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ON MAXIMAL SPACELIKE HYPERSURFACES IN A LORENTZIAN MANIFOLD

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ABSTRACT. We prove a Bernstein-type property for maximal spacelike hypersurfaces in a Lorentzian manifold.

§ 1. Introduction

The object of this note is to prove the following

THEOREM A. *Let N be a Lorentzian manifold satisfying the strong energy condition. Let M be a complete maximal spacelike hypersurface in N. Suppose that N is locally symmetric and has nonnegative spacelike sectional curvature. Then M is totally geodesic.*

For the terminology in the theorem, see Section 2.

It has been proved by Calabi [2] (for $n \leqq 4$) and Cheng-Yau [4] (for all n) that a complete maximal spacelike hypersurface in the flat Minkowski $(n+1)$-space L^{n+1} is totally geodesic. In particular, the only entire nonparametric maximal spacelike hypersurfaces in L^{n+1} are spacelike hyperplanes. This is remarkable since the Euclidean counterpart, the Bernstein theorem, holds only for $n \leqq 7$: the entire nonparametric minimal hypersurfaces in the Euclidean space $\boldsymbol{R}^{n+1}$, $n \leqq 7$, are hyperplanes (cf. [8]).

Theorem A implies, for instance, that a complete maximal spacelike hypersurface in the Einstein static universe is totally geodesic. In the proof of Theorem A, a refinement of a Bernstein-type theorem of Choquet-Bruhat [5, 6] will be also given.

§ 2. Definitions

First we set up our terminology and notation. Let $N=(N, \bar{g})$ be a

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Lorentzian manifold with Lorentzian metric $\bar{g}$ of signature $(-, +, \cdots, +)$. N has a uniquely defined torsion-free affine connection $\bar{\nabla}$ compatible with the metric $\bar{g}$. N is said to satisfy the *strong energy condition* (the timelike convergence condition in Hawking-Ellis [7]) if the Ricci curvature $\overline{\mathrm{Ric}}$ of N is positive semidefinite for all timelike vectors, that is, if $\overline{\mathrm{Ric}}\,(v, v) \geqq 0$ for every timelike vector $v \in TN$ (cf. [1, 6]). N is called *locally symmetric* if the curvature tensor $\bar{R}$ of N is parallel, that is, $\bar{\nabla}\bar{R} = 0$. We say that N has *nonnegative spacelike sectional curvature* if the sectional curvature $\bar{K}(u \wedge v)$ of N is nonnegative for every nondegenerate tangent 2-plane spanned by spacelike vectors $u, v \in TN$.

Let M be a hypersurface immersed in N. M is said to be *spacelike* if the Lorentzian metric $\bar{g}$ of N induces a Riemannian metric g on M. For a spacelike M there is naturally defined the second fundamental form (the extrinsic curvature) S of M. M is called *maximal* spacelike if the mean (extrinsic) curvature $H = \mathrm{Tr}\, S$, the trace of S, of M vanishes identically. M is maximal spacelike if and only if it is extremal under the variations, with compact support through spacelike hypersurfaces, for the induced volume. M is said to be *totally geodesic* (a moment of time symmetry) if the second fundamental form S vanishes identically.

§ 3. Local formulas

Let M be a spacelike hypersurface in a Lorentzian $(n+1)$-manifold $N = (N, \bar{g})$. We choose a local field of Lorentz orthonormal frames $e_0, e_1, \cdots, e_n$ in N such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. Let $\omega_0, \omega_1, \cdots, \omega_n$ be its dual frame field so that the Lorentzian metric $\bar{g}$ can be written as $\bar{g} = -\omega_0^2 + \sum_i \omega_i^2$.*) Then the connection forms $\omega_{\alpha\beta}$ of N are characterized by the equations

$$
\begin{aligned}
d\omega_i &= -\sum_k \omega_{ik} \wedge \omega_k + \omega_{i0} \wedge \omega_0\,, \\
d\omega_0 &= -\sum_k \omega_{0k} \wedge \omega_k\,, \qquad \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0\,.
\end{aligned} \tag{1}
$$

The curvature forms $\bar{\Omega}_{\alpha\beta}$ of N are given by

$$
\begin{aligned}
\bar{\Omega}_{ij} &= d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} - \omega_{i0} \wedge \omega_{0j}\,, \\
\bar{\Omega}_{0i} &= d\omega_{0i} + \sum_k \omega_{0k} \wedge \omega_{ki}\,,
\end{aligned} \tag{2}
$$

*) We shall use the summation convention with Roman indices in the range $1 \leqq i, j, \cdots \leqq n$ and Greek in $0 \leqq \alpha, \beta, \cdots \leqq n$.

and we have

$$\bar{\Omega}_{\alpha\beta} = \frac{1}{2} \sum_{\gamma,\delta} \bar{R}_{\alpha\beta\gamma\delta}\omega_\gamma \wedge \omega_\delta \,, \tag{3}$$

where $\bar{R}_{\alpha\beta\gamma\delta}$ are components of the curvature tensor $\bar{R}$ of N.

