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ISOMETRY INVARIANT CLOSED GEODESIC ON A NONPOSITIVELY CURVED MANIFOLD

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§0. Introduction

In this paper we wish to study the isometry invariant geodesic on a non-positively curved manifold from a point of view of the displacement function.

For an isometry f on a compact connected Riemannian manifold M a geodesic c is called f-invariant geodesic if $f(c(t)) = f(c(t+1))$ for any $t \in R$. And one point geodesic is called a trivial geodesic. (This is also a fixed point of f.) The isometry invariant geodesic was introduced by K. Grove ([5]) who studied it by using the infinite dimensional critical point theory and the Gromoll-Meyer type theorem was proved by Grove and Tanaka (cf. [7], [11], [12]). At the same time the good results were obtained for $\pi_1(M) = 1$ (cf. [3], [6]). However for $\pi_1(M) \neq 1$ the mathematical phenomena for such a geodesic is not the same as the above case (cf. [1], [4], [6]). Here our method is different from their case because our manifold is topologically very simple but $\pi_1(M) \neq 1$. Our motivation comes from the works of T. Sunada (cf. [9], [10]) and V. Ozols ([8]).

In this paper we always assume that M is a compact connected manifold with nonpositive sectional curvature. Let $F(f)$ be the fixed point set of an isometry f on M. For the existence problem of such a geodesic the case of $\sharp F(f) < \infty$ (finite fixed points) is essential because (1) if $F(f) = \phi$, there always exists such a geodesic (2) if $\dim(F(f)) \geqq 1$, then $F(f)$ is a totally geodesic submanifold of M which implies the existence (in particular it is a f-fixed geodesic).

Thus our first main result is as follows

EXISTENCE CRITERION THEOREM. *Let f be an isometry of finite order*

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k ($\geqq 1$), let $F(f) = \{p_1, p_2, \cdots, p_r\}$ be the set of the finite fixed points of f and $G_{p_i} = \{\mu \circ \tilde{f}_i \circ \mu^{-1} \mid \mu \in \Gamma\}$ where a covering isometry $\tilde{f}_i$ of f has a fixed point on the fibre of p_i and $\Gamma = \pi_1(M)$, then the following statements are mutually equivalent.

(1) *There does not exist a non-trivial f-invariant geodesic.*

(2) *$\tilde{f}^k = 1$ for any covering isometry $\tilde{f}$ of f.*

(3) *The set of the covering isometries of $\tilde{f}$ is just $\bigcup_{i=1}^{r} G_{p_i}$.*

Since f has the finite order k, every non-trivial f-invariant geodesic is closed. This assumption is not so special because the Bochner's theorem implies that every isometry has a finite order under the condition of the negative Ricci curvature.

Now let $\mathrm{Geo}(M, f)$ be the set of the f-invariant closed geodesics. Then in a natural sense $\mathrm{Geo}(M, f) = \bigcup_{\gamma \in \Gamma} \mathfrak{g}^f_{[\gamma]}$ where $\mathfrak{g}^f_{[\gamma]}$ is the set of the free homotopic closed geodesic corresponding to the conjugate class $[\gamma]$ for $\gamma \in \Gamma$. And let $\mathrm{Crit}(\tilde{f})$ be the set of $d^2_{\tilde{f}}$-critical points for the distance function $d_{\tilde{f}}(x) = d(x, \tilde{f}(x))$ for an isometry $\tilde{f}$ on $\tilde{M}$ which is the universal covering space of M. For any covering isometry $\tilde{f}$ of f with $f^k = 1$ we have $\tilde{f}^k = \gamma$ for some $\gamma \in \Gamma$ and so let G_γ be the set of the covering isometries $\tilde{f}$ with $\tilde{f}^k = \gamma$. Now we define an equivalence relation on G_γ such that $\tilde{f} \sim \tilde{g}$ for any $\tilde{f}, \tilde{g}$ in G_γ if and only if there exists $\xi \in \Gamma_\gamma$ with $\tilde{g} = \xi \circ \tilde{f} \circ \xi^{-1}$. We write the equivalence class of $\tilde{f}$ by $\langle \tilde{f} \rangle$.

STRUCTURE THEOREM *Let f be an isometry of finite order. Then $\mathfrak{g}^f_{[\gamma]}$ is homeomorphic to $\bigcup_{\langle \tilde{f} \rangle} \mathrm{Crit}(\tilde{f})/\Gamma_{\tilde{f}}$ where $\tilde{f} \in G_\gamma$ and $\Gamma_{\tilde{f}} = \{\gamma \in \Gamma \mid \gamma \circ \tilde{f} = \tilde{f} \circ \gamma\}$. Thus we have $\mathrm{Geo}(M, f)$ is homeomorphic to $\bigcup_{\gamma \in \Gamma} \bigcup_{\langle \tilde{f} \rangle} \mathrm{Crit}(\tilde{f})/\Gamma_{\tilde{f}}$. Moreover the above homeomorphism is a diffeomorphism if each $\mathrm{Crit}(\tilde{f})$ is a submanifold without boundary.*

This structure theorem is an extension of the Sunada's one ([10]) to the case of Z_k-action. Moreover we have the similar theorem for a general isometry f in Section 2.

And Section 3 provides an interesting example.

