T. Murai
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BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS OF CALDERON TYPE, III

TAKAFUMI MURAI

§1. Introduction

In this paper we investigate the boundedness of Cauchy kernels. The Cauchy kernel associated with a locally integrable real-valued function $\theta(x)$ is defined by

$$(1)\qquad \mathfrak{C}[\theta](x, y) = (1 + i\theta(y))/\{(x - y) + i(\Theta(x) - \Theta(y))\}\,,$$

where $\Theta(x) = \int_0^x \theta(z)\,dz$. This kernel plays an important role in harmonic analysis on the graph $\{(x, \Theta(x));\ x \in (-\infty, \infty)\}$. For $p > 1$ and a non-negative function $\omega(x)$, let L^p_ω denote the space of functions $f(x)$ with $\|f\|_{p\omega} = \left\{\int_{-\infty}^{\infty} |f(x)|^p \omega(x)\,dx\right\}^{1/p} < \infty$. In the case $\omega(x) \equiv 1$, we write simply L^p and $\|\cdot\|_p$. We say that $\mathfrak{C}[\theta]$ is of type (p, ω) if, for any $f \in L^p_\omega$,

$$(2)\qquad \mathfrak{C}[\theta]f(x) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|x-y|<1/\varepsilon} \mathfrak{C}[\theta](x, y)f(y)\,dy$$

exists almost everywhere (a.e.) and $\|\mathfrak{C}[\theta]\|_{p\omega} = \sup\{\|\mathfrak{C}[\theta]f\|_{p\omega}/\|f\|_{p\omega};\ 0 < \|f\|_{p\omega} < \infty\} < \infty$. We also write $\|\mathfrak{C}[\theta]\|_p$ in the case $\omega(x) \equiv 1$. We say that $\omega(x)$ satisfies the Muckenhoupt (A_p) condition if

$$(\mathrm{A}_p)\qquad \sup_I\, (m_I\omega)(m_I\omega^{-1/(p-1)})^{p-1} < \infty\,,$$

where "$\sup_I$" denotes the supremum over all finite intervals I and $m_I\omega = (1/|I|)\int_I \omega(x)\,dx$ ($|I|$: the measure of I). It is well-known that Calderón-Zygmund kernels are of type (p, ω) if $\omega(x)$ satisfies (A_p) ([2]). We shall show that the analogous property is valid for some Cauchy kernels. We say that a locally integrable function $f(x)$ is of bounded mean oscillation if $\|f\|_{\mathrm{BMO}} = \sup_I m_I|f - m_I f| < \infty$. The space BMO of functions of bounded mean oscillation, modulo constants, is a Banach space with norm $\|\cdot\|_{\mathrm{BMO}}$. We show

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THEOREM 1. *If $\theta \in BMO$, then $\mathfrak{C}[\theta]$ is of type (p, ω) for any $\omega(x)$ with* (A_p).

It is necessary to study whether "$\theta \in$ BMO" is a sharp condition for which $\mathfrak{C}[\theta]$ is of type (p, ω). In this paper we work only with $p=2$ and $\omega(x) \equiv 1$. Let us define the distance $d(\theta)$ between $\theta(x)$ and 0 by the supremum of $\|\mathfrak{C}[\theta+u]-\mathfrak{C}[0]\|_2$ over all real numbers u. As a response to this subject, we show

THEOREM 2. $\lim_{t\to 0} d(t\theta)=0$ *if and only if* $\theta \in BMO$.

§2. Notation and Lemmas

We use C for absolute constants. The value of C differs in general from one occasion to another. Let L^∞ denote the Banach space of functions $f(x)$ with norm $\|f\|_\infty = \text{ess.}\sup_x |f(x)| < \infty$ and L^1_{loc} the totality of locally integrable functions. The maximal function of $f \in L^1_{\text{loc}}$ is defined by $f^*(x) = \sup_{x\in I} m_I|f|$, where "$\sup_{x\in I}$" is the supremum over all finite intervals I containing x. For a measurable set E in $(-\infty, \infty)$, $\chi_E(x)$ denotes the characteristic function of E. Given $0<\varepsilon<\eta$ and a real number y, we put $\chi^{(y)}_{\varepsilon,\eta}(x) = \chi^{(y)}_{\varepsilon}(x) - \chi^{(y)}_{\eta}(x)$, where $\chi^{(y)}_{s}(x) = \chi_{[y-s,y+s]^c}(x)$ $(s = \varepsilon, \eta)$. We write $\mathfrak{D}[\theta](x, y) = \mathfrak{C}[\theta](x, y) - \mathfrak{C}[0](x, y)$. For $f \in L^1_{\text{loc}}$, we put $\mathfrak{D}^*[\theta]f(x) = \sup\{|\mathfrak{D}[\theta](\chi^{(x)}_{\varepsilon,\eta}f)(x)|;\ 0<\varepsilon<\eta\}$. The norm $\|\mathfrak{D}^*[\theta]\|_{p\omega}$ is defined analogously as $\|\mathfrak{C}[\theta]\|_{p\omega}$. Here are some lemmas necessary for the proofs of our theorems.

