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Nagoya Math. J.
Vol. 88 (1982), 133-154

EXCEPTIONALLY RAMIFIED MEROMORPHIC FUNCTIONS WITH A NON-ENUMERABLE SET OF ESSENTIAL SINGULARITIES

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§1. Introduction

In the complex function theory, Picard's Great Theorem plays an essential and important role. It is well-known as generalizations of this theorem that in a neighborhood of an isolated essential singularity, a meromorphic function cannot be exceptionally ramified (see W. Gross [2]) and that even it cannot be normal (see O. Lehto and K. I. Virtanen [7]). We are therefore interested in the behaviour of meromorphic functions with non-isolated essential singularities as well as in generalizations of the Gross' result. Several approaches in this direction have been made by G. af Hällström [3], S. Kametani [4], K. Noshiro [13], K. Matsumoto [8], [9], [10], [11], [12], S. Toppila [15], etc..

As for the functions with "more than two Picard exceptional values", K. Matsumoto ([10], [11]) has given sufficient conditions on Cantor sets E whose complements do not admit such functions. One of his basic results is

THEOREM A. *Let E be a Cantor set with successive ratios ξ_n satisfying the condition*

$$\xi_{n+1} = o(\xi_n^2) ,$$

then the domain complementary to E does not admit meromorphic functions with "more than two Picard exceptional values" at each singularities.

Having been inspired by this theorem, we are led to ask whether there is a Cantor set admitting no meromorphic functions with weaker conditions, such as "exceptionally ramified" (or "normal"). An exceptionally ramified

Received October 31, 1980.

meromorphic function is defined as follows: A meromorphic function f on the extended complex plane $\hat{C}$ is said to be exceptionally ramified, if there exist w_k, $1 \leqq k \leqq q$, in $\hat{C}$ such that the multiplicities $\ell_{k,j}$ of the roots $z_{k,j}$ of the equation $f(z) = w_k$ satisfy

$$\ell_{k,j} \geqq \nu_k \qquad \text{except finite } j\text{'s}\,,$$

for a sequence of integers $\nu_k \geqq 2$ with the property

$$\sum_{k=1}^{q}\left(1 - \frac{1}{\nu_k}\right) > 2\,. \tag{1.1}$$

Our main theorem is stated as follows:

THEOREM. *Let E be a Cantor set with successive ratios ξ_n satisfying the condition*

$$\xi_{n+1} = o(\xi_n^5)\,, \tag{1.2}$$

then the domain complementary to E admits no exceptionally ramified meromorphic functions with E as the set of essential singularities.

The author wishes to express her deep gratitude to Professor Kikuji Matsumoto who made many valuable suggestions during the course of this work, and she thanks Professor Masayuki Itô for his help in preparing the manuscript.

§ 2. Preliminaries

2.1. Introducing the chordal distance $\chi(w, \zeta)$ on $\hat{C}$, we denote by $|S|$ the diameter of a subset S in $\hat{C}$. Let Δ be a τ-ply connected domain bounded by positively oriented analytic curves $\{\Gamma_i\}_{i=1,2,\dots,\tau}$, Γ_i: $z = z_i(t)$ $(a \leqq t \leqq b)$ and let f be meromorphic on the closure $\bar{\Delta}$ of Δ. For ζ_1, $\zeta_2 \notin f(\Gamma_i)$, $O(\Gamma_i; \zeta_1, \zeta_2)$ denotes the variation of $(1/2\pi) \arg (f(z) - \zeta_1)/(f(z) - \zeta_2)$ as z describes the curve Γ_i positively once.

We shall deal with an exceptionally ramified meromorphic function f on $\bar{\Delta}$ with q totally ramified values $\{w_k\}_{k=1,2,\dots,q}$ satisfying the following three conditions:

(1) There exist mutually disjoint simply connected sectionally analytic domains $\{D_j\}_{j=1,\dots,\alpha}$, $1 \leqq \alpha \leqq \tau$, with

$$|D_j| < \tfrac{1}{2} \min_{k \neq m} \chi(w_k, w_m) \tag{2.1}$$

and the images $\{f(\Gamma_i)\}_{i=1,\dots,\tau}$ are covered with $\{D_j\}_{j=1,\dots,\alpha}$, each D_j containing

$f(\Gamma_i)$ for at least one i.

(2) The number $\nu(w, f, \Delta)$ of roots of the equation $f(z) = w$ in Δ is $\geqq 1$, for $w \in \hat{C} - \bigcup_{j=1}^{\alpha} \overline{D}_j$.

(3) f has no ramified values on each boundary $\partial D_j \equiv C_j$.

Here the multiplicity is always taken into account.