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§1. Existence

Let M be a compact connected manifold with nonpositive sectional curvature and f be an isometry of M. Here we use the following notations.

$\tilde{M}$: the universal covering space of M, $\pi: \tilde{M} \to M$ is the canonical projection.

$\tilde{f}$: a covering isometry of f.

$\mathrm{Crit}(\tilde{f})$: the set of all critical points of $d_{\tilde{f}}^2(x) = d^2(x, \tilde{f}(x))$ where d is the Riemannian distance function on $\tilde{M}$.

$F(\tilde{f})$: the fixed point set of $\tilde{f}$.

Note that $F(\tilde{f}) \subset \mathrm{Crit}(\tilde{f})$. The following theorem was proved by Ozols ([8]).

LEMMA 1 (Ozols's theorem).

(1) $x \in \mathrm{Crit}(\tilde{f}) - F(\tilde{f})$ *if and only if* $\tilde{f}$ *preserves the minimizing geodesic from* x *to* $\tilde{f}(x)$.

(2) *If* $F(\tilde{f}) \neq \phi$, *then* $\mathrm{Crit}(\tilde{f}) = F(\tilde{f})$.

(3) *If* ξ: $\tilde{M} \to \tilde{M}$ *is an isometry, then* $\mathrm{Crit}(\xi \circ \tilde{f} \circ \xi^{-1}) = \xi(\mathrm{Crit}(\tilde{f}))$.

If c is an f-invariant geodesic, then there exists a lifted geodesic $\tilde{c}$ which is $\tilde{f}$-invariant for some covering isometry $\tilde{f}$, conversely if $\tilde{c}$ is an $\tilde{f}$-invariant geodesic for some covering isometry $\tilde{f}$, then the projection $\pi \circ \tilde{c}$ is f-invariant. Thus the above (1) suggests us to look into $\mathrm{Crit}(\tilde{f})$ for the existence of the f-invariant geodesic. Of course since $\tilde{M}$ is noncompact $\mathrm{Crit}(\tilde{f}) = \phi$ is possible. However we can prove the following lemma.

LEMMA 2. *Let* f *be an isometry of* M, *then* $\mathrm{Crit}(\tilde{f}) \neq \phi$ *for any covering isometry* $\tilde{f}$ *of* f.

Proof. For any covering isometry $\tilde{h}$ we prove that $d_{\tilde{h}}(x) = d(x, \tilde{h}(x))$ has a minimum on $\tilde{M}$. Without loss of generality we can assume $\inf d_{\tilde{h}} = 0$, if not we consider $d_{\tilde{h}} - a$ where $a = \inf d_{\tilde{h}} > 0$. Let $\{x_n\} \subset \tilde{M}$ be a sequence such that $\lim_{n \to \infty} d_{\tilde{h}}(x_n) = \inf d_{\tilde{h}} = 0$. For a fixed $x \in \tilde{M}$ and for the number $r = 2$(the diameter of M) the set $\{x \in \tilde{M} \mid d(x, x_0) \leqq r\}$ is compact. And so we can find $\gamma_n \in \Gamma$ such that $d(\gamma_n(x_n), x_0) < r$ for every $n \geqq 1$, thus $\{\gamma_n(x_n)\}$ has a limit point, say y. For a sufficiently large n, $d(\tilde{h}(x_n), x_n) = d((\gamma_n \tilde{h} \gamma_n^{-1})\gamma_n(x_n), \gamma_n(x_n))$ is near $\inf d_{\tilde{h}} = 0$ and so $\gamma_n \tilde{h} \gamma_n^{-1}(y)$ is contained in a sufficient small neighborhood of y. On the other hand for any given $b>0$ and for any fixed $x \in \tilde{M}$, the set $\{\tilde{g} \mid \tilde{g}$ is a covering isometry of f such that $d(x, \tilde{g}(x)) \leqq b\}$ is a finite set because $\tilde{g}$ is a covering isometry. Thus there exists $\gamma_0 \in \Gamma$ such that $\gamma_n \tilde{h} \gamma_n^{-1}(y) = \gamma_0 \tilde{h} \gamma_0^{-1}(y)$ for infinitely many n. Then $\inf d_{\tilde{h}}$ is attained at $\gamma_0^{-1}(y)$. This implies $\mathrm{Crit}\,(\tilde{h}) \neq \phi$ for any covering isometry $\tilde{h}$ of f. Q.E.D.

By (2) of Lemma 1 and Lemma 2 we have two possibilities as follows

(a) $F(\tilde{f}) = \phi$ and $\mathrm{Crit}\,(\tilde{f}) \neq \phi$

(b) $F(\tilde{f}) = \phi$ and so $\mathrm{Crit}\,(\tilde{f}) = F(\tilde{f})$.

Thus in order to get the existence theorem we must control the $F(\tilde{f})$. As to the information of $F(\tilde{f})$ there is the E. Cartan's theorem.

LEMMA 3 (E. Cartan) ([2]). *Every compact group of isometries of a complete simply connected Riemannian manifold with nonpositive sectional curvature has a fixed point.*

Now we say "f-translated geodesic" if an f-invariant geodesic is not fixed identically.