LEMMA 3 (The Calderón-Zygmund decomposition: Stein [8, p. 17]). *Let $f \in L^1$ and $\lambda > 0$. Then there exists a sequence $\{J_k\}_{k=1}^\infty$ of mutually disjoint finite intervals such that, with $J = \bigcup_{k=1}^\infty J_k$,*

(3) $\quad |J| \le C\|f\|_1/\lambda, \quad m_{J_k}|f| \le 2\lambda \ (k \ge 1), \quad |f(x)| \le \lambda$ a.e. *in* J^c.

LEMMA 4 (John-Nirenberg [5]). *Let $r \ge 1$, $f \in BMO$ and I be a finite interval. Then* $m_I|f - m_If|^r \le C^r\Gamma(r+1)\|f\|^r_{\text{BMO}}$.

LEMMA 5 (Coifman-Fefferman [2]). *If $\omega(x)$ satisfies (A_p), then there exist $1<q<p$ and a constant B_1 such that, for any $f \in L^q_\omega$,* $\|f^*\|_{q\omega} \le B_1\|f\|_{q\omega}$.

LEMMA 6 (Coifman-Fefferman [2]). *If $\omega(x)$ satisfies (A_p), then there exist two constants $0<\gamma\le 1$ and $B_2 \ge 1$ such that, for any finite interval I and a set E in I, $\omega(E)/\omega(I) \le B_2(|E|/|I|)^\gamma$, where $\omega(F) = \int_F \omega(x)\,dx$ $(F \subset$* $(-\infty, \infty))$.

LEMMA 7 (Murai [6]). *For a real-valued function $\theta(x)$ in L^∞, we put*

$$\mathfrak{K}[\theta](x, y) = \frac{1}{x-y} \exp\left\{-i\frac{\Theta(x)-\Theta(y)}{x-y}\right\}. \tag{4}$$

Then there exists an absolute constant $N \geq 2$ such that, for any $r > 1$, $\|\mathfrak{K}^*[\theta]\|_r \leq \{Cr/(r-1)\}\rho(\|\theta\|_\infty)$, where $\rho(t) = (1+t)^N$ $(t \geq 0)$.

LEMMA 8. *For a real-valued function $\theta(x)$ in L^∞ and a real number u, we put*

$$\begin{aligned}&\mathfrak{T}_n[\theta, u](x, y)\\ &\quad = \left\{\frac{\Theta(x)-\Theta(y)}{x-y}\right\}^n \Big/ \{(x-y) + i(\Theta(x)-\Theta(y)+u(x-y))\}\\ &\qquad\qquad (n = 0, 1).\end{aligned} \tag{5}$$

Then, for any $r > 1$, $\|\mathfrak{T}_n^*[\theta, u]\|_r \leq \{Cr/(r-1)\}\|\theta\|_\infty^n \rho(\|\theta\|_\infty)$ $(n = 0, 1)$.

Proof. We choose an infinitely differentiable function $\psi_n(x)$ so that $\psi_n(x) = x^n$ $(|x| \leq 1)$ and $\int_{-\infty}^\infty |\hat{\psi}_n(t)|\rho(t)\,dt < \infty$, where $\hat{\psi}_n(t) = \int_{-\infty}^\infty e^{-itx}\psi_n(x)\,dx$. We have

$$1/\{(x-y) + i(\Theta(x)-\Theta(y)+u(x-y))\} = \int_0^\infty \mathfrak{K}[s\theta](x, y)e^{-s(1+iu)}\,ds$$

and

$$\psi_n\left\{\frac{\Theta(x)-\Theta(y)}{(x-y)\|\theta\|_\infty}\right\} = C\int_{-\infty}^\infty \hat{\psi}_n(t)\exp\left\{it\frac{\Theta(x)-\Theta(y)}{(x-y)\|\theta\|_\infty}\right\}dt.$$

Hence

$$\begin{aligned}&\mathfrak{T}_n[\theta, u](x, y)\\ &\quad = \|\theta\|_\infty^n \psi_n\left\{\frac{\Theta(x)-\Theta(y)}{(x-y)\|\theta\|_\infty}\right\} \Big/ \{(x-y) + i(\Theta(x)-\Theta(y)+u(x-y))\}\\ &\quad = C\|\theta\|_\infty^n \int_{-\infty}^\infty \hat{\psi}_n(t)\,dt \int_0^\infty \mathfrak{K}[s\theta - t\theta/\|\theta\|_\infty](x, y)e^{-s(1+iu)}\,ds.\end{aligned}$$

