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SYMMETRIC HOMOGENEOUS CONVEX DOMAINS

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Introduction

Let D be a convex domain in the n-dimensional real number space $\boldsymbol{R}^n$, not containing any affine line and $A(D)$ the group of all affine transformations of $\boldsymbol{R}^n$ leaving D invariant. If the group $A(D)$ acts transitively on D, then the domain D is said to be *homogeneous*. From a homogeneous convex domain D in $\boldsymbol{R}^n$, a homogeneous convex cone $V = V(D)$ in $\boldsymbol{R}^{n+1} = \boldsymbol{R}^n \times \boldsymbol{R}$ is constructed as follows (cf. Vinberg [11]):

$$V(D) = \{(tx, t) \in \boldsymbol{R}^n \times \boldsymbol{R};\ x \in D,\ t > 0\}, \tag{0.1}$$

which is called the *cone fitted on* the convex domain D. Let $G(V)$ be the group of all linear automorphisms of V and g_V the canonical $G(V)$-invariant Riemannian metric on V (cf. e.g. [8]). Then a natural imbedding

$$\sigma\colon\ x \in D \longrightarrow (x, 1) \in V(D) \tag{0.2}$$

is equivariant with respect to the groups $A(D)$ and $G(V)$. Therefore, the Riemannian metric $g_D = \sigma^* g_V$ on D induced from (V, g_V) by σ is $A(D)$-invariant. The Riemannian metric g_D is called the *canonical metric* of D. We note that the canonical metric g_D is given from the characteristic function φ_V of V as follows: Let us put $\varphi_D = \varphi_V \circ \sigma$. Then

$$g_D = \sum_{1 \leq i, j \leq n} \frac{\partial^2 \log \varphi_D}{\partial x^i \partial x^j} dx^i dx^j, \tag{0.3}$$

where $(x^1, x^2, \cdots, x^n)$ is a system of affine coordinates of $\boldsymbol{R}^n$.

The purpose of the present paper is to determine (up to affine equivalence) all homogeneous convex domains which are Riemannian symmetric with respect to the canonical metric. The main result obtained is stated as follows. *Every symmetric homogeneous convex domain is affinely equiv-*

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alent to one of the following: *a homogeneous self-dual cone*; *an elementary domain*; *a direct product of some of these domains* (Theorem 4.2). For the definition of an elementary domain, see § 1. In order to prove the above result, we need essentially the theory of T-algebras developed by Vinberg [11], [12]. We remark that for homogeneous convex cones, the above problem has been solved by Rothaus [5], Shima [7] and the author [8], [10].

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§ 1. Homogeneous convex domains and T-algebras

In this section, we recall the construction theorem of homogeneous convex domains in terms of T-algebras. The details for them may be found in [11] or [12].

1.1. Let $\mathfrak{A} = \sum_{1 \leqslant i,j \leqslant r} \mathfrak{A}_{ij}$ be a T-algebra of rank r provided with an involutive anti-automorphism $*$. General elements of $\mathfrak{A}_{ij}$ will be denoted as a_{ij}, b_{ij}, $\cdots$, and also an arbitrary element a of $\mathfrak{A}$ will be written like as a matrix $a = (a_{ij})$, where a_{ij} is the $\mathfrak{A}_{ij}$-component of a. Let us define subsets $T = T(\mathfrak{A})$, $V = V(\mathfrak{A})$ and $X = X(\mathfrak{A})$ of $\mathfrak{A}$ by

$$T = \{t = (t_{ij}) \in \mathfrak{A};\ t_{ii} > 0\ (1 \leqslant i \leqslant r),\ t_{ij} = 0\ (1 \leqslant j < i \leqslant r)\},$$
$$V = \{tt^*;\ t \in T\} \subset X = \{x \in \mathfrak{A};\ x^* = x\}.$$

Then it is known in [11] that V is a homogeneous convex cone in the real vector space X and T is a connected Lie group which acts on V simply transitively as linear transformations by

$$(t, ss^*) \in T \times V \longrightarrow (ts)(ts)^* \in V.$$

Conversely, every homogeneous convex cone is realized in this form up to linear equivalence.

Throughout this paper, we will use the following notation:

$$n_{ij} = \dim \mathfrak{A}_{ij} = \dim \mathfrak{A}_{ji}, \qquad n_i = 1 + \tfrac{1}{2} \sum_{k \neq i} n_{ik} \quad (1 \leqslant i, j \leqslant r);$$
$$\mathrm{Sp}\, a = \sum_{1 \leqslant i \leqslant r} n_i a_{ii}\ (a = (a_{ij}) \in \mathfrak{A}); \tag{1.1}$$
$$e = (e_{ij}), \qquad e_{ij} = \delta_{ij} \quad \text{(Kronecker delta)}.$$

Then the numbers $\{n_{ij}\}$ satisfy the condition

(1.2) $$\max\{n_{ij}, n_{jk}\} \leqslant n_{ik}$$

for all indices $i<j<k$ with $n_{ij}n_{jk} \neq 0$. Moreover, the element e is the unit element of T and also e is contained in V. Hence, the tangent space $T_e(V)$ of V at the point e may be naturally identified with the ambient space X and also with the Lie algebra $\mathfrak{t}$ of T. On the other hand, the Lie algebra $\mathfrak{t}$ may be identified with the subspace $\sum_{1\leqslant i\leqslant j\leqslant r}\mathfrak{A}_{ij}$ of $\mathfrak{A}$ provided with the bracket product: $[a, b] = ab - ba$. A canonical linear isomorphism between $\mathfrak{t}$ and X is given by

$$\xi\colon a \in \mathfrak{t} = \sum_{1\leqslant i\leqslant j\leqslant r}\mathfrak{A}_{ij} \longrightarrow a + a^* \in X = T_e(V)\,.$$

Under this identification, by using the canonical Riemannian metric g_V at the point e, we have an inner product $\langle\ ,\ \rangle$ on $\mathfrak{t}$ as follows:

$$\langle a, b\rangle = g_V(e)(\xi(a), \xi(b))$$

for every $a, b \in \mathfrak{t}$. The inner product $\langle\ ,\ \rangle$ has the following expression:

(1.3) $$\langle a, b\rangle = \mathrm{Sp}\,(\xi(a)\xi(b))$$

for every $a, b \in \mathfrak{t}$ (cf. the formula (34) of [11]). From this, we have

(1.4) $$\langle \mathfrak{A}_{ij}, \mathfrak{A}_{k\ell}\rangle = 0$$

for all indices $i \leqslant j$ and $k \leqslant \ell$ satisfying $(i, j) \neq (k, \ell)$.

