_{23})^G \cup (\tilde{S}g_{13}g_{23})^G$ for some $2 \leqq d \leqq r$. $\tilde{S}^G$ is partly generated by $\{f_1, \cdots, f_d\}$, $(\{f_{d+1}, \cdots, f_r\} \cap \tilde{S}g_{13})^2$, $(\{f_{d+1}, \cdots, f_r\} \cap \tilde{S}g_{23})^2$ and $(\{f_{d+1}, \cdots, f_r\} \cap \tilde{S}g_{13}g_{23})^2$. From these we can easily deduce a contradiction as follows: For example, let us suppose $\rho_i(G) = (\mu_8 | \mu_4; \boldsymbol{O} | \boldsymbol{T})$ in $GL(V_i)$, $i = 1, 2$. Then a minimal system of graded generators of $\tilde{S}^G$ contains seven elements of degree $\leqq 12$ (cf. [14, Sect. 4]). On the other hand $g_{i3} \notin S^G$ $(i = 1, 2)$. So $d = 7 \geqq r$, and $S^G = \tilde{S}^G$. The last equality shows that G^1 contains $\mu_4 \boldsymbol{D}_2$, which conflicts with our assumption.

According to (4.7), we may assume that ρ_1 is primitive and ρ_2 is imprimitive. Let $\{X_1, X_2\}$ be a C-basis of V_1 on which $\rho_1(G)$ is represented as one of the groups listed in [4, (3.6)], and $\{X_3, X_4\}$ a C-basis of V_2 on which $\rho_2(G)$ (resp. $\rho_2(G^2)$) is represented as $G(p, q, 2)$ (resp. $A(u, u, 2)$).

LEMMA 4.8. *$\rho_1(G)$ is not equal to $\mu_{12}\boldsymbol{O}$.*

Proof. Suppose $\rho_1(G) = \mu_{12}\boldsymbol{O}$. Since $[\mu_{12}\boldsymbol{O}, \mu_{12}\boldsymbol{O}] = \boldsymbol{T}$ and $\mathrm{Hom}(\mu_{12}\boldsymbol{O}, \boldsymbol{C}^*) = \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/2\boldsymbol{Z} \oplus \boldsymbol{Z}/3\boldsymbol{Z}$, the subset Ω_1 consisting of all pseudo-reflections of order 3 in $\mu_{12}\boldsymbol{O}$ is two conjugate classes of this group ([12, (3.3)]) and the subset of all pseudo-reflections of order 2 is a union of two conjugate classes Ω_2, Ω_3. ρ_1 induces the maps $\tilde{\rho}_1 : \{\sigma \in \mathrm{Spe}(G) | \mathrm{ord}(\sigma) = 3\} \to \Omega_1$; $\{\sigma \in \mathrm{Spe}(G) | \mathrm{ord}(\sigma) = 2\} \to \Omega_2 \cup \Omega_3$. Let L be the subgroup of G generated by $\{\sigma \in \mathrm{Spe}(G) | \mathrm{ord}(\sigma) = 3\}$. Then L is irreducible primitive in $GL(V_1)$ and furthermore $\rho_1(L) = \mu_6\boldsymbol{T}$. As $\rho_2(L)$ is diagonal, we must have $\rho_1(G^1) \supset \boldsymbol{D}_2 = \rho_1([L, L])$ and hence assume $\rho_1(G^1) = \boldsymbol{D}_2$. Then $2p^2/qu = |G(p, q, 2)|/|A(u, u, 2)| = |\rho_1(G)/\rho_1(G^1)| = 36$. Obviously $p/q = 3$ or 6. Suppose $p/q = 3$ i.e., $\rho_2(G) = G(6u, 2u, 2)$. On the other hand, since $G \subset SL(V)$, we have $(\rho_1(G) \cap SL(V_1))/\rho_1(G^1) \cong (\rho_2(G) \cap SL(V_2))/\rho_2(G^2)$. However $(\rho_1(G) \cap SL(V_1))/\rho_1(G^1) \cong S_3$ and $\rho_2(G) \cap SL(V_2)$ is diagonal, which is a contradiction. Therefore $p/q = 6$ i.e. u is divisible by 2 and $\rho_2(G) = G(6u', u', 2)$ where $u' = u/2$. $\mu_{12}\boldsymbol{T}$ is generated by Ω_1 and one of Ω_i $(i = 2, 3)$, say Ω_2 is so. Put $H = \langle \tilde{\rho}_1^{-1}(\Omega_1), \tilde{\rho}_1^{-1}(\Omega_2) \rangle$. Suppose that every element in $\rho_2(\tilde{\rho}_1^{-1}(\Omega_2))$ is non-diagonal. Then $\rho_2(\tilde{\rho}_1^{-1}(\Omega_3))$ is diagonal. Since $\rho_2(H)/\rho_2(H \cap G^2)$ is abelian, $\rho_2(G)/\rho_2(H \cap G^2)$ is abelian, a contradiction. Thus $\rho_2(\tilde{\rho}_1^{-1}(\Omega_2))$ is diagonal, and $\rho_2(H) = \langle A(2, 1, 2), A(3, 1, 2) \rangle$. Putting $H' = \langle H, G^2 \rangle$, we have $[G : H'] = [\rho_1(G) : \rho_1(H')] = 2$ and $\rho_2(H') = A(6u', u', 2)$. Let χ_i $(1 \leqq i \leqq 3)$ be a linear character of $\mu_{12} \cdot \boldsymbol{T} = \rho_1(H)$ defined by $s_{U_j}(V_1, \rho_1(H), \chi_i) = \delta_{ij}$ $(1 \leqq j \leqq 3)$. Here U_j are inequivalent hyperplanes in V_1 relative to $\rho_1(H)$ such that $\mathscr{I}_{U_3}(\rho_1(H)) - \{1\} \subseteqq \Omega_2$ (cf. Sect. 2). Up to scaler multiplication, any element in a minimal

N^2-graded generating system which does not belong to $C[X_1, X_2] \cup C[X_3, X_4]$ is expressed as $(X_3^{2u'})^a (X_4^{2u'})^b (X_3X_4)^c f_\chi(V_1, \rho_1(H))$ for some $\chi \in \mathrm{Hom}(\rho_1(H), C^*)$. Computing $\deg_1$ of invariants, we may suppose that $X_3^{2u'} f_{\chi_1}$, $X_4^{2u'} f_{\chi_2}$, $(X_3X_4)^3 f_{\chi_3}$ are contained in a minimal system of graded gererators of $S^{H'}$, where f_{χ_i} denotes $f_{\chi_i}(V_1, \rho_1(H))$. Put $f_5 = X_3^{2u'} f_{\chi_1} + X_4^{2u'} f_{\chi_2}$, $f_6 = (X_3^{6u'} - X_4^{6u'})(X_3X_4)^3 f_{\chi_3}$, $f_7 = (X_3X_4)^3 f_{\chi_1} f_{\chi_2}$. Then $\{f_i \mid 1 \leqq i \leqq 7\}$ is a minimal generating set of S^G (this follows from the computation of $\deg_1$ of elements in a generating set). On the other hand, $X_3X_4 f_{\chi_1} f_{\chi_2} f_{\chi_3} \in S^{H'}$, and as $G = \langle H', \varepsilon \rangle$ for some $\varepsilon \in \mathrm{Spe}(G)$ such that $\varepsilon \notin H'$, $\varepsilon(f_{\chi_1} f_{\chi_2} f_{\chi_3}) = f_{\chi_1} f_{\chi_2} f_{\chi_3}$ ([20, (4.3.3)]), which implies $X_3X_4 f_{\chi_1} f_{\chi_2} f_{\chi_3} \in S^G$. But $X_3X_4 f_{\chi_1} f_{\chi_2} f_{\chi_3} \notin C[f_1, \cdots, f_7]$, a contradiction.

LEMMA 4.9. *$\rho_1(G)$ is not equal to $\mu_4 \boldsymbol{O}$.*

Proof. Suppose $\rho_1(G) = \mu_4 \boldsymbol{O}$. Since the order of every pseudo-reflection in $\mu_4 \cdot \boldsymbol{O}$ is equal to 2, $\rho_2(G) = G(p, q, 2) = G(2q, q, 2)$ or $G(q, q, 2)$. We easily see that $\rho_1(G^1)$ is equal to one of $\boldsymbol{D}_2$, $\boldsymbol{T}$ and $\boldsymbol{O}$, and so assume $\rho_1(G^1) = \boldsymbol{D}_2$, which implies $p = 2q$ and $2q = 3u$ (as $S_3 \cong (\rho_1(G) \cap SL(V_1)) / \rho_1(G^1) \cong (\rho_2(G) \cap SL(V_2)) / \rho_2(G^2)$). The subgroup N_1 of $\rho_1(G)$ generated by one of $\rho_1(G)$-conjugate classes in $\rho_1(\mathrm{Spe}(G))$ can be identified with $G(4, 2, 2)$ in $GL(V_1)$ and the subgroup N_2 of $\rho_1(G)$ generated by the other $\rho_1(G)$-conjugate class in $\rho_1(\mathrm{Spe}(G))$ is equal to $(\mu_4 \mid \mu_2; \boldsymbol{O} \mid \boldsymbol{T})$. Put $K_i = \langle \sigma \in \mathrm{Spe}(G) \mid \rho_1(\sigma) \in N_i \rangle$ $(i = 1, 2)$. Because $\rho_1(K_1)/\rho_1(K_1 \cap G^1)$ is abelian, we immediately have $\rho_2(K_1) = A(2, 1, 2)$ and hence $\rho_2(K_2) = G(2q, 2q, 2)$. There are graded elements g_1, g_2, g_3 with $\deg(g_1) = \deg(g_2) = 4$, $\deg(g_3) = 6$ in $C[X_1, X_2]$ which satisfy $C[X_1, X_2]^{G^1} = C[g_1, g_2, g_3]$. Then $S^{G^1 \times G^2} = C[g_1, g_2, g_3, X_3^u, X_4^u, X_3X_4]$ and both elements g_3, X_3X_4 are invariants of K_2. Since $S^{K_1} = C[g_1, g_2, X_3^2, X_4^2, g_3X_3X_4] = C[g_1, g_2, X_3^2, X_4^2] \oplus C[g_1, g_2, X_3^2, X_4^2] g_3X_3X_4$ and $C[g_1, g_2, X_3^2, X_4^2]$ is a G-stable subalgebra, we have $S^G = C[g_1, g_2, X_3^2, X_4^2]^G \oplus C[g_1, g_2, X_3^2, X_4^2]^G g_3X_3X_4$. Therefore $C[g_1, g_2, X_3^2, X_4^2]^G$ is also a complete intersection ([1]). Clearly the natural representations of G on $Cg_1 \oplus Cg_2$ and $CX_3^2 \oplus CX_4^2$ are respectively irreducible imprimitive. Applying (4.3), we see that $C[g_1, g_2, X_3^2, X_4^2]^G$ is not a complete intersection, which is a contradiction.