For each C_j, the inverse image $f^{-1}(C_j)$ of C_j consists of a finite number of simple closed analytic curves $\{\Gamma_k^{(j)}\}_k$ in Δ. Then $\mathscr{F}$ denote the family of all subdomains of Δ which are bounded by some of $\{\Gamma_k^{(j)}\}_{k,j}$. By introducing a partial order into $\mathscr{F}$ by inclusion, we choose a maximal element Δ' of $\mathscr{F}$. The boundary $\partial\Delta'$ consists of a subfamily $\{\Gamma_i'\}_{i=1,\cdots,\tau'}(\tau' \leqq \tau)$ of $\{\Gamma_k^{(j)}\}_{k,j}$. We may assume that Γ_i' is positively oriented with respect to Δ'. Denoting by $j(i)$ the number j with $C_j \supset f(\Gamma_i')$, we assume that $C_{j(i)}$, $i = 1, \cdots, \tau'$, form a subset $\{C_j\}_{j=1,\cdots,\alpha'}$ of $\{C_j\}_{j=1,\cdots,\alpha}$. For $\zeta_0 \in \hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j$, $\zeta_{j(i)} \in D_{j(i)}$, we set

$$s_i = O(\Gamma_i'; \zeta_0, \zeta_{j(i)}) .$$

Since Δ' is maximal in $\mathscr{F}$, we see that

$$s_i > 0 \qquad (i = 1, 2, \cdots, \tau')$$

and

$$\nu(\zeta, f, \Delta') \geqq 1 \qquad \text{for } \zeta \in \hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j .$$

Since the Riemannian image $\tilde{S}$ of Δ' under f may be viewed as a covering surface of $\hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j$, the exact value of the Euler characteristic $\rho(\Delta')$ of Δ':

$$\begin{aligned}\rho(\Delta') &= \rho(\tilde{S}) \\ &= \sum_{j=1}^{\alpha'} \rho(D_j)\nu(\zeta_j, f, \Delta') + \rho\Big(\hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j\Big)\nu(\zeta_0, f, \Delta') + v ,\end{aligned}$$

for $\zeta_0 \in \hat{C} - \bigcup_{j=1}^{\alpha'} \overline{D}_j$, $\zeta_j \in D_j$, where v denotes the ramification index of $\tilde{S}$, that is,

$$\tau' - 2 = -\sum_{j=1}^{\alpha'} \nu(\zeta_j, f, \Delta') + (\alpha' - 2)\nu(\zeta_0, f, \Delta') + v .$$

Hence

$$2\nu(\zeta_0, f, \Delta') - 2 = \sum_{j=1}^{\alpha'} \{\nu(\zeta_0, f, \Delta') - \nu(\zeta_j, f, \Delta')\} - \tau' + v .$$

The argument principle proves

$$\nu(\zeta_0, f, \Delta') - \nu(\zeta_j, f, \Delta') = \sum_{i=1}^{\tau'} O(\Gamma'_i; \zeta_0, \zeta_j) , \tag{2.2}$$

so that

$$\begin{aligned} 2\nu(\zeta_0, f, \Delta') - 2 &= \sum_{i=1}^{\tau'} \left(\sum_{j=1}^{\alpha'} O(\Gamma'_i; \zeta_0, \zeta_j) - 1 \right) + v \\ &= \sum_{i=1}^{\tau'} (O(\Gamma'_i; \zeta_0, \zeta_{j(i)}) - 1) + v \\ &= \sum_{i=1}^{\tau'} (s_i - 1) + v . \end{aligned}$$

Putting $n = \nu(\zeta_0, f, \Delta')$, we have

LEMMA 1.

$$2n - 2 = \sum_{i=1}^{\tau'} (s_i - 1) + v . \tag{2.3}$$

2.2. Using Lemma 1 and (1.1) we shall show that Δ and Δ' are at least triply connected.

Let m_k denote the number of roots $z_{k,j}$ of the equation $f(z) = w_k$ restricted to Δ' and let $l_{k,j}$ be the multiplicities of $z_{k,j}$, $j = 1, 2, \cdots, m_k$. For a totally ramified value w_k $(1 \leqq k \leqq q)$, we write

$$N_k = \begin{cases} \{i \mid j(i) = j_k\}, & \text{if } w_k \in D_{j_k} \text{ for some } j_k , \\ \emptyset & , \text{ if } w_k \notin \bigcup_{j=1}^{\alpha'} D_j \end{cases}$$

and

$$\sigma_k = \text{the number of } N_k .$$

Obviously, by (2.1)

$$N_k \cap N_m = \emptyset , \qquad \text{if } k \neq m$$

and

$$0 \leqq \sigma_1 + \sigma_2 + \cdots + \sigma_q \leqq \tau' . \tag{2.4}$$

Since $O(\Gamma'_i; \zeta_0, \zeta_{j_k}) = 0$ for i with $j(i) \neq j_k$, i.e. $i \notin N_k$, the equality

