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ON THE RANK OF CM-TYPE

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In the present note, we prove that every simple CM-type is non-degenerate (i.e. the rank is maximal) if the dimension of corresponding abelian varieties is a prime. This follows directly from the argument of Tankeev [5], in which he has treated the 5-dimensional case.

Recently, S. G. Tankeev and K. A. Ribet have established similar results for more general types of abelian varieties (see [3], [4], [6]).

1. Let K be a CM-field (i.e. a totally imaginary quadratic extension of a totally real number field). We regard K as a subfield of $\boldsymbol{C}$. Let L be the Galois closure of K over $\boldsymbol{Q}$, and put

$$G = \mathrm{Gal}(L/\boldsymbol{Q}), \quad H = \mathrm{Gal}(L/K), \quad d = [K:\boldsymbol{Q}]/2\,.$$

We can canonically identify the embeddings of K into $\boldsymbol{C}$ with the cosets $H\backslash G$ (G acts on K on the right). We denote the complex conjugation by ρ, which belongs to the center of G.

Let S be a subset of $H\backslash G$ such that

$$H\backslash G = S \cup S\rho \quad \text{(disjoint union)}.$$

The pair (K, S) or the triple (G, H, S) is called a *CM-type*. Put

$$\tilde{S} = \{g \in G \mid Hg \in S\}\,.$$

We say that a CM-type (K, S) is *simple* if

$$H = \{g \in G \mid g\tilde{S} = \tilde{S}\}\,.$$

Let

$$H' = \{g \in G \mid \tilde{S}g = \tilde{S}\}\,,$$

and K' be the corresponding CM-field. Let S' be the subset of $H'\backslash G$ which is induced by the inverses of the elements of $\tilde{S}$. Then the pair (K', S') is also a CM-type, which is called the *dual* of (K, S).

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From now on, we consider simple CM-types only.

2. Now we define the rank of a CM-type (K, S).

Let X be the free abelian group generated by the cosets of $H\backslash G$, so every element in X is expressed as a formal sum: $\sum n_\sigma \sigma$ with $\sigma \in H\backslash G$, $n_\sigma \in Z$. Similarly X' denotes the corresponding object for $H'\backslash G$. Define a homomorphism $\phi: X \to X'$ by

$$\sigma \longmapsto \sum_{\tau \in S'} \tau\sigma$$

where the product $\tau\sigma$ is taken between two elements of G which are sent to τ and σ respectively. (Note that ϕ is well defined by the definition of H, H' and S'.)

Then we define the rank of (K, S) or the rank of (G, H, S) by the rank of the image of ϕ (cf. Kubota [1], Ribet [2]). The next proposition summarizes some properties of the rank of CM-type:

PROPOSITION A. *Assume that* (K, S) *is a simple CM-type, then*

(a) $\operatorname{rank}(K, S) = \operatorname{rank}(K', S')$

(b) *For any* $g \in G$, (K, Sg) *is also a CM-type and* $\operatorname{rank}(K, Sg) = \operatorname{rank}(K, S)$

(c) $\max(2 + \log_2 d,\ 2 + \log_2 d') \leqq \operatorname{rank}(K, S) \leqq \min(d + 1,\ d' + 1)$
(*where* $d' = [K' : Q]/2$)

(a), (b) and the second inequality in (c) are immediate from the definition of rank (see [1], [2]). The first inequality in (c) is due to Ribet [2].

If $\operatorname{rank}(K, S) = d + 1$, then we say that the CM-type (K, S) is *non-degenerate*. In particular, if $d \leqq 3$, then every simple CM-type is non-degenerate because of (c) of the proposition.

3. From now on, we assume $d =$ prime. Let $S(d)$ be the symmetric group of degree d. A homomorphism $\mu: S(d) \to \operatorname{Aut}((Z/2Z)^d)$ is defined by the natural permutation. By this action we can make the semidirect product:

$$W_d = (Z/2Z)^d \times_\mu S(d).$$

Then by Tankeev [5], we can regard G as a subgroup of W_d and ρ as the element $(1, 1, \cdots, 1) \in (Z/2Z)^d \subset W_d$. G acts on $H\backslash G$ transitively, so G contains an element of order d, say g_0. Then the cyclic subgroup G_0

generated by ρg_0 forms a complete set of representatives of $H\backslash G$. We identify these elements with the corresponding cosets.

The following proposition is essential:

PROPOSITION B (Tankeev [5]). *(G, H, S), G_0 and g_0 being as above, assume*

$$S \neq \langle g_0 \rangle, \quad \langle \rho g_0 \rangle.$$

Then

$$\operatorname{rank}(G, H, S) \geqq \operatorname{rank}(G_0, \{1\}, S).$$

(Note that S can be naturally regarded as a CM-type for G_0.)

4. In this section we shall prove the next theorem:

THEOREM. *Every simple CM-type with $d = $ prime is non-degenerate.*

As we have remarked at the end of n°2, we may assume that d is an odd prime.

First we prove a lemma which has been announced by F. Hazama.

LEMMA. *Let G_0 be a cyclic group of order $2d$ ($d = $ odd prime). Then every simple CM-type $(G_0, \{1\}, S)$ is non-degenerate.*

We assume that $(G_0, \{1\}, S)$ is degenerate (i.e. rank $< d+1$). Then by Kubota [1], there exists an odd character χ of G_0 such that

$$(*) \qquad \sum_{\sigma \in S} \chi(\sigma) = 0$$

(where "odd" means $\chi(\rho) = -1$ for the element ρ of order 2).

The order of χ must be $2d$ from $(*)$. Let γ be a generator of G_0, then $\zeta = \chi(\gamma)$ is a primitive $2d$-th root of unity. Hence $(*)$ must be of the form:

$$\pm (\zeta^{d-1} - \zeta^{d-2} + \cdots + 1) = 0$$

This implies that the CM-type is not simple, and the lemma follows.

Viewing this lemma, Proposition B implies our theorem when $S \neq \langle g_0 \rangle$, $\langle g_0 \rangle \rho$. So we assume $S = \langle g_0 \rangle$. (Note that S and $S\rho$ have the same rank.)

G induces permutations among the CM-types on K, and H' is the stabilizer of S (see n°1), hence $2d'$ is the cardinality of the orbit containing S. If $d' = 1$, then

$$\text{rank}\,(K, S) = \text{rank}\,(K', S') = d' + 1 = 2\,.$$

This contradicts (c) of Proposition A. If $d' > 1$, there exists $g \in G$ such that $Sg \neq S$, $S\rho$. But (K, Sg) is non-degenerate because of Proposition B. By (b) of Proposition A, $\text{rank}\,(K, S) = \text{rank}\,(K, Sg)$. This implies our theorem.

Remark. The rank of CM-type is nothing but the dimension of "Mumford-Tate group" of corresponding abelian varieties. It is known that if this dimension is "maximal" (i.e. the CM-type is non-degenerate in our sense) then the Hodge conjecture is true for these abelian varieties. For details, see [4], [5].

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