LEMMA 4.10. *$\rho_1(G)$ is not equal to $(\mu_{12} \mid \mu_6; \boldsymbol{O} \mid \boldsymbol{T})$.*

Proof. Suppose $\rho_1(G) = (\mu_{12} \mid \mu_6; \boldsymbol{O} \mid \boldsymbol{T})$. Since orders of pseudo-reflections in $(\mu_{12} \mid \mu_6; \boldsymbol{O} \mid \boldsymbol{T})$ are 2 and 3, $N = \langle \sigma \in \mathrm{Spe}(G) \mid \mathrm{ord}(\sigma) = 3 \rangle$ satisfies $\rho_1(N) = \mu_6 \cdot \boldsymbol{T}$ and $\rho_2(N) = A(3, 1, 2)$. Thus $[\mu_6 \cdot \boldsymbol{T}, \mu_6 \cdot \boldsymbol{T}] = \boldsymbol{D}_2$ is contained in $\rho_1(G^1)$, and we assume $\rho_1(G^1) = \boldsymbol{D}_2$. Let g_1, g_2, g_3 be graded elements in

$C[X_1, X_2]$ with $\deg(g_1) = \deg(g_2) = 4$, $\deg(g_3) = 6$ such that $C[X_1, X_2]^{G^1} = C[g_1, g_2, g_3]$. In $GL(V_1)$, $\boldsymbol{D}_2 = G(4, 2, 2) \cap SL(V_1)$ where $G(4, 2, 2)$ is defined on a C-basis of V_1. By [13, (4.2)], $C[X_1, X_2]^{G^1} = C[X_1, X_2]^{G(4,2,2)}[f_{\det}(V_1, G(4, 2, 2))]$ which shows that $C[g_1, g_2] = C[X_1, X_2]^{G(4,2,2)}$ and $g_3 = f_{\det}(V_1, G(4, 2, 2))$ (up to scaler multiplication). Obviously $B = C[g_1, g_2, X_3, X_4]$ is a G-stable subalgebra over which S is integral. Because the degrees of $(\mu_{12} | \mu_6; \boldsymbol{O} | \boldsymbol{T})$ are 6 and 24, g_3 is an invariant of G and hence B^G is a C.I. Put $W_1 = Cg_1 \oplus Cg_2$. $W_2 = V_2$, $W = W_1 \oplus W_2$ and let $\theta\colon G \to GL(W)$ (resp. $\theta_i\colon G \to GL(W_i)$, $i = 1, 2$) be the representation of G on W (resp. W_i). Both $\theta_i(G)$ are reflection groups in $GL(W_i)$ and moreover, as $|\theta_1(G)| = 18$, $\theta_1(G)$ is irreducible imprimitive. Suppose that, for an element σ, $\theta(\sigma)$ is a pseudo-reflection in $GL(W)$. If $\theta_2(\sigma) = 1$, $\sigma \in G^1$, and so $\theta_1(\sigma) = 1$. For some $\tau \in G^1$, $\rho_1(\sigma) = \rho_1(\tau)$, which shows $\sigma\tau^{-1}$ is a pseudo-reflection of G. Therefore $\theta(G)$ is contained in $SL(W)$ and, applying (4.7), we must have $\theta_1(G_{[W_2]}) \supset [\theta_1(G), \theta_1(G)] \neq 1$, which is a contradiction.

LEMMA 4.11. *$\rho_1(G)$ is not equal to $(\mu_4 | \mu_2; \boldsymbol{O} | \boldsymbol{T})$.*

Proof. Suppose $\rho_1(G) = (\mu_4 | \mu_2; \boldsymbol{O} | \boldsymbol{T})$. Since the degrees of $(\mu_4 | \mu_2; \boldsymbol{O} | \boldsymbol{T})$ are 6 and 8, as in the proof of (4.10), we can easily show $\rho_1(G^1) \neq \boldsymbol{D}_2$. $(\mu_4 | \mu_2; \boldsymbol{O} | \boldsymbol{T})$ contains only pseudo-reflections of order 2 and hence $(p, q) = (2q, q)$ or (q, q), which conflicts with the isomorphism $\boldsymbol{T}/\rho_1(G^1) = (\rho_1(G) \cap SL(V_1))/\rho_1(G^1) \cong (\rho_2(G) \cap SL(V_2))/\rho_2(G^2)$.

Let us complete the proof of (4.1). Assume that G^1 is trivial or of order 2. If $\rho_1(G)$ contains a pseudo-reflection of order $\neq 2$, putting $L = \langle \sigma \in \mathrm{Spe}(G) | \mathrm{ord}(\sigma) \neq 2 \rangle$ and using [4, (3.6)], we see that $\rho_1(L)$ is irreducible primitive and $\rho_2(L)$ is diagonal, which implies $\rho_1(G^1) \supset \rho_1([L, L]) \supset H \cong \boldsymbol{D}_2$ for a subgroup H. Hence, by (4.9) and (4.11), $\rho_1(G) = \mu_4 \boldsymbol{I}$ (cf. [loc. cit., (3.6)]). Clearly $(p, q) = (2q, q)$ or (q, q) and this conflicts with the isomorphism $(\boldsymbol{I}/\langle -1 \rangle$ or $\boldsymbol{I} \cong)$ $(\rho_1(G) \cap SL(V_1))/\rho_1(G^1) \cong (\rho_2(G) \cap SL(V_2))/\rho_2(G_2)$. Consequently $\rho_1(G^1) = \boldsymbol{D}_2$. By (4.8), (4.9), (4.10) and (4.11), the Shephard-Todd number of $\rho_1(G)$ is not greater than 11, and $\rho_1(G)$ contains a pseudo-reflection of order 4 (cf. [4, (3.16)]). Then, putting $L' = \langle \sigma \in \mathrm{Spe}(G) | \mathrm{ord}(\sigma) = 4 \rangle$, we see that $\rho_2(L')$ is diagonal and $\rho_1(L')$ is irreducible primitive (precisely, is conjugate to $(\mu_8 | \mu_4; \boldsymbol{O} | \boldsymbol{T})$). Thus $\rho_1(G^1)$ $(\supset [\rho_1(L'), \rho_1(L')])$ contains a subgroup which is conjugate to $\boldsymbol{T}$, a contradiction.

§ 5. The classification

In this section we shall prove

THEOREM 5.1. *Suppose that G is irredundant in $GL(V)$. Moreover suppose that $n > 4$ if G is irreducible imprimitive in $GL(V)$ and that $n > 10$ if G is irreducible primitive in $GL(V)$. Then S^G is a C.I. if and only if the following conditions are satisfied:*

(1) *G is generated by special elements in $GL(V)$.*

(2) *$(\mathscr{R}(V; \tilde{G}), \mathscr{R}(V; \tilde{G}) \cap G, V)$ is a CI-triplet.*

(3) *For each $1 \leqq i \leqq m$:*

Case A "*$\mathscr{R}(V; \tilde{G})$ is irreducible in $GL(V_i)$*".

If $\rho_i(\mathscr{R}(V; \tilde{G})) \neq \rho_i(G)$ (i.e. G_i is not generated by pseudo-reflections), up to conjugacy, the groups $\rho_i(G)$, $\rho_i(\mathscr{R}(V; \tilde{G}))$, $\rho_i(G^i)$ are listed in one of lines of Table II.

Case B "*$\rho_i(\mathscr{R}(V; \tilde{G}))$ is reducible in $GL(V_i)$ and not abelian (i.e. not diagonalizable)*".

(i) *$n_i = 4$.*

(ii) *$\rho_i(G)/\rho_i(\mathscr{R}(V; \tilde{G}))$ is conjugate in $GL(\oplus_{i=1}^4 Ch_i)$ to one of the groups listed in Table* I *or can be extended to a CI-triplet in $GL(\oplus_{i=1}^4 Ch_i)$ where $\{h_1, \cdots, h_4\}$ is a $G/\mathscr{R}(V; \tilde{G})$-linearized regular system of graded parameters of* $\mathrm{Sym}\,(V_i)^{\mathscr{R}(V;\tilde{G})}$.

(iii) *For any nonzero $x \in V_i$ with $\dim (V_i)_{(G^i)_x} = 3$ (for this notation, see Sect.* 2), *$(G^i)_x$ is extended to a CI-triplet in $GL((V_i)_{(G^i)_x})$ or conjugate to one of the groups listed in* [29, *Sect.* 3].

(iv) *If, for an irreducible $C\mathscr{R}(V; \tilde{G})$-submodule U of V_i, $(G^i)_{[U]}$ (for this notation, see Sect.* 2) *is not contained in $\mathscr{R}(V; \tilde{G})$, up to conjugacy, the groups $\rho_i(\mathscr{R}(V; \tilde{G}))_{[U]}$, $\rho_i(G)_{[U]}$ and $\rho_i(G^i)_{[U]}$ (stabilizers, cf. Sect.* 2), *respectively agree, in $GL((V_i)_{\rho_i(\mathscr{R}(V;\tilde{G}))_{[U]}})$ $(\cong GL_2(C))$, with $\rho_i(\mathscr{R}(V; \tilde{G}))$, $\rho_i(G)$ and $\rho_i(G^i)$ listed in one of the lines with $n_i = 2$ of Table* II.

Case C "*$\rho_i(\mathscr{R}(V; \tilde{G}))$ is reducible in $GL(V_i)$ and non-trivial abelian*".