$$n = \sum_{j=1}^{m_k} \ell_{k,j} + \sum_{i \in N_k} s_i \tag{2.5}$$

comes from (2.2), whence we have

$$n \geqq \nu_k m_k + \sigma_k \qquad (k = 1, 2, \cdots, q)\ , \tag{2.6}$$

because $l_{k,j} \geqq \nu_k$ and $s_i \geqq 1$ for $i \in N_k$.
Hence

$$n \sum_{k=1}^{q} \frac{1}{\nu_k} \geqq \sum_{k=1}^{q} m_k\ . \tag{2.7}$$

From (2.3) and (2.5), it follows that

$$\begin{aligned} 2n - 2 &= \sum_{i=1}^{\tau'} (s_i - 1) + v \\ &\geqq \sum_{k=1}^{q} \sum_{i \in N_k} (s_i - 1) + \sum_{k=1}^{q} \sum_{j=1}^{m_k} (\ell_{k,j} - 1) \\ &= qn - \sum_{k=1}^{q} m_k - \sum_{k=1}^{q} \sigma_k\ , \end{aligned} \tag{2.8}$$

so that

$$\sum_{k=1}^{q} m_k + \sum_{k=1}^{q} \sigma_k - 2 \geqq (q-2)n\ . \tag{2.8$'$}$$

Using (1.1), (2.7) and (2.8)$'$, we obtain

$$\sum_{k=1}^{q} m_k + \sum_{k=1}^{q} \sigma_k - 2 \geqq (q-2)n > n \sum_{k=1}^{q} \frac{1}{\nu_k} \geqq \sum_{k=1}^{q} m_k \tag{2.9}$$

and hence, by (2.4),

$$\tau \geqq \tau' \geqq \sum_{k=1}^{q} \sigma_k \geqq 3\ . \tag{2.10}$$

Thus we have the following

LEMMA 2. *A simply, or doubly, connected domain $\varDelta$ does not admit any exceptionally ramified meromorphic functions satisfying the conditions* (1), (2) *and* (3).

§ 3. Classification of covering surfaces generated by exceptionally ramified meromorphic functions

3.1. For approach it is essential to determine all covering surfaces generating by an exceptionally ramified meromorphic function f with three totally ramified values on a triply connected domain $\varDelta(q = 3$ and $\tau = 3)$. With this choice of q and τ, the inequalities (2.9) and (2.10) imply

(3.1) $$n = m_1 + m_2 + m_3 + 1$$

and

(3.2) $$\tau = \tau' = \sigma_1 + \sigma_2 + \sigma_3 = 3 .$$

The inequality in (2.8) should be equality, so that f cannot have any ramified value other than $\{w_k\}_{k=1,2,3}$. By (3.2), each D_j $(1 \leqq j \leqq \alpha')$ contains one of the $\{w_k\}_{k=1,2,3}$. Since both Δ and Δ' are triply connected, each component of $\Delta - \Delta'$ is a ring domain. The image of a component under f is contained in one of the $\{D_j\}_{j=1,\dots,\alpha'}$. Consequently $\alpha = \alpha'$.

Combining (3.1) with (2.6), we have

(3.3) $$m_1 + m_2 + m_3 + 1 \geqq \nu_k m_k + \sigma_k, \qquad k = 1, 2, 3 .$$

There are four possibilities:

(i) $$m_1 \geqq 1 , \quad m_2 \geqq 1 , \quad m_3 \geqq 1 .$$

(ii) $$m_1 \geqq 1 , \quad m_2 \geqq 1 , \quad m_3 = 0 .$$

(iii) $$m_1 \geqq 1 , \quad m_2 = m_3 = 0 .$$

(iv) $$m_1 = m_2 = m_3 = 0 .$$

Case (i). By (3.2) and (3.3), we have

(3.4) $$0 \geqq (\nu_1 - 3)m_1 + (\nu_2 - 3)m_2 + (\nu_3 - 3)m_3 .$$

From (1.1) and (3.4), follow

$$\nu_1 = 2 , \quad \nu_2 \geqq 3 \quad \text{and} \quad \nu_3 \geqq 4 .$$

From (3.3) and (3.4) follows

(3.5) $$1 \geqq (\nu_2 - 4)m_2 + (\nu_3 - 4)m_3 .$$

By (1.1), the following two possibilities occur

(i_α) $$\nu_2 = 4 , \qquad \nu_3 \geqq 5$$

(i_β) $$\nu_2 = 3 , \qquad \nu_3 \geqq 7 .$$

Case (i_α). From (3.5), $m_3 = 1$ and $\nu_3 = 5$ follow. Hence by (3.3), there are the following possibilities:

(a) $$m_1 = 2 , \qquad m_2 = 1 .$$

(b) $$m_1 = 3 , \qquad m_2 = 1 .$$

(c) $$m_1 = 4 , \qquad m_2 = 2 .$$

In each case, the numbers n, $\ell_{k,j}$, σ_k, s_i are determined by (2.5), (3.1) and (3.2). Since $\sum_{i=1}^{3} s_i \geqq 3$, the case (a) does not occur.

