A. Noda
Nagoya Math. J.
Vol. 94 (1984), 137-164

LÉVY'S BROWNIAN MOTION; TOTAL POSITIVITY STRUCTURE OF $M(t)$-PROCESS AND DETERMINISTIC CHARACTER

AKIO NODA

§1. Introduction

Let $X = \{X(A);\ A \in Q\}$ be a Lévy's Brownian motion with the basic time parameter space Q, where Q is taken to be the n-dimensional metric space $Q^{n,\kappa}$ of constant curvature ($2 \leq n \leq \infty$, $-\infty < \kappa < \infty$), i.e., Q is one of

(a) Euclidean space for $\kappa = 0$, (b) sphere for $\kappa > 0$ and

(c) real hyperbolic space for $\kappa < 0$.

The increment $X(A) - X(B)$ is, by definition, Gaussian in distribution and has mean 0 and variance $d(A, B)$, the distance between A and B. The existence of such a Gaussian random field is well known ([3], [4], [16] and [23]).

To investigate a Lévy's Brownian motion (mainly for $\kappa = 0$), P. Lévy introduced, in [10], the $M(t)$-process:

$$M(t) = \mu(O\,|\,S_t) - X(O)\,,$$

where $S_t = \{A \in Q;\ d(A, O) = t\}$ and $\mu(O\,|\,S_t)$ is the conditional expectation of $X(O)$ given the values $\{X(A);\ A \in S_t\}$. The Gaussian process $\{M(t);\ t \in [0, T)\}$ with $T = \infty$ for $\kappa \leq 0$, $= \pi/2\sqrt{\kappa}$ for $\kappa > 0$, has the well-known expression in terms of the average of $X(A) - X(O)$ over the sphere S_t in Q. P. Lévy obtained further interesting representations of the $M(t)$-process and with the help of them he discussed Markov properties as well as non-deterministic properties of a Lévy's Brownian motion with $\kappa = 0$. In line with his approach we shall investigate the detailed structure of $M(t)$ to develop the theory of a Lévy's Brownian motion for every κ, for which we expect plenty of interesting probabilistic properties to be discovered.

In the present paper we actually find the following two properties:

Received February 28, 1983.

(i) The total positivity structure of the $M(t)$-process in the odd-dimensional case $n = 2\nu + 1$ and in the infinite-dimensional case $n = \infty$;

(ii) The deterministic character of a Lévy's Brownian motion with the infinite-dimensional parameter space $Q^{\infty,\kappa}$.

We shall first discuss the topic (i). A centered Gaussian process $Y = \{Y(t);\ t \in [0, T)\}$ $(Y(0) = 0)$ with covariance function R is said to be (strictly) totally positive if for any r, for all $0 < t_1 < \cdots < t_r < T$ and for all $0 < s_1 < \cdots < s_r < T$,

$$R\begin{pmatrix} t_1, \cdots, t_r \\ s_1, \cdots, s^r \end{pmatrix} \equiv \det\,[R(t_i, s_j)] \underset{(>)}{\geq} 0$$

holds (Definition 2). A Gaussian process with independent increments is totally positive. Our first main result is to prove that the $M(t)$-process is totally positive for $n = 2\nu + 1$ or $n = \infty$ (Theorem 2).

The following discussions in the odd-dimensional case $n = 2\nu + 1$ (Section 4) illustrate the idea of the proof of the theorem as well as the particular structure of the $M(t)$-process behind the total positivity.

The $M(t)$-process satisfies the stochastic differential equation

$$\text{(I)} \qquad LM(t) = w_0(t)\dot{B}(t), \qquad 0 < t < T,$$

with a white noise $\dot{B}(t)$, some positive function $w_0(t)$ and the differential operator L of order $\nu + 1$ expressed in terms of positive functions $\{w_i(t)\}_{i=1}^{\nu+1}$ as follows:

$$L = \frac{d}{dt}\frac{1}{w_1(t)}\frac{d}{dt}\cdots\frac{1}{w_\nu(t)}\frac{d}{dt}\frac{1}{w_{\nu+1}(t)}.$$

As an equivalent statement to the equation (I), we obtain the canonical representation of $M(t)$:

$$\text{(II)} \qquad M(t) = \int_0^t F(t, u)w_0(u)dB(u)\,,$$

where the canonical kernel $F(t, u)$ is the Green's function associated with L:

$$F(t, u) = w_{\nu+1}(t)\int_u^t w_\nu(y_1)\int_u^{y_1}\cdots\int_u^{y_{\nu-1}} w_1(y_\nu)dy_\nu\cdots dy_1\,.$$

Noting that the total positivity of the process $(d/dt)(1/w_{\nu+1}(t))\,M(t)$ implies the same for $M(t)$, it is easily seen from the formulae (I) and (II) that the $M(t)$-process is totally positive.

We also find the total positivity structure for some related Gaussian processes: The derivative $M'(t)$ of the $M(t)$-process, and the $M_{m,j}(t)$-processes which were introduced by H. P. McKean, Jr. [15] for $\kappa = 0$.

In the even-dimensional case $n = 2\nu$, it is proved that such a total positivity structure for the $M(t)$-process does not hold (Remark 2 in Section 4).

We then proceed to discuss the infinite-dimensional case $n = \infty$ (Section 3). To be surprised, the strict total positivity property holds in the infinite-dimensional case. Set $E(\tau, u) = e^{\tau u}$. We can choose a function $\tau(t)$ on $[0, T]$ such that the $M(t)$-process is expressed in the form

$$\text{(III)} \qquad M(t) = \int_{-\infty}^{0} (1 - E(\tau(t), u))\tilde{Z}(du)/\sqrt{2}\ ,$$

where $\tilde{Z}(du)$ is a Gaussian random measure with mean 0 and variance $\tilde{\gamma}(du)$, which is the spectral measure (discussed in Section 2) of a Lévy's Brownian motion. Since the kernel $E(\tau, u)$ is strictly totally positive ([7]), it now follows from the formula (III) that the processes $M(t)$ and $M'(t)$ are both strictly totally positive.

Our second topic is concerned with the prediction problems. For our purpose the expression (III) of $M(t)$ is of great advantage. Given a strictly increasing sequence $\{t_i\}_{i=1}^{\infty}$ in $(0, T)$, we shall consider the problem whether the mean square error

$$\sigma_M^2(t\,|\,\{t_i\}_{i=1}^{\infty}) = \inf_{\sum_i a_i = 1, r<\infty} E\Big[(M(t) - \sum_{i=1}^{r} a_i M(t))^2\Big]$$

is zero or not, where the infimum is taken over all coefficients $\{a_i\}_{i=1}^{r}$ for any $r < \infty$ such that $\sum_{i=1}^{r} a_i = 1$. If we observe

$$\sigma_M^2(t\,|\,\{t_i\}_{i=1}^{\infty}) = \inf_{\sum_i a_i = 1, r<\infty} \int_{-\infty}^{0} \Big(e^{\tau(t)u} - \sum_{i=1}^{r} a_i e^{\tau(t_i)u}\Big)^2 \tilde{\gamma}(du)/2\ ,$$

that comes from (III), then we see that our problem can be solved by using the theory of Müntz-Szász type approximation (see [14], [21] and [27]).

From these considerations on the $M(t)$-process given in Section 3, we find in Section 5 the deterministic character of a Lévy's Brownian motion itself. This character can be illustrated by introducing the set

$$K(e) = \{A \in Q;\ \sigma(A\,|\,e) = 0\}$$

for a non-empty subset e of Q, where $\sigma^2(A\,|\,e) = E[(X(A) - \mu(A\,|\,e))^2]$ is the mean square error of $X(A)$ given the values $\{X(B); B \in e\}$. At each point $A \in K(e)$, the random variable $X(A)$ is predictable without error. Our second main result, Theorem 4 states that, with a particular choice of $e = \bigcup_{i=1}^{N} S_{t_i}$ $(1 \leq N \leq \infty)$, we have

$$K\Big(\bigcup_{i=1}^{N} S_{t_i}\Big) = \begin{cases} Q & \text{if } N = \infty \quad \text{for } \kappa \neq 0 \text{ or if } \sum_{i=1}^{N} t_i^{-2} = \infty \quad \text{for } \kappa = 0 \,, \\ \bigcup_{i=1}^{N} S_{t_i} & \text{otherwise} \,. \end{cases}$$

This improves the Lévy's result [13] which says that $K(e) = Q$ for every $e \in Q$ containing an interior point.

The author is grateful to Professor T. Hida for his encouragement and valuable advice.

§ 2. Preliminaries

In this section, we shall first give a definition of Lévy's Brownian motion, the parameter space of which is more general than the one in the definition originally given by P. Lévy [10]. Such a Brownian motion is a most important example of a Gaussian random field with isotropic increments. As is well known, a structure function is always associated with such a Gaussian random field, and the function determines the probability distribution of the field under the additional assumption that $X(O) = 0$. We shall therefore be interested in the exact form of the spectral measure of the structure function of a Lévy's Brownian motion. It will be given in Theorem 1, which is due to G. M. Molčan [16], as well as in Proposition 1. In the final part of this section we shall give a definition of total positivity for Gaussian processes.