For each $\sigma \in G^i$,

TABLE II

$\rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G)$	$\rho_i(G^i)$	Conditions
$G(p, p, 2)$	$\langle \rho_i(\mathscr{R}(V; \tilde{G})), \gamma_1 \rangle$	$\rho_i(G) \cap SL(V_i)$	$b > 1$
$\mu_4 \boldsymbol{D}_2$	$\mu_4 \boldsymbol{T}$	$\rho_i(G) \cap SL(V_i)$	

$\mu_6 \boldsymbol{T}$	$\mu_6 \boldsymbol{O}$	$\rho_i(G) \cap SL(V_i)$	
$G(p, p, 3)$	$\langle \rho_i(\mathscr{R}(V; \tilde{G})), -1 \rangle$	$\rho_i(G) \cap SL(V_i)$	$p \in 2\boldsymbol{Z} + 1$
$G(p, q, 3)$	$\langle \rho_i(\mathscr{R}(V; \tilde{G})), \gamma_2 \rangle$	$\langle G(p, qq', 3) \cap SL(V_i), \gamma_2 \rangle$	$p > 1$
$G(3, 3, 3)$	$\langle \rho_i(\mathscr{R}(V; \tilde{G})), \Gamma_1 \rangle$	$\rho_i(G) \cap SL(V_i)$	
$W(H_3)$	$\mu_3 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$W(L_3)$	$\mu_9 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$W(M_3)$	$\mu_9 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$W(J_3(4))$	$\mu_3 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$G(p, q, 4)$	$\langle \rho_i(\mathscr{R}(V; \tilde{G})), \gamma_3 \rangle$	$\langle G(p, qq', 4) \cap SL(V_i), \gamma_3 \rangle$	$p > 1$
$W(D_4)$	$\langle \rho_i(\mathscr{R}(V; \tilde{G})), \Gamma_2 \rangle$	$\rho_i(G) \cap SL(V_i)$	
$W(A_4)$	$\mu_2 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$W(H_4)$	$\mu_4 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$W(F_4)$	$\mu_4 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\mu_4[W(F_4), W(F_4)]$	
$W(F_4)$	$\langle \rho_i(\mathscr{R}(V; \tilde{G})), \Gamma_3 \rangle$	$\rho_i(G) \cap SL(V_i)$	
$W(L_4)$	$\mu_{12} \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$EW(N_4)$	$\mu_8 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$W(A_5)$	$\mu_2 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	
$W(E_6)$	$\mu_2 \rho_i(\mathscr{R}(V; \tilde{G}))$	$\rho_i(G) \cap SL(V_i)$	

$\gamma_1 = \operatorname{diag}[\zeta_{2^b}, -\zeta_{2^b}^{-1}]$; $\gamma_2 = \operatorname{diag}[\zeta_{3p}^{-2}, \zeta_{3p}, \zeta_{3p}]$; $\gamma_3 = \operatorname{diag}[\zeta_{2^b}, \zeta_{2^b}, \zeta_{2^b}^{-1}, \zeta_{2^b}^{-1}]$;
$\Gamma_1 = (\mu_u W(L_3)) \cap SL(V_i)$ $(u = 1, 9)$ or $[W(L_3) \cap SL(V_i), W(L_3) \cap SL(V_i)]$;
$\Gamma_2 = \mu_{2u}[W(F_4), W(F_4)]$ $(u = 1, 2)$; $\Gamma_3 = \mu_4, \lambda, \zeta_4 \lambda$ or $\mu_4 \lambda$ $(W(F_4) \not\ni \lambda \in SL_4(\boldsymbol{R})$, $\lambda^2 = 1$, $\lambda W(F_4) = W(F_4)\lambda)$; $G(3, 3, 3) \subset W(M_3)$; $W(L_3) \subset W(M_3)$;
$W(D_4) \subset W(F_4)$; $N \ni q' = p/q$ or $p/2q$; $N \ni b$, $2^{b-1} \| p$

$$\prod_{s_{jk} \neq 0} s_{jk} = 1 ,$$

where s_{jk} $(1 \leqq j, k \leqq n_i)$ *are entries of the matrix* $[s_{jk}]$ *of* $\rho_i(\sigma)$ *afforded by a* $\boldsymbol{C}$*-basis on which* $\rho_i(\mathscr{R}(V; \tilde{G}))$ *is represented as a diagonal group and* G^i *is conjugate, in* $GL(V_i)$, *to one of* $G(p, p, n_i) \cap SL(V_i)$ $(p > 1, n_i > 2)$ $\langle G(p, p, 4) \cap SL(V_i), \operatorname{diag}[\zeta_{2^b}, \zeta_{2^b}, \zeta_{2^b}^{-1}, \zeta_{2^b}^{-1}] \rangle$ $(2^{b-1} \| p, b \geqq 1, n_i = 4)$, *the groups in Table* I, $\langle G(p, p, 3) \cap SL(V_i), \operatorname{diag}[\zeta_{3p}^{-2}, \zeta_{3p}, \zeta_{3p}] \rangle$ $(p \geqq 2, n_i = 3)$, $\langle G(p, p, 3) \cap SL(V_i), \operatorname{diag}[\zeta_{7p}, \zeta_{7p}^2, \zeta_{7p}^{-3}] \rangle$ $(p > 1, n_i = 3)$.

Case D "$\rho_i(\mathscr{R}(V; \tilde{G})) = 1$".

$m = 1$ *and* G *can be extended to a CI-triplet in* $GL(V)$ (i.e., $G = G^{\#} \cap SL(V)$ *for a finite reflection subgroup* $G^{\#}$ *of* $GL(V)$ *in which all orders of pseudo-reflections are equal to the index* $[G^{\#}:G]$).

Remark 5.2. The conditions in Case B of (3) of (5.1) can be replaced by a concrete classification of some subgroups in $GL(V_i)$. However it is rather complicated.

For convenience sake, put $\mathscr{R} = \mathscr{R}(V; \tilde{G})$ and $\Lambda = \{\chi \in \mathrm{Hom}(\tilde{G}, C^*) \mid \chi(G) = 1\}$. We suppose that G is irredundant in $GL(V)$, $n > 4$ if G is irreducible imprimitive and $n > 10$ if G is irreducible primitive, and furthermore may suppose that G is generated by special elements. If S^G is a C.I., then $G \supseteqq [\tilde{G}, \tilde{G}]$ (i.e. $\rho_i(G^i) \supseteqq [\rho_i(G), \rho_i(G)]$) and, for each $1 \leqq i \leqq m$, both $\mathrm{Sym}(V_i)^G$ (cf. [21, (5.2)]) and $\mathrm{Sym}(V_i)^{G^i}$ ([14, (2.6)]) are also C.I.'s (cf. (2.4)). Conversely if $\mathscr{R} \cap G \supseteqq [\mathscr{R}, \mathscr{R}]$, one easily sees $G \supseteqq [\tilde{G}, \tilde{G}]$, since $\mathscr{R} = \mathscr{R} \cap G_1 \times \cdots \times \mathscr{R} \cap G_m$.

LEMMA 5.3. *Suppose that $f_\chi(V, \tilde{G}) \in s_\chi^{\tilde{G}}$ for all $\chi \in \Lambda$. Then S^G is a C.I. if and only if $(\mathscr{R}, \mathscr{R} \cap G, V)$ is a CI-triplet and all* $\mathrm{Sym}(V_i)^G$ $(1 \leqq i \leqq m)$ *are C.I.'s.*

Proof. By the above observations, (in case of "if" part or in case of "only if" part of this lemma,) we always have $S^G = \oplus_{\chi \in \Lambda} S_\chi^{\tilde{G}} = \oplus_{\chi \in \Lambda} S^{\tilde{G}} f_\chi(V, \tilde{G})$ (cf. [21]). Since $f_\chi(V, \tilde{G}) = f_\chi(V, \mathscr{R})$, $S^{\mathscr{R} \cap G} = \oplus_{\chi \in \Lambda} S_\chi^{\mathscr{R}} = \oplus_{\chi \in \Lambda} S^{\mathscr{R}} f_\chi(V, \mathscr{R})$ and therefore $S^G/(SV)^{\tilde{G}} S^G \cong S^{\mathscr{R} \cap G}/(SV)^{\mathscr{R}} S^{\mathscr{R} \cap G}$. Clearly $S^{\tilde{G}}$ is a C.I. if and only if $\mathrm{Sym}(V_i)^G$ $(1 \leqq i \leqq m)$ are C.I.'s. The closed fibre of the flat morphism $(S_{SV})^{\tilde{G}} \to (S_{SV})^G$ is isomorphic to that of the flat morphism $(S_{SV})^{\mathscr{R}} \to (S_{SV})^{\mathscr{R} \cap G}$ and hence the assertion follows from [1].

In order to prove (5.1), by (5.3) we need only to show that (a) if S^G is a C.I., then the condition (3) in (5.1) holds, and (b) if the condition (3) in (5.1) holds, then $\mathrm{Sym}(V_i)^G$ is a C.I. and $f_\chi(V_i, \rho_i(G)) \in \mathrm{Sym}(V_i)^{G^i}$ for each $1 \leqq i \leqq m$ and all $\chi \in \mathrm{Hom}(\rho_i(G), C^*)$ with $\chi(\rho_i(G^i)) = 1$, because $f_\chi(V, \tilde{G}) = \prod_{i=1}^m f_\chi(V_i, G_i)$ and $f_\chi(V, \tilde{G}) \in S_\chi^{\mathscr{R}}$ for $\chi \in \mathrm{Hom}(\tilde{G}, C^*)$ ([21]). So let us fix $1 \leqq i \leqq m$ and divide the proof of the above assertions into the cases as follows:

Case A "$\mathscr{R}$ is irreducible in $GL(V_i)$". Since the "not if" part follows immediately from [21], we may suppose that S^G is a C.I. (in the proof of the last assertion in (b), we do not use this assumption, and use the first assertion in (b)) and $\rho_i(G) \neq \rho_i(\mathscr{R})$ (then $\rho_i(G^i) \not\subseteq \rho_i(\mathscr{R}) \cap SL(V_i)$). It should be noted that $f_{\det^{-1}}(V_i, \rho_i(G)) \in \mathrm{Sym}(V_i)^{G^i}$ ([21, 25]).