Using Lemma 7, $\rho(s+t) \leq \rho(s)\rho(t)$ and $\rho(st) \leq \rho(s)\rho(t)$, we have

$$\begin{aligned}\|\mathfrak{T}_n^*[\theta, u]\|_r &\leq C\|\theta\|_\infty^n \int_{-\infty}^\infty |\hat{\psi}_n(t)|\,dt \int_0^\infty \|\mathfrak{K}^*[s\theta - t\theta/\|\theta\|_\infty]\|_r e^{-s}\,ds\\ &\leq \{Cr/(r-1)\}\|\theta\|_\infty^n \int_{-\infty}^\infty |\hat{\psi}_n(t)|\,dt \int_0^\infty \rho(s\|\theta\|_\infty + |t|)e^{-s}\,ds\\ &\leq \left\{[Cr/(r-1)]\int_{-\infty}^\infty |\hat{\psi}_n(t)|\rho(t)\,dt \int_0^\infty \rho(s)e^{-s}\,ds\right\}\|\theta\|_\infty^n \rho(\|\theta\|_\infty).\end{aligned} \tag{6}$$

§ 3. Proof of Theorem 1

Let $\theta \in \mathrm{BMO}$ and let $\omega(x)$ satisfy (A_p). Since $\mathfrak{C}[0]$ is of type (p, ω) ([2]), it is sufficient to show that $\mathfrak{D}[\theta]$ is of type (p, ω). To do this we show that $\mathfrak{D}^*[\theta]$ is of type (p, ω); once this is known, a standard argument easily shows that (2) exists a.e. for any $f \in L^p_\omega$, and hence the (p, ω)-ness of $\mathfrak{D}[\theta]$ follows. For $s \geq 1$, we define

$$(7) \qquad \Lambda_s(f)(x) = (f^s)^*(x)^{1/s}, \qquad \Gamma_s(f)(x) = \sup_{x \in I} \{m_I(|\theta - m_I\theta||f|)^s\}^{1/s} .$$

Note that $f^*(x) \leq \Lambda_s(f)(x)$ and $\Gamma_1(f)(x) \leq \Gamma_s(f)(x)$ $(s \geq 1)$. We choose r so that $1 < r < p/q$, where q is the number associated with p in Lemma 5.

Given $f \in L^p_\omega$ with compact support, we now prove the following good λ inequality:

$$(8) \qquad \omega(x; \mathfrak{D}^*[\theta]f(x) > 2\lambda,\ \Xi(x) \leq \delta\lambda) \leq 4^{-p}\omega(x; \mathfrak{D}^*[\theta]f(x) > \lambda) \quad (\lambda > 0)\,,$$

where $\Xi(x) = \rho(\|\theta\|_{\mathrm{BMO}})\{\Gamma_r(f)(x) + \|\theta\|_{\mathrm{BMO}}\Lambda_r(f)(x)\}$ and a constant δ is determined later. Given $\lambda > 0$, we put

$$(9) \qquad U(\lambda) = \{x; \mathfrak{D}^*[\theta]f(x) > \lambda\}\,, \qquad \sigma(\lambda) = \omega(U(\lambda))\,.$$

Then we can write $U(\lambda) = \bigcup_{k=1}^\infty I_k$ with a sequence $\mathfrak{M}(\lambda) = \{I_k\}_{k=1}^\infty$ of mutually disjoint finite open intervals. To prove (8), it is sufficient to show that, for any $I \in \mathfrak{M}(\lambda)$,

$$(10) \qquad |x \in I;\ \mathfrak{D}^*[\theta]f(x) > 2\lambda,\ \Xi(x) \leq \delta\lambda| \leq B_3|I|\,,$$

where $B_3 = (4^pB_2)^{-1/\gamma}$ (γ, B_2: the constants in Lemma 6); once this is known, Lemma 6 gives $\omega(x \in I;\ \mathfrak{D}^*[\theta]f(x) > 2\lambda, \Xi(x) \leq \delta\gamma) \leq 4^{-p}\omega(I)$ $(I \in \mathfrak{M}(\lambda))$, and hence adding in $I \in \mathfrak{M}(\lambda)$, we obtain (8).

Let $I = (a, b) \in \mathfrak{M}(\lambda)$. If $\Xi(x) > \delta\lambda$ in I, nothing is to be proved. Assuming that $\Xi(\xi) \leq \delta\lambda$ for some $\xi \in I$, we prove

$$(11) \qquad |x \in I;\ \mathfrak{D}^*[\theta]f(x) > 2\lambda| \leq B_3|I|\,.$$

We have, with $\chi(x) = \chi_{\tilde{I}}(x)$ $(\tilde{I} = (a - 3|I|, a + 3|I|))$,

$$(12) \qquad \begin{aligned} &|x \in I;\ \mathfrak{D}^*[\theta]f(x) > 2\lambda| \\ &\leq |x \in I;\ \mathfrak{D}^*[\theta](\chi f)(x) > \lambda/2| + |x \in I;\ \mathfrak{D}^*[\theta](\chi^c f)(x) > 3\lambda/2| \\ &\quad (= P_1 + P_2, \text{ say}). \end{aligned}$$