Parameter spaces $Q^{n,\kappa}$ that we shall be concerned with are now introduced $(2 \leq n \leq \infty,\ -\infty < \kappa < \infty)$:

$$Q^{n,\kappa} = \begin{cases} \Big\{A = (a_0, a_1, a_2, \cdots);\ a_0 = 0,\ \sum_{j=1}^{n} a_j^2 < \infty,\ a_j = 0 \text{ for } j \geq n+1\Big\} \\ \qquad\qquad \text{for } \kappa = 0 \,, \\ \Big\{A = (a_0, a_1, a_2, \cdots);\ (a_0 + 1/\sqrt{\kappa})^2 + \sum_{j=1}^{n} a_j^2 = 1/\kappa \,, \\ \qquad\qquad a_j = 0 \text{ for } j \geq n+1\Big\} \qquad \text{for } \kappa > 0 \,, \\ \Big\{A = (a_0, a_1, a_2, \cdots);\ a_0 > 0,\ (a_0 + 1/\sqrt{|\kappa|})^2 - \sum_{j=1}^{n} a_j^2 = 1/|\kappa|, \\ \qquad\qquad a_j = 0 \text{ for } j \geq n+1\Big\} \qquad \text{for } \kappa < 0 \,. \end{cases}$$

The distance $d(A, B)$ on $Q^{n,\kappa}$ is defined by the formula

$$d(A, B) = \begin{cases} \left(\sum_{j=1}^{n} (a_j - b_j)^2\right)^{1/2} & \text{for } \kappa = 0\,, \\ \cos^{-1}\left[\kappa\left\{(a_0 + 1/\sqrt{\kappa})(b_0 + 1/\sqrt{\kappa}) + \sum_{j=1}^{n} a_j b_j\right\}\right] \Big/ \sqrt{\kappa} & \\ \qquad\qquad \text{for } \kappa > 0\,, & \\ \cosh^{-1}\left[|\kappa|\left\{(a_0 + 1/\sqrt{|\kappa|})(b_0 + 1/\sqrt{|\kappa|}) - \sum_{j=1}^{n} a_j b_j\right\}\right] \Big/ \sqrt{|\kappa|} & \\ \qquad\qquad \text{for } \kappa < 0\,. & \end{cases}$$

DEFINITION 1. (i) Let $Q^{n,\kappa}$ (often denoted by Q simply) be taken to be a parameter space. A Gaussian system $X = \{X(A);\ A \in Q^{n,\kappa}\}$ is called a *Gaussian random field with isotropic increments* if $X(A) - X(B)$ has mean 0 and variance $r(d(A, B))$ with some function r on $[0, \infty)$. The function r is called the *structure function* of X.

(ii) A Gaussian random field X with isotropic increments is called a *Lévy's Brownian motion* if the structure function r is given by $r(t) = t$.

We denote by $\prod^{n,\kappa}$ the class of structure functions that are continuous. Structure functions naturally request certain properties which can easily be expressed in terms of spherical functions on Q. In order to discuss the spectral representation of $r \in \prod$, it suffices for us to treat the following three cases:

(a) $\kappa = 0$, (b) $\kappa = 1$ and (c) $\kappa = -1$.

It is well known ([19], [20], [26], [4], [3], [1] and [17]) that each $r \in \Pi$ can be represented in the form

$$(1) \qquad r(t) = \int_{\Lambda \setminus \{0\}} (1 - \Phi_\lambda(t)) d\Gamma(\lambda) + c\Psi(t)\,,$$

where (*i*) the family $\{\Phi_\lambda(t);\ \lambda \in \Lambda\}$ consists of spherical functions on Q. The exact form of $\Phi_\lambda(t)$ depends on the choice of κ as is explained below (cf. [24]).

(a) $\kappa = 0$.

$$\Phi_\lambda(t) = \begin{cases} \int_0^\pi e^{i\lambda t \cos\theta} \sin^{n-2}\theta d\theta / I_{n-2} & \text{for } n < \infty\,, \\ \lim_{m\to\infty} \Phi_\lambda^{m,0}(\sqrt{m}\, t) = e^{-\lambda^2 t^2}/2 & \text{for } n = \infty\,, \end{cases}$$

with $\lambda \in \Lambda = [0, \infty)$, and

$$I_{n-2} = \int_0^\pi \sin^{n-2}\theta d\theta = \Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}/\Gamma(n/2)\,.$$

(b) $\kappa = 1$.

$$\Phi_\lambda(t) = \begin{cases} \int_0^\pi (\cos t - i \sin t \cos \theta)^\lambda \sin^{n-2} \theta d\theta / I_{n-2} & \text{for } n < \infty , \\ \lim_{m\to\infty} \Phi_\lambda^{m,1}(t) = (\cos t)^\lambda & \text{for } n = \infty , \end{cases}$$

with $\lambda \in \Lambda = \{0, 1, 2, \cdots\}$.

(c-n) $\kappa = -1$ and $n < \infty$.

$$\Phi_\lambda(t) = \int_0^\pi (\cosh t - \sinh t \cos \theta)^\lambda \sin^{n-2} \theta d\theta / I_{n-2} ,$$

with $\lambda \in \Lambda = \Lambda_I \cup \Lambda_{II}$, $\Lambda_I = \{\lambda \in C;\ \lambda = -(n-1)/2 + iy,\ y \geq 0\}$ and $\Lambda_{II} = [-(n-1)/2, 0]$.

(c-∞) $\kappa = -1$ and $n = \infty$.

$$\Phi_\lambda(t) = \lim_{m\to\infty} \Phi_\lambda^{m,-1}(t) = (\cosh t)^\lambda ,$$

with $\lambda \in \Lambda = (-\infty, 0]$.

(ii) The function Ψ in (1) has the following expression:

(a) $\kappa = 0$. $\Psi(t) = t^2/2$.

(b) $\kappa = 1$. $\Psi(t) = 0$.

(c) $\kappa = -1$.

$$\Psi(t) = \begin{cases} \int_0^\pi \log (\cosh t - \sinh t \cos \theta) \sin^{n-2} \theta d\theta / I_{n-2} & \text{for } n < \infty , \\ \lim_{m\to\infty} \Psi^{m,-1}(t) = \log (\cosh t) & \text{for } n = \infty . \end{cases}$$

(iii) The number c is non-negative and γ is a non-negative measure on $\Lambda \backslash \{0\}$ with the following conditions:

(b) $\kappa = 1$. $\gamma(\Lambda \backslash \{0\}) < \infty$.

(a) $\kappa = 0$ or (c) $\kappa = -1$. For each neighborhood U of 0 in Λ,

$$\gamma(\Lambda \backslash U) < \infty \text{ and } \int_{U \backslash \{0\}} |\lambda|^{2+\kappa} d\gamma(\lambda) < \infty \text{ hold} .$$

We call (c, γ) the *spectral measure* of $r \in \Pi$.

We are now in a position to give the exact form of the spectral measure (c, γ) of $r(t) = t$, the structure function of a Lévy's Brownian motion, in the cases $n = 2\nu + 1$ (odd-dimension) and $n = \infty$ (infinite-dimension). In the finite-dimensional cases the following results are known ([26], [4] and [16]).

THEOREM 1. *For $n = 2\nu + 1$, the exact form of the spectral measure (c, γ) of $r(t) = t$ is given as follows*:

(a) $\kappa = 0$. $c = 0$ *and $\gamma(d\lambda)$ is absolutely continuous on Λ with the density function* $f(\lambda) = 2\nu!/(\sqrt{\pi}\,\Gamma(\nu + 1/2)\lambda^2)$.

(b) $\kappa = 1$. $c = 0$ *and* $\gamma(d\lambda) = \sum_{m=1}^{\infty} b_m \delta_{\{2m-1\}}(d\lambda)$, *with Dirac measure* $\delta_{\{a\}}$ *at a point* $a \in R$, *where*

$$b_m = \frac{4\nu!}{\sqrt{\pi}\,\Gamma(\nu + 1/2)} \frac{(2m)(2m+2)\cdots(2m+2\nu-2)(2m+\nu-1)}{(2m-1)^2(2m+1)\cdots(2m+2\nu-3)(2m+2\nu-1)^2} .$$

(c) $\kappa = -1$. $c = 1$ *and $\gamma(d\lambda)$ is absolutely continuous on Λ_I with the density function $f(y)$ for $\lambda = -\nu + iy \in \Lambda_I$ and has the form*

$$\sum_{m=1}^{[(\nu-1)/2]} b_m \delta_{\{-2m\}}(d\lambda) \qquad on\ \Lambda_{II} ,$$

where

$$f(y) = \frac{2(2\nu)!!}{\pi(2\nu-1)!!} \times \begin{cases} \dfrac{(y^2+(\nu-1)^2)(y^2+(\nu-3)^2)\cdots y^2}{(y^2+\nu^2)^2(y^2+(\nu-2)^2)\cdots(y^2+1)} & (\nu \text{ odd}) , \\[2ex] \dfrac{(y^2+(\nu-1)^2)(y^2+(\nu-3)^2)\cdots(y^2+1)}{(y^2+\nu^2)^2(y^2+(\nu-2)^2)\cdots(y^2+2^2)} & (\nu\ even) , \end{cases}$$

$$b_m = (\nu - 2m)\binom{\nu}{m}^2 \Big/ 2m(\nu - m)\binom{2\nu}{2m} .$$

It deserves to be mentioned that in the case $n = \infty$ we have a limit form of the result in Theorem 1 for each κ.

PROPOSITION 1. *For $n = \infty$, the exact form of the spectral measure (c, γ) of $r(t) = t$ is given as follows*:

(a) $\kappa = 0$. $c = 0$ *and $\gamma(d\lambda)$ is absolutely continuous on Λ with the density function* $f(\lambda) = \sqrt{2/\pi}\,\lambda^{-2}$.

(b) $\kappa = 1$. $c = 0$ *and* $\gamma(d\lambda) = \sum_{m=1}^{\infty} b_m \delta_{\{2m-1\}}(d\lambda)$ *with*

$$b_m = (2m-3)!!/(2m-2)!!\,(2m-1) .$$

(c) $\kappa = -1$. $c = 1$ *and* $\gamma(d\lambda) = \sum_{m=1}^{\infty} b_m \delta_{\{-2m\}}(d\lambda)$ *with*

$$b_m = (2m-1)!!/(2m)!!\,(2m) .$$

Proof. In view of the explicit expression of $\Phi_\lambda^{\infty,\kappa}$ and $\Psi^{\infty,\lambda}$, it suffices to prove the following formulae:

(a) $\kappa = 0$. $\sqrt{2/\pi}\int_0^\infty (1 - e^{-\lambda^2 t^2/2})\lambda^{-2}dt = t$;

(b) $\kappa = 1$. $\sum_{m=1}^\infty \frac{(2m-3)!!}{(2m-2)!!}\frac{1-x^{2m-1}}{2m-1} = \cos^{-1} x$;

(c) $\kappa = -1$. $\sum_{m=1}^\infty \frac{(2m-1)!!}{(2m)!!}\frac{1-x^{-2m}}{2m} = \cosh^{-1} x - \log x$.

In fact, these formulae can easily be shown.