Subcase 1 "$\rho_i(\mathscr{R})$ is primitive and $n_i = 2$". Assume that $\rho_i(\mathscr{R}) = (\mu_{ab} \mid \mu_a; H \mid \rho_i(\mathscr{R}) \cap SL(V_i))$ for some subgroup H of $SL(V_i)$ and natural numbers a, *b*. Since $\langle H, \rho_i(G)\cap SL(V_i)\rangle$ is a finite group containing H and $\rho_i(G)\cap SL(V_i)$ as normal subgroups respectively, we must have $H\supseteqq \rho_i(G)\cap SL(V_i)$ or $\rho_i(G)\cap SL(V_i)\supseteqq H$. Our assumption and this imply $\rho_i(G)\supseteqq \mu_{ab}\cdot(\rho_i(\mathscr{R})\cap SL(V_i))$, which shows $\rho_i(G)=\rho_i(\mathscr{R})$ (cf. [4, (3.6)]). Therefore $\rho_i(\mathscr{R})$ may be identified with $\mu_a\cdot(\rho_i(\mathscr{R})\cap SL(V_i))$ for a natural number a. Because $\mu_a\cdot(\rho_i(G)\cap SL(V_i))$ is not a reflection group, $a=6$ and the groups $\rho_i(G)\cap SL(V_i)$ and $\rho_i(\mathscr{R})\cap SL(V_i)$ can be regarded as $\boldsymbol{O}$ and $\boldsymbol{T}$ respectively. Because $\boldsymbol{O}\supseteqq \rho_i(G^i)\supsetneqq \boldsymbol{D}_2=[\mu_6\cdot\boldsymbol{T}, \mu_6\cdot\boldsymbol{T}]$ (cf. (4.1)) and $\rho_i(G^i)\not\subseteq \rho_i(\mathscr{R})\cap SL(V_i)$, $\rho_i(G^i)=\boldsymbol{O}=\rho_i(G)\cap SL(V_i)$. $f_{\det}(V_i, \rho_i(G))$ is a graded element of degree 8 in $\mathrm{Sym}(V_i)$ which is an invariant of $\boldsymbol{O}$ (in fact $f_{\det}(V_i,\rho_i(G))=f_{\det}(V_i,\mu_6\cdot\boldsymbol{T})$ is a unique nonzero invariant of degree 8 of $\boldsymbol{T}$ (up to constant multiple) and $\boldsymbol{O}$ has a graded nonzero invariant of degree 8). If a linear character χ of $\rho_i(G)$ satisfies $\chi(\rho_i(G^i))=1$, $\chi=\det^u$ on V_i for some $u\in \boldsymbol{N}$. Clearly

$$f_{\det^u}(V_i,\rho_i(G))=\begin{cases}1 & \text{if } u\equiv 0 \bmod 3\\ f_{\det}(V_i,\rho_i(G)) & \text{if } u\equiv 1 \bmod 3\\ f_{\det^{-1}}(V_i,\rho_i(G)) & \text{if } u\equiv 2 \bmod 3\end{cases}$$

for $u\in\boldsymbol{N}$ and hence the rest of the assertions follows.

Subcase 2 "$\rho_i(\mathscr{R})$ is a primitive Coxeter group ($n_i>2$)". Let $\sigma\in\rho_i(G^i)$ be any special element which does not belong to $\mathscr{R}$ and let $(V_i)_{\boldsymbol{R}}$ be a G-stable real structure of V_i. $\rho_i(\mathscr{R})$ may be regarded as a subgroup of $GL((V_i)_{\boldsymbol{R}})$. Since $\rho_i(\mathscr{R})$ is absolutely irreducible in $GL((V_i)_{\boldsymbol{R}})$ and $\sigma\rho_i(\mathscr{R})=\rho_i(\mathscr{R})\sigma$, for some $c\in\boldsymbol{C}^*$, $c\cdot\sigma$ belongs to $GL((V_i)_{\boldsymbol{R}})$. By [2, p. 232, Exc. 16] and [4] we can similarly show the assertion as in the next case.

Subcase 3 "$\rho_i(\mathscr{R})=W(L_3)$". $\rho_i(\mathscr{R})$ can be regarded as a subgroup of $W(M_3)$ generated by all pseudo-reflections of order 3 in $W(M_3)$. For a special element $\sigma\in\rho_i(G^i)$ with $\sigma\notin\rho_i(\mathscr{R})$, by [4, (5.14)], there are a natural number a and $\tau\in W(M_3)$ such that $\sigma=\zeta_a\cdot\tau$ and $\dim V_i(\tau,\zeta_a^{-1})=1$. Since the degrees of $W(L_3)$ are 6, 9, 12 and $\mathrm{Sym}(V_i)^{\mathscr{R}}$ is divisorially unramified over $\mathrm{Sym}(V_i)^G$ ([7]), exactly one of ζ_a^6, $\pm\zeta_a^9$, ζ_a^{12} is equal to 1. Moreover, as $\det(\tau)\in\mu_6=\det(W(M_3))$, $a=9$. There are regular elements μ of $W(M_3)$ and μ' of $W(L_3)$ of order 9 ([4, (4.16)]) satisfying $\dim V_i(\mu,\zeta_a^{-1})=\dim V_i(\mu',\zeta_a^{-1})=1$ ([19, (4.2), (ii)]). Then μ and μ' are conjugate to τ in $W(M_3)$, and as $W(L_3)$ is normal in $W(M_3)$, $\tau\in W(L_3)$, i.e. $\rho_i(G)=\mu_9W(L_3)$. Using $\deg(f_{\det}(V_i,\rho_i(G)))=12$ and $\sigma\in SL(V_i)$, we see that $f_{\det}(V_i,\rho_i(G))$ is an invariant of σ ([19]). The assertion in (b) follows from the fact "$\mathrm{Hom}(W(L_3),\boldsymbol{C}^*)=\{1,\det,\det^{-1}\}$" and $f_{\det^{-1}}(V_i,\rho_i(G))=f_{\det}(V_i,\rho_i(G))^2$.

Subcase 4 "$\rho_i(\mathscr{R}) = W(M_3)$". Let σ be any special element in $\rho_i(G^i)$ such that $\sigma \notin \rho_i(\mathscr{R})$. By [4, (5.14)], $\sigma = \zeta_a \cdot \tau$ for some $\tau \in W(M_3)$ with $\dim V_i(\tau, \zeta_a^{-1}) = 1$. Since the degrees of $W(M_3)$ are 6, 12, 18 and $\det(\tau) \in \mu_6$, we have $a = 9$ or 18 and by [19, § 4], find τ, which is regular, in $W(M_3)$. The rest of the assertions follows from [21] and the following computation of the degrees of $f_{\det^j}(V_i, \rho_i(G))$; $\deg(f_{\det^j}(V_i, \rho_i(G)) = 21$, if $j = 1$; $= 24$, if $j = 2$; $= 9$, if $j = 3$; $= 12$, if $j = 4$; $= 33$; if $j = 5$ (cf. [4, (4.16)]).

Subcase 5 "$\rho_i(\mathscr{R})$ is a primitive complex reflection group ($n_i > 2$)". Using [loc. cit., (5.14)], we can prove the assertion by the similar method as in Subcase 3.

Subcase 6 "ρ_i is monomial and $n_i = 2$". Let $\{X_1, X_2\}$ be a $\boldsymbol{C}$-basis on which $\rho_i(G)$ is monomial and $\rho_i(\mathscr{R})$ agrees with $G(p, q, 2)$. Let σ be a special element in G^i which does not belong to $\mathscr{R}$. Then, on $\{X_1, X_2\}$, $\rho_i(\sigma) = \mathrm{diag}\,[c, d] \cdot (1, 2)[2]$ for some $c, d \in \boldsymbol{C}$ with $cd = -1$. Assume $p/q > 2$. By $\rho_i(G) = \langle \rho_i(G^i), \rho_i(\mathrm{Spe}\,(G))\rangle$, we find an element γ in $\mathrm{Spe}\,(G)$ with $\mathrm{ord}\,(\rho_i(\gamma)) = \mathrm{ord}\,(\gamma) > 2$ such that $\rho_i(\gamma)$ is diagonal on $\{X_1, X_2\}$. Put $L = G_{[(\oplus_{j\neq i} V_j)\langle\gamma\rangle]}$ (the stabilizer) and choose an element Z from V satisfying $(\gamma - 1)(\oplus_{j\neq i} V_j) = \boldsymbol{C}Z$. Clearly $\rho_i(L)$ is irreducible and is not conjugate to $\langle \mathrm{diag}\,[\zeta_a, \zeta_a^{-1}], (1, 2)[2]\rangle$ ($a \geqq 2$) (it should be noted that, in [29, Theorem 1], these groups are deleted). Because $\boldsymbol{C}[X_1, X_2, Z]^L$ is a C.I., by [29, Theorem 1], $\rho_i(L)$ contains $\mathrm{diag}\,[-1, 1]$, which implies p/q is even. Then from the equality "$\gamma(f_{\det^{-1}}(V_i, \rho_i(G))) = f_{\det^{-1}}(V_i, \rho_i(G))$ (this polynomial can be identified with $(X_1X_2)^{p/q-1}(X_1^p - X_2^p)$)" it follows that $c^p = d^p = 1$. Hence if $p \neq q$, $\rho_i(\sigma) \in \rho_i(\mathscr{R}) = G(p, q, 2)$, which conflicts with our choice. We see that $p = q$ and moreover, by the invariance of $f_{\det^{-1}}(V_i, \rho_i(G))$, p is even. In $G(p, q, 2)$ there are exactly two equivalent classes in $\mathscr{H}(V_i, G(p, q, 2))$ (cf. [12]). Since $X_1^{p/2} - X_2^{p/2}$ and $X_1^{p/2} + X_2^{p/2}$ are relative invariants of $G(p, q, 2)$, for any χ in $\mathrm{Hom}\,(\rho_i(G), \boldsymbol{C}^*)$ with $\chi \neq 1$ and $\chi(\rho_i(G^i)) = 1$, $f_\chi(V_i, \rho_i(G))$ can be identified with one of the polynomials $X_1^{p/2} - X_2^{p/2}$, $X_1^{p/2} + X_2^{p/2}$ and $X_1^p - X_2^p$. Obviously $\sigma^2 \in \mathscr{R}$, which implies $\sigma(f_\chi(V_i, \rho_i(G))) = \pm f_\chi(V_i, \rho_i(G))$. However $c^p = d^p = -1$ and hence $\chi = \det^{-1}$ on $\rho_i(G)$, i.e. $\rho_i(G^i) = \rho_i(G) \cap SL(V_i)$.