PROPOSITION 1. *There does not exist a non-trivial f-translated geodesic if the group generated by the covering isometries of f is compact.*

PROPOSITION 2. *If $\Gamma_{\tilde{f}} \neq 1$ for some covering isometry $\tilde{f}$ where $\Gamma_{\tilde{f}} = \{\mu \in \Gamma \mid \mu \circ \tilde{f} = \tilde{f} \circ \mu\}$, there exists a non-trivial f-translated geodesic or a non-trivial f-fixed geodesic.*

Proof. First of all we note the known result that $F(\tilde{f})$ is connected ([2]).

Proof of Proposition 1. If there is a non-trivial f-translated geodesic c, then we have a non-trivial $\tilde{f}$-invariant geodesic $\tilde{c}$ covering c for some covering isometry $\tilde{f}$. If $\tilde{c}$ is not identically fixed by $\tilde{f}$, $\tilde{c}(0) = x \in \mathrm{Crit}\,(\tilde{f}) - F(\tilde{f})$ which contradicts (2) of Lemma 1 because $F(\tilde{f}) \neq \phi$ by Lemma 3. Since $F(\tilde{f})$ is connected there are only identically fixed geodesics.

Proof of Proposition 2. By the hypothesis there is a non-trivial $\mu \in \Gamma_{\tilde{f}}$ such that $\mu \circ \tilde{f} \circ \mu^{-1} = \tilde{f}$. Suppose that $\tilde{f}$ has a fixed point p, then $\mu(p) = q$ is also a fixed point of $\tilde{f}$ because $\tilde{f}(q) = \mu \circ \tilde{f} \circ \mu^{-1}(\mu(p)) = \mu(p) = q$. Since μ has no fixed point, q is distinct from p. If $\sharp F(f) < \infty$, then this is impossible because $F(\tilde{f})$ is connected. Thus by Lemma 2 we have the case (a) which implies the existence of a non-trivial f-translated geodesic by (1) of Lemma 1. The other case is $\dim F(f) \geqq 1$ and so there exists a non-trivial f-fixed geodesic. Q.E.D.

In the proof of Proposition 2 we see the following fact

COROLLARY 1. *Suppose* $\sharp F(f) < \infty$, *then* $F(\tilde{f}) = \phi$ *if* $\Gamma_{\tilde{f}} \neq 1$.

PROPOSITION 3. *Let f be an isometry of M which is homotopic to the identity. Then we have* (1) *there exists a covering isometry $\tilde{f}$ such that it is homotopic to the identity and $d_{\tilde{f}}(x) = d(x, \tilde{f}(x))$ is constant and* (2) *f has no fixed point with $0 < \sharp F(f) < \infty$.*

COROLLARY 2. *If f is homotopic to the identity, then there exists a non-trivial f-translated geodesic or a non-trivial f-fixed geodesic.* (*This has been proved by Grove* [6] *in more general case.*)

Proof. We prove Proposition 3.

By the covering homotopy property we can take a covering homotopy $H: I \times \tilde{M} \to \tilde{M}$ such that $H(0, \cdot) = \mathrm{id}_{\tilde{M}}$ and $H(1, \cdot) = \tilde{f}$ with $\pi \circ \tilde{f} = f \circ \pi$. Then we can see that $\tilde{f}$ commutes with any $\gamma \in \Gamma$. Consider $(\gamma \circ H_t)(x)$, $(H_t \circ \gamma)(x)$ for any $x \in \tilde{M}$, then these are paths starting at $\gamma(x)$. Since $\pi(\gamma \circ H_t) = \pi(H_t \circ \gamma) = F_t \circ \pi$ where F_t is the homotopy connecting id_M and f, these paths are the lifts of $(F_t \circ \pi)(x)$ at the same starting point $\gamma(x)$. By the uniqueness of lifting we have $\gamma \circ H_t = H_t \circ \gamma$ for any $(t, x) \in I \times \tilde{M}$ which implies $\gamma \circ \tilde{f} = \tilde{f} \circ \gamma$. We show $d_{\tilde{f}} =$ constant for this $\tilde{f}$. Since $\tilde{M}$ is a covering space on which Γ acts as the covering transformation, there is a fundamental domain V such that $\bar{V}$ is compact and there is $\alpha \in \Gamma$ with $\alpha(x) \in \bar{V}$ for each $x \in \tilde{M}$. By the commutativity $d(x, \tilde{f}(x)) = d(\alpha(x), \alpha(\tilde{f}(x))) = d(\alpha(x), \tilde{f}(\alpha(x)))$ and thus we have $d_{\tilde{f}}(x) < \infty$ for any $x \in \tilde{M}$ because of the compactness of $\bar{V}$. It is known that $d_{\tilde{f}}$ is a convex function and so $d_{\tilde{f}}$ is constant.

Next we prove the second part. As seeing in the proof of (1) there is $\tilde{f}$ such that $\tilde{f}$ commutes with Γ. Since any covering isometry $\tilde{h}$ of f is obtained by $\tilde{h} = \alpha \circ \tilde{f}$ for some $\alpha \in \Gamma$, we have $\Gamma_{\tilde{h}} \neq 1$ which implies $F(\tilde{h}) = \phi$ by Corollary 1. However it is impossible because there exists at least one covering isometry $\tilde{g}$ with $F(\tilde{g}) \neq \phi$.