In what follows we shall use the following notation and terminology (cf. [7]). For a kernel $K(x, y)$, $x \in I$, $y \in J$ $(I, J \subset R)$, set

$$K\begin{pmatrix} x_1, \cdots, x_r \\ y_1, \cdots, y_r \end{pmatrix} = \det [K(x_i, y_j)] \begin{pmatrix} x_1 < \cdots < x_r,\ y_1 < \cdots < y_r \\ x_i \in I,\ y_i \in J,\ i = 1, \cdots, r \end{pmatrix}.$$

A kernel $K(x, y)$ is said to be *totally positive* if for any r, for all $x_1 < \cdots < x_r$ and for all $y_1 < \cdots < y_r$, we have

$$K\begin{pmatrix} x_1, \cdots, x_r \\ y_1, \cdots, y_r \end{pmatrix} \geq 0 .$$

If strict positivity always holds, then we say that $K(x, y)$ is *strictly totally positive*.

DEFINITION 2, Let $Y = \{Y(t); t \in [0, T)\}$ $(Y(0) = 0)$ be a centered Gaussian process with covariance function $R(t, s)$. The Gaussian process Y is said to be (*strictly*) *totally positive* if $R(t, s)$, $(t, s) \in (0, T)^2$, is (strictly) totally positive.

We note the following property of a totally positive Gaussian process Y with the non-degenerate condition:

$$R\begin{pmatrix} t_1, \cdots, t_r \\ t_1, \cdots, t_r \end{pmatrix} > 0 \qquad \text{for all } 0 < t_1 < \cdots < t_r < T .$$

The conditional expectation $E[Y(t_0)|Y(t_i); 1 \leq i \leq r]$ $(0 < t_0 < t_1 < \cdots < t_r < T)$ is expressed in the form

$$E[Y(t_0)|Y(t_i); 1 \leq i \leq r] = \sum_{i=1}^r a_i Y(t_i) ,$$

where the uniquely determined coefficients $\{a_i\}_{i=1}^r$ satisfy

$$(-1)^{i-1} a_i \geq 0 , \qquad i = 1, \cdots, r .$$

In case Y is strictly totally positive, strict positivity holds for every i.

§ 3. $M(t)$-process; infinite-dimensional case

This section and the next are devoted to discuss the detailed structure of the $M(t)$-process of a Lévy's Brownian motion. In the first half of this section we shall obtain the spectral representation of the $M(t)$-process (Proposition 2) as a consequence of Theorem 1 and Proposition 1, and then prove our first main result, Theorem 2, in the infinite-dimensional case $n = \infty$. In the second half of this section we shall give a solution to a certain prediction problem concerning the $M(t)$-process for $n = \infty$ (Proposition 4). This fact is closely connected with the famous Müntz-Szász theorem (see, for example, [14], [21] and [27]) and will play a key role in the proof of Theorem 4 in Section 5.

We recall the definition of the $M(t)$-process. Let $X = \{X(A);\ A \in Q^{n,\kappa} = Q\}$ be a Lévy's Brownian motion. Denote by S_t the sphere in Q with radius t and center at the origin $O = (0, 0, \cdots) \in Q$: $S_t = \{A \in Q;\ d(A, O) = t\}$. Using the notations $\mu(A \mid e) = E[X(A) \mid X(B);\ B \in e]$ and $\sigma^2(A \mid e) = E[(X(A) - \mu(A \mid e))^2]$, introduced in Section 1, for a non-empty subset e of Q, we define

$$\overline{M}(t) = \mu(O \mid S_t) \quad \text{and} \quad M(t) = \overline{M}(t) - X(O)\,.$$

The Gaussian process $\{M(t);\ t \in [0, T]\}$ is called the *$M(t)$-process*. In view of the isotropic property of X, we have the following expression of $M(t)$ for $n < \infty$:

$$M(t) = \int_{S_t} (X(A) - X(O))\,d\sigma_t(A)\,, \tag{2}$$

where σ_t is the uniform probability measure on the sphere S_t. For $n = \infty$, it is easily seen that

$$M(t) = \lim_{m \to \infty} M^{m,\kappa}(t) \tag{2'}$$

(cf. [10] and [2]).

Now the covariance function $\Gamma(t, s) = E[M(t)M(s)]$ is calculated by the use of the spectral representation (1) of $r(t) = t$. For $n < \infty$, we have

$$\begin{aligned}\Gamma(t, s) &= \int_{S_t}\int_{S_s} E[(X(A) - X(O))(X(B) - X(O))]\,d\sigma_t(A)\,d\sigma_s(B) \\ &= \frac{1}{2}\left\{t + s - \int_{S_t} d(A, B_0)\,d\sigma_t(A)\right\} \qquad (B_0 \text{ is fixed in } S_s \text{ arbitrarily.})\end{aligned}$$

$$= \frac{1}{2}\Big\{t + s - \int_{\Lambda\setminus\{0\}}\int_{S_t}(1 - \Phi_\lambda(d(A, B_0)))d\gamma(\lambda)d\sigma_t(A) \\ - c\int_{S_t}\Psi(d(A, B_0))d\sigma_t(A)\Big\}$$

$$= \frac{1}{2}\Big\{t + s - \int_{\Lambda\setminus\{0\}}(1 - \Phi_\lambda(t)\Phi_\lambda(s))d\gamma(\lambda) - c(\Psi(t) + \Psi(s))\Big\},$$

where we have made use of the formulae for Φ_λ and Ψ:

$$\int_{S_t}\Phi_\lambda(d(A, B_0))d\sigma_t(A) = \Phi_\lambda(t)\Phi_\lambda(s),$$

$$\int_{S_t}\Psi(d(A, B_0))d\sigma_t(A) = \Psi(t) + \Psi(s).$$

Again applying the formula (1) with $r(t) = t$, $\Gamma(t, s)$ is finally expressed in the form

$$\Gamma(t, s) = \int_{\Lambda\setminus\{0\}}(1 - \Phi_\lambda(t))(1 - \Phi_\lambda(s))d\gamma(\lambda)/2. \tag{3}$$

For $n = \infty$, the formula (2′) enables us to obtain $\Gamma(t, s)$ as the limit of the corresponding quantity in the finite-dimensional cases:

$$\Gamma(t, s) = \lim_{m\to\infty}\Gamma^{m,\kappa}(t, s)$$

$$= \lim_{m\to\infty}\int_{\Lambda^{m,\kappa}\setminus\{0\}}(1 - \Phi_\lambda^{m,\kappa}(t))(1 - \Phi_\lambda^{m,\kappa}(s))d\gamma^{m,\kappa}(\lambda)/2.$$

It can easily be shown by Theorem 1 and Proposition 1 that the formula (3) also holds in the infinite-dimensional case.

The following proposition is an immediate consequence of the above expression (3) of $\Gamma(t, s)$.

PROPOSITION 2. *For every pair* (n, κ) $(2 \le n \le \infty,\ -\infty < \kappa < \infty)$, *the* $M(t)$*-process can be expressed in the form*

$$M(t) = \int_{\Lambda\setminus\{0\}}(1 - \Phi_\lambda(t))Z(d\lambda)/\sqrt{2}, \tag{4}$$

where $Z(d\lambda)$ *is a Gaussian random measure with mean* 0 *and variance* $\gamma(d\lambda)$, *which is the spectral measure of a Lévy's Brownian motion.*

From now onward in this section we shall discuss the case $n = \infty$. Introduce the following changes of variables $t \in [0, T)$ and $\lambda \in \Lambda\setminus\{0\}$:

$$\tau = \tau(t) = \begin{cases} t^2/2 \\ -\log(\cos t) \\ \log(\cosh t), \end{cases} \qquad u = u(\lambda) = \begin{cases} -\lambda^2 & \text{for } \kappa = 0, \\ -\lambda & \text{for } \kappa = 1, \\ \lambda & \text{for } \kappa = -1. \end{cases} \tag{5}$$

Then we have $\Phi_\lambda(t) = e^{\tau u}$ with $\tau \in [0, \infty)$ and $u \in (-\infty, 0)$; the spectral measure $\gamma(d\lambda)$ changes into the measure $\tilde{\gamma}(du)$ on $(-\infty, 0)$ and the spectral representation (4) of $M(t)$ now takes the form

$$(4') \qquad M(t) = \int_{-\infty}^{0} (1 - e^{\tau u}) \tilde{Z}(du)/\sqrt{2} \ ,$$

where $\tilde{Z}(du)$ is a Gaussian random measure with mean 0 and variance $\tilde{\gamma}(du)$.

Now noting that the derivative $M'(t)$ of $M(t)$ has the expression

$$M'(t) = \tau'(t) \int_{-\infty}^{0} e^{\tau u} |u| \tilde{Z}(du)/\sqrt{2} \ ,$$

the covariance function $\gamma(t, s) \equiv E[M'(t)M'(s)] = (\partial^2/\partial t \partial s)\Gamma(t, s)$ is given by

$$\gamma(t, s) = \tau'(t)\tau'(s) \int_{-\infty}^{0} E(\tau(t), u) E(\tau(s), u) u^2 \tilde{\gamma}(du)/2 \ ,$$

where $E(\tau, u) = e^{\tau u}$. It is known ([7]) that the exponential kernel $E(\tau, u)$ is strictly totally positive.