Subcase 7 "$\rho_i(\mathscr{R})$ is imprimitive, $\rho_i(G)$ is primitive and $n_i = 2$". According to [4, (2.13)] we see that $\rho_i(\mathscr{R})$ is conjugate to $G(4, 2, 2)$ or $G(2, 1, 2)$ in $GL(V_i)$. In both cases, each orbit in $\mathscr{H}(V_i, \rho_i(\mathscr{R}))$ under the action of $\rho_i(\mathscr{R})$ consists of two hyperplanes, and so, because $\rho_i(G) = \rho_i(\mathscr{R})\rho_i(G^i)$ is not monomial, $\rho_i(G)$ acts transitively on $\mathscr{H}(V_i, \rho_i(\mathscr{R}))$. Let σ be any element in $\mathrm{Spe}(G)$ such that $\rho_i(\sigma) \neq 1$. Putting $L' = G_{[(\oplus_{j\neq i} V_j^{\langle\sigma\rangle}]}$, we easily see that $\rho_i(L') = \rho_i(G)$, $\dim V_{L'} = 3$ and $\mathrm{Sym}(V_{L'})^{L'}$ is a C.I., Then, by [29], $\rho_i(L')$ is conjugate to $\mu_4 \cdot \boldsymbol{T}$ and $\rho_i(G^i) = \rho_i(G) \cap SL(V_i)$. If a nontrivial $\chi \in \mathrm{Hom}(\rho_i(G), C^*)$ satisfies $\chi(\rho_i(G^i)) = 1, f_\chi(V_i, \rho_i(G)) = f_{\det^{-1}}(V_i, \rho_i(G))$, which shows the assertion in (b).

Subcase 8 "ρ_i is monomial and $n_i > 2$". Let $\boldsymbol{X} = \{X_1, \cdots, X_{n_i}\}$ be a C-basis of V_i on which $\rho_i(G)$ is monomial and $\rho_i(\mathscr{R})$ is identified with $G(p, q, n_i)$. Since $\rho_i(G^i) \supseteqq [\rho_i(\mathscr{R}), \rho_i(\mathscr{R})] = G(p, p, n_i) \cap SL(V_i)$, $\prod_{\boldsymbol{X}}(\rho_i(G^i))$ is isomorphic to S_{n_i} or A_{n_i}. Suppose $\rho_i(G^i) - \rho_i(\mathscr{R})$ contains $\sigma = \mathrm{diag}\,[a\ b, c, 1, \cdots, 1] \cdot (1, 2)[n_i]$ satisfying (1) $ab = -1$, $c = 1$ or (2) $ab = 1$, $c = -1$. Using $\sigma G(p, p, n_i)\sigma^{-1} = G(p, p, n_i)$, we easily see $a^p = b^p = c^p = 1$ if $n_i > 3$ or if $n_i = 3$ and $c = 1$. In this case p/q is odd, and hence $\sigma(f_{\det^{-1}}(V_i, \rho_i(G))) = -f_{\det^{-1}}(V_i, \rho_i(G))$, which is a contradiction. Consequently $n_i = 3$, $c = -1$ and $b = a^{-1}$. When p is even, exchanging σ, we may suppose $c = 1$. Thus it should be assumed that p is odd. By [29], we can identify $\rho_i(G^i)$ with $\langle G(p, p, 3) \cap SL(V_i), \mathrm{diag}\,[-1, -1, -1]\,(1, 2)[3]\rangle$. Assume $p \neq q$. Then there is an element τ in $\mathrm{Spe}(G)$ such that $\rho_i(\tau) = \mathrm{diag}\,[\zeta_u, 1, 1]$ with $u \geqq 2$. Putting $H = G_{[(\oplus_{j\neq i} V_j^{\langle\tau\rangle}]}$, we see $\rho_i(H)$ is equal to $\langle \rho_i(G^i), \rho_i(\mu)\rangle$ or $\langle \rho_i(G^i), \rho_i(\mu), (1, 2)[3]\rangle$, since H is generated by special elements. Here μ is an element of $\mathrm{Spe}(G)$ such that $\langle \mu \rangle \ni \tau$. In both cases, by a direct computation, $\mathrm{emb}\,(\mathrm{Sym}\,(V_H)^H) \geqq 8$, a contradiction. Consequently $\rho_i(G) = \langle G(p, p, 3), -1\rangle$, $\rho_i(G^i) \supseteqq \rho_i(G) \cap SL(V_i)$ and $f_{\det}(V_i, \rho_i(G))$ is an invariant of $\rho_i(G^i)$. For the rest of cases, by [8, 29], we infer that the assertion holds.

Subcase 9 "$\rho_i(G)$ is not monomial, $\rho_i(\mathscr{R})$ is imprimitive and $n_i > 2$". $\rho_i(\mathscr{R})$ may be identified with $G(3, 3, 3)$ or $G(2, 2, 4)$ (cf. [4, (2.13)]). Suppose $\rho_i(\mathscr{R}) = G(3, 3, 3)$ and regard $\rho_i(G)$ is a subgroup of $\mu_\infty \cdot W(M_3)$. Because $\rho_i(G^i)$ is irreducible primitive and $\mathrm{Sym}\,(V_i)^{G^i}$ is a C.I., by [29], $\rho_i(G^i)$ is in $(\mu_9 W(L_3)) \cap SL(V_i) = (\mu_9 W(M_3)) \cap SL(V_i)$. Clearly $f_{\det^{-1}}(V_i, \rho_i(G)) = f_{\det^3}(V_i, W(M_3))$ is an invariant of $W(L_3) \cap SL(V_i)$, and the assertion follows from [29]. We can similarly treat the case "$\rho_i(\mathscr{R}) = W(D_4)$".

Case B "$\rho_i(\mathscr{R})$ is reducible and not abelian". Suppose that S^G is a C.I. Then, as $\mathrm{Sym}\,(V_i)^G$ is a C.I., by [14, (4.3)] (the circumstance of [14, (4.3)] is somewhat different from our present circumstance, but its proof is applicable), $n_i = 4$. Let $\{X_1, X_2, X_3, X_4\}$ be a C-basis of V_i on which matrices are always defined and suppose that $CX_1 \oplus CX_2$ and $CX_3 \oplus CX_4$ are irreducible $C\mathscr{R}$-submodules of V_i. Denote by H the decomposition group of $\mathrm{Sym}\,(V_i)(X_1, X_2)$ under the action of $\rho_i(G)$, and let ψ_1: $H \rightarrow$

$GL(CX_1 \oplus CX_2)$ and ψ_2: $H \to GL(CX_3 \oplus CX_4)$ be the natural representations of H. We may suppose that $\rho_i(G) = \langle H, (1, 3)(2, 4)[4] = \gamma \rangle$, and there are canonical isomorphisms $\psi_1(H) \cong \psi_2(H)$ and $\psi_1(\rho_i(\mathscr{R})) \cong \psi_2(\rho_i(\mathscr{R}))$. Clearly $\rho_i(\mathscr{R})$ is the direct product of $\mathrm{Ker}\, \psi_1 \cap \rho_i(\mathscr{R})$ and $\mathrm{Ker}\, \psi_2 \cap \rho_i(\mathscr{R})$. Moreover H is generated by the union of $\rho_i(G^i)_{[\{X_1, X_2\}]}$, $\rho_i(G^i)_{[\{X_3, X_4\}]}$, $\rho_i(\mathscr{R})$,

$$L_1 = \left\{ \begin{bmatrix} F & \\ & F^{-1} \end{bmatrix} \middle| F \in GL_2(C) \right\} \cap \rho_i(G^i)$$

and $L_2 = \{\beta \in \rho_i(G^i) \cap H \mid \psi_1(\beta)$ and $\psi_2(\beta)$ are pseudo-reflections in $GL_2(C)\}$. If $H/\rho_i(\mathscr{R})$ is abelian, the assertion (a) is evident and so we assume $H/\rho_i(\mathscr{R})$ is not abelian. If, for a normal subgroup G' of $\rho_i(G)$ generated by some pseudo-reflections, the pair of degrees of $\psi_1(G')$ is consisting of distinct numbers, $H/\rho_i(\mathscr{R})$ is abelian, because $\psi_1(H)/\psi_1(G')$ and $\psi_2(H)/\psi_2(G')$ act faithfully on $C[X_1, X_2]^{G'}$ and $C[X_3, X_4]^{G'}$ respectively. Suppose that $\psi_1(\rho_i(\mathscr{R}))$ is primitive. Then since the degrees of $\psi_1(\rho_i(\mathscr{R}))$ are equal, by [4, (3.6)], $\psi_1(\rho_i(\mathscr{R}))$ is identified with one of $\mu_{12} \cdot \boldsymbol{T}$, $\mu_{24} \cdot \boldsymbol{O}$ and $\mu_{60} \cdot \boldsymbol{I}$ in $GL_2(C)$. Let N be a subgroup of $\rho_i(G)$ generated by all pseudo-reflections of order 3 in $\rho_i(G)$. The pair of the degrees of $\psi_1(N)$ is consisting of distinct numbers ([4]), which is a contradiction. Thus $\psi_1(\rho_i(\mathscr{R}))$ is imprimitive, and furthermore, by [4, (2.13)], $\psi_1(\rho_i(\mathscr{R}))$ (resp. $\psi_2(\rho_i(\mathscr{R}))$) may be identified with $\mu_4 \cdot \boldsymbol{D}_2$ on the C-basis $\{X_1, X_2\}$ (resp. $\{X_3, X_4\}$). Using a classification of finite subgroups of $GL_2(C)$ (cf. [4, (3.1)]) and our assumption on $\psi_1(H)/\psi_1(\rho_i(\mathscr{R}))$, we easily see that $\psi_1(H)$ is equal to $\mu_{2u} \cdot \boldsymbol{O}$ or $(\mu_{4u} \mid \mu_{2u}; \boldsymbol{O} \mid \boldsymbol{T})$ on $\{X_1, X_2\}$, where $u \in \boldsymbol{N}$ is even. There are homogeneous polynomials g_1, g_2 (resp. g_3, g_4) in $C[X_1, X_2]$ (resp. $C[X_3, X_4]$) such that $\gamma(g_1) = g_3$, $\gamma(g_2) = g_4$ and $\{g_1, g_2, g_3, g_4\}$ is a $G/\mathscr{R}$-linearized regular system of graded parameters of $\mathrm{Sym}\,(V_i)^{\mathscr{R}}$. Let φ_1: $H/\rho_i(\mathscr{R}) \to GL(Cg_1 \oplus Cg_2)$ and φ_2: $H/\rho_i(\mathscr{R}) \to GL(Cg_3 \oplus Cg_4)$ be the canonical representations. Moreover, since $\varphi_j(H/\rho_i(\mathscr{R}))$ $(j = 1, 2)$ are metabelian groups, we may suppose that $\rho_i(G)/\rho_i(\mathscr{R})$ is monomial on the C-basis $\boldsymbol{g} = \{g_1, g_2, g_3, g_4\}$ and $\mathscr{R}(Cg_1 \oplus Cg_2; H/\rho_i(\mathscr{R}))$ (resp. $\mathscr{R}(Cg_3 \oplus Cg_4; H/\rho_i(\mathscr{R}))$) is represented as a diagonal group or $G(p, q, 2)$ on $\{g_1, g_2\}$ (resp. $\{g_3, g_4\}$).