First we estimate P_1. Let $I^* = (a - 4|I|, a + 4|I|)$. We have, with

$g(x)=(\theta(x)-m_{I*}\theta)\chi_{I*}(x)$ and $G(x)=\int_0^x g(z)\,dz$.

$$\mathfrak{D}[\theta](x,y)$$
$$(13)\qquad =i\left\{g(y)-\frac{G(x)-G(y)}{x-y}\right\}\Big/\{(x-y)+i(G(x)-G(y)+m_{I*}\theta(x-y))\}$$
$$(x\in I,\ y\in\tilde{I})\ .$$

Since $\|g\|_1\le C\|\theta\|_{\mathrm{BMO}}|I|$, the Calderón-Zygmund decomposition shows that there exists a sequence $\{J_k\}_{k=1}^{\infty}$ of mutually disjoint finite intervals such that, with $J=\bigcup_{k=1}^{\infty}J_k$,

$$(14)\qquad \begin{aligned}&|J|\le(B_3/4)|I|,\quad m_{J_k}|g|\le C\|\theta\|_{\mathrm{BMO}}\quad(k\ge 1)\\ &|g(x)|\le C\|\theta\|_{\mathrm{BMO}}\quad \text{a.e. in } J^c\ .\end{aligned}$$

We put $h(x)=g(x)$ $(x\in I_*-(\bigcup_{k\in\Lambda}J_k))$, $h(x)=m_{J_k}g$ $(x\in J_k,\ k\in\Lambda)$ and $h(x)=0$ $(x\in I_*^c)$, where $\Lambda=\{k;\ J_k\subset I^*\}$ and I_* is the smallest interval containing $\bigcup_{k\in\Lambda}J_k$. We may assume $I_*\supset\tilde{I}$, adding small intervals if necessary. Put $H(x)=\int_d^x h(z)\,dz$ (d: a point in $J-I$). Then we have, for any $x\in I$, $y\in\tilde{I}$,

$$(15)\qquad \begin{aligned}\mathfrak{D}[\theta](x,y)&=i\mathfrak{T}_0[h,m_{I*}\theta](x,y)g(y)-i\mathfrak{T}_1[h,m_{I*}\theta](x,y)\\ &\quad+i\{\mathfrak{T}_0[g,m_{I*}\theta](x,y)-\mathfrak{T}_0[h,m_{I*}\theta](x,y)\}g(y)\\ &\quad-i\left\{\frac{G(x)-G(y)}{x-y}-\frac{H(x)-H(y)}{x-y}\right\}\mathfrak{T}_0[g,m_{I*}\theta](x,y)\\ &\quad-i\left\{\frac{H(x)-H(y)}{x-y}\right\}\{\mathfrak{T}_0[g,m_{I*}\theta](x,y)-\mathfrak{T}_0[h,m_{I*}\theta](x,y)\}\\ &\quad(=\mathfrak{D}_1(x,y)+\mathfrak{D}_2(x,y)+\cdots+\mathfrak{D}_5(x,y),\ \text{say})\ .\end{aligned}$$

Since $\|h\|_\infty\le C\|\theta\|_{\mathrm{BMO}}$, Lemma 8 shows that

$$(16)\qquad \begin{aligned}&|x\in I;\ \mathfrak{D}_1^*(\chi f)(x)>\lambda/10|\le|x;\ \mathfrak{T}_0^*[h,m_{I*}\theta](g\chi f)(x)>\lambda/10|\\ &\le\{C\|\mathfrak{T}_0^*[h,m_{I*}\theta]\|_r\|g\chi f\|_r/\lambda\}^r\\ &\le\{[Cr/(r-1)]\rho(\|\theta\|_{\mathrm{BMO}})\Gamma_r(f)(\xi)/\lambda\}^r|I|\le\{C\delta r/(r-1)\}^r|I|\ .\end{aligned}$$

We have analogously

$$(17)\qquad \begin{aligned}&|x\in I;\ \mathfrak{D}_2^*(\chi f)(x)>\lambda/10|\le\{C\|\mathfrak{T}_1^*[h,m_{I*}\theta]\|_r\|\chi f\|_r/\lambda\}^r\\ &\le\{[Cr/(r-1)]\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\Lambda_r(f)(\xi)/\lambda\}^r|I|\le\{C\delta r/(r-1)\}^r|I|\ .\end{aligned}$$

Let $J^*=\bigcup_{k\in\Lambda}J_k^*$, where J_k^* is the open interval with the same midpoint as J_k and length $2|J_k|$. Then $|J^*|\le(B_3/2)|I|$. We have, for any $x\in I-J^*$,

$$\begin{aligned}
\mathfrak{D}_3^*(\chi f)(x) &\le \int_{-\infty}^{\infty} |\mathfrak{T}_0[g, m_{I*}\theta](x,y) - \mathfrak{T}_0[h, m_{I*}\theta](x,y)||(g\chi f)(y)|\,dy \\
&\le \int_{-\infty}^{\infty} \left\{\left|\int_y^x (g(z) - h(z))\,dz\right| \Big/ (x-y)^2\right\} |(g\chi f)(y)|\,dy \\
&\le \sum_{k\in\Lambda} \int_{J_k} \left\{\left(\int_{J_k} |g(z) - h(z)|\,dz\right) \Big/ (x-y)^2\right\} |(g\chi f)(y)|\,dy \\
&\le \|\theta\|_{\mathrm{BMO}} \sum_{k\in\Lambda} |J_k| \int_{J_k} |(g\chi f)(y)|/(x-y)^2\,dy \; (= \mathfrak{E}(x), \text{ say}) .
\end{aligned}$$

