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ON DEFORMATIONS OF HOPF MAPS AND HYPERGEOMETRIC SERIES

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Introduction

Let $\boldsymbol{R}^n$ denote the Euclidean space of dimension $n \geqq 1$ with the standard inner product $\langle x, y\rangle$ and the norm $Nx = \langle x, x\rangle$. We shall denote by $d\omega_{n-1}$ the volume element of the unit sphere $S^{n-1} = \{x \in \boldsymbol{R}^n; Nx = 1\}$ normalized so that the volume of S^{n-1} is 1.

With each continuous map $f: S^{n-1} \to \boldsymbol{R}^m$, we shall associate a function $f^\#(z)$ of a complex variable z by

$$f^\#(z) = \int_{S^{n-1}} e^{z(f(x))} d\omega_{n-1} .$$

Clearly $f^\#(z)$ is an entire function and its Taylor expansion is given by

$$f^\#(z) = \sum_{k=0}^{\infty} N_k(f) \frac{z^k}{k!}$$

where

$$N_k(f) = \int_{S^{n-1}} N(f(x))^k d\omega_{n-1} .$$

When f is spherical, i.e. when f maps S^{n-1} in S^{m-1}, we have $f^\#(z) = e^z$. When we are given a family $\{f_t\}$, $0 \leqq t \leqq 1$, of maps: $S^{n-1} \to \boldsymbol{R}^m$ such that f_0 is spherical, we have a family $\{f_t^\#\}$ of entire functions beginning with $f_0^\# = e^z$ and ending with some advanced function $f_1^\#$.

Here is an illustrative example: consider the family

$$f_t(x) = (x_1^2 - x_2^2, 2(1 + t)^{1/2} x_1 x_2), \qquad 0 \leqq t \leqq 1 .$$

The map f_0: $S^1 \to \boldsymbol{R}^2$ is spherical since it is the squaring $x \mapsto x^2$ in $\boldsymbol{C} = \boldsymbol{R}^2$. Passing to the polar coordinates, we have $N(f_t(x)) = 1 + \mathrm{t} \sin^2 2\theta$ and

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$$f_t^*(z)e^{-z} = \frac{1}{2\pi}\int_0^{2\pi} e^{zt\sin^2 2\theta}d\theta = \sum_{k=0}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots\quad 2k}\frac{(tz)^k}{k!}$$
$$= {}_1F_1(1/2; 1; tz),$$

a Kummer's hypergeometric series.

The purpose of this paper is to find similar relations between certain deformations of classical Hopf fibrations $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$ and Kummer's hypergeometric series.

§ 1. Prerequisites

As for proofs of formulas below, see our earlier paper [2].

The symbols $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}, \boldsymbol{O}$ denote the set of integers, rational numbers, real numbers, complex numbers, Hamilton's quaternions and Cayley's octonions, respectively. The set of nonnegative real numbers is denoted by $\boldsymbol{R}_+$. For a subset M of $\boldsymbol{R}$, we put $M_+ = M \cap \boldsymbol{R}_+$. The set of all $(m \times n)$-matrices over a field K is written $K_{m,n}$. If $m = n$, we write K_m for $K_{m,m}$. For a symmetric matrix $A \in K_n$ and vectors $x \in K^n$, we put $A[x] = {}^t xAx$, the quadratic form of A. For $a \in \boldsymbol{C}$, $k \in \boldsymbol{Z}_+$, the Appell's symbol is:

$$(a, k) = \begin{cases} a(a+1)\cdots(a+k-1) & \text{if } k \geqq 1 \\ 1 & \text{if } k = 0. \end{cases}$$

We have the duplication formula: $(2a, 2k) = 4^k(a, k)(a + 1/2, k)$. For $a = (a_1, \cdots, a_p) \in \boldsymbol{C}^p$, $b = (b_1, \cdots, b_q) \in \boldsymbol{C}^q$, the (generalized) hypergeometric series is defined by

$${}_pF_q(a; b; z) = \sum_{k=0}^{\infty}\frac{(a_1, k)\cdots(a_p, k)}{(b_1, k)\cdots(b_q, k)}\frac{z^k}{k!}.$$

${}_2F_1$ and ${}_1F_1$ are also called Gauss' and Kummer's series, respectively. For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \boldsymbol{R}^n$ and $\nu \in \boldsymbol{Z}_+$, the numbers $b_\nu(2; \lambda)$ are defined by the generating relation:

$$\sum_{\nu=0}^{\infty} b_\nu(2; \lambda)t^\nu = \prod_{i=1}^{n}(1 - 4\lambda_i t)^{-1/2}. \tag{1.1}$$