We are now ready to prove the following

THEOREM 2 (*the case* $n = \infty$). *The following Gaussian processes are all strictly totally positive*:

(i) *The* $M(t)$*-process;*

(ii) *The* $M'(t)$*-process.*

Proof. The assertion (i) follows easily from (ii) if we note the relation $\Gamma(t, s) = \int_0^t \int_0^s \gamma(u, v) du dv$. We shall prove (ii). Since $\tau'(t) > 0$ on $(0, T)$, it suffices to prove that the kernel

$$K(\tau, \sigma) = \int_{-\infty}^{0} E(\tau, u) E(\sigma, u) m(du) \ , \quad m(du) = u^2 \tilde{\gamma}(du) \ , \quad 0 < \tau, \ \sigma < \infty \ ,$$

is strictly totally positive. We make use of the basic composition formula (see [7]): Let $K_1(x, y)$, $K_2(y, z)$ and $K_3(x, z)$ ($x \in I$, $y \in J$, $z \in L$; $I, J, L \subset R$) be kernels satisfying

$$K_3(x, z) = \int_J K_1(x, y) K_2(y, z) d\mu(y)$$

with some measure μ on J. Then, for any r, for all $x_1 < \cdots < x_r$ and for all $z_1 < \cdots < z_r$, we have

$$(6) \qquad K_3\begin{pmatrix} x_1, \cdots, x_r \\ z_1, \cdots, z_r \end{pmatrix} = \int \cdots \int_{y_1 < \cdots < y_r} K_1\begin{pmatrix} x_1, \cdots, x_r \\ y_1, \cdots, y_r \end{pmatrix} K_2\begin{pmatrix} y_1, \cdots, y_r \\ z_1, \cdots, z_r \end{pmatrix} \times d\mu(y_1) \cdots d\mu(y_r) .$$

It follows from this formula (6) that for any r, for all $\tau_1 < \cdots < \tau_r$ and for all $\sigma_1 < \cdots < \sigma_r$,

$$K\begin{pmatrix} \tau_1, \cdots, \tau_r \\ \sigma_1, \cdots, \sigma_r \end{pmatrix} = \int \cdots \int_{u_1, < \cdots < u_r} E\begin{pmatrix} \tau_1, \cdots, \tau_r \\ u_1, \cdots, u_r \end{pmatrix} E\begin{pmatrix} \sigma_1, \cdots, \sigma_r \\ u_1, \cdots, u_r \end{pmatrix} \times m(du_1) \cdots m(du_r) > 0 ,$$

which completes the proof.

From the strict total positivity of $M'(t)$, we obtain an interesting property of sign changes of the conditional correlation function $\rho(A, O | \bigcup_{i=1}^r S_{t_i}) \equiv E[\xi(A | \bigcup_{i=1}^r S_{t_i})\xi(O | \bigcup_{i=1}^r S_{t_i})]$ $(0 < t_1 < \cdots < t_r < T)$, where $\xi(A | e)$, $e \subset Q$, is defined by

$$\xi(A | e) = \begin{cases} (X(A) - \mu(A | e))/\sigma(A | e) & \text{if } \sigma(A | e) > 0 , \\ 0 & \text{if } \sigma(A | e) = 0 . \end{cases}$$

PROPOSITION 3. *For $n = \infty$ and every κ, and for all $0 < t_1 < \cdots < t_r < T$, we have*

$$(-1)^i \rho\Big(A, O \Big| \bigcup_{j=1}^r S_{t_j}\Big) > 0 \qquad \textit{if } t_i < d(A, O) < t_{i+1}$$
$$(i = 0, 1, \cdots, r;\ t_0 = 0,\ t_{r+1} = T) .$$

The detailed discussion of the conditional correlation function $\rho(A, O | \bigcup_{j=1}^r S_{t_j})$ is seen in [18] and so the proof of Proposition 3 is omitted here.

In the rest of this section we investigate a certain prediction problem concerning the $M(t)$-process. For a non-empty subset e of $(0, T)$ and for $t \in [0, T] \backslash e$, we put

$$\mu_M(t | e) = E[\overline{M}(t) | \overline{M}(s);\ s \in e] .$$

We note that $\mu_M(t | e)$ may be calculated by assuming that $\overline{M}(s_0) = 0$ with some point $s_0 \in e$, because the quantity $\mu_M(t | e)$ depends only on $\{\overline{M}(u) - \overline{M}(v) = M(u) - M(v);\ 0 \leq u,\ v < T\}$. The predictability that we are now concerned with is to determine whether the mean square error

$$\sigma_M^2(t | e) = E[(\overline{M}(t) - \mu_M(t | e))^2]$$

is zero or not. While, the analytic property of the $M(t)$-process, which was first noted by P. Lévy [11] for $\kappa=0$ and is easily seen from the expression (4′) of $M(t)$, tells us that $\sigma_M(t|e)=0$ for any $t\notin e$ if a subset e has an accumulation point in $(0, T)$. This leads us to consider the case $e=\{t_i\}_{i=1}^N$ with a strictly increasing sequence $\{t_i\}_{i=1}^N$ in $(0, T)$ $(1\leq N\leq\infty)$.

PROPOSITION 4. *For $n=\infty$ and every κ, and for a strictly increasing sequence $\{t_i\}_{i=1}^N$ in $(0, T)$, we have*

$$\sigma_M(t|\{t_i\}_{i=1}^N)\begin{cases}=0 & \text{if } \sum_{i=1}^N h(t_i)=\infty\,,\\ >0 & \text{if } \sum_{i=1}^N h(t_i)<\infty\,,\end{cases}$$

for any $t\notin\{t_i\}_{i=1}^N$, where

$$h(t)=\begin{cases}t^{-2} & \text{for } \kappa=0\,,\\ 1 & \text{for } \kappa\neq 0\,.\end{cases}$$

Proof. We first note the following formulae of the mean square error:

$$\begin{aligned}\sigma_M^2(t|\{t_i\}_{i=1}^N) &= \inf_{\substack{\sum_i a_i=1,\\ r=N<\infty \text{ or } r<N=\infty}} E\left[(\overline{M}(t)-\sum_{i=1}^r a_i\overline{M}(t_i))^2\right]\\ &= \inf_{\substack{\sum_i a_i=1,\\ r=N<\infty \text{ or } r<N=\infty}} E\left[(M(t)-\sum_{i=1}^r a_i M(t_i))^2\right]\,,\end{aligned}$$

where the infimum is taken over all coefficients $\{a_i\}_{i=1}^r$ such that $\sum_{i=1}^r a_i=1$ with $r=N$ if $N<\infty$ or any $r<\infty$ if $N=\infty$. In view of the expression (4′) of $M(t)$ we can also write

$$\begin{aligned}\sigma_M^2(t|\{t_i\}_{i=1}^N) &= \inf_{\substack{\sum_i a_i=1,\\ r=N<\infty \text{ or } r>N=\infty}} \int_{-\infty}^0 (e^{\tau u}-\sum_{i=1}^r a_i e^{\tau_i u})^2\tilde{\gamma}(du)/2\\ &= \inf_{\substack{\{a_i\}_{i=2}^r,\\ r=N<\infty \text{ or } r<N=\infty}} \int_{-\infty}^0 \left\{\left(e^{\tau u}-e^{\tau_1 u}\right)-\sum_{i=2}^r a_i(e^{\tau_i u}-e^{\tau_1 u})\right\}^2\tilde{\gamma}(du)/2\,,\end{aligned}$$

where we put $\tau=\tau(t)$ and $\tau_i=\tau(t_i)$ (see (5)).

In case $N<\infty$, the problem is easily solved: Noting that the support of the spectral measure $\tilde{\gamma}(du)$ contains an infinite number of points (Proposition 1), we always have

$$\sigma_M(t|\{t_i\}_{i=1}^N)>0 \qquad \text{for any } t\notin\{t_i\}_{i=1}^N\,.$$

Next consider the case $N=\infty$. We first prove that $\sigma_M(t|\{t_i\}_{i=1}^\infty)=0$ for any $t\notin\{t_i\}_{i=1}^\infty$ in the following two cases:

$$\text{(i)}\ \sum_{i=1}^{\infty}\tau_i^{-1}=\infty \text{ for every } \kappa \text{ and } \quad \text{(ii)}\ \sum_{i=1}^{\infty}\tau_i^{-1}<\infty \text{ for } \kappa\neq 0.$$

Our aim is to prove that $H(\{\tau_i\}_{i=1}^{\infty})$ coincides with the whole space $H\equiv L^2((-\infty,0),\ d\tilde{\gamma}(u)/2)$, where we denote by $H(\{\tau_i\}_{i=1}^{\infty})$ the closed subspace of H generated by the system of functions $\{e^{\tau_i u}-e^{\tau_1 u};\ i=2,3,\cdots\}$. Indeed, this is a stronger assertion than what is requested to prove.

Suppose that $H(\{\tau_i\}_{i=1}^{\infty})\subsetneq H$ were true. Then we can choose a non-zero function $f(u)\in H$ such that

$$(f(u), e^{\tau_i u}-e^{\tau_1 u})_H=0 \qquad \text{for all } i=2,3,\cdots.$$

For a complex number z in $C_+\equiv\{z\in C;\ \mathrm{Re}\,z>0\}$, consider the function

$$F(z)=\int_{-\infty}^{0}(e^{zu}-e^{\tau_1 u})f(u)d\tilde{\gamma}(u)/2\,,$$

which is holomorphic in C_+ with real zeros $\{\tau_i\}_{i=1}^{\infty}$. Taking an arbitrarily small number $\varepsilon\in(0,\tau_1)$, we see that the function

$$F'(z)=\int_{-\infty}^{0}e^{zu}f(u)ud\tilde{\gamma}(u)/2\,,$$

for $z\in C_\varepsilon\equiv\{z\in C;\ \mathrm{Re}\,z\geq\varepsilon\}$, is bounded and has real zeros $\{s_i\}_{i=1}^{\infty}$ such that $\tau_i<s_i<\tau_{i+1}$.

Now we have to deal with the two cases (i) and (ii) separately. The case (i) $\sum_{i=1}^{\infty}\tau_i^{-1}=\infty$ for every κ is easily treated; indeed, we have $\sum_{i=1}^{\infty}s_i^{-1}=\infty$ and this implies that $F'(z)\equiv 0$ i.e. $F(z)\equiv 0,\ z\in C_\varepsilon$ (see, for example, [27] p. 85). We have thus proved that $f(u)\equiv 0$ i.e. $H(\{\tau_i\}_{i=1}^{\infty})=H$.

On the other hand, in the case (ii) $\sum_{i=1}^{\infty}\tau_i^{-1}<\infty$ for $\kappa\neq 0$, we see that the support of $\tilde{\gamma}$ consists of negative (odd for $\kappa>0$, or even for $\kappa<0$) integers $\{u_m\}_{m=1}^{\infty}$ (Proposition 1). Hence the function

$$F'(z)=\sum_{m=1}^{\infty}e^{zu_m}f(u_m)u_m b_m/2$$

is a bounded holomorphic almost periodic function in C_ε. By the Bohr theorem (see [9] p. 270), it follows that the zeros of such a function $F'(z)$ are in the region $\{z\in C_\varepsilon;\ \mathrm{Re}\,z\leq c\}$ with some $c>0$. This contradicts the fact that we have real zeros $\{s_i\}_{i=1}^{\infty}$ of $F'(z)$ such that $\lim_{i\to\infty}s_i=\infty$. We have thus proved that $H(\{\tau_i\}_{i=1}^{\infty})=H$ also in the case (ii).