Claim "If σ is an element of H such that g_3, g_4 are relative invariants of σ, then g_1 and g_2 are also relative invariants of σ". We may suppose that σ belongs to one of $\rho_i(G^i)_{[\{X_1, X_2\}]}$, $\rho_i(G^i)_{[\{X_3, X_4\}]}$, L_1 and L_2. If $\sigma \in L_1 \cup \rho_i(G^i)_{[\{X_1, X_2\}]}$, the assertion is evident. Suppose $\psi_1(H) = \mu_{2u} \cdot \boldsymbol{O}$. $\mathscr{R}(CX_1 \oplus CX_2; H)$ is equal to $\mu_4 \cdot \boldsymbol{O}$, $\mu_8 \cdot \boldsymbol{O}$, $\mu_{12} \cdot \boldsymbol{O}$ or $\mu_{24} \cdot \boldsymbol{O}$ in $GL_2(C)$ and hence $\mathscr{R}(Cg_1 \oplus Cg_2; H/\rho_i(\mathscr{R}))$ and $\mathscr{R}(Cg_3 \oplus Cg_4; H/\rho_i(\mathscr{R}))$ are regarded as one of the groups

$G(3,3,2)$, $G(6,6,2)$, $G(3,1,2)$, $G(6,2,2)$ in $GL_2(C)$. If $\sigma\in L_2$, by the definition of $\rho_i(\mathscr{R})$, $\operatorname{ord}(\varphi_1(\sigma\rho_i(\mathscr{R})))=\operatorname{ord}(\varphi_2(\sigma\rho_i(\mathscr{R})))$, which implies our assertion. So we assume $\sigma\in\rho_i(G^i)_{[\{X_3,X_4\}]}$. Then $\psi_1(\sigma)\in\psi_1(H)\cap SL_2(C)=O\subsetneqq\mathscr{R}(CX_1\oplus CX_2;H)$, and $\operatorname{ord}(\varphi_1(\sigma\rho_i(\mathscr{R})))=1$, 2 or 3. Since $\varphi_1(\sigma\rho_i(\mathscr{R}))$ is not a pseudo-reflection in $GL(Cg_1\oplus Cg_2)$ and belongs to $\mathscr{R}(Cg_1\oplus Cg_2;H/\rho_i(\mathscr{R}))$, $\varphi_1(\sigma\rho_i(\mathscr{R}))$ is diagonal. We now suppose $\psi_1(H)=(\mu_{4u}\,|\,\mu_{2u};\,\boldsymbol{O}\,|\,\boldsymbol{T})$. $\mathscr{R}(CX_1\oplus CX_2;H)$ is identified with $\mu_4\cdot\boldsymbol{D}_2$, $\mu_{12}\cdot\boldsymbol{T}$, $(\mu_8\,|\,\mu_4;\,\boldsymbol{O}\,|\,\boldsymbol{T})$ or $(\mu_{24}\,|\,\mu_{12};\,\boldsymbol{O}\,|\,\boldsymbol{T})$. $\mathscr{R}(Cg_1\oplus Cg_2;H/\rho_i(\mathscr{R}))$ and $\mathscr{R}(Cg_3\oplus Cg_4;H/\rho_i(\mathscr{R}))$ may be regarded as one of a diagonal group, $G(3,3,2)$ and $G(3,1,2)$. We can similarly show this claim.

By Claim, $\prod_g(\rho_i(G)/\rho_i(\mathscr{R}))=\langle(Cg_1,Cg_2)(Cg_3,Cg_4),(Cg_1,Cg_3)(Cg_2,Cg_4)\rangle$, which proves (ii) of (3). For any nonzero $x\in V_i$, $\operatorname{Sym}((V_i)_{(G^i)_x})^{G^i_x}$ is a C.I., and hence (iii) of (3) is satisfied ([29]). (iv) of (3) follows immediately from the assertion in Case A (we can replace G and G^i by $G_{[\{X_3,X_4\}]}$ and $G^i_{[\{X_3,X_4\}]}$ respectively and apply the assertion (3) in Case A). Thus the proof of (a) is completed.

Next we suppose that the condition (3) in (5.1) holds. The first part of the assertion (b) is evident. Let χ be a non-trivial linear character of $\rho_i(G)$ satisfying $\chi(\rho_i(G^i))=1$ and put $f_\chi^{(1)}=f_\chi(CX_1\oplus CX_2,\mathscr{R})$ and $f_\chi^{(2)}=f_\chi(CX_3\oplus CX_4,\mathscr{R})$. Then $f_\chi(V_i,\rho_i(G))=f_\chi(V_i,\rho_i(\mathscr{R}))=f_\chi^{(1)}f_\chi^{(2)}$ in S and, if $f_\chi^{(1)}$ is regarded as a polynomial $g(X_1,X_2)$ with the variables X_1, X_2, $f_\chi^{(2)}$ can be identified with $g(X_3,X_4)$. Let σ be any element in $(\rho_i(G^i)_{[\{X_1,X_2\}]}\cup\rho_i(G^i)_{[\{X_3,X_4\}]}\cup L_1\cup L_2)-\rho_i(\mathscr{R})$. It suffices to show $\sigma(f_\chi(V_i,\rho_i(G)))=f_\chi(V_i,\rho_i(G))$. If $\sigma\in L_1$, this assertion is trivial (note that $f_\chi(V_i,\rho_i(G))$ is a relative invariant of $\rho_i(G)$). On the other hand, if $\sigma\in\rho_i(G^i)_{[\{X_3,X_4\}]}$, by (iv) of (3) $f_\chi^{(1)}=f_{\det^u}(CX_1\oplus CX_2,\mathscr{R})$ for some $u\in N-\{0\}$, which shows $\sigma(f_\chi^{(1)})=f_\chi^{(1)}$ (cf. the proof in Case A, $n_i=2$). Finally, suppose $\sigma\in L_2$. $\langle\psi_1(\rho_i(\mathscr{R})),\psi_1(\sigma)\rangle$ and $\langle\psi_2(\rho_i(\mathscr{R})),\psi_2(\sigma)\rangle$ are reflection groups in $GL_2(C)$ which properly contain $\psi_1(\rho_i(\mathscr{R}))\cong\psi_2(\rho_i(\mathscr{R}))$. If $\langle\psi_1(\rho_i(\mathscr{R})),\psi_1(\sigma)\rangle$ is primitive and $\langle\psi_2(\rho_i(\mathscr{R})),\psi_2(\sigma)\rangle$ is imprimitive, as in the proof in Subcase 7 in Case A, we see $f_\chi^{(1)}=f_{\det^{-1}}(CX_1\oplus CX_2,\psi_1(\rho_i(\mathscr{R})))$, and hence $f_\chi^{(2)}=f_{\det^{-1}}(CX_3\oplus CX_4,\psi_2(\rho_i(\mathscr{R})))$. Since $f_{\det^{-1}}(V_i,\rho_i(G))$ is a $\det^{-1}$-invariant of $\rho_i(G)$, in this case, the assertion follows. So we assume that $\langle\psi_j(\rho_i(\mathscr{R})),\psi_j(\sigma)\rangle$, $\psi_j(\rho_i(\mathscr{R}))$ $(j=1,2)$ are simultaneously primitive or imprimitive in $GL_2(C)$.

Subcase 1 "$(\sigma-1)(CX_1\oplus CX_2)=(\sigma_1-1)V_i$ and $(\sigma-1)(CX_3\oplus CX_4)=(\sigma_2-1)V_i$ for some $\sigma_1,\sigma_2\in\mathscr{R}$". Suppose $\langle\psi_1(\rho_i(\mathscr{R})),\psi_1(\sigma)\rangle$ (resp. $\langle\psi_2(\rho_i(\mathscr{R})),\psi_2(\sigma)\rangle$) is monomial on the C-basis $\{X_1,X_2\}$ (resp. $\{X_3,X_4\}$) and especially $\psi_1(\rho_i(\mathscr{R}))$ (resp. $\psi_2(\rho_i(\mathscr{R}))$) is represented as $G(p,q,2)$ on $\{X_1,X_2\}$ (resp.