1.2. For a T-algebra $\mathfrak{A} = \sum_{1\leqslant i,j\leqslant r}\mathfrak{A}_{ij}$ of rank r $(r \geqslant 2)$, we define subsets $T_0 \subset T$, $X_0 \subset X$ and $D = D(\mathfrak{A}) \subset V = V(\mathfrak{A})$ by

$$T_0 = \{t = (t_{ij}) \in T;\ t_{rr} = 1\}\,, \qquad X_0 = \{x = (x_{ij}) \in X;\ x_{rr} = 0\}$$

and

(1.5) $$D = D(\mathfrak{A}) = \{x = (x_{ij}) \in V(\mathfrak{A});\ x_{rr} = 1\} = V(\mathfrak{A}) \cap (X_0 + e)\,,$$

respectively. Then it is known in Vinberg [11] that the domain $D(\mathfrak{A})$ is a homogeneous convex domain in the affine subspace $X_0 + e$ satisfying the condition $V(D) = V(\mathfrak{A})$ and T_0 is a closed subgroup of T acting on D simply transitively as affine transformations by

$$(t, ss^*) \in T_0 \times D \longrightarrow (ts)(ts)^* \in D\,.$$

Conversely, every homogeneous convex domain is realized in this form up to affine equivalence.

Let us define a subspace $\mathfrak{t}_0$ of $\mathfrak{t}$ by

$$\mathfrak{t}_0 = \{t = (t_{ij}) \in \mathfrak{t};\ t_{rr} = 0\}\,.$$

Then $\mathfrak{t}_0$ is the Lie subalgebra of $\mathfrak{t}$ corresponding to the subgroup T_0 of T. Similarly as in the case of homogeneous convex cones, we can identify the Lie algebra $\mathfrak{t}_0$ with the tangent space $T_e(D)$ of D at the point e and also with the vector space X_0 by the following linear isomorphism:

$$\xi_0\colon a \in \mathfrak{t}_0 \longrightarrow a + a^* \in X_0 = T_e(D)\,.$$

From the canonical metric g_D of D at the point e, we have an inner product $\langle\ ,\ \rangle_0$ on $\mathfrak{t}_0$ by

$$\langle a, b\rangle_0 = g_D(e)(\xi_0(a), \xi_0(b))$$

for every $a, b \in \mathfrak{t}_0$. Since the equivariant imbedding $\sigma\colon D(\mathfrak{A}) \to V(\mathfrak{A})$ defined by (0.2) is the inclusion mapping, we have the following relations:

$$\xi_0(a) = \xi(a) \quad \text{and} \quad \langle a, b\rangle_0 = \langle a, b\rangle$$

for every $a, b \in \mathfrak{t}_0$. Therefore, we can identify $\langle\ ,\ \rangle_0$ with $\langle\ ,\ \rangle$ restricted to the subspace $\mathfrak{t}_0$, and we may omit the subscript in $\langle\ ,\ \rangle_0$.

EXAMPLE ([11]). We now give a typical example of homogeneous convex domains. Let $(\ ,\ )$ be an inner product on the real number space $\boldsymbol{R}^n$. Then the domain $D(n+1)$ defined by

$$D(n+1) = \{(x, y) \in \boldsymbol{R} \times \boldsymbol{R}^n;\ x - (y, y) > 0\}$$

is a homogeneous convex domain in $\boldsymbol{R}^{n+1}$. This domain is called the *elementary domain* of dimension $n+1$. The domain $D(n+1)$ is constructed from a T-algebra as follows: Let us take a T-algebra $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{22} + \mathfrak{A}_{12} + \mathfrak{A}_{21}$ of rank two with $\mathfrak{A}_{12} = \boldsymbol{R}^n$. Then, the cone $V(\mathfrak{A})$ is the $(n+2)$-dimensional circular cone

$$C(n+2) = \left\{x = \begin{pmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{pmatrix} \in X(\mathfrak{A});\ x_{11}x_{22} - (x_{12}, x_{12}) > 0,\ x_{22} > 0\right\},$$

where $x_{11}x_{22}$ is a usual multiplication of real numbers $x_{ii} \in \mathfrak{A}_{ii} = \boldsymbol{R}$ ($i = 1, 2$), and the domain $D(\mathfrak{A})$ given by (1.5) is the elementary domain $D(n+1)$.

1.3. On the direct product of homogeneous convex domains, we have the following

PROPOSITION 1.1. *Let D_i be a homogeneous convex domain in the real number space $\boldsymbol{R}^{n_i}$ ($i = 1, 2$). Then the product domain $D = D_1 \times D_2$ is a* *homogeneous convex domain in* $\boldsymbol{R}^n = \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$ $(n = n_1 + n_2)$ *and the canonical metric* g_D *of* D *is the product Riemannian metric of* g_{D_i} $(i = 1, 2)$.