Proof of Corollary 2. In the proof of (1) there is $\tilde{f}$ such that $\tilde{f}$ commutes with Γ and so $\Gamma_{\tilde{f}} \neq 1$. We have the conclusion by Proposition 2. Q.E.D.

As looking in the introduction the existence problem is essential in the case of $\sharp F(f) < \infty$. From now on we fix our consideration in this case. If $F(f) = \{p_1, p_2, \cdots, p_r\}$, then there is a covering isometry $\tilde{f}_i$ such that it has a fixed point on $\pi^{-1}(p_i)$. Thus we consider the following covering isometry class

$$G_{p_i} = \{\mu \circ \tilde{f}_i \circ \mu^{-1} \mid \mu \in \Gamma\}.$$

LEMMA 4. *A covering isometry $\tilde{g}$ of f has a fixed point if and only if $\tilde{g} \in G_{p_i}$ for some i.*

Proof. The fixed point of $\tilde{g}$ must be in $\pi^{-1}(p_i)$ for some i, now let it be x. If y is a fixed point of $\tilde{f}_i$, $\tilde{h} = \gamma \circ \tilde{f}_i \circ \gamma^{-1}$ is a covering isometry with the fixed point x for $\gamma \in \Gamma$ such that $\gamma(y) = x$. Thus $\tilde{g}$ and $\tilde{h}$ has the same fixed point and so $\tilde{g} = \tilde{h}$ by the uniqueness of covering. Therefore we have $\tilde{g} \in G_{p_i}$. Conversely it is almost same way to show that every element of G_{p_i} for any i has a fixed point. Q.E.D.

LEMMA 5. *Let f be an isometry of finite order k ($\geqq 1$). Then a covering isometry $\tilde{f}$ of f has a fixed point if and only if $\tilde{f}^k = 1$. (It is not necessary the assumption of the finiteness of $F(f)$.)*

Proof. Suppose $\tilde{f}$ has a fixed point but $\tilde{f}^k = \gamma \neq 1$. Let x be a fixed point of $\tilde{f}$, then x is also a fixed point of $\tilde{f}^k$ and so it is a fixed point of γ. However this is impossible because γ is fixed point free. Conversely if $\tilde{f}^k = 1$, $\tilde{f}$ has a fixed point by Lemma 3. Q.E.D.

Thus we can set up the existence criterion as follows

EXISTENCE CRITERION THEOREM. *Let f be an isometry of a nonpositively curved compact manifold M such that $f^k = 1$ for some integer $k \geqq 1$ and $F(f) = \{p_1, p_2, \cdots, p_r\}$, then the following statements are mutually equivalent.*

(1) *There does not exist a non-trivial f-invariant geodesic.*

(2) *$\tilde{f}^k = 1$ for any covering isometry $\tilde{f}$ of f.*

(3) *The set of covering isometries of $\tilde{f}$ is just $\bigcup_{i=1}^{r} G_{p_i}$.*

In particular assume only the finiteness of the fixed point of f, then (1) *and* (3) *are equivalent.*

Proof. (1) $\Rightarrow$ (2). Suppose that there is a covering $\tilde{f}$ with $\tilde{f}^k = 1$, then $\tilde{f}$ has no fixed point by Lemma 5 and so the case (a) occurs which implies the existence of a non-trivial f-invariant geodesic. This is impossible.

(2) $\Rightarrow$ (3). If there is a covering $\tilde{f}$ such that $\tilde{f} \notin \bigcup_{i=1}^{r} G_{p_i}$, $\tilde{f}$ has no fixed point by Lemma 4. However it is impossible by Lemma 5.

(3) $\Rightarrow$ (1). Suppose that there exists a non-trivial f-invariant geodesic, then the case (b) occurs by the same argument as Proposition 1. By Lemma 4 it is impossible.

The second part is also same. Q.E.D.

§2. Structure

We use here the following notations.

Geo (M, f): the set of all f-invariant geodesics.

$\mathfrak{g}_{[\gamma]}^{f}$: the free homotopy class of Geo (M, f) corresponding to the conjugate class $[\gamma]$ of $\gamma \in \Gamma$.

Then in a natural sense Geo $(M, f) = \bigcup_{\gamma} \mathfrak{g}_{[\gamma]}^{f}$ and we introduce it's topology from $\Lambda(M, f)$ which was used in [5], [7]. Here we consider the relation between $\mathfrak{g}_{[\gamma]}^{f}$ and Crit $(\tilde{f})$.

Let f be an isometry of finite order k ($\geqq 1$).

G_{γ}: the set of covering isometries $\tilde{f}$ with $\tilde{f}^{k} = \gamma$ ($\gamma \in \Gamma$). Now we define an equivalence relation on G_{γ} such that $\tilde{f}$ is equivalent $\tilde{g}$ for any $\tilde{f}, \tilde{g} \in G_{\gamma}$ if and only if there is $\xi \in \Gamma_{\gamma}$ with $\tilde{g} = \xi \circ \tilde{f} \circ \xi^{-1}$. The equivalence class of $\tilde{f}$ is written by $\langle \tilde{f} \rangle$.

The main theorem of this section is the following.

