Y. Furuta
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# A NORM RESIDUE MAP FOR CENTRAL EXTENSIONS OF AN ALGEBRAIC NUMBER FIELD

YOSHIOMI FURUTA

Let $K$ be a finite Galois extension of an algebraic number field $k$ with $G = \mathrm{Gal}(K/k)$, and $M$ be a Galois extension of $k$ containing $K$. We denote by $K^*_{M/k}$ resp. $\hat{K}_{M/k}$ the genus field resp. the central class field of $K$ with respect to $M/k$. By definition, the field $K^*_{M/k}$ is the composite of $K$ and the maximal abelian extension over $k$ contained in $M$. The field $\hat{K}_{M/k}$ is the maximal Galois extension of $k$ contained in $M$ satisfying the condition that the Galois group over $K$ is contained in the center of that over $k$. Then it is well known that $\mathrm{Gal}(\hat{K}_{M/k}/K^*_{M/k})$ is isomorphic to a factor group of the Schur multiplicator $H^{-3}(G, Z)$, and is isomorphic to $H^{-3}(G, Z)$ when $M$ is sufficiently large. In this case we call $M$ abundant for $K/k$ (See Heider [3, §4] and Miyake [6, Theorem 5]).

Let $G$ be abelian with a decomposition $G = G_1 \times \cdots \times G_r$ to cyclic factors such that the order of $G_i$ is divisible by that of $G_j$ for $i < j$. Then the Schur multiplicator $H^{-3}(G, Z)$ is isomorphic to the second exterior power of $G$, and hence isomorphic to $\oplus \sum_{i<j} G_j$.

Corresponded with the above decomposition of $H^{-3}(G, Z)$, we show in Section 3 that the central class field $\hat{K}_{M/k}$ is the composite of central class fields over bicyclic subextensions of $K/k$ when $K$ is abelian over $k$ and $M$ is abundant for $K/k$ (Proposition 5). Then in Section 4 we define a mapping $\Psi_{M/K/k}$ via Artin's reciprocity map, which is a surjective homomorphism from a group of certain ideals of $k$ to $\oplus \sum_{i<j} G_j \cong \Lambda(G)$ (Theorem). The mapping $\Psi_{M/K/k}$ describes the prime decomposition in $\hat{K}_{M/k}/K^*_{M/k}$. On the other hand, in Section 2 we define a surjective homomorphism $\varphi_{M/K/k}$ from $\Lambda(G)$ to $\mathrm{Gal}(\hat{K}_{M/k}/K^*_{M/k})$ by means of canonical cocycles of class field theory. The mapping $\Psi_{M/K/k}$ is regarded as the inverse of $\varphi_{M/K/k}$.

When $K$ is bicyclic biquadratic over the rational number field, the

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mapping $\Psi$ is given explicitly in [2] by using solutions of certain quadratic diophantine equations.

## §1. Commutator factors of group extensions

Let $G$ be a finite abelian group, and $\mathfrak{G}$ be a group extension of an abelian group $A$ by $G: \mathfrak{G}/A \cong G$. Let $\{U_\sigma\}$ be a system of representatives of $G$ in $\mathfrak{G}$, and $\{C_{\sigma,\tau}\}$ be the factor system: $U_\sigma U_\tau = U_{\sigma\tau} C_{\sigma,\tau}$ for $\sigma, \tau \in G$. Denote by $I_G$ the augmentation ideal of the group ring $Z[G]$. Denote further by $\Lambda(G)$ the second exteror power of $G$. Then it is well known that $H^{-3}(G, Z) \cong \Lambda(G)$ (See, for instance, Razar [7, Lemma 5]). The following fact is also probably well known, but we prove it here because it is fundamental in this paper.