$\{X_3, X_4\}$). Because $g(X_1, X_2)$ is a relative invariant of $G(p, q, 2)$, there is a polynomial $g'(X_1, X_2) \in C[X_1, X_2]$ and an element $v \in N$ such that $g(X_1, X_2) = (X_1X_2)^v g'(X_1, X_2)$ and $g'(X_1, X_2)$ is not divisible by X_1 and by X_2 in $C[X_1, X_2]$. If $\psi_1(\sigma)$ is not diagonal, $\mathrm{ord}(\sigma) = 2$, and so $\psi_1(\sigma) = \psi_1(\rho_i(\sigma_1))$ and $\psi_2(\rho) = \psi_2(\rho_i(\sigma_2))$, which implies $\sigma = \rho_i(\sigma_1)\rho_i(\sigma_2) \in \rho_i(\mathscr{R})$. Therefore $\psi_j(\sigma)$ $(j = 1, 2)$ are diagonal. Since $\sigma(g'(X_1, X_2)) = g'(X_1, X_2)$ and $\sigma(g'(X_3, X_4)) = g'(X_3, X_4)$, $\sigma(f_\chi(V_i, \rho_i(G))) = \det(\psi_1(\sigma))^v f_\chi^{(1)} \det(\psi_2(\sigma))^v f_\chi^{(2)} = f_\chi(V_i, \rho_i(G))$. Suppose $\langle \psi_j(\rho_i(\mathscr{R})), \psi_j(\sigma)\rangle$ $(j = 1, 2)$ are primitive in $GL_2(C)$. Since $\psi_1(\sigma) \notin \psi_1(\rho_i(\mathscr{R}))$ (if $\psi_1(\sigma) \in \psi_1(\rho_i(\mathscr{R}))$, $\sigma \in \rho_i(\mathscr{R})$) and $(\psi_1(\sigma) - 1)(CX_1 \oplus CX_2) = (\psi_1(\rho_i(\sigma_1)) - 1)(CX_1 \oplus CX_2)$, by a classification in [4, (3.5)], we see that $\mathrm{ord}(\psi_1(\sigma))$ $(=\mathrm{ord}(\sigma)) = 4$, $\mathrm{ord}(\sigma_1) = \mathrm{ord}(\sigma_2) = 2$ and $\sigma^2 = \rho_i(\sigma_1\sigma_2)$. In any primitive 2-dimensional reflection group, the set of all pseudo-reflections of order 4 is empty or a conjugate class. Thus $\psi_1(\rho_i(\mathscr{R}))$ does not have a pseudo-reflection of order 4, and using [4, (3.5)] again, we can identify $\psi_1(\rho_i(\mathscr{R}))$ with $\mu_{12}\cdot T$. By the definition of $f_\chi^{(1)}$ (cf. [20, (4.3.3)]), $\sigma(f_\chi^{(1)})/f_\chi^{(1)} = (\sigma(L_{U'}(CX_1 \oplus CX_2, \rho_i(\mathscr{R})))/L_{U'}(CX_1 \oplus CX_2, \rho_i(\mathscr{R})))^{s_{U'}(CX_1\oplus CX_2, \rho_i(\mathscr{R}), \chi)} = 1$ if $\chi(\rho_i(\sigma_1)) = 1$; $=\sigma(f_{\det^{-1}}(CX_1 \oplus CX_2, \mathscr{R}))/f_{\det^{-1}}(CX_1 \oplus CX_2, \mathscr{R})$ otherwise, where U' is the reflecting hyperplane in $CX_1 \oplus CX_2$ associated to $\psi_1(\rho_i(\sigma_1))$ i.e. $\mathscr{I}_{U'}(\rho_i(\mathscr{R})) = \langle \rho_i(\sigma_1)\rangle$. Similarly $\sigma(f_\chi^{(2)})/f_\chi^{(2)} = 1$ if $\chi(\rho_i(\sigma_2)) = 1$; $= \sigma(f_{\det^{-1}}(CX_3 \oplus CX_4, \mathscr{R}))/f_{\det^{-1}}(CX_3 \oplus CX_4, \mathscr{R})$ otherwise, and therefore, observing $1 = \chi(\sigma^2) = \chi(\rho_i(\sigma_1))\chi(\rho_i(\sigma_2))$ and $\sigma(f_{\det^{-1}}(V_i, \rho_i(G))) = \det(\sigma)^{-1} f_{\det^{-1}}(V_i, \rho_i(G)) = f_{\det^{-1}}(V_i, \rho_i(G))$ (cf. [21]), we always have $\sigma(f_\chi(V_i, \rho_i(G)) = f_\chi(V_i, \rho_i(G))$.

Subcase 2 "$(\sigma - 1)(CX_1 \oplus CX_2) = (\sigma_1 - 1)V_i$ for some $\sigma_1 \in \mathscr{R}$ and $(\sigma - 1)(CX_3 \oplus CX_4) \neq (\tau - 1)V_i$ for every $\tau \in \mathscr{R}$". Since $\sigma(f_\chi^{(2)}) = f_\chi^{(2)}$ ([20, (4.3.3)]), we need only to show $\sigma(f_\chi^{(1)}) = f_\chi^{(1)}$. Suppose $\langle \psi_j(\rho_i(\mathscr{R})), \psi_j(\sigma)\rangle$ $(j = 1, 2)$ are primitive in $GL_2(C)$. Then, as in Subcase 1, we similarly have $\sigma(f_\chi^{(1)})/f_\chi^{(1)} = 1$ if $\chi(\rho_i(\sigma_1)) = 1$; $= \sigma(f_{\det^{-1}}(CX_1 \oplus CX_2, \mathscr{R}))/f_{\det^{-1}}(CX_1 \oplus CX_2, \mathscr{R})$ otherwise. Thus the assertion follows from the equality $f_{\det^{-1}}(V_i, \rho_i(G)) = \sigma(f_{\det^{-1}}(V_i, \rho_i(G))) = \sigma(f_{\det^{-1}}(CX_1 \oplus CX_2, \mathscr{R}))f_{\det^{-1}}(CX_3 \oplus CX_4, \mathscr{R})$. Next, suppose that $\langle \psi_1(\rho_i(\mathscr{R})), \psi_1(\sigma)\rangle$ (resp. $\langle \psi_2(\rho_i(\mathscr{R})), \psi_2(\sigma)\rangle$) is monomial on $\{X_1, X_2\}$ (resp. $\{X_3, X_4\}$ and especially $\psi_1(\rho_i(\mathscr{R}))$ (resp. $\psi_2(\rho_i(\mathscr{R}))$) is represented as $G(p, q, 2)$ on $\{X_1, X_2\}$ (resp. $\{X_3, X_4\}$), where $p, q \in N$ with $q|p$. Since $\psi_1(\sigma) \notin \psi_1(\rho_i(\mathscr{R}))$ and $(\psi_1(\sigma) - 1)(CX_1 \oplus CX_2) = (\psi_1(\rho_i(\sigma_1)) - 1)(CX_1 \oplus CX_2)$, $\psi_1(\sigma)$ is diagonal on $\{X_1, X_2\}$, and using our assumption in this case, we easily see that $\psi_2(\sigma)$ is not diagonal on $\{X_3, X_4\}$, which requires $\mathrm{ord}(\sigma) = 2$. Obviously it may be assumed that $\mathrm{diag}[-1, 1, a, a^{-1}]\cdot(3,4)[4]$ for some $a \in C^*$, and hence p/q is odd $(\geqq 3)$. Because $\rho_i([\mathscr{R}, \mathscr{R}]) \subseteqq \rho_i(G^i)$, $\psi_2(\rho_i(G^i)_{X_2})$

is irreducible and not conjugate to $\langle \operatorname{diag}[\zeta_u, \zeta_u^{-1}], (1, 2)[2]\rangle$ $(u \in N - \{0\})$ in $GL_2(C)$. Applying [29, Theorem 1] to $\rho_i(G^i)_{X_2}$ (cf. (iii) of (3)), we see $\operatorname{diag}[-1, 1, -1, 1] \in \rho_i(G^i)_{X_2}$, which implies p/q is even (cf. (iv) of (3)). This is a contradiction.

Subcase 3 "$(\sigma - 1)(CX_1 \oplus CX_2) \neq (\tau - 1)V_i$ and $(\sigma - 1)(CX_3 \oplus CX_4) \neq (\tau - 1)V_i$ for every $\tau \in \mathscr{R}$". Clearly $f_\chi^{(1)}$ and $f_\chi^{(2)}$ are invariants of τ. Thus the assertion follows.

Case C "$\rho_i(\mathscr{R})$ is reducible and non-trivial abelian". Let $X = \{X_1, \cdots, X_{n_i}\}$ be a C-basis of V_i on which $\rho_i(\mathscr{R})$ is diagonal and every matrix is defined. $\rho_i(G^i)$ is a transitively imprimitive group with the complete system $\{CX_1, \cdots, CX_{n_i}\}$ of imprimitivities and $\rho_i(\mathscr{R}) = \langle \operatorname{diag}[\zeta_c, 1, \cdots, 1], \cdots, \operatorname{diag}[1, \cdots, 1, \zeta_c]\rangle$ for some $c \in N$ with $c \geqq 2$. Hence $\{f_\chi(V_i, \rho_i(G)) \mid \chi \in \operatorname{Hom}(\rho_i(G), C^*), \chi(\rho_i(G^i)) = 1\} \subseteqq \{(X_1 \cdots X_{n_i})^v \mid 0 \leqq v < c\}$. The last assertion of (b) follows immediately from the condition (3) and so we assume S^G is a C.I. and G is a minimal counter-example for the assertion that $X_1 \cdots X_{n_i} \in S^{G^i}$ with respect to $|G|$. Then it may be seen that $i = 1$, $m = 2$ and $\dim V_2 = 1$ (in fact, for an element $\sigma \in \operatorname{Spe}(G)$ with $\operatorname{ord}(\rho_i(\sigma)) > 1$, $G_{[(\oplus_{j \neq i} V_j)^{\langle\sigma\rangle}]}$ is also a counter-example).