Case (b). Since $n = 6$, we have

$$\sum_{j=1}^{3} \ell_{1,j} + \sum_{i\in N_1} s_i = \ell_{2,1} + \sum_{i\in N_2} s_i = \ell_{3,1} + \sum_{i\in N_3} s_i = 6 .$$

This implies

$$\begin{cases} n = 6,\ \ell_{1,j} = 2 \text{ for } j = 1, 2, 3,\ \ell_{2,1} = 4,\ \ell_{3,1} = 5 , \\ \sigma_1 = 0,\ \sigma_2 = 2,\ \sigma_3 = 1,\ \{s_i\}_{i\in N_2} = \{1, 1\} , \\ \{s_i\}_{i\in N_3} = \{1\} . \end{cases}$$

This covering surface is said to be of class 1.

Case (c). Similarly as above, we have

$$\begin{cases} n = 8,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 4,\ \ell_{2,1} = \ell_{2,2} = 4 , \\ \ell_{3,1} = 5,\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i\in N_3} = \{1, 1, 1\} . \end{cases}$$

This covering surface is said to be of class 2.

Case (i_β). The inequality (3.3) with (3.2) gives

$$m_1 \geqq 4m_3 . \tag{3.6}$$

From (3.3), it follows that

$$2(m_3 + 1) \geqq m_2 , \tag{3.7}$$

so that by (3.3), (3.6) and (3.7), we have

$$m_3 = 1,\ 2 \text{ or } 3.$$

Hence, using the inequalities (3.3) and (3.6) again, we have seven possibilities:

(d) $m_1 = 4 ,\quad m_2 = 2 ,\quad m_3 = 1 .$

(e) $m_1 = 4 ,\quad m_2 = 3 ,\quad m_3 = 1 .$

(f) $m_1 = 5 ,\quad m_2 = 3 ,\quad m_3 = 1 .$

(g) $m_1 = 6 ,\quad m_2 = 4 ,\quad m_3 = 1 .$

(*h*) $m_1 = 8 ,\quad m_2 = 5 ,\quad m_3 = 2 .$

(i) $m_1 = 9 ,\quad m_2 = 6 ,\quad m_3 = 2 .$

(j) $m_1 = 12 ,\quad m_2 = 8 ,\quad m_3 = 3 .$

In each case, the numbers n, $\ell_{k,j}$, σ_k and s_i are determined as follows:

Case (d).
$$\begin{cases} n = 8, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 4 , \\ \ell_{2,1} = \ell_{2,2} = 3, \ \ell_{3,1} = 7 , \\ \sigma_1 = 0, \ \sigma_2 = 2, \ \sigma_3 = 1 , \\ \{s_i\}_{i \in N_2} = \{1, 1\}, \ \{s_i\}_{i \in N_3} = \{1\} . \end{cases}$$

This covering surface is said to be of class 3.

Case (e).
$$\begin{cases} n = 9, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 4 , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 3, \ \ell_{3,1} = 7 , \\ \sigma_1 = 1, \ \sigma_2 = 0, \ \sigma_3 = 2 , \\ \{s_i\}_{i \in N_1} = \{1\} \ \{s_i\}_{i \in N_3} = \{1, 1\} . \end{cases}$$

This covering surface is said to be of class 4.

Case (f).
$$\begin{cases} n = 10, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 5 , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 3, \ \ell_{3,1} = 7 , \\ \sigma_1 = 0, \ \sigma_2 = 1, \ \sigma_3 = 2 , \\ \{s_i\}_{i \in N_2} = \{1\}, \ \{s_i\}_{i \in N_3} = \{1, 2\} , \end{cases}$$
$$\begin{cases} n = 10, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 5 , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 3, \ \ell_{3,1} = 8 , \\ \sigma_1 = 0, \ \sigma_2 = 1, \ \sigma_3 = 2 , \\ \{s_i\}_{i \in N_2} = \{1\}, \ \{s_i\}_{i \in N_3} = \{1, 1\} , \end{cases}$$
$$\begin{cases} n = 10, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 5 , \\ \{\ell_{2,1}, \ell_{2,2}, \ell_{2,3}\} = \{3, 3, 4\}, \ \ell_{3,1} = 7 , \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 1\} . \end{cases}$$

These covering surfaces are said to be of classes 5, 6 and 7, respectively.