PROPOSITION 1. *Let $A$, $G$ and $\mathfrak{G}$ be as above, and for $\sigma \wedge \tau \in \Lambda(G)$ let*

$$\varphi(\sigma \wedge \tau) \equiv C_{\sigma,\tau} C_{\tau,\sigma}^{-1} \quad \mod I_G A .$$

*Then $\varphi$ induces a surjective homomorphism of $\Lambda(G)$ to $[\mathfrak{G}, \mathfrak{G}]/I_G A$, where $[\mathfrak{G}, \mathfrak{G}]$ is the commutator subgroup of $\mathfrak{G}$.*

*Proof.* Let $a, b \in A$ and $\sigma, \tau \in G$. Then since $A$ and $G$ are abelian, we have $(U_\sigma a)^{-1}(U_\tau b)^{-1}(U_\sigma a)(U_\tau b) \equiv C_{\sigma,\tau} C_{\tau,\sigma}^{-1} a^{\tau-1}(b^{\sigma-1})^{-1} \equiv C_{\sigma,\tau} C_{\tau,\sigma}^{-1} \mod I_G A$. Hence if we put $\varphi_1(\sigma, \tau) = C_{\sigma,\tau} C_{\tau,\sigma}^{-1} \mod I_G A$, $\varphi_1$ defines a mapping of $G \times G$ onto $[\mathfrak{G}, \mathfrak{G}]/I_G A$. $\varphi_1$ is alternative and bilinear. In fact since $C_{\sigma\tau,\rho} C_{\sigma,\tau}^{\rho} = C_{\sigma,\tau\rho} C_{\tau,\rho}$ for any $\sigma, \tau, \rho \in G$, we have

$$\frac{C_{\sigma\tau,\rho}}{C_{\sigma,\rho} C_{\tau,\rho}} = \frac{C_{\sigma,\tau\rho}}{C_{\sigma,\tau}^{\rho} C_{\sigma,\rho}} = \frac{C_{\sigma,\tau\rho}}{C_{\sigma,\tau} C_{\sigma,\rho}} \cdot C_{\sigma,\tau}^{1-\rho}$$
$$\equiv \frac{C_{\sigma,\tau\rho}}{C_{\sigma,\tau} C_{\sigma,\rho}} \quad \mod I_G A .$$

The first form is symmetric for $\sigma$ and $\tau$, and the last form is so for $\tau$ and $\rho$, because $G$ is abelian. Hence we have

$$\frac{C_{\sigma\rho,\tau}}{C_{\sigma,\tau} C_{\rho,\tau}} \equiv \frac{C_{\tau,\sigma\rho}}{C_{\tau,\sigma} C_{\tau,\rho}} \quad \mod I_G A .$$

This implies that $\varphi_1$ is alternative bilinear map, and the proposition is implied.

*Remark.* It is easy to see that $\varphi$ does not depend on the choice of the factor system $\{C_{\sigma,\tau}\}$.

## § 2. Mapping $\varphi_{M/K/k}$

We apply Proposition 1 to an abelian Galois tower $M/K/k$, which means that both $M/K$ and $K/k$ are abelian extensions and $M/k$ is a Galois extension of algebraic number fields. Put $\mathfrak{G} = \mathrm{Gal}(M/k)$, $A = \mathrm{Gal}(M/K)$ and $G = \mathrm{Gal}(K/k)$. Then the canonical cocycle $\xi_{K/k}$ of class field theory gives a factor system for $\mathfrak{G}/A \cong G$. For any algebraic number field $L$, we denote by $J_L$ the idele group of $L$, and by $L^\times$ the group of principal ideles of $L$. Denote further by $H(L'/L)$ the subgroup of $J_L$ corresponding to $L'$ by class field theory when $L'$ is a Galois extension of $L$: $H(L'/L) = L^\times \cdot N_{L'/L}J_{L'}$.

Now we define a mapping $\varphi_{M/K/k}$ of the second exterior power $\Lambda(G)$ of $G$ to $J_k/H(\hat{K}_{M/k}/K)$ by

$$\varphi_{M/K/k}(\sigma \wedge \tau) \equiv \xi_{K/k}(\sigma, \tau)\xi_{K/k}(\tau, \sigma)^{-1} \qquad \mathrm{mod}\ H(\hat{K}_{M/k}/K) \tag{1}$$

for any $\sigma, \tau \in G$. Then it follows from Proposition 1 that $\varphi_{M/K/k}$ induces a surjective homomorphism of $\Lambda(G)$ to $G(\hat{K}_{M/k}/K^*_{M/k})$ via Artin's resiprocity map for $H(K^*_{M/k}/K)H/(\hat{K}_{M/k}/K) \cong G(\hat{K}_{M/k}/K^*_{M/k})$.