Proof. Let us put the subgroup $A_0(D)$ of $A(D)$ by $A_0(D) = A(D_1) \times A(D_2)$. Then $A_0(D)$ acts transitively on D and D is a homogeneous convex domain in $\boldsymbol{R}^n$. We now define a function $\psi\colon V(D) \to \boldsymbol{R}$ by

$$\psi(tx, t) = t\varphi_1(tx_1, t)\varphi_2(tx_2, t)$$

for every $(tx, t) \in V(D)$ and $x = (x_1, x_2) \in D = D_1 \times D_2$, where φ_i is the characteristic function of $V_i = V(D_i)$ $(i = 1, 2)$. We want to show that the function ψ satisfies the condition

$$\psi(g(tx, t)) = \psi(tx, t)/|\det g| \tag{1.6}$$

for every $g \in G(V)$ and $(tx, t) \in V(D)$. In fact, from a property of the characteristic function φ_i (cf. [11]), we have

$$\psi(\lambda(tx, t)) = \psi(tx, t)/\lambda^{n+1}$$

for every $\lambda > 0$ and $(tx, t) \in V(D)$. In general, for each affine transformation B on $\boldsymbol{R}^m$, we denote by $\tilde{B}$ the natural extension of B as a linear transformation on $\boldsymbol{R}^{m+1} = \boldsymbol{R}^m \times \boldsymbol{R}$. Then we have

$$\begin{aligned}\psi(\tilde{A}(tx, t)) &= \psi(tAx, t)\\ &= t\varphi_1(tA_1x_1, t)\varphi_2(tA_2x_2, t) = t\varphi_1(\tilde{A}_1(tx_1, t))\varphi_2(\tilde{A}_2(tx_2, t))\\ &= t\varphi_1(tx_1, t)\varphi_2(tx_2, t)/|(\det \tilde{A}_1 \det \tilde{A}_2)| = \varphi(tx, t)/|\det \tilde{A}|\end{aligned}$$

for every $A = (A_1, A_2) \in A_0(D)$ and $(tx, t) \in V(D)$. On the other hand, the subgroup of $G(V)$ generated by $\tilde{A}_0(D)$ and the similarity transformations acts on V transitively. Therefore, the function ψ satisfies the condition (1.6), and we can write $\varphi_V = c\psi$ by a positive number c. Hence,

$$\varphi_V(x, 1) = c\psi(x, 1) = c\varphi_1(x_1, 1)\varphi_2(x_2, 1)\,,$$

which means

$$\varphi_D(x) = c\varphi_{D_1}(x_1)\varphi_{D_2}(x_2)$$

for every $x = (x_1, x_2) \in D = D_1 \times D_2$. Therefore, by (0.3), we have $g_D = g_{D_1} \times g_{D_2}$. q.e.d.

From the definition (0.1), we can easily see the following

PROPOSITION 1.2. *Let* V_0 *be a homogeneous convex cone and* D *a homogeneous convex domain. Then the cone* $V(V_0 \times D)$ (resp. $V(V_0)$) *fitted on*

$V_0 \times D$ (resp. V_0) is the product cone $V_0 \times V(D)$ (resp. $V_0 \times \boldsymbol{R}^+$), where $\boldsymbol{R}^+$ is the cone of all positive real numbers.

§ 2. Connection and curvature for the canonical metric

In this section, we study some of basic properties of the Riemannian connection for the canonical metric on a homogeneous convex domain. Let $D = D(\mathfrak{A})$ (resp. $V = V(\mathfrak{A})$) be the homogeneous convex domain (resp. cone) corresponding to a T-algebra $\mathfrak{A} = \sum_{1 \leqslant i,j \leqslant r} \mathfrak{A}_{ij}$ of rank r $(r \geqslant 2)$ (cf. (1.5)).

2.1. The connection function β and the curvature tensor R for the canonical metric g_D are described in terms of the Lie algebra $\mathfrak{t}_0$ and the inner product $\langle\ ,\ \rangle$ as follows (cf. Nomizu [4]):

$$\beta\colon \mathfrak{t}_0 \times \mathfrak{t}_0 \longrightarrow \mathfrak{t}_0\,,$$
$$2\langle \beta(a, b), c\rangle = \langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle$$

and

$$\begin{aligned} &R\colon \mathfrak{t}_0 \times \mathfrak{t}_0 \times \mathfrak{t}_0 \longrightarrow \mathfrak{t}_0\,, \\ &R(a, b, c) = R(a, b)c = \beta(a, \beta(b, c)) - \beta(b, \beta(a, c)) - \beta([a, b], c) \end{aligned} \tag{2.1}$$

for every $a, b, c \in \mathfrak{t}_0$. Furthermore, the connection function α for the canonical metric g_V on the homogeneous convex cone $V = V(\mathfrak{A}) = V(D)$ is given by the Lie algebra $\mathfrak{t}$ and the inner product $\langle\ ,\ \rangle$ as follows:

$$\begin{aligned} &\alpha\colon \mathfrak{t} \times \mathfrak{t} \longrightarrow \mathfrak{t}\,, \\ &2\langle \alpha(a, b), c\rangle = \langle [c, a], b\rangle + \langle [c, b], a\rangle + \langle [a, b], c\rangle \end{aligned} \tag{2.2}$$

for every $a, b, c \in \mathfrak{t}$. Now, let us put

$$e_i = \frac{1}{2\sqrt{n_i}} e_{ii}\,, \tag{2.3}$$

where $e_{ii} = 1$ is the unit element of the subalgebra $\mathfrak{A}_{ii} = \boldsymbol{R}$ $(1 \leqslant i \leqslant r)$. Then by (1.1) and (1.3), we have

$$\|e_i\| = 1\,,$$

where $\|\ ,\ \|$ is the norm with respect to the inner product $\langle\ ,\ \rangle$.

We first prove the following

LEMMA 2.1. *The connection functions α and β satisfy the following relations*:

(1) $\beta(x, y) = \alpha(x, y)$ *for every* $x \in \mathfrak{A}_{ij}$ $(i \leqslant j)$, $y \in \mathfrak{A}_{k\ell}$ $(k \leqslant \ell)$ $((i, j) \neq (k, \ell))$.

(2) $\beta(x, y) = \alpha(x, y) + \dfrac{1}{2\sqrt{n_r}}\langle x, y\rangle e_r = \dfrac{1}{2\sqrt{n_i}}\langle x, y\rangle e_i$ *for every* $x, y \in \mathfrak{A}_{ir}$ $(1 \leqslant i \leqslant r-1)$.

(3) $\beta(x, y) = \alpha(x, y) = \dfrac{\langle x, y\rangle}{2}\Big(\dfrac{1}{\sqrt{n_i}}e_i - \dfrac{1}{\sqrt{n_j}}e_j\Big)$ *for every* $x, y \in \mathfrak{A}_{ij}$ $(1 \leqslant i < j \leqslant r-1)$.

(4) $\beta(e_i, x) = 0$ *for every* $x \in \mathfrak{t}_0$ *and* $1 \leqslant i \leqslant r-1$.