Finally, for $\kappa=0$, we prove that $\sigma_M(t\,|\,\{t_i\}_{i=1}^{\infty})>0$ for any $t\notin\{t_i\}_{i=1}^{\infty}$ if $\sum_{i=1}^{\infty}\tau_i^{-1}<\infty$ i.e. if $\sum_{i=1}^{\infty}h(t_i)<\infty$. Noting that the spectral measure $\tilde{\gamma}(du)$ has the continuous density $\tilde{f}(u)$, we can find three numbers a, b and

c such that $-\infty < a < b < 0$ and $\tilde{f}(u) \geq 2c > 0$ for $a \leq u \leq b$. It follows that

$$\sigma_M^2(t \mid \{t_i\}_{i=1}^{\infty}) \geq \inf_{\substack{\{a_i\}_{i=1}^r, \\ \text{any } r<\infty}} \int_{-\infty}^{0} \left(e^{\tau u} - \sum_{i=1}^{r} a_i e^{\tau_i u}\right)^2 \tilde{f}(u) du/2$$

$$\geq c \inf_{\substack{\{a_i\}_{i=1}^r, \\ \text{any } r<\infty}} \int_{a}^{b} \left(e^{\tau u} - \sum_{i=1}^{r} a_i e^{\tau_i u}\right)^2 du .$$

Now we appeal to the Müntz-Szász theorem ([14] and [21]) to see that the last expression is always positive for $\tau \notin \{\tau_i\}_{i=1}^{\infty}$. This completes the proof of Proposition 4.

Remark 1. In the proof of Proposition 4, we have used only the fact that the spectral measure $\tilde{\gamma}$ of a Lévy's Brownian motion has a continuous density function for $\kappa = 0$ and is discrete with the maximum point in the support of $\tilde{\gamma}$ for $\kappa \neq 0$. The exact form of $\tilde{\gamma}$ (Proposition 1) plays no role, which means that Proposition 4 can be extended to some other Gaussian random fields with isotropic increments. In this paper, however, we content ourselves with giving considerations only on a Lévy's Brownian motion.

§4. *M*(*t*)-process; odd-dimensional case

This section is devoted to the investigation of the total positivity structure of $M(t)$-processes and of some Gaussian processes derived from them in the case of odd-dimensional parameter: $n = 2\nu + 1$. We shall begin with a review of the canonical representation of $M(t)$ (Proposition 5) due to S. Takenaka, I. Kubo and H. Urakawa [23]. By using the representation we shall obtain the stochastic differential equation satisfied by the $M(t)$-process (Proposition 5′). Alternative approach, more appropriate to investigate $M(t)$-processes, to the equation will be given in the cases (a) $\kappa = 0$ and every ν; (b) $\kappa = 1$ and $\nu = 2$. We use the spectral representation (4) of $M(t)$ established in Section 3. Some other Gaussian processes involving $M'(t)$-processes and McKean's $M_{m,j}(t)$-processes are also discussed on the same lines, and the total positivity is proved for these processes.

We begin with the canonical representation of $M(t)$.

PROPOSITION 5 ([23]). *For $n = 2\nu + 1$ and every κ, the canonical representation of $M(t)$ is given by*

(7) $$M(t)=\int_0^t P(g(u)/g(t))w_0(u)dB(u)\,, \qquad 0<t<T\,,$$

where $B(t)$ *is a Brownian motion,* $w_0(t)=\sqrt{(2\nu)!/2}\,(c(t))^\nu$,

$$P(x)=\int_x^1 (1-u^2)^{\nu-1}du/(2\nu-2)!!\,, \qquad 0<x<1\,,$$

and

$$g(t)=\begin{cases} t \\ \tan t \\ \tanh t\,, \end{cases} \qquad c(t)=\begin{cases} 1 & \text{for } \kappa=0\,, \\ \cos t & \text{for } \kappa=1\,, \\ \cosh t & \text{for } \kappa=-1\,. \end{cases}$$

We now have the expression

(8) $$P(g(u)/g(t))=w_{\nu+1}(t)\int_u^t w_\nu(y_1)\int_u^{y_1}\cdots\int_u^{y_{\nu-1}} w_1(y_\nu)dy_\nu\cdots dy_1\,,$$

with the positive functions $\{w_i(t)\}_{i=1}^{\nu+1}$ given by

$$\begin{cases} w_1(t)=g'(t)\,, \quad w_i(t)=g'(t)g(t) \quad (2\le i\le \nu)\,, \\ w_{\nu+1}(t)=(g(t))^{1-2\nu}\,. \end{cases}$$

Define a differential operator L of order $\nu+1$ by

$$L=\frac{d}{dt}\frac{1}{w_1(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{w_{\nu+1}(t)}\,.$$

Then L can be expressed in the form

$$L=\frac{d}{dt}D_1D_2\cdots D_\nu\,,$$

where

$$D_i=\{g'(t)(g(t))^{2i-2}\}^{-1}\frac{d}{dt}\{g(t)\}^{2i-1}\,.$$

The canonical kernel $P(g(u)/g(t))$ is proved to be the Green's function associated with L. This fact enables us to paraphrase Proposition 5 in the following form (cf. [6]).

PROPOSITION 5′. *For* $n=2\nu+1$ *and every* κ, *the* $M(t)$*-process is a* $(\nu+1)$*-ple Markov Gaussian process in the restricted sense determined by the stochastic differential equation*

(9) $$LM(t)=w_0(t)\dot{B}(t)\,, \qquad 0<t<T\,,$$

where $\dot{B}(t)$ is a white noise.

We now give an alternative proof of Proposition 5′ using the spectral representation (4) of $M(t)$ in the following cases:

(a) $\kappa = 0$ and every ν; (b) $\kappa = 1$ and $\nu = 2$.

The calculations for $\kappa = 1$ and general ν would be so complicated that we have to content ourselves with treatment only in the particular case $\nu = 2$. It is interesting to note that there are significant differences between the proofs in (a) and (b) depending on the curvatures of the parameter spaces.

The proof of the equation (9) uses the following formula for the spherical function Φ_λ, the proof of which will be given in Appendix.

LEMMA 1. *For $\kappa = 0$ or 1,*

$$D_\nu \Phi_\lambda^{2\nu+1,\kappa}(t) = (2\nu - 1)\left\{\frac{\kappa\lambda + 2\nu}{2(\kappa\lambda + \nu)}\Phi_\lambda^{2\nu-1,\kappa}(t) + \frac{\kappa\lambda}{2(\kappa\lambda + \nu)}\Phi_{\lambda+2}^{2\nu-1,\kappa}(t)\right\} \tag{10}$$
$$(\nu = 1, 2, \cdots).$$

(a) The proof of the equation (9) for $\kappa = 0$.

Applying the formula (10) to the spectral representation (4) of $M(t)$, we get

$$D_1 D_2 \cdots D_\nu M(t) = (2\nu - 1)!! \int_0^\infty \{1 - \cos \lambda t\} Z(d\lambda)/\sqrt{2}\ .$$

Set

$$W(d\lambda) = \sqrt{\frac{\pi(2\nu - 1)!!}{2(2\nu)!!}}\,\lambda Z(d\lambda)\,.$$

Then we have

$$LM(t) = w_0(t)\sqrt{2/\pi}\int_0^\infty \sin \lambda t\, W(d\lambda)\,.$$

It follows from Theorem 1 that $E[W^2(d\lambda)] = d\lambda$, and making use of the Parseval identity for the sine transform, it can easily be shown that $\sqrt{2/\pi}\int_0^\infty \sin \lambda t\, W(d\lambda)$ is a white noise. Hence the proof is completed.

(b) The proof of the equation (9) for $\kappa = 1$ and $\nu = 2$.

In the same way as in (a), we apply the formula (10) to (4) to obtain

$$LM(t) = \frac{3}{2^{5/2}}\sum_{\substack{\lambda = 2m-1,\\ m\geq 1}} \sin \lambda t\left\{\frac{\lambda(\lambda + 4)}{\lambda + 1}Z_\lambda + (\lambda - 2)(\lambda + 2)\left(\frac{1}{\lambda + 1} + \frac{1}{\lambda - 1}\right)\right.$$
$$\left.\times Z_{\lambda-2} + \frac{(\lambda - 4)\lambda}{\lambda - 1}Z_{\lambda-4}\right\},$$

where $Z_\lambda = 0$ for $\lambda < 0$. Set

$$\tilde{Z}_\lambda = \sqrt{\frac{3\pi}{2}} \frac{\lambda(\lambda+4)}{2^2} Z_\lambda \quad \text{and} \quad \eta_\lambda = (\tilde{Z}_\lambda + \tilde{Z}_{\lambda-2})/(\lambda+1)\,, \quad \lambda = 2m-1\,.$$

Then we have

$$LM(t) = 2\sqrt{3\pi}\cos t \sum_{m=1}^{\infty} \eta_{2m-1} \sin 2mt\,.$$

From Theorem 1 it follows that $E[\tilde{Z}_{2m-1}^2] = 2m(2m+2)$, which implies that the η_{2m-1} satisfy the following conditions:

$$E[\eta_{2m-1}^2] = 2,\ E[\eta_{2m-1}\eta_{2m+1}] = 1 \text{ and } E[\eta_{2m-1}\eta_{2m+2j-1}] = 0$$

for all $j \geq 2$. In terms of a mutually independent standard Gaussian random sequence $\{\xi_{2m-1}\}_{m=0}^{\infty}$, the η_{2m-1} can be represented in the form

$$\eta_{2m-1} = \xi_{2m-1} + \xi_{2m-3}\,, \qquad m = 1, 2, \cdots.$$

Hence we have

$$LM(t) = w_0(t)\frac{2}{\sqrt{\pi}} \sum_{m=0}^{\infty} \xi_{2m-1} \sin(2m+1)\,t\,.$$

Noting that $\{(2/\sqrt{\pi})\sin(2m+1)t\}_{m=0}^{\infty}$ is a C.O.N.S. in L^2 ([0, T], dt), it can be shown that $(2/\sqrt{\pi})\sum_{m=0}^{\infty}\xi_{2m-1}\sin(2m+1)t$ is a white noise on $(0, T)$. The proof is thus completed.