In particular, we have

$$b_\nu(2; 1_n) = \frac{4^\nu(n/2, \nu)}{\nu!} \qquad \text{for } 1_n = (1, \cdots, 1) \in \boldsymbol{Z}_+^n. \tag{1.2}$$

When $\lambda = (\lambda_1, \cdots, \lambda_n)$ is the set of eigenvalues of a quadratic form $q(x)$ on $\boldsymbol{R}^n$, we have

$$(1.3) \qquad \int_{S^{n-1}} q(x)^{\nu} d\omega_{n-1} = \frac{b_{\nu}(2; \lambda)}{b_{\nu}(2; 1_n)} .$$

For a continuous map $f: S^{n-1} \to \boldsymbol{R}^m$, we put

$$(1.4) \qquad f_{\nu}(\xi) = \int_{S^{n-1}} \langle \xi, f(x) \rangle^{\nu} d\omega_{n-1}, \qquad \xi \in \boldsymbol{R}^m ,$$

$$(1.5) \qquad \sigma_{\nu}(f) = \int_{S^{m-1}} f_{\nu}(\xi) d\omega_{m-1} , \qquad \nu \in \boldsymbol{Z}_+ .$$

Then, we have

$$(1.6) \qquad N_k(f) = \int_{S^{n-1}} N(f(x))^k d\omega_{n-1} = \frac{b_k(2; 1_m)}{b_k(2; 1)} \sigma_{2k}(f) \qquad k \in \boldsymbol{Z}_+ .$$

§ 2. Quadratic maps of type (S)

Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a quadratic map. By definition, each component $f_i(x)$, $1 \leqq i \leqq m$, of $f(x)$ is a quadratic form on $\boldsymbol{R}^n$ and we can write $f_i(x) = A_i[x]$ with a symmetric matrix A_i in $\boldsymbol{R}_n$. We shall obtain a general formula for the number $N_k(f)$. In view of (1.4), (1.5), (1.6), we shall consider $f_{2k}(\xi)$ and $\sigma_{2k}(f)$ in order. Since $\langle \xi, f(x) \rangle = \xi_1 f_1(x) + \cdots + \xi_m f_m(x) = \xi_1 A_1[x] + \cdots + \xi_m A_m[x]$, we have

$$(2.1) \qquad \langle \xi, f(x) \rangle = A[x] \qquad \text{with } A = \xi_1 A_1 + \cdots + \xi_m A_m .$$

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ be the eigenvalues of A. Then, by (1.3), (1.4) we have

$$(2.2) \qquad f_{2k}(\xi) = \frac{b_{2k}(2; \lambda)}{b_{2k}(2; 1_n)} .$$

From (1.3), (1.5), (1.6), (2,2), it follows that

$$N_k(f) = \beta_k \int_{S^{m-1}} b_{2k}(2; \lambda) d\omega_{m-1}$$

where

$$\beta_k = \frac{b_k(2; 1_m)}{b_k(2; 1) b_{2k}(2; 1_n)} .$$

Using (1.2) three times and the duplication formula for Appell's symbol twice, we can determine β_k explicitly and we get

$$(2.3)\qquad N_k(f) = \frac{(m/2, k)k!}{4^{2k}(n/4, k)((n+2)/4, k)} \int_{S^{m-1}} b_{2k}(2; \lambda) d\omega_{m-1}.$$

In view of (1.1), the main problem is to determine the eigenvalues of the symmetric matrix $A = \xi_1 A_1 + \cdots + \xi_m A_m$. In order to facilitate the argument, let us make the following assumptions on the quadratic map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ in terms of the matrix A in (2, 1).

(2.4) Definition. We shall say that a quadratic map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is of type (S) if the following conditions are satisfied:
(S1) n is even: $n = 2p$,
(S2) the trace of A is zero,
(S3) $A^2 = a1_{2p}$ where $a = a(\xi)$ is a positive definite quadratic form on $\boldsymbol{R}^m$.