Proof. We first remark that the connection functions α and β satisfy the identity

$$\beta(a, b) = \alpha(a, b) - \langle \alpha(a, b), e_r\rangle e_r \tag{2.4}$$

for every $a, b \in \mathfrak{t}_0$. By (2.2), we have

$$2\langle \alpha(x_{ij}, y_{k\ell}), e_r\rangle = \langle [e_r, x_{ij}], y_{k\ell}\rangle + \langle [e_r, y_{k\ell}], x_{ij}\rangle + \langle [x_{ij}, y_{k\ell}], e_r\rangle\,. \tag{2.5}$$

On the other hand, by the conditions (2.3) and $[a, b] = ab - ba$ for every $a, b \in \mathfrak{t}_0$ (cf. (1.1) and (1.2) of [8]), we get

$$[e_r, x_{ij}] = \frac{1}{2\sqrt{n_r}}(\delta_{ir} - \delta_{jr})x_{ij}$$

and

$$[x_{ij}, y_{k\ell}] = \delta_{jk}x_{ij}y_{k\ell} - \delta_{i\ell}y_{k\ell}x_{ij}\,.$$

Therefore, from (2.5), we have

$$\langle \alpha(x_{ij}, y_{k\ell}), e_r\rangle = 0$$

for all indices $i \leqslant j$ and $k \leqslant \ell$ satisfying $(i, j) \neq (k, \ell)$. From this and the identity (2.4), we get the identity (1). By Lemma 2.2 of [8], we have

$$\alpha(x, y) = \frac{\langle x, y\rangle}{2}\Big(\frac{1}{\sqrt{n_i}}e_i - \frac{1}{\sqrt{n_j}}e_j\Big)$$

for all $x, y \in \mathfrak{A}_{ij}$ $(1 \leqslant i < j \leqslant r)$. Combining this with (2.4), we get the identities (2) and (3). The identity (4) follows from (1) and the condition $\alpha(e_i, \mathfrak{t}) = 0$ (cf. (1.12) of [10]). q.e.d.

2.2. We now consider $D = (D, g_D)$ as a Riemannian submanifold of $V = (V, g_V)$. Then, from the above lemma, we have the following

THEOREM 2.2. *The mean curvature of a homogeneous convex domain*

D at the point e with respect to the unit normal e_r is equal to

$$\sum_{1\leqslant i\leqslant r-1} n_{ir}/(2\sqrt{n_r}(1-\sum_{1\leqslant i\leqslant j\leqslant r} n_{ij}))\,.$$

Proof. Let $\gamma\colon \mathfrak{t}_0\times\mathfrak{t}_0\to\mathfrak{A}_{rr}$ be the second fundamental form at the point e. Then,

$$\gamma(x,y)=\alpha(x,y)-\beta(x,y)$$

for every $x, y\in\mathfrak{t}_0$ (cf. § 3 of chap. VII in [2]). Let us put a symmetric linear mapping $h\colon\mathfrak{t}_0\to\mathfrak{t}_0$ by $\langle h(x),y\rangle=\langle\gamma(x,y),e_r\rangle$ for every $x,y\in\mathfrak{t}_0$. Then, using Lemma 2.1, we have

$$\begin{aligned}&\langle h(x_{ij}),y_{k\ell}\rangle=0\ ((i,j)\neq(k,\ell)),\ \langle h(x_{ij}),y_{ij}\rangle=0\quad(1\leqslant i\leqslant j\leqslant r-1)\,,\\ &\langle h(x_{ir}),y_{ir}\rangle=\frac{-1}{2\sqrt{n_r}}\langle x_{ir},y_{ir}\rangle\quad(1\leqslant i\leqslant r-1)\,.\end{aligned}\tag{2.6}$$

By (2.6), the principal curvatures (the eigenvalues of the linear mapping h) are 0 and $\dfrac{-1}{2\sqrt{n_r}}$ of multiplicities

$$\sum_{1\leqslant i\leqslant j\leqslant r-1} n_{ij}\quad\text{and}\quad\sum_{1\leqslant i\leqslant r-1} n_{ir}\,,$$

respectively. Hence, we get

$$\operatorname{trace} h=\frac{-1}{2\sqrt{n_r}}\sum_{1\leqslant i\leqslant r-1} n_{ir}\,.$$

On the other hand, the mean curvature H of D with respect to the unit normal e_r is given by the following formula (cf. § 5 of chap. VII in [2]):

$$H=\operatorname{trace} h/\dim D\,.$$

Therefore,

$$H=\sum_{1\leqslant i\leqslant r-1} n_{ir}/(2\sqrt{n_r}(1-\sum_{1\leqslant i\leqslant j\leqslant r} n_{ij}))\,.\qquad\text{q.e.d.}$$

From the above theorem, we have

THEOREM 2.3. *For a homogeneous convex domain D and the cone V fitted on D, the following three conditions are equivalent:*

(1) *(D, g_D) is a totally geodesic submanifold of (V, g_V).*

(2) *(D, g_D) is a minimal submanifold of (V, g_V).*

(3) *D is affinely equivalent to a convex cone and V is the product cone of D and $\boldsymbol{R}^+$.*

Proof. The implication (1) $\rightarrow$ (2) is clear (cf. § 8 of chap. VII in [2]). We now show that the implication (2) $\rightarrow$ (3) holds. By Theorem 2.2, $n_{ir} = 0$ for every index i $(1 \leqslant i \leqslant r-1)$. Therefore, $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{A}_{rr}$, where $\mathfrak{A}_0 = \sum_{1 \leqslant i,j \leqslant r-1} \mathfrak{A}_{ij}$ is a T-ideal of $\mathfrak{A}$ (cf. [1]). From the construction theorem of homogeneous convex cones stated in Section 1, we have

$$V(\mathfrak{A}) = V(\mathfrak{A}_0) \times V(\mathfrak{A}_{rr}),$$

where $V(\mathfrak{A}_{rr}) = \{x_{rr} \in \mathfrak{A}_{rr};\ x_{rr} > 0\} = \boldsymbol{R}^+$. By (1.5), the domain $D(\mathfrak{A})$ is affinely equivalent to $V(\mathfrak{A}_0)$. Hence, the condition (3) holds. The implication (3) $\rightarrow$ (1) follows from Proposition 1.1. q.e.d.