We restrict these forms to M. Then

$$\omega_0 = 0 \,, \tag{4}$$

and the induced Riemannian metric g of M is written as $g = \sum_i \omega_i^2$. From formulas (1)–(4), we obtain the structure equations of M

$$\begin{aligned} d\omega_i &= -\sum_k \omega_{ik} \wedge \omega_k \,, \qquad \omega_{ij} + \omega_{ji} = 0 \,, \\ d\omega_{ij} &= -\sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{i0} \wedge \omega_{0j} + \bar{\Omega}_{ij} \,, \\ \Omega_{ij} &= d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \frac{1}{2} \sum_{k,\ell} R_{ijk\ell}\omega_k \wedge \omega_\ell \,, \end{aligned} \tag{5}$$

where Ω_{ij} and $R_{ijk\ell}$ denote the curvature forms and the components of the curvature tensor R of M, respectively. We can also write

$$\omega_{i0} = \sum_j h_{ij}\omega_j \,, \tag{6}$$

where h_{ij} are components of the second fundamental form $S = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j$ of M. Using (6) in (5) then gives the Gauss formula

$$R_{ijk\ell} = \bar{R}_{ijk\ell} - (h_{ik}h_{j\ell} - h_{i\ell}h_{jk}) \,. \tag{7}$$

Let h_{ijk} denote the covariant derivative of h_{ij} so that

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj} \,. \tag{8}$$

Then, by exterior differentiating (6), we obtain the Coddazi equation

$$h_{ijk} - h_{ikj} = \bar{R}_{0ijk} \,. \tag{9}$$

Next, exterior differentiate (8) and define the second covariant derivative of h_{ij} by

$$\sum_\ell h_{ijk\ell}\omega_\ell = dh_{ijk} - \sum_\ell h_{\ell jk}\omega_{\ell i} - \sum_\ell h_{i\ell k}\omega_{\ell j} - \sum_\ell h_{ij\ell}\omega_{\ell k} \,.$$

Then we obtain the Ricci formula

$$h_{ijk\ell} - h_{ij\ell k} = \sum_m h_{mj}R_{mik\ell} + \sum_m h_{im}R_{mjk\ell} \,. \tag{10}$$

Let us now denote the covariant derivative of $\bar{R}_{\alpha\beta\gamma\delta}$, as a curvature tensor of N, by $\bar{R}_{\alpha\beta\gamma\delta;\varepsilon}$. Then restricting on M, $\bar{R}_{0ijk;\ell}$ is given by

$$\bar{R}_{0ijk;\ell} = \bar{R}_{0ijk\ell} - \bar{R}_{0i0k}h_{j\ell} - \bar{R}_{0ij0}h_{k\ell} - \sum_m \bar{R}_{mijk}h_{m\ell}, \tag{11}$$

where $\bar{R}_{0ijk\ell}$ denote the covariant derivative of $\bar{R}_{0ijk}$ as a tensor on M so that

$$\sum_\ell \bar{R}_{0ijk\ell}\omega_\ell = d\bar{R}_{0ijk} - \sum_\ell \bar{R}_{0\ell jk}\omega_{\ell i} - \sum_\ell \bar{R}_{0i\ell k}\omega_{\ell j} - \sum_\ell \bar{R}_{0ij\ell}\omega_{\ell k}.$$

The Laplacian Δh_{ij} of the second fundamental form h_{ij} is defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}.$$

From (9) we then obtain

$$\Delta h_{ij} = \sum_k h_{kijk} + \sum_k \bar{R}_{0ijkk}, \tag{12}$$

and from (10)

$$h_{kijk} = h_{kikj} + \sum_m h_{mi}R_{mkjk} + \sum_m h_{km}R_{mijk}. \tag{13}$$

Replace h_{kikj} in (13) by $h_{kkij} + \bar{R}_{0kikj}$ (by (9)) and substitute the right hand side of (13) into h_{kijk} in (12). Then we obtain

$$\begin{aligned} \Delta h_{ij} &= \sum_k (h_{kkij} + \bar{R}_{0kikj} + \bar{R}_{0ijkk}) \\ &\quad + \sum_k (\sum_m h_{mi}R_{mkjk} + \sum_m h_{km}R_{mijk}). \end{aligned} \tag{14}$$