We shall apply the formula (10) to the investigation of the $M_{m,j}(t)$-processes for $\kappa = 0$. Take the C.O.N.S. $\{h_{m,j}(\xi);\, 1 \leqq j \leqq d_m,\, m = 0, 1, \cdots\}$ in $L^2(S^{n-1}, d\sigma(\xi))$, where the $h_{m,j}$ $(1 \leq j \leq d_m)$ are the spherical harmonics of degree m. Then,

$$M_{m,j}(t) = \int_{S^{n-1}} (X(t\xi) - X(O))h_{m,j}(\xi)d\sigma(\xi)\,, \qquad t > 0\,. \tag{11}$$

These processes are mutually independent Gaussian processes and $M_{0,1}(t)$ for $m = 0$ coincides with the $M(t)$-process. It is known ([25]) that $M_{m,j}(t)$ with $m \geq 1$ has the following representation similar to (4):

$$M_{m,j}(t) = \int_0^\infty \Psi_\lambda^m(t) Z_{m,j}(d\lambda)/\sqrt{2}\,, \tag{12}$$

where

$$\Psi_\lambda^m(t) = c_m(\lambda t)^m \Phi_\lambda^{2m+n,0}(t)$$

with some constant c_m, and $Z_{m\ j}(d\lambda)$ is a Gaussian random measure with mean 0 and variance $\gamma(d\lambda)$.

We are ready to prove the following

PROPOSITION 6. *For $n = 2\nu + 1$ and $\kappa = 0$, the $M_{m,j}(t)$-process is a $(\nu + 1)$-ple Markov Gaussian process in the restricted sense determined by the stochastic differential equation*

$$L_m M_{m,j}(t) = w_{0,m}(t)\dot{B}_{m,j}(t)\,, \qquad t > 0\,, \tag{13}$$

with a white noise $\dot{B}_{m,j}(t)$ and

$$L_m = \frac{d}{dt}\frac{1}{w_{1,m}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{w_{\nu+1,m}(t)}\,,$$

where $\{w_{i,m}(t);\ 0 \leq i \leq \nu + 1\}$ are the positive functions given by

$$\begin{cases} w_{0,m}(t) = \sqrt{(2\nu)!/2}\ \nu t^{m-1}\,, \quad w_{i,m}(t) = t \quad (1 \leq i \leq \nu)\,, \\ w_{\nu+1,m}(t) = t^{1-2\nu-m}\,. \end{cases}$$

Proof. Noting that $L_m = t^{2m-2}D_m D_{m+1}\cdots D_{m+\nu}t^{-m}$, and applying the formula (10) to (12) to obtain

$$\begin{aligned} L_m M_{m,j}(t) = c_m(2m + 2\nu - 1)(2m + 2\nu - 3)\cdots(2m - 1)t^{2m-2} \\ \times\int_0^\infty \Phi_\lambda^{2m-1,0}(t)\lambda^m Z_{m,j}(d\lambda)/\sqrt{2}\,. \end{aligned}$$

Set

$$W_{m,j}(d\lambda) = \left\{2^{m-3/2}\Gamma\left(m - \frac{1}{2}\right)\frac{2\nu!}{\sqrt{\pi}\,\Gamma(\nu + 1/2)}\right\}^{-1/2}\lambda^m Z_{m,j}(d\lambda)\,.$$

Then we have

$$E[W_{m,j}^2(d\lambda)] = f_m(\lambda)d\lambda\,, \qquad f_m(\lambda) = \{2^{m-3/2}\Gamma(m - \tfrac{1}{2})\}^{-1}\lambda^{2m-2}\,,$$

and

$$L_m M_{m,j}(t) = c'_m f_m(t)\int_0^\infty \Phi_\lambda^{2m-1,0}(t)W_{m,j}(d\lambda)$$

with some constant c'_m. By using the Parseval identity for the Hankel transform

$$\hat{\phi}(\lambda) = \int_0^\infty \Phi_\lambda^{2m-1,0}(t)\phi(t)f_m(t)dt\,,$$

it can be shown that $\sqrt{f_m(t)}\int_0^\infty \Phi_\lambda^{2m-1,0}(t)W_{m,j}(d\lambda)$ is a white noise. We denote it by $\dot{B}_{m,j}(t)$. With this notation we have

$$L_m M_{m,j}(t) = c_m'' t^{m-1}\dot{B}_{m,j}(t)$$

with some constant c_m''. Evaluating the variance of $M_{m,j}(t)$, we get $c_m'' = \sqrt{(2\nu)!/2\nu}$, which completes the proof.

The canonical representation of $M_{m,j}(t)$ follows immediately from the equation (13) (cf. [15]):

$$M_{m,j}(t) = \int_0^t F_m(t, u)w_{0,m}(u)dB_{m,j}(u)\,, \tag{14}$$

with the Green's function $F_m(t, u)$ associated with L_m:

$$F_m(t, u) = w_{\nu+1,m}(t)\int_u^t w_{\nu,m}(y_1)\int_u^{y_1}\cdots\int_u^{y_{\nu-1}} w_{1,m}(y_\nu)dy_\nu\cdots dy_1\,. \tag{15}$$

We now return to the $M(t)$-process for every κ. In view of the relation

$$\frac{d}{dt}D_i = g'(t)D_{i+1/2}(g'(t))^{-1}\frac{d}{dt'}\,,$$

the differential operator L in (9) can be rewritten in the form

$$L = g'(t)D_{3/2}D_{5/2}\cdots D_{\nu+1/2}(g'(t))^{-1}\frac{d}{dt}\,.$$

This enables us to obtain the following equation for the $M'(t)$-process:

$$\tilde{L}M'(t) = \tilde{w}_0(t)\dot{B}(t)\,,\qquad 0 < t < T\,, \tag{16}$$

with

$$\tilde{L} = \frac{d}{dt}\frac{1}{\tilde{w}_1(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{\tilde{w}_\nu(t)}\,,$$

where $\{\tilde{w}_i(t);\, 0\le i\le\nu\}$ are the positive functions given by

$$\begin{cases}\tilde{w}_0(t) = g(t)w_0(t)\,, & \tilde{w}_i(t) = g'(t)g(t)\quad (1\le i\le \nu-1)\,,\\ \tilde{w}_\nu(t) = g'(t)(g(t))^{-2\nu}\,. & \end{cases}$$

We have thus proved the following

PROPOSITION 7. *For $n = 2\nu + 1$ and every κ, the $M'(t)$-process is a ν-ple Markov Gaussian process in the restricted sense determined by the stochastic differential equation* (16).

The canonical representation of $M'(t)$ is given by

$$M'(t) = \int_0^t \tilde{F}(t, u)\tilde{w}_0(u)dB(u)\,, \tag{17}$$

with the Green's function $\tilde{F}(t, u)$ associated with $\tilde{L}$:

$$\tilde{F}(t, u) = \tilde{w}_\nu(t)\int_u^t \tilde{w}_{\nu-1}(y_1)\int_u^{y_1} \cdots \int_u^{y_{\nu-2}} \tilde{w}_1(y_{\nu-1})dy_{\nu-1} \cdots dy_1\,. \tag{18}$$

Now the total positivity for the Gaussian processes $M(t)$, $M'(t)$ and $M_{m,j}(t)$ follows from Propositions 5′, 7 and 6, respectively, if we note the following simple fact: Let Gaussian processes $Y_1(t)$ and $Y_2(t)$ ($Y_1(0) = Y_2(0) = 0$) satisfy the relation

$$\frac{1}{v_1(t)}\frac{d}{dt}\frac{1}{v_2(t)}Y_2(t) = Y_1(t)\,, \qquad 0 < t < T,$$

with positive functions $v_1(t)$ and $v_2(t)$ on $[0, T)$. If $Y_1(t)$ is totally positive, then the same for $Y_2(t)$.

THEOREM 2 (the case $n = 2\nu + 1$). *The following Gaussian processes are all totally positive*:

(i) *$M(t)$ for every κ;*

(ii) *$M'(t)$ for every κ;*

(iii) *$M_{m,j}(t)$ for $\kappa = 0$ ($m \geq 1$).*

It is noted that $M''(t)$ is not totally positive ($n \geq 5$). In fact, it can easily be proved thar $E[M''(t)M''(s)] < 0$ for small t and large s.

For the Gaussian processes in Theorem 2, some specific properties can be discussed more precisely. We discuss only the $M'(t)$-process, since the others can be treated similarly.

THEOREM 3. *Let $n = 2\nu + 1$ and κ be arbitrary. For any r, for all $0 < t_1 \leq t_2 \leq \cdots \leq t_r < T$ such that $p_i \equiv \max\{j;\, t_i = t_{i-1} = \cdots = t_{i-j}\} \leq \nu - 1$, and for all $0 < s_1 \leq s_2 \leq \cdots \leq s_r < T$ such that $q_i \equiv \max\{j;\, s_i = s_{i-1} = \cdots = s_{i-j}\} \leq \nu - 1$, we have*

$$\gamma\begin{pmatrix} t_1, \cdots, t_r \\ s_1, \cdots, s_r \end{pmatrix} \equiv \det\left[\left(\frac{\partial}{\partial t_i}\right)^{p_i}\left(\frac{\partial}{\partial s_j}\right)^{q_j}\gamma(t_i, s_j)\right] \geq 0\,, \tag{19}$$

and strict positivity holds if and only if

$$\max(t_{i-\nu}, s_{i-1}) < \min(t_i, s_i)\,, \qquad i = \nu + 1, \cdots, r\,. \tag{20}$$

For the proof of Theorem 3 we use the following fact for the kernel $\tilde{F}(t, u)$ expressed in the form (18).

LRMMA 2 ([18]). *For any r, for all $0 < t_1 \leq t_2 \leq \cdots t_r < T$ such that $p_i \leq \nu - 1$, and for all $0 < u_1 < u_2 < \cdots < u_r < T$, we have*

$$\tilde{F}\binom{t_1, \cdots, t_r}{s_1, \cdots, s_r} \geq 0,$$

and strict positivity holds if and only if

$$t_{i-\nu} < u_i < t_i, \qquad i = 1, 2, \cdots, r,$$

where for $i \leq \nu$ only the right-hand inequality is relevant.