Since

$$\int_{I-J^*} \mathfrak{E}(x)\,dx \le C\|\theta\|_{\mathrm{BMO}} \|g\chi f\|_1 \le C\rho(\|\theta\|_{\mathrm{BMO}}) \Gamma_1(f)(\xi)|I| \le C\delta\lambda|I| ,$$

we have

$$|x \in I - J^*;\ \mathfrak{D}_3^*(\chi f)(x) > \lambda/10| \le C\delta|I| . \tag{18}$$

Since

$$\mathfrak{D}_4^*(\chi f)(x) \le \|\theta\|_{\mathrm{BMO}} \sum_{k\in\Lambda} |J_k| \int_{J_k} |(\chi f)(x)|/(x-y)^2\,dy \qquad (x \in I - J^*) ,$$

we have

$$|x \in I - J^*;\ \mathfrak{D}_4^*(\chi f)(x) > \lambda/10| \le (10/\lambda) C\|\theta\|_{\mathrm{BMO}} f^*(\xi)|I| \le C\delta|I| . \tag{19}$$

In the same manner as $\mathfrak{D}_3^*$, we have

$$|x \in I - J^*;\ \mathfrak{D}_5^*(\chi f)(x) > \lambda/10| \le C\delta|I| . \tag{20}$$

Consequently, we have, by (16), $\cdots$, (20) and $|J^*| \le (B_3/2)|I|$,

$$P_1 \le \{C\delta r/(r-1) + C\delta + B_3/2\}|I| . \tag{21}$$

Next we estimate P_2. Let $x \in I$. Since $\mathfrak{D}^*[\theta]f(a) \le \lambda$, we have, for any $0 < \varepsilon < \eta$,

$$\begin{aligned}
|\mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(x)}\chi^c f)(x)| &\le |\mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(x)}\chi^c f)(x) - \mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(a)}\chi^c f)(a)| \\
&\quad + |\mathfrak{D}[\theta](\chi_{\varepsilon,\eta}^{(a)}\chi^c f)(a)| \\
&\le |\mathfrak{D}[\theta](\chi_{\varepsilon}^{(x)}\chi^c f)(x) - \mathfrak{D}[\theta](\chi_{\varepsilon}^{(a)}\chi^c f)(a)| \\
&\quad + |\mathfrak{D}[\theta](\chi_{\eta}^{(x)}\chi^c f)(x) - \mathfrak{D}[\theta](\chi_{\eta}^{(a)}\chi^c f)(a)| + \lambda \\
&\quad (= Q_\varepsilon + Q_\eta + \lambda, \text{ say}) .
\end{aligned} \tag{22}$$

Let $V_1 = (x-\varepsilon, x+\varepsilon)^c \cap (a-\varepsilon, a+\varepsilon)^c$, $V_2 = (x-\varepsilon, x+\varepsilon)^c \backslash (a-\varepsilon, a+\varepsilon)^c$ and $V_3 = (a-\varepsilon, a+\varepsilon)^c \backslash (x-\varepsilon, x+\varepsilon)^c$. Then

$$
\begin{aligned}
Q_\varepsilon \le & \int_{V_1} |\mathfrak{D}[\theta](x,y) - \mathfrak{D}[\theta](a,y)|\,|(\chi^c f)(y)|\,dy \\
& + \int_{V_2} |\mathfrak{D}[\theta](x,y)|\,|(\chi^c f)(y)|\,dy + \int_{V_3} |\mathfrak{D}[\theta](a,y)|\,|(\chi^c f)(y)|\,dy \qquad (23)\\
& (= Q_{\varepsilon 1} + Q_{\varepsilon 2} + Q_{\varepsilon 3},\ \text{say})\,.
\end{aligned}
$$