(2.5) Proposition. *Suppose that $f: \boldsymbol{R}^{2p} \to \boldsymbol{R}^m$ is a quadratic map of type* (S). *Notation being as in* (2.4), *let $\lambda = (\lambda_1, \cdots, \lambda_{2p})$ be the set of eigenvalues of A. Then, after a necessary arrangement of λ_i's, we have $\lambda_1 = \cdots = \lambda_p = \sqrt{a}$ and $\lambda_{p+1} = \cdots = \lambda_{2p} = -\sqrt{a}$.*

Proof. Let T be an orthogonal matrix in $\boldsymbol{R}_{2p}$ such that

$$ {}^tTAT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{2p} \end{pmatrix}.$$

By (S3) we have $({}^tTAT)^2 = {}^tTA^2T = a1_{2p}$ and so $\lambda_i^2 = a$ for all i, $1 \leqq i \leqq 2p$. Our assertion then follows at once from (S2). Q.E.D.

Now, back to the formula (2.3), if f is of type (S), we have, by (1.1),

$$\sum_{\nu=0}^{\infty} b_\nu(2; \lambda)t^\nu = (1 - 4\sqrt{a}\,t)^{-p/2}(1 + 4\sqrt{a}\,t)^{-p/2}$$
$$= (1 - 4^2at^2)^{-p/2} = \sum_{k=0}^{\infty} \left(\frac{p}{2}, k\right) \frac{4^{2k}a^k}{k!} t^{2k},$$

and so

$$(2.6)\qquad b_{2k}(2; \lambda) = \left(\frac{p}{2}, k\right) \frac{4^{2k}a^k}{k!}, \qquad k \in \boldsymbol{Z}_+.$$

Combining (2.3) and (2.6), with $n = 2p$, we get

$$(2.7)\qquad N_k(f) = \frac{(m/2, k)}{((p+1)/2, k)} \int_{S^{m-1}} a^k d\omega_{m-1} \qquad \text{when } f \text{ is of type (S)}.$$

Call $\mu = (\mu_1, \cdots, \mu_m)$ the eigenvalues of $a = a(\xi)$; by (S3) all μ_i are positive. From (1.2), (1.3), (2.7), it follows that

$$(2.8) \qquad N_k(f) = \frac{k!}{4^k((p+1)/2, k)} b_k(2; \mu) \qquad \text{when } f \text{ is of type (S)}.$$

A further determination of $N_k(f)$ depends on μ, the eigenvalues of $a = a(\xi)$, via (1.1), again.

As an illustrative example, let us consider the case where $p = 1$, $m = 2$, i.e. the case of a pair of binary quadratic forms:

$$(2.9) \qquad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} \alpha_1 x_1^2 + 2\beta_1 x_1 x_2 - \alpha_1 x_2^2 \\ \alpha_2 x_1^2 + 2\beta_2 x_1 x_2 - \alpha_2 x_2^2 \end{pmatrix}.$$

Thus, we have

$$A_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1 & -\alpha_1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \beta_2 & -\alpha_2 \end{pmatrix}$$

and $A = \xi_1 A_1 + \xi_2 A_2$. Clearly the trace of A is zero. If we assume that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, one verifies easily that f is of type (S) with

$$A^2 = a(\xi) = (\alpha_1^2 + \beta_1^2)\xi_1^2 + 2(\alpha_1\alpha_2 + \beta_1\beta_2)\xi_1\xi_2 + (\alpha_2^2 + \beta_2^2)\xi_2^2,$$

this being positive definite. The characteristic polynomial of the matrix of a(ξ) is

$$(2.10) \qquad t^2 - (\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2)t + (\alpha_1\beta_2 - \alpha_2\beta_1)^2.$$

Now, by (1.1), we have

$$\sum_{k=0}^{\infty} b_k(2; \mu)t^k = (1 - 4\mu_1 t)^{-1/2}(1 - 4\mu_2 t)^{-1/2}$$
$$= (1 - 4(\mu_1 + \mu_2)t + \mu_1\mu_2(4t)^2)^{-1/2} = (1 - 2xz + z^2)^{-1/2}$$

with $x = (\mu_1 + \mu_2)/2(\mu_1\mu_2)^{1/2}$, $z = 4(\mu_1\mu_2)^{1/2}t$. In view of the well-known generating relation of the Legendre polynomials;

$$(1 - 2xz + z^2)^{-1/2} = \sum_{k=0}^{\infty} P_k(x)z^k,$$

we have

$$(2.11) \qquad b_k(2; \mu) = 4^k(\mu_1\mu_2)^{k/2} P_k\left(\frac{\mu_1 + \mu_2}{2(\mu_1\mu_2)^{1/2}}\right).$$

From (2.8), (2.10), (2.11), it follows that

$$N_k(f) = |\alpha_1\beta_2 - \alpha_2\beta_1|^k P_k\left(\frac{\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2}{2|\alpha_1\beta_2 - \alpha_2\beta_1|}\right).$$