The Proof of Theorem 3. Noting the basic composition formula (6) and the fact that

$$\gamma(t, s) = \int_0^{\min(t,s)} \tilde{F}(t, u)\tilde{F}(s, u)\tilde{w}_0^2(u)du,$$

the inequality (19) follows from

$$\gamma\binom{t_1, \cdots, t_r}{s_1, \cdots, s_r} = \int \cdots \int_{u_1 < \cdots < u_r} \tilde{F}\binom{t_1, \cdots, t_r}{u_1, \cdots, u_r}\tilde{F}\binom{s_1, \cdots, s_r}{u_1, \cdots, u_r} \prod_{i=1}^{r} \tilde{w}_0^2(u_i)du_i,$$

which is to be non-negative by Lemma 2. Moreover, we see that strict positivity holds if and only if the subset U of $(0, T)^r$ has strictly positive Lebesgue measure, where

$$U = \{(u_1, \cdots, u_r) \in (0, T)^r;\ u_1 < \cdots < u_r,\ t_{i-\nu} < u_i < t_i,\\ s_{i-\nu} < u_i < s_i,\ i = 1, \cdots, r\}.$$

The last statement is equivalent to (20), hence the proof is completed.

As a consequence of Theorem 3 for $M'(t)$, we obtain the following property of the conditional correlation function

$$\rho(A, O | e_0),\ e_0 = \left(\bigcup_{i=1}^{r} S_{t_i}\right) \cup \left(\bigcup_{a \leq t < b} S_t\right), \qquad 0 < t_1 < \cdots < t_r < a < b \leq T.$$

PROPOSITION 8. *For $n = 2\nu + 1$ and every κ, and for all $0 < t_1 < \cdots < t_r < a < b \leq T$, we have*

$$(-1)^i \rho(A, O | e_0) > 0 \qquad \text{if } t_i < d(A, O) < t_{i+1}$$

$(i = 0, 1, \cdots, r;\ t_0 = 0,\ t_{i+1} = a)$, and

$$\rho(A, O | e_0) = 0 \qquad \text{if } a \leq d(A, O) < T.$$

The proof of Proposition 8 was given in [18], so is omitted.

Remark 2. In the even-dimensional case $n = 2\nu$, the $M(t)$-process is not totally positive. This can be seen from the following consideration.

First the case $\kappa = 0$. Instead of $M(t)$, consider the stationary Gaussian process $Y(t) \equiv e^{-t}M(e^{2t})$, $-\infty < t < \infty$. The spectral density of $Y(t)$ was computed by T. Hida [5]. By using the Schoenberg theorem (see [7], Chapter 7), it can be shown that $Y(t)$ is not totally positive, which implies the same for $M(t)$.

Next the case $\kappa > 0$. Suppose that $M(t)$ were totally positive for some $\kappa_0 > 0$. Then, for every $\kappa > 0$, $M(t)$ is totally positive. Since $\Gamma^{n,\kappa}(t, s)$ depends upon κ continuously, we obtain, as the limit $\kappa \downarrow 0$, the total positivity of $\Gamma^{n,0}(t, s)$. This is not the case and we have proved that $M(t)$ is not totally positive for every $\kappa > 0$. The same for the case $\kappa < 0$.

§ 5. Deterministic character of Lévy's Brownian motion

This final section will be concerned with a Lévy's Brownian motion $X = \{X(A); A \in Q\}$ with the infinite-dimensional parameter space $Q = Q^{\infty,\kappa}$. The set $K(e)$, introduced in Section 1, will be investigated. In particular, with the choice of $e = \bigcup_{i=1}^{N} S_{t_i}$ for a strictly increasing sequence $\{t_i\}_{i=1}^{N}$ $(1 \leq N \leq \infty)$, we shall give the exact form of $K(\bigcup_{i=1}^{N} S_{t_i})$ (Theorem 4). This result includes an improvement of the Lévy's result ([13]). We shall finally discuss some developments of Theorem 4.

We begin with the Lévy's result which was proved for $\kappa = 0$ in [13]. His proof can easily be modified for $\kappa \neq 0$.

PROPOSITION 9 ([13]). *For $n = \infty$ and every κ, we have $K(e) = Q$ for any subset e of Q containing an interior point.*

We are now in a position to prove our second main result.

THEOREM 4. *For $n = \infty$ and every κ, and for a strictly increasing sequence $\{t_i\}_{i=1}^{N}$ with $0 < t_i < T$ $(1 \leq N \leq \infty)$, we have*

$$K\Big(\bigcup_{i=1}^{N} S_{t_i}\Big) = \begin{cases} Q & \text{if } \sum_{i=1}^{N} h(t_i) = \infty\,, \\ \bigcup_{i=1}^{N} S_{t_i} & \text{if } \sum_{i=1}^{N} h(t_i) < \infty\,, \end{cases}$$

where $h(t)$ is the function introduced in Proposition 4.

Proof. First consider the case $\sum_{i=1}^{N} h(t) < \infty$. Suppose that $\sigma(A \,|\, \bigcup_{i=1}^{N} S_{t_i})$

$= 0$ were true for some point $A \notin \bigcup_{i=1}^{N} S_{t_i}$. Then, by the isotropic property of a Lévy's Brownian motion X, $\sigma(A' \mid \bigcup_{i=1}^{N} S_{t_i}) = 0$ must hold for every A' such that $d(A', O) = d(A, O) \equiv t$. This fact implies that $M(t)$ is measurable with respect to the σ-field generated by $\{X(B);\ B \in \bigcup_{i=1}^{N} S_{t_i}\}$. Furthermore, we see again from the isotropic property of X that $M(t)$ is actually measurable with respect to the σ-field generated by $\{M(t_i)\}_{i=1}^{N}$, i.e., $\sigma_M(t \mid \{t_i\}_{i=1}^{N}) = 0$. Proposition 4, however, tells us that $\sigma_M(t \mid \{t_i\}_{i=1}^{N}) > 0$, which is a contradiction. Hence we have $\sigma(A \mid \bigcup_{i=1}^{N} S_{t_i}) > 0$ for any $A \notin \bigcup_{i=1}^{N} S_{t_i}$, i.e., $K(\bigcup_{i=1}^{N} S_{t_i}) = \bigcup_{i=1}^{N} S_{t_i}$.

Next consider the case $\sum_{i=1}^{\infty} h(t_i) = \infty$ $(N = \infty)$. We shall first show that $\sigma(O \mid \bigcup_{i=1}^{\infty} S_{t_i}) = 0$. From the isotropic property of X it follows that

$$\mu\left(O \,\middle|\, \bigcup_{i=1}^{\infty} S_{t_i}\right) = E[X(O) \mid \overline{M}(t_i);\ i = 1, 2, \cdots] = \mu_M(0 \mid \{t_i\}_{i=1}^{\infty}) .$$

Hence by Proposition 4, we get

$$\sigma\left(O \,\middle|\, \bigcup_{i=1}^{\infty} S_{t_i}\right) = \sigma_M(0 \mid \{t_i\}_{i=1}^{\infty})\Big) = 0 .$$

Now we take an arbitrarily point A with $0 < d(A, O) \equiv t < t_1$ and show that $\sigma(A \mid \bigcup_{i=1}^{\infty} S_{t_i}) = 0$. Take such a motion α on Q that carries the point A to the origin O and O to $A_0 = (a_0(t), a_1(t), 0, 0, \cdots) \in Q$, where

$$a_0(t) = \begin{cases} 0 \\ \cos t - 1 \\ \cosh t - 1 , \end{cases} \qquad a_1(t) = \begin{cases} t & \text{for } \kappa = 0 , \\ \sin t & \text{for } \kappa = 1 , \\ \sinh t & \text{for } \kappa = -1 . \end{cases}$$

Set $Q_1 = \{B = (b_0, b_1, b_2, \cdots) \in Q;\ b_1 = 0\}$. Then, we see that

$$\alpha S_{t_i} \cap Q_1 = \{B \in Q_1;\ \alpha^{-1} B \in S_{t_i}\} = \{B \in Q_1;\ d(B, O) = \tilde{t}_i\} ,$$

where

$$\tilde{t}_i = \begin{cases} \sqrt{t_i^2 - t^2} & \text{for } \kappa = 0 , \\ \cos^{-1}(\cos t_i / \cos t) & \text{for } \kappa = 1 . \\ \cosh^{-1}(\cosh t_i / \cosh t) & \text{for } \kappa = -1 , \end{cases}$$

Since we have $\sum_{i=1}^{\infty} h(\tilde{t}_i) = \infty$, it follows that $\sigma(O \mid \bigcup_{i=1}^{\infty} \alpha S_{t_i} \cap Q_1) = 0$. By using the relation

$$\sigma(\alpha \tilde{A} \mid \alpha e) = \sigma(\tilde{A} \mid e) \qquad (\tilde{A} \in Q,\ e \in Q) ,$$

we have

$$\sigma\Big(A\Big|\bigcup_{i=1}^{\infty} S_{t_i}\Big) \leq \sigma\Big(\alpha^{-1}O\Big|\bigcup_{i=1}^{\infty} S_{t_i}\cap\alpha^{-1}Q_1\Big) = \sigma\Big(O\Big|\bigcup_{i=1}^{\infty}\alpha S_{t_i}\cap Q_1\Big) = 0\,.$$

This proves that $K(\bigcup_{i=1}^{\infty} S_{t_i}) \supset \bigcup_{0\leq t\leq t_1} S_t$.

Now Proposition 9 leads us to conclude that

$$K\Big(\bigcup_{i=1}^{\infty} S_{t_i}\Big) = K\Big(\Big(\bigcup_{i=1}^{\infty} S_{t_i}\Big)\cup\Big(\bigcup_{0\leq t<t_1} S_t\Big)\Big) = Q\,,$$

which completes the proof.

We shall give a simple development of Theorem 4. Set

$$Q_m = \{B = (b_0, b_1, b_2, \cdots)\in Q;\, b_1 = b_2 = \cdots = b_m = 0\} \quad (1\leq m<\infty).$$

The following facts were first discovered by P. Lévy [10] for $\kappa = 0$. We can now prove the same result for the cases $\kappa = 1$ and $\kappa = -1$.