Case (g).
$$\begin{cases} n = 12, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6 , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4, \ \ell_{3,1} = 7 , \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 3\} , \end{cases}$$
$$\begin{cases} n = 12, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6 , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4, \ \ell_{3,1} = 7 , \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 2, 2\} , \end{cases}$$
$$\begin{cases} n = 12, \ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6 , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4, \ \ell_{3,1} = 8 , \\ \sigma_1 = \sigma_2 = 0, \ \sigma_3 = 3, \ \{s_i\}_{i \in N_3} = \{1, 1, 2\} , \end{cases}$$

$$\begin{cases} n = 12,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 6\ , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 4,\ \ell_{3,1} = 9\ , \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 1\}\ . \end{cases}$$

These covering surfaces are said to be of classes 8, 9, 10 and 11, respectively.

Case (h).
$$\begin{cases} n = 16,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 8\ , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 5,\ \ell_{3,1} = \ell_{3,2} = 7\ , \\ \sigma_1 = 0,\ \sigma_2 = 1,\ \sigma_3 = 2\ , \\ \{s_i\}_{i \in N_2} = \{1\},\ \{s_i\}_{i \in N_3} = \{1, 1\}\ . \end{cases}$$

This covering surface is said to be of class 12.

Case (i).
$$\begin{cases} n = 18,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 9\ , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 6,\ \ell_{3,1} = \ell_{3,2} = 7\ , \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 2\}\ , \end{cases}$$
$$\begin{cases} n = 18,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 9\ , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 6,\ \{\ell_{3,1}, \ell_{3,2}\} = \{7, 8\}\ , \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 1\}\ . \end{cases}$$

These covering surfaces are said to be of classes 13 and 14, respectively.

Last case (j).
$$\begin{cases} n = 24,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 12\ , \\ \ell_{2,j} = 3 \text{ for } j = 1 \text{ to } 8,\ \ell_{3,j} = 7 \text{ for } j = 1 \text{ to } 3\ , \\ \sigma_1 = \sigma_2 = 0,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 1\}\ . \end{cases}$$

This covering surface is said to be of class 15.

3.2. *Case* (ii). The inequality (3.3) yields

$$(3.8) \qquad m_1 + m_2 + 1 \geqq \nu_k m_k + \sigma_k\ , \qquad k = 1,\ 2\ .$$

From (1.1), the following possibilities occur:

$$(\text{ii}_\alpha) \qquad \nu_1 = 2,\ \nu_2 \geqq 3\ .$$
$$(\text{ii}_\beta) \qquad \nu_1 \geqq 3,\ \nu_2 \geqq 3\ .$$

Case (ii$_\alpha$). The inequality (3.8) implies the following five possibilities:

$$(\text{k}) \qquad m_1 = 1\ , \quad m_2 = 1\ , \quad \sigma_1 = \sigma_2 = 0\ .$$
$$(\text{l}) \qquad m_1 = 2\ , \quad m_2 = 1\ , \quad \sigma_1 = \sigma_2 = 0\ .$$

(m) $m_1 = 2\,,\quad m_2 = 1\,,\quad \sigma_1 = 0\,,\quad \sigma_2 = 1\,.$

(n) $m_1 = 1\,,\quad m_2 = 1\,,\quad \sigma_1 = 1\,,\quad \sigma_2 = 0\,.$

(o) $m_1 = 3\,,\quad m_2 = 2\,,\quad \sigma_1 = \sigma_2 = 0\,.$

Using (2.5), (3.1) and (3.2) in each case, we have:

Case (k). $\begin{cases} n = 3,\ \ell_{1,1} = \ell_{2,1} = 3,\ \sigma_3 = 3\,, \\ \{s_i\}_{i \in N_3} = \{1, 1, 1\}\,. \end{cases}$

Case (l). $\begin{cases} n = 4,\ \ell_{1,1} = \ell_{1,2} = 2,\ \ell_{2,1} = 4,\ \sigma_3 = 3\,, \\ \{s_i\}_{i \in N_3} = \{1, 1, 2\}\,. \end{cases}$

Case (m). $\begin{cases} n = 4,\ \ell_{1,1} = \ell_{1,2} = 2,\ \ell_{2,1} = 3,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{1\},\ \{s_i\}_{i \in N_3} = \{1, 3\}\,, \end{cases}$

$\begin{cases} n = 4,\ \ell_{1,1} = \ell_{1,2} = 2,\ \ell_{2,1} = 3,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{1\},\ \{s_i\}_{i \in N_3} = \{2, 2\}\,. \end{cases}$

Case (n). $\begin{cases} n = 3,\ \ell_{1,1} = 2,\ \ell_{2,1} = 3,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_1} = \{1\},\ \{s_i\}_{i \in N_3} = \{1, 2\}\,. \end{cases}$

Case (o). $\begin{cases} n = 6,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 3\,, \\ \ell_{2,1} = \ell_{2,2} = 3,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 1, 4\}\,, \end{cases}$

$\begin{cases} n = 6,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 3\,, \\ \ell_{2,1} = \ell_{2,2} = 3,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{1, 2, 3\}\,, \end{cases}$

$\begin{cases} n = 6,\ \ell_{1,j} = 2 \text{ for } j = 1 \text{ to } 3, \\ \ell_{2,1} = \ell_{2,2} = 3,\ \sigma_3 = 3,\ \{s_i\}_{i \in N_3} = \{2, 2, 2\}\,. \end{cases}$

These covering surfaces are said to be of classes 16 to 23, respectively.