From (7), (11) and (14) we then obtain

$$\begin{aligned} \Delta h_{ij} &= \sum_k h_{kkij} + \sum_k \bar{R}_{0kik;j} + \sum_k \bar{R}_{0ijk;k} \\ &\quad + \sum_k (h_{kk}\bar{R}_{0ij0} + h_{ij}\bar{R}_{0k0k}) \\ &\quad + \sum_{m,k} (h_{mj}\bar{R}_{mkik} + 2h_{mk}\bar{R}_{mijk} + h_{mi}\bar{R}_{mkjk}) \\ &\quad - \sum_{m,k} (h_{mi}h_{mj}h_{kk} + h_{km}h_{mj}h_{ik} - h_{km}h_{mk}h_{ij} - h_{mi}h_{mk}h_{kj}). \end{aligned} \tag{15}^{*)}$$

Now we assume that N is locally symmetric, that is, $\bar{R}_{\alpha\beta\gamma\delta;\varepsilon} = 0$ and that M is maximal in N, so that $\sum_k h_{kk} = 0$. Then, from (15) we obtain

$$\begin{aligned} \sum_{i,j} h_{ij}\Delta h_{ij} &= \sum_{i,j,k} h_{ij}^2\bar{R}_{0k0k} + \sum_{i,j,k,m} 2(h_{ij}h_{mj}\bar{R}_{mkik} + h_{ij}h_{mk}\bar{R}_{mijk}) \\ &\quad + (\sum_{i,j} h_{ij}^2)^2. \end{aligned} \tag{16}$$

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*) This is the Lorentzian version of the well-known formula established, for example, in [8].

§ 4. Proof of Theorem A

Theorem A is an immediate consequence of the following

THEOREM B. *Let* $N = (N, \bar{g})$ *be a locally symmetric Lorentzian* $(n+1)$-*manifold and* M *be a complete maximal spacelike hypersurface in* N. *Assume that there exist constants* c_1, c_2 *such that*

(i) $\overline{\mathrm{Ric}}(v, v) \geqq c_1$ *for all timelike vectors* $v \in TN$,

(ii) $\bar{K}(u \wedge v) \geqq c_2$ *for all nondegenerate tangent* 2-*planes spanned by spacelike vectors* $u, v \in TN$, *and*

(iii) $c_1 + 2nc_2 \geqq 0$.

Then M *is totally geodesic.*

To prove Theorem B, we first note

LEMMA 1. *Under the assumptions of Theorem* B,

$$\frac{1}{2}\Delta\Big(\sum_{i,j} h_{ij}^2\Big) \geqq \Big(\sum_{i,j} h_{ij}^2\Big)^2 . \tag{17}$$

Proof. For any point $p \in M$, we may choose our frame $\{e_1, \cdots, e_n\}$ at p so that $h_{ij} = \lambda_i \delta_{ij}$. Then, by assumption (ii) of Theorem B, we have at p

$$\begin{aligned}
\sum_{i,j,k,m} & 2(h_{ij}h_{mj}\bar{R}_{mkik} + h_{ij}h_{mk}\bar{R}_{mijk}) \\
&= \sum_{i,k} 2(\lambda_i^2 \bar{R}_{ikik} + \lambda_i\lambda_k \bar{R}_{kiik}) \\
&= \sum_{i,k} (\lambda_i - \lambda_k)^2 \bar{R}_{ikik} \geqq c_2 \sum_{i,k} (\lambda_i - \lambda_k)^2 \\
&= 2c_2\Big(n \sum_i \lambda_i^2 - \Big(\sum_i \lambda_i\Big)^2\Big) = 2nc_2 \sum_{i,j} h_{ij}^2 .
\end{aligned}$$

Also we have by assumption (i)

$$\sum_k \bar{R}_{0k0k} \geqq c_1 .$$

It then follows from (16) and assumption (iii) that

$$\begin{aligned}
\frac{1}{2}\Delta\Big(\sum_{i,j} h_{ij}^2\Big) &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij}\Delta h_{ij} \\
&\geqq (c_1 + 2nc_2)\Big(\sum_{i,j} h_{ij}^2\Big) + \Big(\sum_{i,j} h_{ij}^2\Big)^2 \\
&\geqq \Big(\sum_{i,j} h_{ij}^2\Big)^2 .
\end{aligned}$$

Let $u = \sum_{i,j} h_{ij}^2$ be the squared of the length of the second fundamental form of M. The proof of Theorem B is complete if we show that u vanishes identically. Recall that from (17), u satisfies

$$\Delta u \geqq 2u^2 . \tag{18}$$

Then, by the maximum principle, the result is immediate provided M is compact.