2.3. Finally in this section, we investigate a geometric property of an elementary domain. By calculating the curvature tensor, we have the following

PROPOSITION 2.4. *The elementary domain $D(n+1)$ in $\boldsymbol{R}^{n+1}$ is a simply connected hyperbolic space form of the sectional curvature $-1/(2(n+2))$.*

Proof. Since $D = D(n+1)$ is a homogeneous convex domain, D is simply connected and complete. Hence, in order to prove the above statement, it suffices to show that the sectional curvature of D is constant and equal to $-1/(2(n+2))$. As was stated in Example of Section 1, we may assume that D is constructed from a T-algebra $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{22} + \mathfrak{A}_{12} + \mathfrak{A}_{21}$ of rank two with $n_{12} = n$ and $\mathfrak{t}_0 = \mathfrak{A}_{11} + \mathfrak{A}_{12}$. Let us take arbitrary orthonormal vectors $x = x_{11} + x_{12}$ and $y = y_{11} + y_{12} \in \mathfrak{t}_0$. Then, by (1.4),

$$\|x_{11}\|^2 + \|x_{12}\|^2 = \|y_{11}\|^2 + \|y_{12}\|^2 = 1 \quad \text{and} \quad \langle x_{11}, y_{11}\rangle + \langle x_{12}, y_{12}\rangle = 0\,. \tag{2.7}$$

By using Lemmas 1.1 and 2.2 of [8], the formula (4) of Lemma 2.1 and the condition (2.1), we have

$$\begin{aligned} R(x_{11}, y_{12}, y_{11}) &= -\beta([x_{11}, y_{12}], y_{11}) = \frac{-1}{2\sqrt{n_1}}\langle x_{11}, e_1\rangle \beta(y_{12}, y_{11}) \\ &= \frac{1}{4n_1}\langle x_{11}, y_{11}\rangle y_{12}\,. \end{aligned}$$

From the formulas (2) of Lemma 2.1 and (2.1), we get

$$R(x_{12}, y_{11}, y_{12}) = \frac{1}{4n_1}\langle x_{12}, y_{12}\rangle y_{11}$$

and

$$R(x_{12}, y_{12}, y_{12}) = \frac{-1}{4n_1}\|y_{12}\|^2 x_{12} + \frac{1}{4n_1}\langle x_{12}, y_{12}\rangle y_{12}\,.$$

Furthermore, using Bianchi's identity and the above formulas, we have the following identities:

$$R(x_{11}, y_{12}, y_{12}) = \frac{-1}{4n_1}\|y_{12}\|^2 x_{11}\,, \qquad R(x_{12}, y_{11}, y_{11}) = \frac{-1}{4n_1}\|y_{11}\|^2 x_{12}$$

and

$$R(x_{12}, y_{12}, y_{11}) = 0\,.$$

On the other hand, $R(x, y, y) = R(x_{11}, y_{12}, y_{11}) + R(x_{11}, y_{12}, y_{12}) + R(x_{12}, y_{11}, y_{11}) + R(x_{12}, y_{11}, y_{12}) + R(x_{12}, y_{12}, y_{11}) + R(x_{12}, y_{12}, y_{12})$. Hence, using the above formulas and the condition (2.7), we have

$$\langle R(x, y, y), x\rangle = \frac{-1}{4n_1}\,,$$

where $n_1 = 1 + (n/2)$ (cf. (1.1)). q.e.d.

Every simply connected hyperbolic space form is Riemannian symmetric (cf. e.g. [2]). Therefore, from the above proposition we have the following

COROLLARY 2.5. *An elementary domain is Riemannian symmetric with respect to the canonical metric.*

Remark. It is known in Shima [6] that the sectional curvature of a homogeneous convex domain D is strictly negative if and only if D is affinely equivalent to an elementary domain.

§ 3. Necessary conditions for a domain to be symmetric

In this section, we give necessary conditions for a homogeneous convex domain $D = D(\mathfrak{A})$ to be Riemannian symmetric with respect to the canonical metric in terms of the T-algebra $\mathfrak{A} = \sum_{1\leqslant i,j\leqslant r}\mathfrak{A}_{ij}$ corresponding to D (cf. (1.5)).

From now on, we will consider exclusively the canonical Riemannian metric of a homogeneous convex domain. So, for the sake of brevity, the terminology *with respect to the canonical metric* may be omitted.

3.1. We first remark that a homogeneous convex domain D is simply connected and complete. Hence, D is Riemannian symmetric if and only if the following identity

$$\beta(x, R(y, z, w)) = R(\beta(x, y), z, w) + R(y, \beta(x, z), w) + R(y, z, \beta(x, w)) \tag{3.1}$$

holds for every x, y, z and $w \in \mathfrak{t}_0$ (cf. [4]).

LEMMA 3.1. *If a homogeneous convex domain D is Riemannian symmetric, then the following three conditions are satisfied*:

(1) $n_{ik} \leqslant n_{ij}$ *holds for every triple* (i, j, k) *of indices* $1 \leqslant i < j < k \leqslant r - 1$ *satisfying* $n_{jk} \neq 0$.

(2) $n_{jk} \leqslant n_{ij}$ *holds for every triple* (i, j, k) *of indices* $1 \leqslant i < j < k \leqslant r - 1$ *satisfying* $n_{ik} \neq 0$.

(3) $n_{ij}n_{jr} = 0$ *holds for every pair* (i, j) *of indices* $1 \leqslant i < j \leqslant r - 1$.