Case (ii$_\beta$). The inequality (3.8) yields $\sigma_1 = \sigma_2 = 0$ and $m_1 = m_2 = 1$, that is, the case (k).

Case (iii). The inequality (3.3) yields $m_1 = 1$. Hence we have

$$\begin{cases} n = 2,\ \ell_{1,1} = 2,\ \sigma_1 = 0,\ \sigma_2 = 1,\ \sigma_3 = 2\,, \\ \{s_i\}_{i \in N_2} = \{2\},\ \{s_i\}_{i \in N_3} = \{1, 1\}\,. \end{cases}$$

This covering surface is said to be of class 24.

Case (iv). We have easily

$$n = 1\,,\quad \{s_i\}_{i \in N_k} = \{1\} \qquad \text{for } k = 1, 2, 3\,.$$

This surface covers univalently the base domain $\hat{C} - \bigcup_{j=1}^{3} \bar{D}_j$ and is said to be of class 25.

Summing up the above discussion, we state the following

LEMMA 3. *Let Δ be a triply connected domain bounded by analytic curves $\{\Gamma_i\}_{i=1,2,3}$ and let f be exceptionally ramified meromorphic on $\bar{\Delta}$ with three totally ramified values $\{w_k\}_{k=1,2,3}$ and satisfy the conditions* (1), (2) *and* (3).

Then, for the above domain Δ' mentioned, we have:

1°) *Δ' is a triply connected subdomain of Δ, and the covering surface generated by f restricted to Δ' belongs to one of the* 25 *classes (see Table* 1).

2°) *f has no ramified values other than $\{w_k\}_{k=1,2,3}$ in Δ'.*

3°) *Each component of $\Delta - \Delta'$ is doubly connected and its image is contained in one of the $\{D_j\}_{j=1,\dots,\alpha}$ ($\alpha \leqq 3$).*

4°) *Each D_j contains one of the $\{w_k\}_{k=1,2,3}$.*

Table 1

class	ν_1	ν_2	ν_3	m_1 $\ell_{1,j}$	m_2 $\ell_{2,j}$	m_3 $\ell_{3,j}$	n	σ_1 $\{s_i\}_{i\in N_1}$	σ_2 $\{s_i\}_{i\in N_2}$	σ_3 $\{s_i\}_{i\in N_3}$
1	2	4	5	3 $\ell_{1,j}=2$	1 $\ell_{2,1}=4$	1 $\ell_{3,1}=5$	6	0	2 {1, 1}	1 {1}
2	2	4	5	4 $\ell_{1,j}=2$	2 $\ell_{2,j}=4$	1 $\ell_{3,1}=5$	8	0	0	3 {1, 1, 1}
3	2	3	7	4 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	8	0	2 {1, 1}	1 {1}
4	2	3	7	4 $\ell_{1,j}=2$	3 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	9	1 {1}	0	2 {1, 1}
5	2	3	7	5 $\ell_{1,j}=2$	3 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	10	0	1 {1}	2 {1, 2}
6	2	3	7	5 $\ell_{1,j}=2$	3 $\ell_{2,j}=3$	1 $\ell_{3,1}=8$	10	0	1 {1}	2 {1, 1}
7	2	3	7	5 $\ell_{1,j}=2$	3 $\{\ell_{2,1}, \ell_{2,2}, \ell_{2,3}\}$ $=\{3, 3, 4\}$	1 $\ell_{3,1}=7$	10	0	0	3 {1, 1, 1}
8	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	12	0	0	3 {1, 1, 3}
9	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=7$	12	0	0	3 {1, 2, 2}
10	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=8$	12	0	0	3 {1, 1, 2}
11	2	3	7	6 $\ell_{1,j}=2$	4 $\ell_{2,j}=3$	1 $\ell_{3,1}=9$	12	0	0	3 {1, 1, 1}