For simplicity, put $\varDelta = |\alpha_1\beta_2 - \alpha_2\beta_1|$ and $\sigma = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2$. Then, we have

$$f^{\#}(z) = \sum_{k=0}^{\infty} N_k(f)\frac{z^k}{k!} = \sum_{k=0}^{\infty} P_k\left(\frac{\sigma}{2\varDelta}\right)\frac{(\varDelta z)^k}{k!}$$
$$= e^{\sigma z/2}{}_0F_1\left(;1;\left(\frac{z}{4}\right)^2(\sigma^2 - 4\varDelta^2)\right).^{1)}$$

If, in particular, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 = (1+t)^{1/2}$, then $\sigma = 2 + t$, $\varDelta = (1+t)^{1/2}$ and we get

$$f^{\#}(z) = f_t^{\#}(z) = e^{(1+t/2)z}{}_0F_1\left(;1;\left(\frac{tz}{4}\right)^2\right)$$

which is consistent with the formula $f_t^{\#}(z) = e^z{}_1F_1(1/2;1;tz)$ of the example in the introduction as can be verified directly.

§ 3. Deformations of Hopf maps

Throughout this section, we shall denote by X one of the algebras $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$, $\boldsymbol{O}$, of real numbers, complex numbers, Hamilton's quaternions and Cayley's octonions, respectively. Using the standard basis, X may be identified with the Euclidean space, $\boldsymbol{R}^p$, $p = 1, 2, 4, 8$, with the inner product $\langle x, y\rangle$ and the norm $Nx = \langle x, x\rangle = \bar{x}x = x\bar{x}$ where $x \mapsto \bar{x}$ is the standard involution of X. We put $Tx = \bar{x} + x$, the trace of x. Then, we have

$$\langle x, y\rangle = 1/2T(\bar{x}y). \tag{3.1}$$

The following properties of the trace

$$T(xy) = T(yx), \qquad T((xy)z) = T(x(yz)) \tag{3.2}$$

are very useful because the algebra X itself is not necessarily commutative and associative.

Let f_t, $t > -1$, be the quadratic map $\boldsymbol{R}^{2p} = X \times X \to \boldsymbol{R}^{1+p} = \boldsymbol{R} \times X$ defined by

$$f_t(z) = (Nx - Ny, 2(1+t)^{1/2}xy), \qquad z = (x, y) \in \boldsymbol{R}^{2p} = X \times X. \tag{3.3}$$

1) See [1] p. 233, line 2.

When $t = 0$, f_0 induces a map of S^{2p-1} onto S^p which is the classical Hopf fibration.

(3.4) PROPOSITION. *For each* $t > -1$, *the map* $f_t: \boldsymbol{R}^{2p} \to \boldsymbol{R}^{1+p}$ *defined by* (3.3) *is of type* (S).

Proof. Let A_t be the symmetric matrix in $\boldsymbol{R}_{2p}$ such that

$$\langle \zeta, f_t(z) \rangle = A_t[z]\,, \quad z = (x, y) \in X \times X\,, \quad \zeta = (\xi, \eta) \in \boldsymbol{R} \times X\,.$$

Substituting (3.3) in the left hand side, we have

$$\langle \zeta, f_t(z) = \xi(Nx - Ny) + 2(1 + t)^{1/2}\langle \eta, xy \rangle\,. \tag{3.5}$$

By (3.1), (3.2), we have

$$\langle \eta, xy \rangle = \tfrac{1}{2}T(\bar{\eta}(xy)) = \tfrac{1}{2}T(\overline{xy}\eta) = \tfrac{1}{2}T(\bar{y}(\bar{x}\eta)) = \tfrac{1}{2}T(\bar{x}(\eta\bar{y})) = \langle x, \eta\bar{y} \rangle\,. \tag{3.6}$$

Since, for each $\eta \in X$, the map $y \mapsto \eta\bar{y}$ is a linear endomorphism of X, there is a matrix $B(\eta)$ in $\boldsymbol{R}_p$ such that

$$\eta\bar{y} = B(\eta)y\,, \qquad \text{for all } y \in X\,. \tag{3.7}$$

Hence, from (3.5), (3.6), it follows that

$$\langle \zeta, f_t(z) \rangle = \xi(Nx - Ny) + 2(1 + t)^{1/2}\langle x, B(\eta)y \rangle\,.$$

From this, one verifies easily that

$$\langle \zeta, f_t(z) \rangle = A_t[z] \quad \text{with} \quad A_t = \begin{pmatrix} \xi 1_p & (1 + t)^{1/2}B(\eta) \\ (1 + t)^{1/2}B(\eta) & -\xi 1_p \end{pmatrix}.$$