STRUCTURE THEOREM I. *Let f be an isometry of finite order k on a nonpositively curved compact manifold M.*

(1) *If $\mathfrak{g}_{[\gamma]}^{f} \neq \phi$ for some $\gamma \in \Gamma$, then $\mathfrak{g}_{[\gamma]}^{f}$ is homeomorphic to $\bigcup_{\langle \tilde{f} \rangle}$ Crit $(\tilde{f})/\Gamma_{\tilde{f}}$ where $\tilde{f} \in G_{\gamma}$ and $\Gamma_{\tilde{f}} = \{\alpha \in \Gamma \mid \alpha \circ \tilde{f} = \tilde{f} \circ \alpha\}$.*

(2) *If each* Crit $(\tilde{f})$ *is a submanifold without boundary, then the homeomorphism in* (1) *is a diffeomorphism.*

Remark. Since Crit $(\tilde{f})$ is a totally geodesic submanifold with possibly non-smooth boundary (see [8]), $\mathfrak{g}_{[\gamma]}^{f}$ is also a submanifold with boundary by the theorem. In particular the case of (2) implies that $\mathfrak{g}_{[\gamma]}^{f}$ is a differentiable manifold.

Of course it is clear $\mathfrak{g}_{[\gamma]}^{f} = F(f)$ for $\gamma = 1$ and the structure of $F(f)$ is well known. Hence we assume $\gamma \neq 1$ from now.

First of all we construct a corresponding Φ: $\cup$ Crit $(\tilde{f}) \to \mathfrak{g}_{[\gamma]}^{f}$ for a covering isometry $\tilde{f}$ with $\tilde{f}^{k} = \gamma$ as follows, By Lemma 5 and (1) of Lemma 1 there exists an $\tilde{f}$-invariant geodesic $\tilde{c}_{p}$ ($p \in$ Crit $(\tilde{f})$, $\tilde{c}_{p}(0) = p$) and so we put $\Phi(p) = \pi \circ \tilde{c}_p = c_x$ $(x = c_x(0))$, then c_x is an f-invariant geodesic and moreover c_x^k is a representation of $[\gamma]$ because $\tilde{c}_p^k$ is γ-invariant, where $c_x^k(t) = c_x(kt)$, $\tilde{c}_p^k(t) = \tilde{c}_v(kt)$. Then it is easy to see the continuity of Φ.

LEMMA 6. *For any* $\tilde{f}$, $\tilde{g} \in G_\gamma$ *we have* $\tilde{f} \sim \tilde{g}$ *if and only if* $\Phi(\mathrm{Crit}\,(\tilde{f})) = \Phi(\mathrm{Crit}\,(\tilde{g}))$.

Proof. If $\tilde{f} \sim \tilde{g}$ in G_γ, there is $\xi \in \Gamma_\gamma$ with $\tilde{g} = \xi \circ \tilde{f} \circ \xi^{-1}$. For any $c_x \in \Phi(\mathrm{Crit}\,(\tilde{f}))$ there is $p \in \mathrm{Crit}\,(\tilde{f})$ with $\Phi(p) = c_x$ and so there is an $\tilde{f}$-invariant geodesic $\tilde{c}_p$ such that $\pi \circ \tilde{c}_p = c_x$. Then $\xi(p) \in \mathrm{Crit}\,(\xi \circ \tilde{f} \circ \xi^{-1}) = \mathrm{Crit}\,(\tilde{g})$ by (3) of Lemma 1. Thus $\xi(\tilde{c}_p) = \tilde{c}_{\xi(p)}$ is a $\tilde{g}$-invariant geodesic and $\Phi(\xi(p)) = \pi \circ \tilde{c}_{\xi(p)} = c_x$ which implies $\Phi(\mathrm{Crit}\,(\tilde{f})) \subset \Phi(\mathrm{Crit}\,(\tilde{g}))$. By the same argument we have $\Phi(\mathrm{Crit}\,(\tilde{g})) \subset \Phi(\mathrm{Crit}\,(\tilde{f}))$ and so $\Phi(\mathrm{Crit}\,(\tilde{f})) = \Phi(\mathrm{Crit}\,(\tilde{g}))$. Conversely we have only to show that if $\tilde{f} \not\sim \tilde{g}$ in G_γ, then $\Phi(\mathrm{Crit}\,(\tilde{f})) \cap \Phi(\mathrm{Crit}\,(\tilde{g})) = \phi$. If not, there is $c_x \in \Phi(\mathrm{Crit}\,(\tilde{f})) \cap \Phi(\mathrm{Crit}\,(\tilde{g}))$ such that there are $\tilde{f}$-invariant $\tilde{c}_p$ and $\tilde{g}$-invariant $\tilde{c}_q$ with $\pi \circ \tilde{c}_p = c_x = \pi \circ \tilde{c}_q$. Then there is $\xi \in \Gamma$ such that $\xi(\tilde{c}_p)$ is a lift of c_x through a point $q = \xi(p)$ and thus $\xi(\tilde{c}_p) = \tilde{c}_q$ because of the uniqueness of the lifting. On the other hand since $q = \xi(p) \in \mathrm{Crit}\,(\xi \circ \tilde{f} \circ \xi^{-1}) \cap \mathrm{Crit}\,(\tilde{g})$, we have $(\xi \circ \tilde{f} \circ \xi^{-1})(q) = \tilde{g}(q)$ and so $\xi \circ \tilde{f} \circ \xi^{-1} = \tilde{g}$ by the uniqueness of covering isometry. And $\xi \circ \gamma \circ \xi^{-1} = \xi \circ \tilde{f}^k \circ \xi^{-1} = (\xi \circ \tilde{f} \circ \xi^{-1})^k = \tilde{g}^k = \gamma$ implies $\xi \in \Gamma_\gamma$. This contradicts $\tilde{f} \not\sim \tilde{g}$.