When $\sigma = ((K/k)/\mathfrak{a})$ and $\tau = ((K/k)/\mathfrak{b})$ for $\mathfrak{a}, \mathfrak{b} \in J_k$, we set

$$\varphi_{M/K/k}(\mathfrak{a} \wedge \mathfrak{b}) = \varphi_{M/K/k}(\sigma \wedge \tau)\,. \tag{2}$$

Then $\varphi_{M/K/k}$ induces a homomorphism of $\Lambda(J_k)$ to $H(K^*_{M/k}/K)/H(\hat{K}_{M/k}/K)$.

For the sake of simplicity, we shall use the following notation in general: Suppose that $H$, $H_1$ and $H_2$ are subgroups of an abelian group $G$, and $H$ contains both $H_1$ and $H_2$. Then by the congruence $\alpha \equiv \beta \bmod H$ for $\alpha \in G/H_1$ and $\beta \in G/H_2$, we mean $a \equiv b \bmod H$, where $a$ and $b$ are representatives of $\alpha$ and $\beta$ in $G$ respectively.

PROPOSITION 2.

(i) *Let $M_1 \supset M_2 \supset K \supset k$ be a Galois tower, and $K/k$ be abelian. Then for any $\mathfrak{a}, \mathfrak{b} \in J_k$ we have*

$$\varphi_{M_2/K/k}(\mathfrak{a} \wedge \mathfrak{b}) \equiv \varphi_{M_1/K/k}(\mathfrak{a} \wedge \mathfrak{b}) \qquad \mathrm{mod}\ H(\hat{K}_{M_2/k}/K)\,.$$

(ii) *Let $M \supset K_1 \supset K_2 \supset k$ be a Galois tower, and suppose that both $K_1/k$ and $K_2/k$ are abelian. Then for any $\mathfrak{a}, \mathfrak{b} \in J_k$ we have*

$$\varphi_{M/K_2/k}(\mathfrak{a} \wedge \mathfrak{b}) \equiv N_{K_1/K_2}\varphi_{M/K_1/k}(\mathfrak{a} \wedge \mathfrak{b}) \qquad \mathrm{mod}\ H(\hat{K}_2/K_2)\,.$$

(iii) *Let $M \supset K \supset k_1 \supset k_2$ be a Galois tower, and suppose that $K/k_2$ is abelian. Then for any $\mathfrak{a}, \mathfrak{b} \in J_{k_1}$, we have*

$$\varphi_{M/K/k_1}(\mathfrak{a} \wedge \mathfrak{b}) \equiv \varphi_{M/K/k_2}(N_{k_1/k_2}\mathfrak{a} \wedge N_{k_1/k_2}\mathfrak{b}) \quad \bmod H(\hat{K}_{M/k_2}/K).$$

*Proof.* (i) The assertion is implied immediately from (1), (2) and $H(\hat{K}_{M_1/k}/K) \subset H(\hat{K}_{M_2/k}/K)$.

(ii) For $i = 1, 2$, denote by $\hat{K}_i$ the central class field of $K_i$ with respect to $M/k$. Put $G_i = \mathrm{Gal}(K_i/k)$, $A_i = \mathrm{Gal}(\hat{K}_i/K_i)$, $\mathfrak{G}_i = \mathrm{Gal}(\hat{K}_i/k)$, and let $C_i$ be a factor set for $\mathfrak{G}_i/A_i \cong G_i$. Let further $U_{\sigma_1}$ resp. $V_{\sigma_2}$ be representatives of $\sigma_1 \in G_1$ resp. $\sigma_2 \in G_2$ in $\mathfrak{G}_1$ resp. $\mathfrak{G}_2$. Put $B = \mathrm{Gal}(\hat{K}_1/\hat{K}_2)$ and $D = \mathrm{Gal}(K_1/K_2)$, and let $W_{\sigma_2}$ be a representative of $\sigma_2 \in G_2$ in $G_1$. Then by Remark after Proposition 1 we may suppose that $V_{\sigma_2} = (U_{W_{\sigma_2}} \bmod B)$. We estimate the norm residue symbol as follows:

$$\begin{aligned}(\varphi_{M/K_1/k}(W_{\sigma_2} \wedge W_{\tau_2}), \hat{K}_2/K_1) &= V_{\sigma_2}^{-1}V_{\tau_2}^{-1}V_{\sigma_2}V_{\tau_2} \\ &= C_2(\sigma_2, \tau_2)C_2(\tau_2, \sigma_2)^{-1} = (\varphi_{M/K_2/k}(\sigma_2 \wedge \tau_2), \hat{K}_2/K_2).\end{aligned}$$

Hence $\varphi_{M/K_2/k}(\sigma_2 \wedge \tau_2) \equiv N_{K_1/K_2}\varphi_{M/K_1/k}(W_{\sigma_2} \wedge W_{\tau_2}) \bmod H(\hat{K}_2/K_2)$. This implies (ii) by setting $\sigma_2 = ((K_2/k)/\mathfrak{a})$ and $\tau_2 = ((K_2/k)/\mathfrak{b})$.

(iii) For $i = 1, 2$, put $\hat{K}_i = \hat{K}_{M/k_i}$, $G_i = \mathrm{Gal}(K/k_i)$, $A_i = \mathrm{Gal}(\hat{K}_i/K)$, $\mathfrak{G}_i = \mathrm{Gal}(\hat{K}_i/k_i)$ and $B = \mathrm{Gal}(\hat{K}_1/\hat{K}_2)$. Let $U_{\sigma_1}$ resp. $V_{\sigma_2}$ be representatives of $\sigma_1 \in G_1$ resp. $\sigma_2 \in G_2$ in $\mathfrak{G}_1$ resp. $\mathfrak{G}_2$. Then we have

$$\begin{aligned}(\varphi_{M/K/k_1}(\sigma_1 \wedge \tau_1), \hat{K}_1/K) &= U_{\sigma_1}^{-1}U_{\tau_1}^{-1}U_{\sigma_1}U_{\tau_1} \\ &\equiv V_{\sigma_1}^{-1}V_{\tau_1}^{-1}V_{\sigma_1}V_{\tau_1} \equiv (\varphi_{M/K/k_2}(\sigma_1 \wedge \tau_1), \hat{K}_2/K) \quad \bmod B.\end{aligned}$$

This implies the assertion, since

$$\sigma_1 = \left(\frac{K/k_1}{\mathfrak{a}}\right) = \left(\frac{K/k_2}{N_{k_1/k_2}\mathfrak{a}}\right) \quad \text{and} \quad \tau_1 = \left(\frac{K/k_1}{\mathfrak{b}}\right) = \left(\frac{K/k_2}{N_{k_1/k_2}\mathfrak{b}}\right)$$

## § 3. Decomposition of $\Lambda(G)$ and central extensions

Let $M/K/k$ be a Galois tower, and put $G = \mathrm{Gal}(K/k)$ and $\mathfrak{G} = \mathrm{Gal}(M/k)$. Then we have

$$\mathrm{Gal}(\hat{K}_{M/k}/K^*_{M/k}) \cong H^{-3}(G, Z)/\mathrm{Def}_{\mathfrak{G}\to G}H^{-3}(\mathfrak{G}, Z). \tag{3}$$

For this isomorphism, see for instance Kuz'min [4, § 4] or Razar [7, Proof of Lemma 3, (b)]. We call $M$ to be *abundant* for $K/k$ when $\mathrm{Gal}(\hat{K}_{M/k}/K^*_{M/k}) \cong H^{-3}(G, Z)$. Then it is known that for any Galois extension $K/k$ there always exists an abelian extension $M/K$ which is abundant for $K/k$.