Proof. We consider the following identity (cf. (3.1)):

$$\begin{aligned} \beta(x_{jk}, R(e_i, x_{ik}, x_{ik})) &= R(\beta(x_{jk}, e_i), x_{ik}, x_{ik}) \\ &\quad + R(e_i, \beta(x_{jk}, x_{ik}), x_{ik}) + R(e_i, x_{ik}, \beta(x_{jk}, x_{ik})) \\ &\qquad (1 \leqslant i < j < k \leqslant r) . \end{aligned} \tag{3.2}$$

We now want to calculate the left hand side of (3.2). From (2.1) and Lemma 2.1, we have

$$R(e_i, x_{ik}, x_{ik}) = \frac{-1}{4n_i}\|x_{ik}\|^2 e_i + \frac{1}{4\sqrt{n_i n_k}}(1 - \delta_{kr})\|x_{ik}\|^2 e_k .$$

Hence,

$$\beta(x_{jk}, R(e_i, x_{ik}, x_{ik})) = \frac{1}{8n_k\sqrt{n_i}}(1 - \delta_{kr})\|x_{ik}\|^2 x_{jk} .$$

On the other hand, the first term of the right hand side of (3.2) is zero since $\beta(x_{jk}, e_i) = \alpha(x_{jk}, e_i) = 0$ (cf. (1.11) of [10] and Lemma 2.1). We next calculate the second and the third terms of the right hand side of (3.2). By Lemma 2.2 of [8], the formulas (1) of Lemma 2.1 and (2.1), we have

$$\begin{aligned} R(e_i, \beta(x_{jk}, x_{ik}), x_{ik}) &= \frac{1}{2}R(e_i, x_{ik}x_{jk}^*, x_{ik}) = \frac{-1}{2}\beta([e_i, x_{ik}x_{jk}^*], x_{ik}) \\ &= \frac{-1}{4\sqrt{n_i}}\beta(x_{ik}x_{jk}^*, x_{ik}) = \frac{1}{8\sqrt{n_i}}(x_{jk}x_{ik}^*)x_{ik} . \end{aligned}$$

Similarly, we get

$$R(e_i, x_{ik}, \beta(x_{jk}, x_{ik})) = \frac{1}{8\sqrt{n_i}}(x_{jk}x_{ik}^*)x_{ik}\,.$$

Hence, the equality

$$(x_{jk}x_{ik}^*)x_{ik} = \frac{1}{2n_k}(1 - \delta_{kr})\|x_{ik}\|^2 x_{jk} \tag{3.3}$$

holds. Putting $k < r$, we have $(x_{jk}x_{ik}^*)x_{ik} = (1/2n_k)\|x_{ik}\|^2 x_{jk}$. Therefore, if $x_{jk} \neq 0$, then the linear mapping: $x \in \mathfrak{A}_{ik} \to x_{jk}x^* \in \mathfrak{A}_{ji}$ is injective. Hence, the condition $n_{jk} \neq 0$ implies that $n_{ik} \leqslant n_{ij}$ holds. If $x_{ik} \neq 0$, then the linear mapping: $x \in \mathfrak{A}_{jk} \to x x_{ik}^* \in \mathfrak{A}_{ji}$ is also injective. This means that the condition $n_{ik} \neq 0$ implies $n_{jk} \leqslant n_{ij}$. Hence, the conditions (1) and (2) hold. Next, putting $k = r$ in (3.3), we have $(x_{jr}x_{ir}^*)x_{ir} = 0$. Taking the traces of the both hand sides of $((x_{jr}x_{ir}^*)x_{ir})x_{jr}^* = 0$, we get

$$\mathrm{Sp}\,((x_{jr}x_{ir}^*)(x_{jr}x_{ir}^*)^*) = \mathrm{Sp}\,((x_{jr}x_{ir}^*)x_{ir})x_{jr}^*) = 0\,,$$

which means that $x_{ir}x_{jr}^* = 0$ for every $x_{ir} \in \mathfrak{A}_{ir}$ and $x_{jr} \in \mathfrak{A}_{jr}$. Let us take arbitrary elements $x_{ij} \in \mathfrak{A}_{ij}$, $x_{jr} \in \mathfrak{A}_{jr}$, and put $x_{ir} = x_{ij}x_{jr}$. Then by using the formulas (1.7) of [8] and (2.4) of [10], we have

$$\frac{1}{2n_j}\|x_{ij}\|^2\|x_{jr}\|^2 = \|x_{ir}\|^2 = \langle x_{ij}x_{jr}, x_{ir}\rangle = \langle x_{ij}, x_{ir}x_{jr}^*\rangle = 0\,.$$

This implies $n_{ij}n_{jr} = 0$. q.e.d.

We next prove the following

LEMMA 3.2. *If a homogeneous convex domain D is Riemannian symmetric, then the following two conditions are satisfied*:

(1) $n_{ik} \leqslant n_{jk}$ *holds for every triple* (i, j, k) *of indices* $1 \leqslant i < j < k \leqslant r$ *satisfying* $n_{ij} \neq 0$.

(2) $n_{ij} \leqslant n_{jk}$ *holds for every triple* (i, j, k) *of indices* $1 \leqslant i < j < k \leqslant r$ *satisfying* $n_{ik} \neq 0$.

Proof. Since $[e_j, x_{ik}] = 0$ (cf. (1.6) of [8]), using (4) of Lemma 2.1 and (2.1) we have

$$R(e_j, x_{ik}, x_{ik}) = R(e_j, x_{ik}, \beta(x_{ij}, x_{ik})) = 0\,.$$

Thus, by (3.1), we get

$$R(\beta(x_{ij}, e_j), x_{ik}, x_{ik}) + R(e_j, \beta(x_{ij}, x_{ik}), x_{ik}) = 0.$$

Similarly as in the proof of Lemma 3.1, we can see that the following formulas

$$R(\beta(x_{ij}, e_j), x_{ik}, x_{ik}) = \frac{-1}{8n_i\sqrt{n_j}} \|x_{ik}\|^2 x_{ij} + \frac{1}{8\sqrt{n_j}} x_{ik}(x_{ik}^* x_{ij})$$

and

$$R(e_j, \beta(x_{ij}, x_{ik}), x_{ik}) = \frac{1}{8\sqrt{n_j}} x_{ik}(x_{ik}^* x_{ij})$$

hold. Therefore, we have

$$x_{ik}(x_{ik}^* x_{ij}) = \frac{1}{2n_i} \|x_{ik}\|^2 x_{ij} .$$

Using this equality in the same way as the proof of Lemma 3.1, we obtain the above conditions. q.e.d.