Q.E.D.

Remark. M. Tanaka kindly notified the author of this lemma and of it's usefulness for proving Structure theorems I, II.

Proof of (1). Now we show that for any $c_x \in \mathfrak{g}^f_{[\gamma]}$ there are $\tilde{h} \in G_\gamma$ and an $\tilde{h}$-invariant geodesic $\tilde{c}_p$ with $\pi \circ \tilde{c}_p = c_x$. Let $\tilde{c}_q$ be a lift of c_x which is an $\tilde{g}$-invariant geodesic for some covering isometry $\tilde{g}$ with $\tilde{g}^k = \eta$ $(\eta \in \Gamma)$. Since $\pi \circ \tilde{c}_q = c_x$ is an element of $\mathfrak{g}^f_{[\gamma]}$, $\eta \in [\gamma]$ and so $\eta = \xi \circ \gamma \circ \xi^{-1}$ for some $\xi \in \Gamma$. Put $\tilde{h} = \xi^{-1} \circ \tilde{g} \circ \xi$, then $\tilde{h} \in G_\gamma$. Now take an $\tilde{h}$-invariant geodesic $\tilde{c}_{\xi^{-1}(q)}$, then $\tilde{c}_{\xi^{-1}(q)}$ is satisfied with $\pi \circ \tilde{c}_{\xi^{-1}(q)} = c_x$. Finally by this fact and Lemma 6 we have the surjection $\Phi\colon \bigcup_{\langle \tilde{f} \rangle} \mathrm{Crit}\,(\tilde{f}) \to \mathfrak{g}^f_{[\gamma]}$. Next we show that if $\Phi(p) = \Phi(q)$ for any point $p \neq q \in \mathrm{Crit}\,(\tilde{f})$, then there exists $\mu \in \Gamma_{\tilde{f}}$ such that $\mu(p) = q$. Put $c_x = \Phi(p)$ $(= \Phi(q),\ x = c_x(0))$ and let $\tilde{c}_p$, $\tilde{c}_q$ are the lifts of c_x $(\tilde{c}_p(0) = p, \tilde{c}_q(0) = q)$. Since the both lifts are $\tilde{f}$-invariant, these are $\tilde{f}^k = \gamma$-invariant. Thus there exists $\mu \in \Gamma_\gamma$ such that $\mu(p) = q$ by the fact discussed in [10]. By the way $\tilde{f}^{-1} \circ \mu \circ \tilde{f} = \xi$ for some ξ because the covering isometry normalizes Γ. Since $\mu \circ \tilde{f}(p) = \tilde{f} \circ \mu(p)$ at p, we have

$\xi(p)=\mu(p)$ and so $\xi=\mu$ because Γ has no fixed point. Hence $\tilde{f}\circ\mu=\mu\circ\tilde{f}$ and Φ induce the one to one corresponding $\bar{\Phi}$: $\bigcup_{\langle \tilde{f}\rangle}\mathrm{Crit}(\tilde{f})/\Gamma_{\tilde{f}}\to \mathfrak{g}^{f}_{[r]}$. The continuity of $\bar{\Phi}$ and $\bar{\Phi}^{-1}$ is clear from the construction. Q.E.D.

For the proof of (2) we need some lemmas. The next lemma is proved as same as Theorem 1.3.8 in [8] and so we omit here the proof.

LEMMA 7. *Suppose that* $\mathrm{Crit}(\tilde{f})$ *satisfies the assumption of* (2) *in the theorem. If* $\tilde{X}_p$ *is a tangent vector of* $\mathrm{Crit}(\tilde{f})$ *at* p *which is transversal to the* $\tilde{f}$*-invariant geodesic* $\tilde{c}_p$, *then the surface* H: $R\times(-\varepsilon,\varepsilon)\to\mathrm{Crit}(\tilde{f})$ *defined by* $H(s,t)=\tilde{c}_{\exp(t\tilde{X}_p)}(s)$ *is totally geodesic and it's curvature is zero.*

LEMMA 8. *Suppose that* $\mathrm{Crit}(\tilde{f})$ *satisfies the assumption of* (2) *in the theorem. Then* Φ: $\mathrm{Crit}(\tilde{f})\to\Lambda(M,f)$ *is a smooth immersion.*