PROPOSITION 3. *Let $M/L/K/k$ be a Galois tower. If $M$ is abundant for $L/k$, then $M$ is also abundant for $K/k$.*

*Proof.* Put $G = \mathrm{Gal}(L/k)$, $G_1 = \mathrm{Gal}(K/k)$ and $\mathfrak{G} = \mathrm{Gal}(M/k)$. If $M$ is abundant for $L/k$, then $\mathrm{Def}_{\mathfrak{G}\to G}H^{-3}(\mathfrak{G}, Z) = 1$ by (3). Since $\mathrm{Def}_{\mathfrak{G}\to G_1} = \mathrm{Def}_{G\to G_1}\circ \mathrm{Def}_{\mathfrak{G}\to G}$, the proposition is proved.

The following Proposition is easily obtained.

PROPOSITION 4. *Let $L_i$ be a central extension of a Galois extension $K_i/k$ for $i = 1, 2$. Then*

(i) *$L_1 \cap L_2$ is a central extension of $K_1 \cap K_2/k$,*

(ii) *$L_1L_2$ is a central extension of $K_1K_2/k$.*

We call extensions $K_1, \cdots, K_r$ over $k$ disjoint when $K_i \cap K_{j_1} \cdots K_{j_t} = k$ for any $i$ and any $j_s$ $(s = 1, \cdots, t)$ such that $K_i \neq K_{j_s}$.

Now let $M/K/k$ be as before a Galois tower, and assume that $K/k$ is abelian. Put $G = \mathrm{Gal}(K/k)$, and let

$$G = G_1 \times \cdots \times G_r \tag{4}$$

be a decomposition to the direct product by cyclic factors $G_i$ of $G$ such that the order of $G_i$ is divisible by the order of $G_j$ when $i < j$. Denote by $G_i \wedge G_j$ the subgroup of $\Lambda(G)$ generated by all elements $\sigma \wedge \tau$ such that $\sigma \in G_i$ and $\tau \in G_j$. Then

$$G_i \wedge G_j \cong G_j \qquad \text{for } i < j\,, \tag{5}$$

and

$$\Lambda(G) \cong \oplus \sum_{i<j} (G_i \wedge G_j) \cong \oplus \sum_{i<j} G_j\,, \tag{6}$$

where the sum is taken over all pairs $(i, j)$ satisfying $i < j$ for $i, j = 1, \cdots, r$. This corresponds to Lyndon [5, Theorem 6].

Let $K_i$ be the subfield of $K$ corresponding to $G/G_i$ over $k$, and put $K_{ij} = K_iK_j$. Hence $\mathrm{Gal}(K_i/k) \cong G_i$ and $\mathrm{Gal}(K_{ij}/k) \cong G_i \times G_j$.

PROPOSITION 5. *Notation being as above, assume that $M$ is abundant for $K/k$. Let $\hat{K}_{ij}$ be the central class field of $K_{ij}$ with respect to $M/k$. Then we have*

$$\hat{K}_{M/k} = \prod_{i<j} \hat{K}_{ij} \quad (\textit{disjoint over } K^*_{M/k}),$$
$$\mathrm{Gal}(\hat{K}_{ij}/K^*_{M/k}) \cong G_i \wedge G_j \cong G_j \cong \mathrm{Gal}(K_j/k) \qquad \textit{for } i < j\,.$$

*Proof.* Put $A = \mathrm{Gal}(M/K)$, $\mathfrak{G} = \mathrm{Gal}(M/k)$ and $\mathfrak{G}_i = \mathrm{Gal}(M/\prod_{t\neq i} K_t)$ for $i = 1, \cdots, r$. Since $M$ is abundant for $K/k$, the mapping $\varphi$ defined in

Proposition 1 gives an isomorphism $\Lambda(G) \cong [\mathfrak{G}, \mathfrak{G}]/I_G A$. For $i < j$, put

$$\Lambda_{ij}(G) = \oplus \sum_{\substack{s<t \\ (s,t)\neq(i,j)}} G_s \wedge G_t .$$

