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Nagoya Math. J.
Vol. 92 (1983), 175–177

DESCENT OF P-PROPERTY BY PROPER SURJECTIVE MORPHISMS

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It is well known that the quasi-exellent property (i.e. G-ring and J-II) ascends by finite type morphisms. On the other hand Greco showed in [2, Proposition 4.1 and Proposition 2.3] that G-property (i.e. formal fibers are geometrically regular) does not descend by finite type morphisms, although J-II property (i.e. any finite type algebra has regular locus open) does. Then he proposed the following [2, Remark 3.8 (iii)]

CONJECTURE (Greco). Let $f: X \to Y$ be a proper surjective morphism of locally noetherian schemes. If X is quasi-excellent, then Y is also quasi-excellent.

Recently, B. Bellaccini proved this conjecture in the case Y is a $\boldsymbol{Q}$-scheme [1, Theorem 2.1]. Her proof uses Hironaka's disingulalization theorem, hence she needs the assumption of zero-characteristic, and both of the properties G-ring and J-II must be assumed on X.

Here we prove a slightly stronger theorem than the Conjecture of Greco, that is, we prove the descent of the property of formal fibers without open locus conditions.

All rings are assumed to be commutative noetherian rings with identity. We use freely the notation and terminology of [3] and [4].

Let $\boldsymbol{P}$ be a local property which descend by faithfully flat ring homomorphisms of noetherian rings such as regular, normal, reduced, Cohen Macaulay, Gorenstein, R_n, S_n etc. ([4, (21.C), (21.E)] and [6, Theorem 1]).

THEOREM. *Let $f: X \to Y$ be a proper surjective morphism of locally noetherian schemes X and Y. If the formal fibers of $\mathcal{O}_{X,x}$ are geometrically $\boldsymbol{P}$ for all $x \in X$, then the same holds for Y.*

Proof. Considering the base change $\operatorname{Spec} \mathcal{O}_{Y,y} \to Y$, we may assume

Received October 15, 1982.

$Y = \operatorname{Spec} A$ with $(A, \mathfrak{m})$ a local ring. Now it is sufficient to show that the ring $\hat{A} \otimes_A k(\mathfrak{q}) \otimes_{k(\mathfrak{q})} K$ has property $\boldsymbol{P}$ for any $\mathfrak{q} \in \operatorname{Spec} A$ and for any finite field extension K of $k(\mathfrak{q}) = A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$. Take a finite A-algebra B which is a domain with field of quotients K. Then

$$\hat{A} \otimes_A K = (\hat{A} \otimes_A B) \otimes_B K = \hat{B} \otimes_B K = (\hat{B}_{\mathfrak{n}_1} \times \cdots \times \hat{B}_{\mathfrak{n}_r}) \otimes_B K$$

by [5, (17.7) and (17.8)], where $\mathfrak{n}_1, \cdots, \mathfrak{n}_r$ are the maximal ideals of B. Therefore, considering the base change $\operatorname{Spec} B_{\mathfrak{n}_i} \to Y$, we may assume that A is a local integral domain with field of quotients K and it is sufficient to show that the ring $\hat{A} \otimes_A K$ has property $\boldsymbol{P}$. To see this it is sufficient to show that $\hat{A}_{\mathfrak{p}}$ has property $\boldsymbol{P}$ for any $\mathfrak{p} \in \operatorname{Spec} \hat{A}$ such that $\mathfrak{p} \cap A = 0$.

Since f is surjective, there exists a point x in X which is mapped by f to the generic point of Y. Considering the closed subscheme of X defined by x instead of X, we may assume X itself is a integral scheme with function field, say L. Then we can regard K as a subfield of L through f.

Since $\mathfrak{p} \cap K = 0$, there exists $\mathfrak{P}' \in \operatorname{Spec}(\hat{A} \otimes_A L)$ such that $\mathfrak{P}' \cap \hat{A} = \mathfrak{p}$.