We now assume that M is noncompact and complete. We will modify the maximum principle argument as in [4]. Take a point $p \in M$, and let r denote the geodesic distance on M from p with respect to the induced Riemannian metric. For $a > 0$, let $B_a(p) = \{x \in M \mid r(x) < a\}$ be the geodesic ball of radius a and center p.

LEMMA 2. *For any $a > 0$, there exists a constant c depending only on n such that*

$$u(x) \leqq \frac{ca^2(1 + |c_2|^{1/2}a)}{(a^2 - r(x)^2)^2} \tag{19}$$

for all $x \in B_a(p)$.

Proof. Assuming that u is not identically zero on $B_a(p)$, we consider the function

$$f(x) = (a^2 - r(x)^2)^2 u(x) , \qquad x \in B_a(p) .$$

Then f attains a nonzero maximum at some point $q \in B_a(p)$, for the closure of $B_a(p)$ is compact since M is complete. As in [§ 2, 3], we may assume that f is C^2 around q. Then we have

$$\nabla f(q) = 0 , \qquad \Delta f(q) \leqq 0 .$$

Hence at q*⁾

$$\frac{\nabla u}{u} = \frac{4r\nabla r}{a^2 - r^2} ,$$

$$\frac{\Delta u}{u} \leqq \frac{|\nabla u|^2}{u^2} + \frac{8r^2}{(a^2 - r^2)^2} + \frac{4(1 + r\Delta r)}{a^2 - r^2} ,$$

from which we obtain

$$\frac{\Delta u}{u}(q) \leqq \frac{24r^2}{(a^2 - r^2)^2}(q) + \frac{4(1 + r\Delta r)}{a^2 - r^2}(q) . \tag{20}$$

On the other hand, according to [Lemma 1, 9], $\Delta r(q)$ is bounded from above by

*⁾ We may concentrate on the case of $q \neq p$ for the proof become simpler when $q=p$.

$$\text{(21)} \qquad \Delta r(q) \leqq \min_{0\leqq k\leqq r(q)} \left[\frac{n-1}{r(q)-k} - \frac{1}{(r(q)-k)^2}\int_k^{r(q)} (t-k)^2 \operatorname{Ric}(\dot\sigma(t), \dot\sigma(t))dt\right],$$

where $\dot\sigma(t)$ is the tangent vector of the minimizing geodesic $\sigma\colon [0, r(q)] \to M$ from p to q and Ric denote the Ricci curvature of M. Also, from (7) and assumption (ii) of Theorem B, $\operatorname{Ric}(\dot\sigma(t), \dot\sigma(t))$ is bounded from below by

$$\text{(22)} \qquad \operatorname{Ric}(\dot\sigma(t), \dot\sigma(t)) \geqq (n-1)c_2,$$

since M is maximal spacelike. From (21) and (22) we then obtain

$$\text{(23)} \qquad r\Delta r(q) \leqq (n-1) + 2(n-1)|c_2|^{1/2} r(q).$$

It follows from (20) and (23) that

$$(a^2 - r(q)^2)^2 u^{-1}\Delta u(q) \leqq 24a^2 + 8na^2(1 + |c_2|^{1/2}a).$$

From (18) we then obtain

$$f(q) = (a^2 - r(q)^2)^2 u(q) \leqq ca^2(1 + |c_2|^{1/2}a),$$

c being a constant depending only on n. Since q is the maximum point of f in $B_a(p)$, this implies that

$$(a^2 - r(x)^2)^2 u(x) \leqq ca^2(1 + |c_2|^{1/2}a)$$

for all $x \in B_a(p)$.

Since M is complete, we may fix x in Inequality (19) and let a tend to infinity. Then we obtain $u(x) = 0$ for all $x \in M$. This completes the proof of Theorem B.

Remark. Let $N = L^{k+1} \times S^{n-k}$ be the product Lorentzian manifold of the flat Minkowski $(k+1)$-space L^{k+1}, $1 \leqq k \leqq n$, and S^{n-k}, a Riemannian $(n-k)$-manifold of positive constant curvature. Then N satisfies the assumptions of Theorem A. The Einstein static space $N = (R, -dt^2) \times S^n$ also satisfies these assumptions.

The Lorentzian $(n+1)$-manifold S_1^{n+1} of constant curvature $c > 0$, called the de Sitter space, satisfies the assumptions of Theorem B (with $c_1 = -cn$, $c_2 = c$). Theorem B then gives a refinement of a theorem of Choquet-Bruhat [Theorem 4.6, 6].

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