3.2. Let us put the set $I = \{1, 2, \cdots, r\}$ and define two subsets I_0 and I_1 of I by

$$I_0 = \{i \in I;\ n_{ir} = 0\} \quad \text{and} \quad I_1 = \{i \in I;\ n_{ir} \neq 0\} ,$$

respectively. Then,

$$I = I_0 \cup I_1 \quad \text{(disjoint)} . \tag{3.4}$$

By making use of the lemmas obtained above, we have

PROPOSITION 3.3. *If a homogeneous convex domain D is Riemannian symmetric, then the following two conditions are satisfied*:

(1) $n_{ij} = 0$ *holds for every pair* (i, j) *of indices* $i \in I_1$ *and* $j \in I$ $(r \neq i \neq j \neq r)$.

(2) *Either* $n_{ik} = n_{jk} = 0$ *or* $n_{ij} = n_{jk} = n_{ik}$ *holds for every triple* (i, j, k) *of indices* $i, j \in I_0$ $(i < j)$, $k \in I$ *satisfying the conditions* $n_{ij} \neq 0$ *and* $k \neq i, j$.

Proof. We now show that the condition (1) holds in the case of $j < i$. In fact, the condition $n_{ij} = 0$ follows from (3) of Lemma 3.1. In the case of $i < j$, we suppose that $n_{ij} \neq 0$. Then, by (1) or (2) of Lemma 3.2, we have $n_{jr} \neq 0$. Again, by (3) of Lemma 3.1, this is a contradiction. Therefore, the condition (1) holds. We proceed to showing (2). Combining the conditions (1) of Lemmas 3.1 and 3.2 with (1.2), we can see that

$$n_{ij} n_{jk} \neq 0 \quad \text{implies} \quad n_{ij} = n_{jk} = n_{ik} \tag{3.5}$$

for every triple (i, j, k) of indices $1 \leqslant i < j < k \leqslant r-1$. We now consider the case of $k < i < j$. If $n_{ik} \neq 0$, then from (3.5), we have the equalities $n_{ij} = n_{jk} = n_{ik}$. If $n_{jk} \neq 0$, then (2) of Lemma 3.1 implies $n_{ij} \leqslant n_{ik} \neq 0$. Again, we have $n_{ij} = n_{jk} = n_{ik}$. Therefore, (2) holds in this case. We next consider the case of $i < k < j$. By (1) of Lemma 3.2, $n_{ik} \neq 0$ implies $n_{ij} \leqslant n_{jk} \neq 0$. Hence, by (3.5), we have $n_{ij} = n_{jk} = n_{ik}$. If $n_{ik} = 0$, then (2) of Lemma 3.1 implies $n_{jk} = 0$. Let us consider the case of $i < j < k < r$. Then, by (2) of Lemmas 3.1 and 3.2, the condition $n_{ik} \neq 0$ implies $n_{ij} = n_{jk} \neq 0$. Hence, by (3.5) and (1.2), we have the equalities $n_{ij} = n_{jk} = n_{ik}$ or $n_{ik} = n_{jk} = 0$. Finally, for $k = r$, $n_{ir} = n_{jr} = 0$ holds, since $i, j \in I_0$. Therefore, the condition (2) holds for every index $k \in I$ with $k \neq i, j$. q.e.d.

§ 4. Symmetric domains

In this section, we determine all symmetric homogeneous convex domains by making use of the results obtained in the preceding sections. Throughout this section, we assume that a homogeneous convex domain D is realized as the domain $D(\mathfrak{A})$ given by (1.5) in terms of a T-algebra $\mathfrak{A} = \sum_{1 \leqslant i,j \leqslant r} \mathfrak{A}_{ij}$ of rank r $(r \geqslant 2)$.

4.1. We first prove

PROPOSITION 4.1. *If a homogeneous convex domain D is Riemannian symmetric and satisfies the condition $n_{ir} \neq 0$ for every index i $(1 \leqslant i \leqslant r-1)$, then the following three conditions are satisfied:*

(1) *$n_{ij} = 0$ holds for every pair (i, j) of indices $1 \leqslant i < j \leqslant r-1$.*

(2) *The domain D is the direct product of the elementary domains $D(n_{ir}+1)$ $(1 \leqslant i \leqslant r-1)$.*

(3) *The cone $V(D)$ fitted on D is given by*

$$V(D) = \left\{ x = (x_{ij}) \in X(\mathfrak{A});\ \begin{matrix} x_{rr}x_{ii} - (x_{ir}, x_{ir}) > 0 \quad (1 \leqslant i \leqslant r-1) \\ x_{rr} > 0 \end{matrix} \right\}. \tag{4.1}$$

Proof. From (1) of Proposition 3.3, we have $n_{ij} = 0$ for every $1 \leqslant i < j \leqslant r-1$. Therefore,

$$X(\mathfrak{A}) = \left\{ \begin{pmatrix} x_{11} & 0 & & x_{1r} \\ & x_{22} & & x_{2r} \\ 0 & & \ddots & \vdots \\ x_{1r}^* & x_{2r}^* & \cdots & x_{rr} \end{pmatrix} \right\} \subset \mathfrak{A}\,.$$

By using the inequalities defining the cone $V(\mathfrak{A})$ in $X = X(\mathfrak{A})$ (cf. Proposition 2 of p. 385 in [11]), we can see that the cone $V(D) = V(\mathfrak{A})$ is given by the form (4.1). From this and (1.5), we have

$$\begin{aligned} D &= \{x \in V(\mathfrak{A});\ x_{rr} = 1\} \\ &= \prod_{1 \leqslant i \leqslant r-1} \{(x_{ii}, x_{ir}) \in \mathfrak{A}_{ii} \times \mathfrak{A}_{ir};\ x_{ii} - (x_{ir}, x_{ir}) > 0\} \\ &= \prod_{1 \leqslant i \leqslant r-1} D(n_{ir} + 1). \end{aligned}$$

q.e.d.

We now porve the main theorem stated in Introduction.

THEOREM 4.2. *A homogeneous convex domain D in $\boldsymbol{R}^n$ is Riemannian symmetric with respect to the canonical metric if and only if D is affinely equivalent to one of the following*:

V_0; $D(m_1) \times D(m_2) \times \cdots \times D(m_k)$ $(m_1 + m_2 + \cdots + m_k = n)$;
$V_0 \times D(m_1) \times D(m_2) \times \cdots \times D(m_k)$ $(\dim V_0 + m_1 + m_2 + \cdots + m_k = n)$,

where V_0 is an arbitrary homogeneous self-dual cone and $D(m_i)$ is the elementary domain of dimension m_i.