class	ν_1	ν_2	ν_3	m_1 $\ell_{1,j}$	m_2 $\ell_{1,j}$	m_3 $\ell_{3,j}$	n	σ_1 $\{s_i\}_{i\in N_1}$	σ_1 $\{s_i\}_{i\in N_2}$	σ_3 $\{s_i\}_{i\in N_3}$
12	2	3	7	8 $\ell_{1,j}=2$	5 $\ell_{2,j}=3$	2 $\ell_{3,j}=7$	16	0	1 {1}	2 {1, 1}
13	2	3	7	9 $\ell_{1,j}=2$	6 $\ell_{2,j}=3$	2 $\ell_{3,j}=7$	18	0	0	3 {1, 1, 2}
14	2	3	7	9 $\ell_{1,j}=2$	6 $\ell_{2,j}=3$	2 $\{\ell_{3,1}, \ell_{3,2}\}=\{7, 8\}$	18	0	0	3 {1, 1, 1}
15	2	3	7	12 $\ell_{1,j}=2$	8 $\ell_{2,j}=3$	3 $\ell_{3,j}=7$	24	0	0	3 {1, 1, 1}
16	2	3		1 $\ell_{1,1}=3$	1 $\ell_{2,1}=3$	0	3	0	0	3 {1, 1, 1}
17	2	3		2 $\ell_{1,j}=2$	1 $\ell_{2,1}=4$	0	4	0	0	3 {1, 1, 2}
18	2	3		2 $\ell_{1,j}=2$	1 $\ell_{2,1}=3$	0	4	0	1 {1}	2 {1, 3}
19	2	3		2 $\ell_{1,j}=2$	1 $\ell_{2,1}=3$	0	4	0	1 {1}	2 {2, 2}
20	2	3		1 $\ell_{1,1}=2$	1 $\ell_{2,1}=3$	0	3	1 {1}	0	2 {1, 2}
21	2	3		3 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	0	6	0	0	3 {1, 1, 4}
22	2	3		3 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	0	6	0	0	3 {1, 2, 3}
23	2	3		3 $\ell_{1,j}=2$	2 $\ell_{2,j}=3$	0	6	0	0	3 {2, 2, 2}
24	2			1 $\ell_{1,1}=2$	0	0	2	0	1 {2}	2 {1, 1}
25				0	0	0	1	1 {1}	1 {1}	1 {1}

§ 4. Key Lemma

4.1. We form a Cantor set in the usual manner. Let $\{\xi_n\}$ be a sequence of positive numbers satisfying $0<\xi_n<2/3$, $n=1, 2, 3, \cdots$. We remove first an open interval of length $(1-\xi_1)$ from the interval $I_{0,1}$: $[-1/2, 1/2]$, so that on both sides there remains a closed interval of length $\xi_1/2\equiv\eta_1$. The remained intervals are denoted by $I_{1,1}$ and $I_{1,2}$. Inductively we remove an open interval of length $(1-2\eta_n)\prod_{p=1}^{n-1}\eta_p$, with $\eta_p=(1/2)\xi_p$ $(p=1, 2, \cdots)$, from each $I_{n-1,k}$, $k=1, 2, \cdots, 2^{n-1}$, so that on both sides there remains a closed interval of length $\prod_{p=1}^{n}\eta_p$. The remained intervals are denoted by

$I_{n,2k-1}$ and $I_{n,2k}$. By repeating this procedure endlessly, we obtain an infinite sequence of closed intervals $\{I_{n,k}\}_{n=1,2,\cdots,\ k=1,2,\cdots,2^n}$. The set given by

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{2^n} I_{n,k}$$

is said to be the Cantor set in the interval $I_{0,1}$ with successive ratios ξ_n.

Set

$$S_{n,k} = \left\{ z \,\middle|\, \prod_{p=1}^{n} \eta_p < |z - z_{n,k}| < \frac{1}{3} \prod_{p=1}^{n-1} \eta_p \right\}$$

and

$$\Gamma_{n,k} = \left\{ z \,\middle|\, |z - z_{n,k}| = \prod_{p=1}^{n-1} \eta_p \sqrt{\frac{\eta_n}{3}} \right\},$$

where $z_{n,k}$ is the midpoint of $I_{n,k}$. Denoting by $\mu_n = \mu(S_{n,k})$ the harmonic modulus of $S_{n,k}$, we have

$$\mu_n = \log \frac{1}{3\eta_n} = \log \frac{2}{3\xi_n}. \tag{4.1}$$

We give Lemma 4 which will be a key of our proof of Theorem.

LEMMA 4. *Let E be the Cantor set with successive ratios ξ_n satisfying the condition*

$$\lim_{n\to\infty} \xi_n = 0.$$

Let f be an exceptionally ramified meromorphic function in the complement E^c. Then, for a sufficiently large $n(\geqq L_1)$, we have with a positive constant M depending only on E and f,

$$|f(\Gamma_{n,k})| < M \exp(-\mu_n/2).$$

In order to prove Lemma 4, we use Lemma 5 due to L. Carleson and K. Matsumoto.