PROPOSITION 10. *Let $n = \infty$ and $\kappa = 0, 1$ or -1. For $A_0 = (a_0(t), a_1(t), 0, 0, \cdots)$ with $d(A_0, Q_m) = t\in(0, T)$, we have*

$$\mu(A_0\,|\,Q_m) = \mu(A_0\,|\,S_t\cap Q_m)$$

and

$$\sigma^2(A_0\,|\,Q_m) = \begin{cases} t/\sqrt{2} & \text{for } \kappa = 0\,,\\ \cos^{-1}(\cos^2 t)/2 & \text{for } \kappa = 1\,,\\ \cosh^{-1}(\cosh^2 t)/2 & \text{for } \kappa = -1\,.\end{cases}$$

In view of Proposition 10 it is easy to prove the following development of Theorem 4.

THEOREM 4′. *For $n = \infty$ and every κ, and for a strictly increasing sequence $\{t_i\}_{i=1}^{N}$ with $0<t_i<T$ $(1\leq N\leq\infty)$, we have*

$$K\Big(\bigcup_{i=1}^{N} S_{t_i}\cap Q_m\Big) = \begin{cases} Q_m & \text{if } \sum_{i=1}^{N} h(t_i) = \infty\,,\\ \bigcup_{i=1}^{N} S_{t_i}\cap Q_m & \text{if } \sum_{i=1}^{N} h(t_i) < \infty\,.\end{cases}$$

Appendix. Proof of Lemma 1

The formula (10) which we must prove can easily be shown for $\nu = 1$, so we consider $\nu\geq 2$ in what follows.

(a) $\kappa = 0$. The spherical function $\Phi_\lambda^{2\nu+1,0}(t)$ is expressed in terms of the Bessel function $J_{\nu-1/2}(x)$ of order $\nu - 1/2$:

$$\Phi_\lambda^{2\nu+1,0}(t) = \Gamma(\nu + 1/2)(\lambda t/2)^{1/2-\nu} J_{\nu-1/2}(\lambda t)\,.$$

By virtue of the formula

$$\frac{d}{dx}\{x^{\nu-1/2}J_{\nu-1/2}(x)\} = x^{\nu-1/2}J_{\nu-3/2}(x),$$

we have

$$\begin{aligned} D_\nu \Phi_\lambda^{2\nu+1,0}(t) &= 2^{\nu-1/2}\Gamma\Big(\nu+\frac{1}{2}\Big)\Big[x^{-2\nu+2}\frac{d}{dx}\{x^{\nu-1/2}J_{\nu-1/2}(x)\}\Big]_{x=\lambda t} \\ &= 2^{\nu-1/2}\Gamma\Big(\nu+\frac{1}{2}\Big)[x^{-\nu+3/2}J_{\nu-3/2}(x)]_{x=\lambda t} = (2\nu-1)\Phi_\lambda^{2\nu-1,0}(t). \end{aligned}$$

(b) $\kappa = 1$. Making a change of variable $y = g(t) = \tan t$, we have

$$\begin{aligned} D_\nu \Phi_\lambda^{2\nu+1,1}(t) &= y^{-2\nu+2}\frac{d}{dy}\int_0^\pi \Big(\frac{1-iy\cos\theta}{\sqrt{1+y^2}}\Big)^\lambda (y\sin\theta)^{2\nu-1}d\theta/I_{2\nu-1} \\ &= (2\nu-1)\Phi_\lambda^{2\nu+1,1}(t) - y^{-2\nu+2}\int_0^\pi \frac{y+i\cos\theta}{-iy\sin\theta(1+y^2)} \\ &\qquad \times \Big\{\frac{d}{d\theta}\Big(\frac{1-iy\cos\theta}{\sqrt{1+y^2}}\Big)^\lambda\Big\}(y\sin\theta)^{2\nu-1}d\theta/I_{2\nu-1} \\ &= (2\nu-1)\Phi_\lambda^{2\nu+1,1}(t) + y^{-2\nu+2}\int_0^\pi \Big(\frac{1-iy\cos\theta}{\sqrt{1+y^2}}\Big)^\lambda \\ &\qquad \times \frac{d}{d\theta}\Big\{\frac{(y+i\cos\theta)(y\sin\theta)^{2\nu-2}}{i(1+y^2)}\Big\}d\theta/I_{2\nu-1} \\ &= (2\nu-1)\{\sin^2 t\,\Phi_\lambda^{2\nu+1,1}(t) + \cos t\,\Phi_{\lambda+1}^{2\nu-1,1}(t)\}. \end{aligned}$$

The spherical function $\Phi_\lambda^{2\nu+1,1}(t)$ is expressed in term of the ultraspherical polynomial $P_\lambda^{(\nu)}(x)$ of degree λ and order ν:

$$\Phi_\lambda^{2\nu+1,1}(t) = P_\lambda^{(\nu)}(x)/P_\lambda^{(\nu)}(1), \qquad x = \cos t.$$

We use the formulae ([22] pp. 81–86)

$$\begin{aligned} 2(\nu-1)(1-x^2)P_\lambda^{(\nu)}(x) &= -(\lambda+1)xP_{\lambda+1}^{(\nu-1)}(x) + (\lambda+2\nu-2)P_\lambda^{(\nu-1)}(x), \\ 2(\nu+\lambda)xP_{\lambda+1}^{(\nu-1)}(x) &= (\lambda+2)P_{\lambda+2}^{(\nu-1)}(x) + (\lambda+2\nu-2)P_\lambda^{(\nu-1)}(x), \end{aligned}$$

to obtain

$$\begin{aligned} &D_\nu \Phi_\lambda^{2\nu+1,1}(t) \\ &\quad = (2\nu-1)\Big\{\frac{\lambda}{2\nu+\lambda-1}xP_{\lambda+1}^{(\nu-1)}(x)/P_{\lambda+1}^{(\nu-1)}(1) + \frac{2\nu-1}{2\nu+\lambda-1}P_\lambda^{(\nu-1)}(x)/P_\lambda^{(\nu-1)}(1)\Big\} \\ &\quad = (2\nu-1)\Big\{\frac{\lambda}{2(\nu+\lambda)}\Phi_{\lambda+2}^{2\nu-1,1}/t) + \frac{2\nu+\lambda}{2(\nu+\lambda)}\Phi_\lambda^{2\nu-1,1}(t)\Big\}, \end{aligned}$$

which completes the proof.

REFERENCES

[1] R. Askey and N. H. Bingham, Gaussian processes on compact symmetric spaces, Z. Wahrsch. und Verw. Gebiete, **37** (1976), 127–143.
[2] S. M. Berman, Isotropic Gaussian processes on the Hilbert sphere, Ann. Probab., **8** (1980), 1093–1106.
[3] J. Faraut et K. Harzallah, Distances hilbertiennes invariantes sur un espace homogène, Ann. Inst. Fourier (Grenoble), **24** (1974), 171–217.
[4] R. Gangolli, Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters, Ann. Inst. H. Poincaré Sect. B, **3** (1967), 121–225.
[5] T. Hida, Canonical representations of Gaussian processes and their applications, Mem. Coll. Sci. Univ. Kyoto, Ser. A, **33**, Math., (1960), 109–155.
[6] T. Hida and M. Hitsuda, Gaussian processes (in Japanese), Kinokuniya Book Store, Tokyo, 1976.
[7] S. Karlin, Total positivity, Vol. 1, Stanford University Press, Stanford, California, 1968.
[8] S. Karlin and Z. Ziegler, Chebyshevian spline functions, SIAM J. Numer. Anal., **3** (1966), 514–543.
[9] B. Ja. Levin, Distribution of zeros of entire functions, Transl. Math. Monographs Vol. 5, Amer. Math. Soc., Providence, Rhode Island, 1964.
[10] P. Lévy, Processus stochastiques et mouvement brownien, Gauthier-Villars, Paris, 1965.
[11] ——, A special problem of Brownian motion, and a general theory of Gaussian random functions, Proc. Third Berkeley Symp. on Math. Statist. and Probability, Vol. 2 (1956), 133–175.
[12] ——, Le mouvement brownien fonction d'un point de la sphère de Riemann, Rend. Circ. Mat. Palermo (2) **8** (1959), 1–14.
[13] ——, Le déterminisme de la fonction brownienne dans l'espace de Hilbert, Ann. Sci. École Norm. Sup., **79** (1962), 377–398; Second mémoire, **80** (1963), 193–212.
[14] W. A. Luxemburg and J. Korevaar, Entire functions and Müntz-Szász type approximation, Trans. Amer. Math. Soc., **157** (1971), 23–37.
[15] H. P. McKean, Jr., Brownian motion with a several-dimensional time, Theory Probab. Appl., **8** (1963), 335–354.
[16] G. M. Molčan, The Markov property of Lévy fields on spaces of constant curvature, Soviet Math. Dokl., **16** (1975), 528–532.
[17] ——, On homogeneous random fields on symmetric spaces of rank one, Theory Probab. Math. Statist., **21** (1980), 143–168.
[18] A. Noda, Lévy's Brownian motion and total positivity, Proc. Fourth Japan-USSR Symp. on Probability Theory and Math. Statist., Lecture Notes in Math., Springer, Berlin (to appear).
[19] I. J. Schoenberg, Metric spaces and completely monotone functions, Ann. of Math., **39** (1938), 811–841.
[20] ——, Positive definite functions on spheres, Duke Math. J., **9** (1942), 96–108.
[21] L. Schwartz, Etude des sommes d'exponentielles, Publications de l'Institut de Mathématique de l'Université de Strasbourg V, Hermann, Paris, 1959.
[22] G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ. Vol. 23, Amer. Math. Soc., Providence, Rhode Island, 1939.
[23] S. Takenaka, I. Kubo and H. Urakawa, Brownian motion parametrized with metric space of constant curvature, Nagoya Math. J., **82** (1981), 131–140.
[24] N. J. Vilenkin, Special functions and the theory of group representations, Transl. Math. Monographs Vol. 22, Amer. Math. Soc., Providence, Rhode Island, 1968.

[25] E. Wong, Stochastic processes in information and dynamical systems, McGraw-Hill, New York, 1971.
[26] A. M. Yaglom, Some classes of random fields in n-dimensional space related to stationary random processes, Theory Probab. Appl., **2** (1957), 273–320.
[27] R. M. Young, An introduction to nonharmonic Fourier series, Academic Press, New York, 1980.

Department of Mathematics
Aichi University of Education
Kariya 448
Japan