Then by taking account of $\mathfrak{G}_s \cdot I_G A/I_G \cdot A$ to be abelian, we have

$$\begin{aligned} \varphi(\Lambda_{ij}(G)) &= \prod_{\substack{s<t \\ (s,t)\neq(i,j)}} [\mathfrak{G}_s, \mathfrak{G}_t] \cdot I_G A/I_G A \\ &= [\mathfrak{G}, \prod_{\substack{s\neq i \\ s\neq j}} \mathfrak{G}_s]/I_G A = [\mathfrak{G}, \mathfrak{G}_{ij}]/I_G A , \end{aligned}$$

where $\mathfrak{G}_{ij} = \mathrm{Gal}(M/K_{ij})$. Hence the intermediate field of $\hat{K}_{M/k}/K^*_{M/k}$ corresponding to $\varphi(\Lambda_{ij}(G))$ is $\hat{K}_{ij}$, and $[\hat{K}_{ij}: K^*_{M/k}] = |G_i \wedge G_j|$. Since the intersection of all $\Lambda_{ij}(G)$ is $\{1\}$ and $\varphi$ is an isomorphism, the intersection of all $\varphi(\Lambda_{ij}(G))$ is also $\{1\}$. Hence $\hat{K}_{M/k} = \prod_{i<j} \hat{K}_{ij}$. Disjointness follows from $[\hat{K}_{M/k}: K^*_{M/k}] = [\prod_{i<j} \hat{K}_{ij}: K^*_{M/k}] \leqq \prod_{i<j} [\hat{K}_{ij}: K^*_{M/k}] = \prod_{i<j} |G_i \wedge G_j| = |\Lambda(G)| = [\hat{K}_{M/k}: K^*_{M/k}]$.

## § 4. Norm residue map $\Psi_{M/K/k}$

Throughout this section, we assume that $K/k$ is abelian, $M/K/k$ is a Galois tower and $M$ is abundant for $K/k$. Put $G = \mathrm{Gal}(K/k)$, $\hat{K} = \hat{K}_{M/k}$ and $K^* = K^*_{M/k}$. Then

$$\mathrm{Gal}(\hat{K}/K^*) \cong H^{-1}(G, C_K) \cong H^{-3}(G, Z) \cong \Lambda(G), \tag{7}$$

where $C_K$ is the idele class group of $K$. In Section 2 we defined the mapping $\varphi_{M/K/k}$ of $\Lambda(G)$ to $J_K/H(\hat{K}/K)$. In the present section we shall study the inverse mapping of $\varphi_{M/K/k}$.

Let notation be as in Section 3. It follows from Proposition 5 and (7) that

$$\begin{aligned} \mathrm{Gal}(\hat{K}/K^*) &\cong \oplus \sum_{i<j} \mathrm{Gal}(\hat{K}_{ij}/K^*) \\ &\cong \oplus \sum_{i<j} \mathrm{Gal}(K_j/k) \cong \Lambda(G). \end{aligned} \tag{8}$$

We denote by $I(K^*/k)$ the group of norms of ideals of $K^*$ to $k$ which are relatively prime to the discriminants of $M/k$. Let $\mathfrak{a} \in I(K^*/k)$, and $\mathfrak{A}$ be an ideal of $K^*$ such that $\mathfrak{a} = N_{K^*/k}\mathfrak{A}$. We define a mapping $\psi$ of $I(K^*/k)$ to $\oplus \sum_{i<j} \mathrm{Gal}(\hat{K}_{ij}/K^*)$ by

$$\psi(\mathfrak{a}) = \oplus \sum_{i<j} \left( \frac{\hat{K}_{ij}/K^*}{\mathfrak{A}} \right). \tag{9}$$

Since $\mathrm{Gal}(\hat{K}/K^*)$ is contained in the center of $\mathrm{Gal}(\hat{K}/k)$, the value of $\psi(\mathfrak{a})$ does not depend on the choice of $\mathfrak{A}$. It follows from (8) and (9) that $\psi$ is a surjective homomorphism.

In order to get the image of $((\hat{K}_{ij}/K^*)/\mathfrak{A})$ by the isomorphism

$$\mathrm{Gal}(\hat{K}_{ij}/K^*) \cong \mathrm{Gal}(K_j/k),$$

we use the following proposition which is a special case of [1, Proposition 5.1].