We have

$$
\begin{aligned}
Q_{\varepsilon 1} &\le \int_{-\infty}^{\infty} |\mathfrak{D}[\theta](x,y) - \mathfrak{D}[\theta](a,y)|\,|(\chi^c f)(y)|\,dy \\
&\le \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} |\mathfrak{D}[\theta](x,y) - \mathfrak{D}[\theta](a,y)|\,|(\partial\chi_k f)(y)|\,dy\,, \qquad (24)
\end{aligned}
$$

where $\partial\chi_k(y)$ denotes the characteristic function of $\tilde{I}_k - \tilde{I}_{k-1}$ and $\tilde{I}_\ell$ is the open interval with midpoint a and length $2^\ell|\tilde{I}|$ ($\ell \ge 0$). Let $\tilde{\theta}_k(y) = (\theta(y) - m_{\tilde{I}_k}\theta)\chi_{\tilde{I}_k}(y)$ ($k \ge 1$). Then we have, for any $x \in I$, $y \in \tilde{I}_k$,

$$
\begin{aligned}
&\mathfrak{D}[\theta](x,y) - \mathfrak{D}[\theta](a,y) \\
&= i\Big[\Big\{\tilde{\theta}_k(y) - \frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz\Big\}\Big/\{(x-y) + i(\Theta(x)-\Theta(y))\} \\
&\quad - \Big\{\tilde{\theta}_k(y) - \frac{1}{a-y}\int_y^a \tilde{\theta}_k(z)\,dz\Big\}\Big/\{(a-y) + i(\Theta(a)-\Theta(y))\}\Big] \\
&= i\tilde{\theta}_k(y)\Big\{\frac{1}{(x-y)+i(\Theta(x)-\Theta(y))} - \frac{1}{(a-y)+i(\Theta(a)-\Theta(y))}\Big\} \\
&\quad - i\Big\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz\Big\}\Big\{\frac{1}{(x-y)+i(\Theta(x)-\Theta(y))} \\
&\qquad\qquad - \frac{1}{(a-y)+i(\Theta(a)-\Theta(y))}\Big\} \\
&\quad - i\Big\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz - \frac{1}{a-y}\int_y^a \tilde{\theta}_k(z)\,dz\Big\}\Big/ \\
&\qquad\qquad \{(a-y) + i(\Theta(a)-\Theta(y))\} \\
&= \tilde{\theta}_k(y)\Big\{\frac{\Theta(x)-\Theta(y)}{x-y} - \frac{\Theta(a)-\Theta(y)}{a-y}\Big\}\Big/ \qquad (25)\\
&\qquad\qquad \Big\{(x-y)\Big(1 + i\frac{\Theta(x)-\Theta(y)}{x-y}\Big)\Big(1 + i\frac{\Theta(a)-\Theta(y)}{a-y}\Big)\Big\} \\
&\quad - i\tilde{\theta}_k(y)(x-a)\Big/\Big\{(x-y)(a-y)\Big(1 + i\frac{\Theta(a)-\Theta(y)}{a-y}\Big)\Big\} \\
&\quad - \Big\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz\Big\}\Big\{\frac{\Theta(x)-\Theta(y)}{x-y} - \frac{\Theta(a)-\Theta(y)}{a-y}\Big\}\Big/ \\
&\qquad\qquad \Big\{(x-y)\Big(1 + i\frac{\Theta(x)-\Theta(y)}{x-y}\Big)\Big(1 + i\frac{\Theta(a)-\Theta(y)}{a-y}\Big)\Big\}
\end{aligned}
$$

$$+ i\left\{\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz\right\}(x-a)\Big/$$
$$\left\{(x-y)(a-y)\left(1+i\frac{\Theta(a)-\Theta(y)}{a-y}\right)\right\}$$
$$- i\left\{\frac{1}{x-y}\int_a^x \tilde{\theta}_k(z)\,dz\right\}\Big/\{(a-y)+i(\Theta(a)-\Theta(y))\}$$
$$+ i\left\{\frac{(x-a)}{(x-y)(a-y)}\int_y^a \tilde{\theta}_k(z)\,dz\right\}\Big/\{(a-y)+i(\Theta(a)-\Theta(y))\}\,.$$

Note that $|m_{\tilde{I}_\ell}\theta - m_{\tilde{I}_{\ell-1}}\theta| \leq C\|\theta\|_{\mathrm{BMO}}$ ($\ell \geq 1$). Since

$$\int_{\tilde{I}} |\tilde{\theta}_k(z)|\,dz \leq \int_{\tilde{I}} |\theta(z) - m_{\tilde{I}_0}\theta|\,dz + |m_{\tilde{I}_0}\theta - m_{\tilde{I}_k}\theta||\tilde{I}| \leq Ck\|\theta\|_{\mathrm{BMO}}|\tilde{I}|\,,$$

we have, for any $x \in I$, $y \in \tilde{I}_k - \tilde{I}_{k-1}$,

$$\left|\frac{\Theta(x)-\Theta(y)}{x-y} - \frac{\Theta(a)-\Theta(y)}{a-y}\right|$$

$$(26)\qquad = \left|\frac{1}{x-y}\int_y^x \tilde{\theta}_k(z)\,dz - \frac{1}{a-y}\int_y^a \tilde{\theta}_k(z)\,dz\right|$$
$$= \left|\frac{1}{x-y}\int_a^x \tilde{\theta}_k(z)\,dz - \frac{(x-a)}{(x-y)(a-y)}\int_y^a \tilde{\theta}_k(z)\,dz\right|$$
$$\leq (C/|\tilde{I}_k|)\int_{\tilde{I}} |\tilde{\theta}_k(z)|\,dz + (C2^{-k}/|\tilde{I}_k|)\int_{\tilde{I}_k} |\tilde{\theta}_k(z)|\,dz \leq C\|\theta\|_{\mathrm{BMO}}k2^{-k}\,.$$