Claim "$|\{j \mid 1 \leqq j \leqq n_1, V^{\langle\sigma\rangle} \ni X_j\}| < n_1 - 2$ for any special element σ in G^1". We suppose that this Claim is false. Then one may suppose $\rho_1(\sigma) = \operatorname{diag}[-1, 1, \cdots, 1]\cdot(1, 2)[n_1] \in \rho_1(G^1)$ for some $\sigma \in G^1$, and by the minimality of G, $\dim V_1 = n_1 = 2$. Because $C[X_1, X_2]^G$ is a C.I., $f_{\det^{-1}}(V_1, G) = (X_1X_2)^{c-1}$ is an anti-invariant of G, which requires c is odd. This conflicts with [29].

Applying [14, (4.2)] to G^1, we have $n_1 = 3$ or 4. By [8, Table II] and our assumption, $\prod_X(\rho_1(G^1))$ is conjugate to neither A_{n_1} nor $\langle (CX_1, CX_2)(CX_3, CX_4), (CX_1, CX_3)(CX_2, CX_4)\rangle$. Suppose $\prod_X(\rho_1(G^1)) = \langle (CX_1, CX_2), (CX_3, CX_4), (CX_1, CX_3)(CX_2\ CX_4)\rangle$ $(n_1 = 4)$. Then $\rho_1(G^1) \ni \operatorname{diag}[1\ 1, -1, 1]\cdot(1, 2)[4]$ and since, on $CX_1 \oplus CX_2$, $G^1_{X_4}$ is not conjugate to $\langle \operatorname{diag}[\zeta_u, \zeta_u^{-1}], (1, 2)[2]\rangle$ in $GL_2(C)$ and $C[X_1, X_2, X_3]^{G^1_{X_4}}$ is a complete intersection, by [29], $\rho_1(G^1) \ni \operatorname{diag}[-1, 1, -1, 1]$. Hence $\rho_1(G^1) \ni \operatorname{diag}[1, -1, 1, 1]\cdot(1, 2)[4]$, which is a contradiction (cf. Claim). By Claim, [8, 29] and the minimality of G, $n_1 = 3$ and $\rho_1(G^1)$ may be identified with $\langle \operatorname{diag}[\zeta_a, \zeta_a^{-1}, 1], \operatorname{diag}[1, 1, -1]\cdot(1, 2)[3], \operatorname{diag}[-1, 1, 1]\cdot(2, 3)[3]\rangle$ where a is an odd natural number and $c \mid a$ in N. Moreover $G = \langle G^1, \gamma\rangle$ for an element $\gamma \in \mathscr{R}$. Clearly $\operatorname{Sym}(V_1)^{G^1} = C[X_1^{2a} + X_2^{2a} + X_3^{2a}, X_1^aX_2^a + X_2^aX_3^a - X_3^aX_1^a, (X_1X_2X_3)^2, X_1X_2X_3(X_1^a - X_2^a + X_3^a), (X_1^a + X_2^a)(X_2^a + X_3^a)(X_3^a - X_1^a)]$ (cf. [29]) and because $\gamma((X_1X_2X_3)^2) = \zeta_c^2(X_1X_2X_3)^2$, $\gamma(X_1X_2X_3(X_1^a$

$- X_2^a + X_3^a)) = \zeta_c X_1 X_2 X_3 (X_1^a - X_2^a + X_3^a)$ and $\gamma(Z) = \zeta_c^{-1} Z$, it follows easily from $(c, 2) = 1$ that S^G is not a C.I., where Z is a nonzero element of V_2, which is a contradiction.

We always conclude $X_1 \cdots X_{n_i} \in S^{G^i}$ if (3) holds or if S^G is a C.I., and so assume $X_1 \cdots X_{n_i} \in S^{G^i}$. Then, if G^i is generated by special elements, $\prod_X (\rho_i(G^i))$ is generated by double transpositions and 3-cycles and does not contain a transposition, i.e. especially if $n_i \leqq 4$, $\prod_X (\rho_i(G^i)) = A_4$ $(n_i = 4)$, $Z/2Z \rtimes S_2$ $(n_i = 4)$ or A_3 $(n_i = 3)$ ([8]). On the other hand, if $\rho_i(G^i)$ is conjugate to the groups "5" or "6" in [8, Table II], we can easily show $\mathrm{emb}(\mathrm{Sym}(V_i)^{G^i}) \geqq 8$, a contradiction. Furthermore, if $\rho_i(G^i)$ is conjugate to $\langle G(p, p, 4) \cap SL(V_i),\ \mathrm{diag}[\zeta_{2^b}, \zeta_{2^b}, \zeta_{2^b}^{-1}, \zeta_{2^b}^{-1}] \rangle$ $(2^{b-1} \| p,\ b \geqq 1)$, $\mathrm{Sym}(V_i)^{G^i}$ is a Gorenstein ring with $\mathrm{emb}(\mathrm{Sym}(V_i)^G) = 6$, and is a C.I.. By our assumption on G^i, $\mathrm{Sym}(V_i)^G$ is a C.I. if and only if $\mathrm{Sym}(V_i)^{G^i}$ is a C.I., since the closed fibre of the flat morphism $(\mathrm{Sym}(V_i)^G)_{(\mathrm{Sym}(V_i)V_i)^G} \to (\mathrm{Sym}(V_i)^{G^i})_{(\mathrm{Sym}(V_i)V_i)^{G^i}}$ is a hypersurface. Therefore the rest of the assertion follows from the above observations, [14, (4.2)] and [29, Theorem 2].

Case D "$\rho_i(\mathscr{R}) = 1$". Claerly $m = 1$. When G is imprimitive, see [14, (4.2)]. When G is primitive, as in the proof of [14, (4.6)], this follows from (4.1).

Thus the proof of (5.1) is completed.

Notes added in proof. There are errors in the author's classification of irreducible groups of dimension $\leqq 10$ and its proof published in LNM 1092 (Springer) and manuscripta math. 48, 163–187 (1984). A revised classification shall be given in a part of a forthcoming paper. Case A of the classification of reducible groups in those notes must be replaced by Case A in (5.1) of this paper. [32] must be added to their references. In [33] the author generalized the result in [26].

References

[1] L. L. Avramov, Flat morphisms of complete intersections, Soviet Math. Dokl., **16** (1975), 1413–1417.

[2] Bourbaki, Groupes et Algèbres de Lie, Chaps. 4, 5 et 6., Hermann, Paris, 1968.

[3] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math., **99** (1977), 447–485.

[4] A. M. Cohen, Finite complex reflection groups, Ann. Sci. Ecole. Norm. Sup., **9** (1976), 379–436.

[5] C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Wiley, New York, 1962.

[6] E. Gottschling, Die Uniformisierbarkeit der Fixpunkte eigentlich diskontinuierlicher Gruppen von biholomorphen Abbildungen, Math. Ann., **169** (1967), 26–54.
[7] A. Grothendieck, Revêtements Etales et Groupes Fondamental (SGA 1), Lect. Notes in Math. No. **224**, Springer, Berlin, 1971.
[8] W. C. Huffman, Imprimitive linear groups generated by elements containing an eigenspace of codimension two, J. Algebra, **63** (1980), 499–513.
[9] V. Kac and K. Watanabe, Finite linear groups whose ring of invariants is a complete intersection, Bull. Amer. Math. Soc., **6** (1982), 221–223.
[10] J. Kempf, The Hochster-Roberts theorem of invariant theory, Michigan Math. J., **26** (1979), 19–32.
[11] J. Lipman and B. Teissier, Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals, Michigan Math. J., **28** (1981), 97–116.
[12] H. Nakajima, Relative invariants of finite groups, J. Algebra, **79** (1982) 218–234.
[13] ——, Rings of finite groups which are hypersurfaces, J. Algebra, **80** (1983), 279–294.
[14] ——, Rings of invariants of finite groups which are hypersurfaces, II, Advances in Math., to appear.
[15] V. L. Popov, The constructive theory of invariants, Math. USSR Izv., **19** (1982), 359–376.
[16] D. Prill, Local classification of quotients of complex manifolds by discontinuous groups, Duke Math. J., **34** (1967), 375–386.
[17] D. Rotillon, Deux contre-exemples à une conjecture de R. Stanley sur les anneaux d'invariants intersections complètes, C. R. Acad. Sci. Paris, **292** (1981), 345–348.
[18] J.-P. Serre, Sur les modules projectifs, Sem. Dubreil-Pisot, 1960/1961.
[19] T. A. Springer, Regular elements of finite reflection groups, Invent. Math., **25** (1974), 159–198.
[20] T. A. Springer, Invariant Theory, Lect. Notes in Math. No. **585**, Springer, Berlin, 1977.
[21] R. Stanley, Relative invariants of finite groups generated by pseudo-reflections, J. Algebra, **49** (1977), 134–148.
[22] ——, Hilbert functions of graded algebras, Advances in Math., **28** (1978), 57–83.
[23] ——, Invariants of finite groups and their applications to combinatorics, Bull. Amer. Math. Soc., **1** (1979), 475–511.
[24] G. C. Shephard and J. A. Todd, Finite unitary reflection groups, Canad. J. Math., **6** (1954), 274–304.
[25] K. Watanabe, Certain invariant subrings are Gorenstein, II, Osaka J. Math., **11** (1974), 379–388.
[26] ——, Invariant subrings which are complete intersections, I (Invariant subrings of finite Abelian groups), Nagoya Math. J., **77** (1980), 89–98.
[27] ——, On conditions for invariant subrings to be complete intersections (in Japanese), Kôkyûroku No. **444**, Research Inst. for Mathematical Sciences of Kyoto Univ., 1981.
[28] ——, Invariant subrings of finite groups which are complete intersections, Commutative Algebra: Analytic Methods, Marcel Dekker, New York, 1982.
[29] K. Watanabe and D. Rotillon, Invariant subrings $C[X, Y Z]$ which are complete intersections, Manuscripta Math., **39** (1982), 339–357.
[30] O. Riemenschneider, Deformationen von Quotienten-singularitäten (nach zyklischen Gruppen), Math. Ann., **209** (1974), 211–248.
[31] S. Goto and K. Watanabe, The embedding dimension and multiplicities of rational singularities which are complete intersections, to appear.
[32] N. L. Gordeev, On the Stanley conjecture and the classification of finite groups

whose algebras of invariants is a complete intersection, Soviet Math. Dokl., **26** (1982), 722–724.

[33] H. Nakajima, Affine torus embeddings which are complete intersections, to appear.

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