PROPOSITION 6. *Let $F/k$ be a cyclic extension with $\mathfrak{g} = \mathrm{Gal}(F/k)$ generated by $\sigma$, $M \supset L \supset F \supset k$ be a Galois tower, and $L/k$ and $M/F$ be abelian. Then*

$$\mathrm{Gal}(\hat{L}_{M/k}/L^*_{M/k}) \cong \mathfrak{D}/C(\mathfrak{H}(L/F))\mathfrak{R} \cong \mathrm{Gal}(F'/F), \tag{10}$$

*where $\mathfrak{D}$ is the ideal class group of $F$ corresponding to $M$, $\mathfrak{H}(L/F)$ is the congruent ideal group of $F$ corresponding to $L$, $C(\mathfrak{H}(L/F))$ is the subgroup of $\mathfrak{D}$ represented by $\mathfrak{H}(L/F)$, $\mathfrak{R}$ is the group of elements $\mathfrak{c}$ of $\mathfrak{D}$ such that $\mathfrak{c}^\sigma = \mathfrak{c}$, and $F'$ is the subfield of $L$ over $F$ corresponding to $C(\mathfrak{H}(L/F))\mathfrak{R}$.*

*The above isomorphism $\mathrm{Gal}(\hat{L}_{M/k}/L^*_{M/k}) \cong \mathrm{Gal}(F'/F)$ is given by*

$$\left(\frac{\hat{L}_{M/k}/L^*_{M/k}}{\mathfrak{A}^*}\right) \longrightarrow \left(\frac{F'/F}{\mathfrak{B}}\right),$$

*where $\mathfrak{A}^*$ is any ideal of $L^*_{M/k}$ prime to the conductor of $M/F$ and $\mathfrak{B}$ is an ideal of $F$ such that $\mathfrak{B}^{\sigma-1} \equiv N_{L^*_{M/k}/F}\mathfrak{A}^* \bmod \mathfrak{H}(M/F)$.*

We apply the above proposition taking $\hat{K}_{ij}$, $K_{ij}$ and $K_i$ instead of $M, L$ and $F$ respectively. Then $L^*_{M/k}$ in the proposition becomes $K^*_{ij}$ and $F'$ becomes $K_{ij}$, because $\mathrm{Gal}(F'/F) \cong G_j \cong \mathrm{Gal}(K_{ij}/K_i)$ and $\mathrm{Gal}(\hat{K}_{ij}/K^*_{ij}) = G_j$ owing to abundantness of $M$ for $K/k$. For $\mathfrak{a} \in I(K^*/k)$, let $\mathfrak{A}_i$ and $\mathfrak{B}_i$ be ideals of $K_i$ such that $N_{K_i/k}\mathfrak{A}_i = \mathfrak{a}$ and $\mathfrak{B}_i^{\sigma_i-1} \equiv \mathfrak{A}_i \bmod \mathfrak{H}(M/K_i)$, where $\sigma_i$ is a generator of the cyclic group $\mathrm{Gal}(K_i/k)$ and $\mathfrak{H}(M/K_i)$ is the ideal group of $K_i$ corresponding to $M$. Let further $\mathfrak{b}_i = N_{K_i/k}\mathfrak{B}_i$. We define a mapping $\Psi_{ij}$ of $I(K^*/k)$ to $G_j$ for $i < j$ by

$$\Psi_{ij}(\mathfrak{a}) = \left(\frac{K_j/k}{\mathfrak{b}_i}\right). \tag{11}$$

Now Proposition 6 implies immediately the following

THEOREM. *Let $K/k$ be an abelian extension with $G = \mathrm{Gal}(K/k)$, and $M$ be a Galois extension over $k$ such that $M$ contains $K$ and abundant for $K/k$. Let $G = G_1 \times \cdots \times G_r$, $K_i$, $K_{ij}$ and $\hat{K}_{ij}$ be as in Section* 3 (4) *and* *after that. Let the notation* $\hat{K}$, $K^*$, $I(K^*/k)$ *and* $\Lambda(G)$ *be as above. We define a mapping* $\Psi_{M/K/k}$ *of* $I(K^*/k)$ *to* $\oplus \sum_{i<j} G_j \cong \Lambda(G)$ *by*

$$\Psi_{M/K/k}(\mathfrak{a}) = \oplus \sum_{i<j} \Psi_{ij}(\mathfrak{a}) \qquad \textit{for } \mathfrak{a} \in I(K^*/k),$$

*where* $\Psi_{ij}$ *is the mapping defined by* (11). *Then* $\Psi_{M/K/k}$ *is a surjective homomorphism of* $I(K^*/k)$ *to* $\oplus \sum_{i<j} G_j$ *which is isomorphic to* $\Lambda(G)$ *and so to* $\mathrm{Gal}(\hat{K}/K^*)$.