Proof. Let us suppose that D is Riemannian symmetric. We first consider the case of $I = I_1$. Then by Proposition 4.1, D is the direct product of $r - 1$ elementary domains. We next consider the case of $I \neq I_1$. Then by (1) of Proposition 3.3, $n_{ij} = 0$ holds for every pair (i, j) of indices $i \in I_0$ and $j \in I_1$. Hence, from this and (3.4), the sets I_0 and I_1 are admissible in the sense of Asano [1]. Therefore, by putting

$$\mathfrak{A}_0 = \sum_{i,j \in I_0} \mathfrak{A}_{ij} \quad \text{and} \quad \mathfrak{A}_1 = \sum_{i,j \in I_1} \mathfrak{A}_{ij}\,,$$

we can see that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are T-ideals of $\mathfrak{A}$ satisfying

$$\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{A}_1 \quad (\text{direct sum})\,.$$

Hence, by Lemma 3 of [1], we have

$$V(\mathfrak{A}) = V(\mathfrak{A}_0) \times V(\mathfrak{A}_1)\,.$$

On the other hand, from (2) of Proposition 3.3, it follows that the kernel of $\mathfrak{A}_0$ coincides with $\mathfrak{A}_0$ (cf. p. 69 of [12] or Lemma 2.2 of [10]). Again, by a result of [12], $V(\mathfrak{A}_0)$ is self-dual. If $I_1 = \{r\}$, then $\mathfrak{A}_1 = \mathfrak{A}_{rr}$ and $V(\mathfrak{A}_1) = \{x_{rr} > 0\} = \boldsymbol{R}^+$ the cone of all positive real numbers. By (1.5), we have

$$D = \{x \in V(\mathfrak{A});\ x_{rr} = 1\} = V(\mathfrak{A}_0) \times \{1\} \subset V(\mathfrak{A}_0) \times \boldsymbol{R}^+\,,$$

and D is affinely equivalent to the self-dual cone $V(\mathfrak{A}_0)$. Finally, if $I_1 = \{i_1, i_2, \cdots, i_k\}$ with $i_1 < i_2 < \cdots < i_k = r$ $(1 < k < r)$, then by (1) of Proposition 3.3, we have

$$n_{i_1 r} n_{i_2 r} \cdots n_{i_{k-1} r} \neq 0 \quad \text{and} \quad n_{i_\lambda i_\mu} = 0 \quad (1 \leqslant \lambda \neq \mu \leqslant k-1).$$

From Proposition 4.1, it follows that the domain $D(\mathfrak{A}_1)$ corresponding to the T-algebra $\mathfrak{A}_1$ is the direct product of the elementary domains $D(n_{i_\lambda r} + 1)$ $(1 \leqslant \lambda \leqslant k-1)$. Hence, by (1.5), we have

$$\begin{aligned} D(\mathfrak{A}) &= \{x \in V(\mathfrak{A}) = V(\mathfrak{A}_0) \times V(\mathfrak{A}_1);\ x_{rr} = 1\} = V(\mathfrak{A}_0) \times D(\mathfrak{A}_1) \\ &= V(\mathfrak{A}_0) \times \prod_{1 \leqslant \lambda \leqslant k-1} D(n_{i_\lambda r} + 1). \end{aligned}$$

Conversely, every homogeneous self-dual cone is Riemannian symmetric (cf. Rothaus [5]). Combining this with Proposition 1.1 and Corollary 2.5, we can see that the sufficient condition in the above statement is satisfied. q.e.d.

Every homogeneous convex cone in $\boldsymbol{R}^n$ $(n \geqslant 2)$ is always reducible as a Riemannian manifold (cf. [3] or [9]). Therefore, from the above theorem, Propositions 1.1 and 2.4, we have the following

COROLLARY 4.3. *A homogeneous convex domain D in $\boldsymbol{R}^n$ $(n \geqslant 2)$ is Riemannian symmetric and irreducible with respect to the canonical metric if and only if D is affinely equivalent to the elementary domain $D(n)$.*

4.2. Finally, we determine all homogeneous convex cones which are to be the cones fitted on symmetric homogeneous convex domains. For this purpose, we employ the following notation: For positive integers $m_1, m_2, \cdots, m_k$, we put

$$V_{m_1, m_2, \cdots, m_k} = \{(x, y, t) \in \boldsymbol{R}^k \times \boldsymbol{R}^m \times \boldsymbol{R};\ t > 0,\ P_i > 0\ (1 \leqslant i \leqslant k)\},$$

where

$$P_i = t x_i - (y_i, y_i), \quad x = (x_1, x_2, \cdots, x_k) \in \boldsymbol{R}^k$$

and

$$y = (y_1, y_2, \cdots, y_k) \in \boldsymbol{R}^m = \boldsymbol{R}^{m_1} \times \boldsymbol{R}^{m_2} \times \cdots \times \boldsymbol{R}^{m_k}.$$

Then it is easy to see that the cone V_{m_1} is the circular cone $C(m_1 + 2)$ (cf. Example in § 1), and for $r > 2$, the cone $V_{n_{1r}, n_{2r}, \cdots, n_{r-1r}}$ is non-self-dual and exactly the one given by (4.1). Combining Theorem 4.2 with Proposition 1.2, we have the following

COROLLARY 4.4. *A homogeneous convex cone V in $\boldsymbol{R}^n$ $(n \geqslant 2)$ is the cone fitted on some symmetric homogeneous convex domain if and only if V is linearly equivalent to one of the following*:

$$V_0 \times \boldsymbol{R}^+;\ V_{m_1,m_2,\dots,m_k}\ (m_1 + m_2 + \cdots + m_k + k + 1 = n);$$
$$V_0 \times V_{m_1,m_2,\dots,m_k}\ (\dim V_0 + m_1 + m_2 + \cdots + m_k + k + 1 = n),$$

where V_0 is an arbitrary homogeneous self-dual cone.

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