By (25) and (26), we have

$$|\mathfrak{D}[\theta](x,y) - \mathfrak{D}[\theta](a,y)|$$
$$\leq C\rho(\|\theta\|_{\mathrm{BMO}})\{|\tilde{\theta}_k(y)| + \|\theta\|_{\mathrm{BMO}}\}k2^{-k}/|\tilde{I}_k| \qquad (x \in I, y \in \tilde{I}_k - \tilde{I}_{k-1})\,,$$

and hence

$$(27)\qquad Q_{\varepsilon 1} \leq C\rho(\|\theta\|_{\mathrm{BMO}})\sum_{k=1}^{\infty}(k2^{-k}/|\tilde{I}_k|)\int_{-\infty}^{\infty}\{|\tilde{\theta}_k(y)| + \|\theta\|_{\mathrm{BMO}}\}|(\partial\chi_k f)(y)|\,dy$$
$$\leq C\rho(\|\theta\|_{\mathrm{BMO}})\left\{\sum_{k=1}^{\infty}k2^{-k}\right\}\{\Gamma_1(f)(\xi) + \|\theta\|_{\mathrm{BMO}}f^*(\xi)\} \leq C\Xi(\xi) \leq C\delta\lambda\,.$$

If $\varepsilon < 2|I|$, then $Q_{\varepsilon 2} = Q_{\varepsilon 3} = 0$. If $\varepsilon \geq 2|I|$, then we have, with $\tilde{\theta}_V(y) = (\theta(y) - m_V\theta)\chi_V(y)$ ($V = (a-\varepsilon, b)$),

$$Q_{\varepsilon 2} = \int_{V_2}\left|\left\{\tilde{\theta}_V(y) - \frac{1}{x-y}\int_y^x \tilde{\theta}_V(z)\,dz\right\}\Big/\right.$$
$$(28)\qquad \left.\{(x-y)+i(\Theta(x)-\Theta(y))\}\right||(\chi^c f)(y)|\,dy$$

$$\leq (C/|V|)\int_{V_2}\{|\tilde{\theta}_V(y)| + \|\theta\|_{\mathrm{BMO}}\}|(\chi^c f)(y)|\,dy$$
$$\leq C\{\Gamma_1(f)(\xi) + \|\theta\|_{\mathrm{BMO}} f^*(\xi)\} \leq C\Xi(\xi) \leq C\delta\lambda\,.$$

We have analogously $Q_{\varepsilon 3} \leq C\delta\lambda$. Consequently, $Q_\varepsilon \leq C\delta\lambda$. In the same manner, we have $Q_\eta \leq C\delta\lambda$. Since $0 < \varepsilon < \eta$ are abritrary, we have, by (22),

$$\mathfrak{D}^*[\theta](\chi^c f)(x) \leq (1 + C\delta)\lambda \qquad (x \in I)\,. \tag{29}$$

Using (21) and (29), we choose δ so small that $P_1 \leq B_3|I|$ and $P_2 = 0$. Then we have (11) according to (12). Hence we obtain (10). This completes the proof of (8).

Now we deduce the (p, ω)-ness of $\mathfrak{D}^*[\theta]$ from (8). By Lemma 4, we have, with a constant $D_1 \geq 1$,

$$\Gamma_r(f)(x) = \sup_{x\in I}\Big\{(1/|I|)\int_I (|\theta(y) - m_I\theta||f(y)|)^r\,dy\Big\}^{1/r}$$
$$\leq \sup_{x\in I}\Big\{(1/|I|)\int_I |\theta(y) - m_I\theta|^{pr/(p-qr)}\,dy\Big\}^{(p-qr)/pr}$$
$$\times \Big\{(1/|I|)\int_I |f(y)|^{p/q}\,dy\Big\}^{q/p} \leq D_1\|\theta\|_{\mathrm{BMO}}\Lambda_{p/q}(f)(x)\,,$$

and hence $\Xi(x) \leq CD_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\Lambda_{p/q}(f)(x)$. We have, by Lemma 5,

$$\begin{aligned}\|\Xi\|_{p\omega} &\leq CD_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|(f^{p/q})^*\|_{q\omega}^{q/p} \\ &\leq CD_1B_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|(f^{p/q})\|_{q\omega}^{q/p} = CD_1B_1\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|f\|_{p\omega}\,.\end{aligned} \tag{30}$$

Let $\kappa(\lambda) = \omega(x;\, \Xi(x) > \lambda)$ $(\lambda > 0)$. Then (8) shows that

$$\sigma(2\lambda) \leq \kappa(\delta\lambda) + 4^{-p}\sigma(\lambda) \qquad (\lambda > 0)\,. \tag{31}$$

As in the proof of Lemma 12 in [7], we can easily verify that the following formal calculus holds true.