Proof. Φ is a composition map of θ: $\mathrm{Crit}(\tilde{f})\to\Lambda(\tilde{M},\tilde{f})$ and $\pi_\sharp$: $\Lambda(\tilde{M},\tilde{f})\to\Lambda(M,f)$ defined $\pi_\sharp\circ\tilde{c}(t)=\pi(\tilde{c}(t))$. At first we show θ is smooth. Let $\theta(p)=\tilde{c}_p$, then we have only to prove $\mathrm{Exp}^{-1}\circ\theta\circ\widetilde{\exp}$ is smooth where $(U_p,\widetilde{\exp}^{-1})$ and $(V_{\tilde{c}_p},\mathrm{Exp}^{-1})$ are local charts of p and $\tilde{c}_p$. For any $\tilde{X}_p\in T_p(\mathrm{Crit}(\tilde{f}))$ with $\widetilde{\exp}(\tilde{X}_p)\in U_p$, $\mathrm{Exp}^{-1}\circ\theta\circ\widetilde{\exp}(\tilde{X}_p)=\mathrm{Exp}^{-1}\circ\theta(q)=\mathrm{Exp}^{-1}(\tilde{c}_q)=\tilde{X}$ is $H_*(\partial/\partial t)(s,t)$ by using Lemma 7 and moreover $\tilde{X}=H_*(\partial/\partial t)(s,0)$ is parallel along $\tilde{c}_p$ and $\tilde{X}(p)=\tilde{X}_p$. Since $\tilde{X}$ is parallel, it is determined uniquely by $\tilde{X}_p$ and $\mathrm{Exp}^{-1}\circ\theta\circ\widetilde{\exp}$: $\tilde{X}_p\to\tilde{X}$ is linear injective which implies θ is smooth and θ_* is injective. Consequently $\Phi=\pi_\sharp\circ\theta$ is smooth because π is a covering map. Moreover by the injectivity of θ_* and the uniqueness of lift we have Φ is also a smooth immersion. Q.E.D.

Proof of (2). Since the smooth immersion Φ in Lemma 8 is actually into $\mathfrak{g}^{f}_{[r]}$ and $\bigcup_{\langle \tilde{f}\rangle}\mathrm{Crit}(\tilde{f})/\Gamma_{\tilde{f}}\cong\mathfrak{g}^{f}_{[r]}$ by (1), $\bar{\Phi}$: $\bigcup_{\langle \tilde{f}\rangle}\mathrm{Crit}(\tilde{f})/\Gamma_{\tilde{f}}\to\mathfrak{g}^{f}_{[r]}\subset\Lambda(M,f)$ is a smooth embedding. Thus $\mathfrak{g}^{f}_{[r]}$ becomes a submanifold of $\Lambda(M,f)$ which implies that $\bar{\Phi}$ is a diffeomorphism. Q.E.D.

Remark. If M is a locally symmetric space of nonpositively curved, then each $\mathrm{Crit}(\tilde{f})$ is an analytic submanifold without boundary. And in this case $\mathrm{Crit}(\tilde{f})=Z^0_{I(\tilde{M})}(\tilde{f})\cdot x$ for $x\in\mathrm{Crit}(\tilde{f})$ where $Z^0_{I(\tilde{M})}(\tilde{f})$ is the identity component of centralizer of $\tilde{f}$ in $I(\tilde{M})$ and $I(\tilde{M})$ is the group of isometries of $\tilde{M}$ ([8]).

Until now our interest was "closed geodesics". However, the above consideration is valid for a general isometry. And so we note the similar structure theorem which was suggested by M. Tanaka.

Let G_f be the set of all covering isometries of f, then we define an equivalence relation in G_f as follows, $\tilde{f} \sim \tilde{g}$ for any $\tilde{f}, \tilde{g} \in G_f$ if and only if there is $\xi \in \Gamma$ such that $\tilde{g} = \xi \circ \tilde{f} \circ \xi^{-1}$. Put $\mathfrak{g}[\tilde{f}] = \{c \in \mathrm{Geo}(M, f) \mid \text{Each lift of } c \text{ is } \xi \circ \tilde{f} \circ \xi^{-1}\text{-invariant for some } \xi \in \Gamma\}$, then the corresponding Φ: $\mathrm{Crit}(\tilde{f}) \to \mathfrak{g}[\tilde{f}]$ is defined also as above. Now our statement is the following

STRUCTURE THEOREM II. *Let f be an isometry on a compact non-positively curved manifold. Then $\bar{\Phi}$: $\bigcup_{\langle \tilde{f} \rangle} \mathrm{Crit}(\tilde{f})/\Gamma_{\tilde{f}} \to \mathfrak{g}[\tilde{f}]$ is a homeomorphism and moreover it is a diffeomorphism if each $\mathrm{Crit}(\tilde{f})$ is a submanifold without boundary.*

§ 3. Example

Let M be a flat torus.

(1) $f = \pi/2$ rotation. Assume the fig. 1 as M. Then $F(f) = \{p_1, p_2\}$, $f^4 = 1$ and $\pi_1(M) = Z \times Z$.