Therefore, f_t satisfies (S1), (S2) of (2.4). Next, we shall show that

$${}^tB(\eta)B(\eta) = B(\eta){}^tB(\eta) = (N\eta)1_p\,. \tag{3.8}$$

In fact, using (3.1), (3.2), (3.7), we see that

$$\begin{aligned} \langle {}^tB(\eta)x, y \rangle &= \langle x, B(\eta)y \rangle = \langle x, \eta\bar{y} \rangle = \tfrac{1}{2}T(\bar{x}(\eta\bar{y})) \\ &= \tfrac{1}{2}T(\bar{y}(\bar{x}\eta)) = \langle y, \bar{x}\eta \rangle = \langle \bar{x}\eta, y \rangle\,, \qquad \text{for all } y \in X\,, \end{aligned}$$

which implies that

$${}^tB(\eta)x = \bar{x}\eta\,. \tag{3.9}$$

From (3.7), (3.9), we have

$$\begin{aligned} {}^tB(\eta)B(\eta)y &= {}^tB(\eta)\eta\bar{y} = (y\bar{\eta})\eta = (N\eta)y\,, \\ B(\eta){}^tB(\eta)x &= B(\eta)\bar{x}\eta = \eta(\bar{\eta}x) = (N\eta)x\,, \end{aligned}$$

which proves (3.8). It follows from (3.8) that

$$A_t^2 = a1_{2p} \quad \text{with} \quad a = a(\zeta) = \xi^2 + (1+t)N\eta\,.$$

Since $a(\zeta)$ is positive definite for $t > -1$, f_t satisfies (S3) of (2.4). Q.E.D.

Having verified that f_t is of type (S), we may use (2.7), with $m = p + 1$, and get

$$N_k(f_t) = \int_{S^p} a^k d\omega_p\,.$$

Since $\xi^2 + N\eta = N\zeta = 1$ on S^p, we have

$$N_k(f_t) = \int_{S^p} (1 + tN\eta)^k d\omega_p = \sum_{j=0}^{k} \binom{k}{j} t^j \int_{S^p} (N\eta)^j d\omega_p\,.$$

Now, the eigenvalues of the quadratic form $N\eta$ on $\boldsymbol{R}^{p+1}$ are $\mu = (0, 1, \cdots, 1) \in \boldsymbol{R}^{p+1}$, and so, by (1.2), (1.3),

$$\int_{S^p} (N\eta)^j d\omega_p = \frac{b_j(2;\mu)}{b_j(2;1_{p+1})} = \frac{j!}{4^j((p+1)/2, j)} b_j(2;\mu)\,.$$

On the other hand, since

$$\prod_{i=1}^{p+1} (1 - 4\mu_i t)^{-1/2} = (1-4t)^{-p/2} = \sum_{j=0}^{\infty} \left(\frac{p}{2}, j\right) \frac{4^j t^j}{j!}\,,$$

by (1.1), we have

$$b_j(2;\mu) = \frac{(p/2, j)4^j}{j!}\,.$$

Therefore, we have

$$\begin{aligned} N_k(f_t) &= \sum_{j=0}^{k} \binom{k}{j} t^j \frac{(p/2, j)}{((p+1)/2, j)} = \sum_{j=0}^{k} \frac{(-k, j)(p/2, j)}{((p+1)/2, j)} \frac{(-t)^j}{j!} \\ &= {}_2F_1\left(-k, \frac{p}{2}; \frac{p+1}{2}; -t\right). \end{aligned}$$

Finally, we have

$$\begin{aligned} f_t^{\sharp}(z) &= \sum_{k=0}^{\infty} N_k(f_t) \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{j=0}^{\infty} \frac{(-k, j)(p/2, j)}{((p+1)/2, j)} \frac{(-t)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{(p/2, j)(-t)^j}{((p+1)/2, j)j!} \sum_{k=0}^{\infty} \frac{(-k, j)}{k!} z^k\,, \end{aligned}$$

where the inner sum is equal to

$$\sum_{k \geqq j} (-1)^j \binom{k}{j} \frac{j!}{k!} z^k = (-1)^j \sum_{k \geqq j} \frac{z^k}{(k-j)!} = (-1)^j z^j e^z .$$

Therefore, we obtain

$$f_t^{\#}(z) = e^z_1 F_1\left(\frac{p}{2}; \frac{p+1}{2}; tz\right).$$

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