Integrating each quantity in (31) by $\lambda^{p-1}d\lambda$ from 0 to infinity, we have, with a constant D_2,

$$\int_0^\infty \lambda^{p-1}\sigma(\lambda)\,d\lambda \leq (C/\delta)^p\int_0^\infty \lambda^{p-1}\kappa(\lambda)\,d\lambda \leq \{(D_2/\delta)\|\theta\|_{\mathrm{BMO}}\rho(\|\theta\|_{\mathrm{BMO}})\|f\|_{p\omega}\}^p\,.$$

This shows that $\|\mathfrak{D}^*[\theta]f\|_{p\omega}/\|f\|_{p\omega} \leq (D_2/\delta)\|\theta\|_{\mathrm{BMO}}\,\rho(\|\theta\|_{\mathrm{BMO}})$. Taking the supremum over all $f \in L^p_\omega$ with compact support, we have

$$\|\mathfrak{D}^*[\theta]\|_{p\omega} \leq (D_2/\delta)\,\|\theta\|_{\mathrm{BMO}}\,\rho(\|\theta\|_{\mathrm{BMO}})\,. \tag{32}$$

This completes the proof of Theorem 1.

§ 4. Proof of Theorem 2

Let $\theta \in \mathrm{BMO}$. Then we have, by (32), $d(t\theta) = O(t)$ $(t \to 0)$, and hence $\lim_{t\to 0} d(t\theta) = 0$. Suppose that $\lim_{t\to 0} d(t\theta) = 0$. We choose $s > 0$ so that $d(s\theta) \le \tau$, where τ is determined later. Given a finite interval I, we denote by w its midpoint. Put $\bar{\theta}(x) = s\theta(x)$, $\tilde{\theta}(x) = \bar{\theta}(x) - m_I\bar{\theta}$ and

$$\mathfrak{G}(x, y) = \{\tilde{\theta}(x) - \tilde{\theta}(y)\} \Big/ \left\{(x - y) + i\int_y^x \tilde{\theta}(z)\,dz\right\}. \tag{33}$$

Since $\mathfrak{G}(x, y)/i$ is the sum of $\mathfrak{D}[\tilde{\theta}](x, y)$ and the dual kernel of $\overline{\mathfrak{D}[\tilde{\theta}](x, y)}$, we have $\|\mathfrak{G}\|_2 \le 2\tau$. We have, for almost all x in I,

$$\begin{aligned}
|I|\{\bar{\theta}(x) - m_I\bar{\theta}\} &= \int_I \{\tilde{\theta}(x) - \tilde{\theta}(y)\}\,dy \\
&= \int_I \mathfrak{G}(x, y)\left\{(x - y) + i\int_y^x \tilde{\theta}(z)\,dz\right\}dy \\
&= (x - w)\mathfrak{G}\chi_I(x) - \mathfrak{G}\{(\cdot - w)\chi_I\}(x) \\
&\quad + i\left\{\int_w^x \tilde{\theta}(z)\,dz\right\}\mathfrak{G}\chi_I(x) - i\mathfrak{G}\left\{\left(\int_w^{\cdot} \tilde{\theta}(z)\,dz\right)\chi_I\right\}(x).
\end{aligned}$$

Thus

$$\begin{aligned}
&|I|\int_I |\bar{\theta}(x) - m_I\bar{\theta}|\,dx \\
&\le \left\{\int_I (x - w)^2\,dx\right\}^{1/2}\|\mathfrak{G}\chi_I\|_2 + \sqrt{|I|}\,\|\mathfrak{G}\{(\cdot - w)\chi_I\}\|_2 \\
&\quad + \left[\int_I \left\{\int_w^x \tilde{\theta}(z)\,dz\right\}^2 dx\right]^{1/2}\|\mathfrak{G}\chi_I\|_2 + \sqrt{|I|}\left\|\mathfrak{G}\left\{\left(\int_w^{\cdot} \tilde{\theta}(z)\,dz\right)\chi_I\right\}\right\|_2 \\
&\le C\left\{1 + (1/|I|)\int_I |\tilde{\theta}(z)|\,dz\right\}\|\mathfrak{G}\|_2|I|^2 \\
&\le C\tau\left\{1 + (1/|I|)\int_I |\bar{\theta}(z) - m_I\bar{\theta}|\,dz\right\}|I|^2.
\end{aligned} \tag{34}$$

Now we choose τ so that $C\tau \le 1/2$. Then $(1/|I|)\int_I |\bar{\theta}(x) - m_I\bar{\theta}|\,dx \le 1/2$. Taking the supremum over all finite intervals I, we obtain $\|\bar{\theta}\|_{\mathrm{BMO}} \le 1/2$, which shows $\theta \in \mathrm{BMO}$. This completes the proof of Theorem 2.

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Department of Mathematics
College of General Education
Nagoya University
Chikusa-ku, Nagoya, 464
Japan