Since $\hat{A} \otimes_A L$ is a localization of a ring $\hat{A} \otimes_A C$ for any A-algebra C with field of fractions L, $\mathfrak{P}'$ corresponds to a point of $X \times_Y \operatorname{Spec} \hat{A}$. Let V be the closed set of $X \times_Y \operatorname{Spec} \hat{A}$ defined by the point corresponding to $\mathfrak{P}'$ and take a closed point z in V. Since $\hat{f}: X \times_Y \operatorname{Spec} \hat{A} \to \operatorname{Spec} \hat{A}$ is a closed map, $\hat{f}(z)$ is a closed point of $\operatorname{Spec} \hat{A}$, that is $\{\mathfrak{m}\hat{A}\}$.

Let $(B, \mathfrak{n})$ be the local ring of X at $p(z)$ where $p: X \times_Y \operatorname{Spec} \hat{A} \to X$ is the projection. Then B is an A-algebra essentially of finite type (because f is a morphism of finite type) and hence $\hat{A} \otimes_A B$ is noetherian. The local ring of $X \times_Y \operatorname{Spec} \hat{A}$ at z is isomorphic to $(\hat{A} \otimes_A B)_{\mathfrak{M}}$ where $\mathfrak{M} = \mathfrak{m}(\hat{A} \otimes_A B) + \mathfrak{n}(\hat{A} \otimes_A B) = \mathfrak{n}(\hat{A} \otimes_A B)$. Now the completion of $(\hat{A} \otimes_A B)_{\mathfrak{M}}$ is isomorphic to $\hat{B}$ by [3, IV, Lemma (7.9.3.1)]. Put $\mathfrak{P} = \mathfrak{P}' \cap (\hat{A} \otimes_A B)$, then $\mathfrak{P} \cap L = 0$ and $\mathfrak{P} \subseteq \mathfrak{M}$ by construction. Hence $\mathfrak{P}(\hat{B} \otimes_B L) \neq \hat{B} \otimes_B L$ and there exists $\mathfrak{Q} \in \operatorname{Spec} \hat{B}$ such that $\mathfrak{P} = (\hat{A} \otimes_A B) \cap \mathfrak{Q}$ and $\mathfrak{Q} \cap B = 0$.

Now the localization $\hat{B}_{\mathfrak{Q}}$ of a formal fiber $\hat{\mathcal{O}}_{X, p(z)} \otimes_B L$ of $\mathcal{O}_{X, p(z)}$ has property $\boldsymbol{P}$. On the other hand $(\hat{A} \otimes_A B)_{\mathfrak{P}} \to \hat{B}_{\mathfrak{Q}}$ is faithfully flat, because $\hat{A} \otimes_A B$ is noetherian and hence

$$(\hat{A} \otimes_A B)_{\mathfrak{M}} \longrightarrow \widehat{(\hat{A} \otimes_A B)_{\mathfrak{M}}} = \hat{B}$$

is flat. Moreover, $\hat{A}_{\mathfrak{p}} \to (\hat{A} \otimes_A B)_{\mathfrak{P}}$ is faithfully flat because $\hat{A} \otimes_A K \to (\hat{A} \otimes_A K) \otimes_K L = \hat{A} \otimes_A L$ is flat and $\mathfrak{p} \cap K = 0$, $\mathfrak{P} \cap L = 0$. Thereore $\hat{A}_{\mathfrak{p}} \to \hat{B}_{\mathfrak{Q}}$ is faithfully flat and, since $\hat{B}_{\mathfrak{Q}}$ has property $\boldsymbol{P}$, $\hat{A}_{\mathfrak{p}}$ has property $\boldsymbol{P}$.

COROLLARY. *The conjecture of Greco holds in the full general sense.*

Proof. If we put "regular" for P in the Theorem, we see that G-property descends. Since the descent of J-II property is known, we have the corollary.

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