The mapping $\Psi_{M/K/k}$ is regarded as the inverse of the mapping $\varphi_{M/K/k}$ defined in Section 2. In fact we have the following proposition.

PROPOSITION 7. *For* $\mathfrak{a} \in J_K$ *denote by* $[\mathfrak{a}]$ *an ideal of* $K$ *such that* $(\mathfrak{a}, \hat{K}/K) = ((\hat{K}/K)/[\mathfrak{a}])$. *Then other notation being as above, we have*

$$\Psi_{ij}(N_{K/k}[\varphi_{M/K/k}(\sigma_i \wedge \sigma_j)]) = \sigma_j .$$

*Proof.* To simplify the notation, we put $\hat{K} = \hat{K}_{M/k}$ and $\varphi = \varphi_{M/K/k}$. Put further $G = \mathrm{Gal}(K/k)$ and $A = \mathrm{Gal}(\hat{K}/K)$. Let $U_{\sigma_i}$ resp. $U_{\sigma_j}$ be representatives of $\sigma_i$ resp. $\sigma_j$ in $\mathrm{Gal}(\hat{K}/k)$. Then by (1) in Section 2, we have

$$\begin{aligned}\left(\frac{\hat{K}/K}{[\varphi(\sigma_i \wedge \sigma_j)]}\right) &= \left(\frac{\hat{K}/K}{[\xi_{K/k}(\sigma_i, \sigma_j)]}\right)\left(\frac{\hat{K}/K}{[\xi_{K/k}(\sigma_j, \sigma_i)]}\right)^{-1} \\ &\equiv U_{\sigma_i}^{-1}U_{\sigma_j}^{-1}U_{\sigma_i}U_{\sigma_j} \equiv U_{\sigma_j}^{\sigma_i - 1} \qquad \mathrm{mod}\ I_G A .\end{aligned}$$

Let $\hat{\mathfrak{B}}$ be an ideal of $\hat{K}$ such that $U_{\sigma_j} = [(\hat{K}/k)/\hat{\mathfrak{B}}]$, the product of the Frobenius automorphisms for the prime factors of $\hat{\mathfrak{B}}$. Then

$$\left(\frac{\hat{K}/K}{[\varphi(\sigma_i \wedge \sigma_j)]}\right) = \left[\frac{\hat{K}/k}{\hat{\mathfrak{B}}}\right]^{\sigma_i - 1} = \left[\frac{\hat{K}/k}{\hat{\mathfrak{B}}^{\sigma_i - 1}}\right] = \left(\frac{\hat{K}/k}{\mathfrak{B}^{\sigma_i - 1}}\right),$$

where $\mathfrak{B} = N_{\hat{K}/K}\hat{\mathfrak{B}}$. Let $\mathfrak{B}_i = N_{K/K_i}\mathfrak{B}$. Then

$$N_{K/K_i}[\varphi(\sigma_i \wedge \sigma_j)] \equiv \mathfrak{B}_i^{\sigma_i - 1} \qquad \mathrm{mod}\ \mathfrak{H}(M/K_i),$$

where $\mathfrak{H}(M/K_i)$ is, as in Proposition 6, the congruent ideal group of $K_i$ corresponding to $M$. Now let $\mathfrak{a} = N_{K/k}[\varphi(\sigma_i \wedge \sigma_j)]$ and $\mathfrak{b} = N_{K_i/k}\mathfrak{B}_i = N_{\hat{K}/k}\hat{\mathfrak{B}}$. Then we have $\Psi_{ij}(\mathfrak{a}) = ((K_j/k)/\mathfrak{b}) = \sigma_j$ by (11). Thus the proposition is proved.

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*Department of Mathematics*
*Kanazawa University*
*Marunouchi, Kanazawa 920*
*Japan*