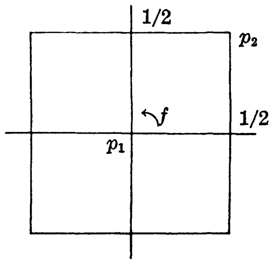

fig. 1

Any covering isometry $\tilde{f}$ of f is

$$\tilde{f}(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ n \end{bmatrix} \qquad \text{for } x \in R^2 = \tilde{M}$$

where $m, n \in Z$. It is easy to check $\tilde{f}^4 = 1$ which implies that there does not exist a nontrivial f-invariant geodesic by our theorem. In this case the set G of the covering isometries is as follows, $G = G_{p_1} \cup G_{p_2}$

$$G_{p_1} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ n \end{bmatrix} \middle| m \equiv n(2) \right\} = \{\mu \circ \tilde{f}_1 \circ \mu^{-1} \mid \mu \in \Gamma\}$$

$$G_{p_2} = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ n \end{bmatrix} \middle| m + n \equiv 1(2) \right\} = \{\mu \circ \tilde{f}_2 \circ \mu^{-1} \mid \mu \in \Gamma\}$$

where $\tilde{f}_1(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot x$, $\tilde{f}_2(x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\Gamma_{\tilde{f}_1} = \Gamma_{\tilde{f}_2} = \{\mathrm{id}\}$.

(2) $f =$ the reflection with respect to x-axis.

$$F(f) = X_1 \cup X_2, \qquad X_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \middle| -1/2 \leqq x \leqq 1/2 \right\} \quad \text{and}$$

$$X_2 = \left\{ \begin{bmatrix} x \\ \pm 1/2 \end{bmatrix} \middle| -1/2 \leqq x \leqq 1/2 \right\}$$

$$f^2 = 1 .$$

Any covering isometry $\tilde{f}$ of f is

$$\tilde{f}(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ n \end{bmatrix} \qquad x \in R^2 = \tilde{M} \quad \text{where} \quad m,\ n \in Z .$$

Now we see the property of the covering isometries. In this case $m = 0$ if and only if $\tilde{f}$ has a fixed point, and so the class G_{id} have the fixed point.

$$G_{\mathrm{id}} = G_1^{\mathrm{id}} \cup G_2^{\mathrm{id}}$$

$$G_1^{\mathrm{id}} = \{\mu \circ \tilde{f}_1 \circ \mu^{-1} | \mu \in \Gamma\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ n \end{bmatrix} \middle| n \equiv 0 \ (2) \right\}$$

$$G_2^{\mathrm{id}} = \{\mu \circ \tilde{f}_2 \circ \mu^{-1} | \mu \in \Gamma\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ n \end{bmatrix} \middle| n \equiv 1 \ (2) \right\}$$

where

$$\tilde{f}_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x,\ \tilde{f}_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and}$$

$$\Gamma_{\tilde{f}_1} = \Gamma_{\tilde{f}_2} = \left\{ \mu \in \Gamma \,|\, \mu(x) = x + \begin{bmatrix} m \\ 0 \end{bmatrix},\ m \in Z \right\}.$$

Then $\mathfrak{g}_{[1]}^{f} =$ {one point invariant geodesics} $= F(f) \cong \mathrm{Crit}\,(\tilde{f}_1)/\Gamma_{\tilde{f}_1} \cup \mathrm{Crit}\,(\tilde{f}_2)/\Gamma_{\tilde{f}_2}$.

On the other hand if $m \neq 0$, $\tilde{f}$ has no fixed point and so $\tilde{f}^2 = \gamma \neq 1$. In this case γ is a following form, $\gamma(x) = x + \begin{bmatrix} 2m \\ 0 \end{bmatrix}$. Then G_γ is

$$G_\gamma = G_1^\gamma \cup G_2^\gamma$$

$$G_1^\gamma = \{\mu \circ \tilde{g}_1 \circ \mu^{-1} | \mu \in \Gamma_\gamma\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ n \end{bmatrix} \middle| n \equiv 0 \ (2) \right\}$$

$$G_2^\gamma = \{\mu \circ \tilde{g}_2 \circ \mu^{-1} | \mu \in \Gamma_\gamma\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ n \end{bmatrix} \middle| n \equiv 1 \ (2) \right\}$$

where

$$\tilde{g}_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ 0 \end{bmatrix}, \qquad \tilde{g}_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} m \\ 1 \end{bmatrix}.$$

The structure theorem implies

$$\mathfrak{g}^f_{[\gamma]} \cong \mathrm{Crit}\,(\tilde{g}_1)/\Gamma_{\tilde{g}_1} \cup \mathrm{Crit}\,(\tilde{g}_2)/\Gamma_{\tilde{g}_2} \qquad \text{for the above } \gamma.$$

More in detail $\mathrm{Crit}\,(\tilde{g}_i) = Z_{E(2)}(\tilde{g}_i)\cdot x$ for $x \in \mathrm{Crit}\,(\tilde{g}_i)$ where $E(2) =$ Euclidean group of isometries of R^2 (see Remark in § 2). Thus we have

$$\mathrm{Crit}\,(\tilde{g}_i) = \left\{ e_i \in E(2) \,|\, e_i(x) = x + \begin{bmatrix} r_i \\ 0 \end{bmatrix},\ r_i \in R \right\}$$

$$\Gamma_{\tilde{g}_i} = \left\{ \mu_i \in \Gamma \,|\, \mu_i(x) = x + \begin{bmatrix} s_i \\ 0 \end{bmatrix},\ s_i \in Z \right\}$$

and finally $\mathrm{Crit}\,(\tilde{g}_i)/\Gamma_{\tilde{g}_i} \cong S^1$ (1-dimensional sphere). Therefore we have $\mathfrak{g}^f_{[\gamma]} \cong S^1 \cup S^1$.

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