LEMMA 5. *Let f be meromorphic in an annulus $\bar{R}$: $1 \leqq |z| \leqq \exp \mu$ $(0 < \mu < \infty)$. If the image $f(\bar{R})$ is contained in the open disc $D(\zeta_0, d)$ with center ζ_0 and radius d $(0 < d < 1/2)$, then by putting $L = \{|z| = \exp \mu/2\}$ we have with some positive constant A depending only on d*

$$|f(L)| < A \exp(-\mu/2),$$

whenever μ is sufficiently large $(\mu \geqq \mu_0)$.

Moreover, we can choose A with

$$A = O(d) \qquad as \quad d \to 0$$

(cf. L. Sario and K. Noshiro [14], 128–129).

4.2. *Proof of Lemma* 4. Since f is exceptionally ramified in E^c, f is normal. Hence, denoting by $d\sigma_{E^c}(z)$ (resp. $d\sigma_{S_{n,k}}$) the element of hyperbolic length of E^c (resp. $S_{n,k}$), we have

$$\{|f'(z)|/(1 + |f(z)|^2)\}|dz| \leqq Cd\sigma_{E^c}(z) \leqq Cd\sigma_{S_{n,k}}(z) ,$$

in E^c with some constant C depending only on f and E (cf. O. Lehto and K. I. Virtanen [7]). Denote by $\zeta = \phi\ (z)$ the conformal mapping of $S_{n,k}$ onto G': $1 < |\zeta| < \exp \mu_n$ and put $g(\zeta) = f(\phi^{-1}(\zeta))$. Both of $d\sigma_{S_{n,k}}(z)$ and $\{|f'(z)|/(1 + |f(z)|^2)\}|dz|$ are conformally invariant, so that

$$\{|g'(\zeta)|/(1 + |g(\zeta)|^2)\}|d\zeta| < Cd\sigma_{G'}(\zeta) = \left\{C\pi/2\mu_n|\zeta| \sin\left(\frac{\pi}{\mu_n} \log|\zeta|\right)\right\}|d\zeta| .$$

Denoting by $L_{n,k}^{(1)}$ and $L_{n,k}^{(2)}$ the inverse images of L'_{ν_0} $:|\zeta| = \exp \nu_0$ and of L''_{ν_0} $:|\zeta| = \exp(\mu_n - \nu_0)$ under ϕ, respectively, we have

$$\begin{aligned}
&\int_{z\in L_{n,k}^{(1)}} \{|f'(z)|/(1 + |f(z)|^2)\}|dz| \\
&= \int_{\zeta\in L'_{\nu_0}} \{|g'(\zeta)|/(1 + |g(\zeta)|^2)\}|d\zeta| \\
&< \int_{\zeta\in L'_{\nu_0}} \left\{C\pi/2\mu_n|\zeta| \sin\left(\frac{\pi}{\mu_n} \log|\zeta|\right)\right\}|d\zeta| \\
&= \int_0^{2\pi} \left\{C\pi/2\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right)\right\}d\theta \\
&= C\pi^2/\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right) .
\end{aligned}$$

Similarly,

$$\int_{z\in L_{n,k}^{(2)}} \{|f'(z)|/(1 + |f(z)|^2)\}|dz| < C\pi^2/\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right) .$$

We take a fixed ν_0 with $\nu_0 > 32C$ and a sufficiently large n with $\mu_n > \bar{\mu} = \max(\mu_0, \nu_0)$ $(n \geqq L_2)$. From

$$C\pi^2/\mu_n \sin\left(\frac{\pi}{\mu_n}\nu_0\right) < \frac{C\pi}{\nu_0} + \frac{\pi}{32} < \frac{\pi}{16}$$

follow

$$|f(L_{n,k}^{(1)})| < \tfrac{1}{16} \quad \text{and} \quad |f(L_{n,k}^{(2)})| < \tfrac{1}{16} .$$

Hence there are discs D_i with $|D_i| < 1/8$ such that $D_i \supset f(L_{n,k}^{(i)})$ $(i = 1, 2)$. Lemma 2 implies therefore that, with the ring domain $T_{n,k}$ bounded by $L_{n,k}^{(1)}$ and $L_{n,k}^{(2)}$,

$$\nu(w, f, T_{n,k}) = 0 \qquad \text{for } w \in \hat{C} - (\bar{D}_1 \cup \bar{D}_2) ,$$

because if $\nu(w, f, T_{n,k}) \geqq 1$ for $w \in \hat{C} - (\bar{D}_1 \cup \bar{D}_2)$, f is not exceptionally ramified.

Consequently,

$$\bar{D}_1 \cap \bar{D}_2 \neq \phi \quad \text{and} \quad f(\bar{T}_{n,k}) \subset \bar{D}_1 \cup \bar{D}_2 .$$

Applying Lemma 5 to f in $T_{n,k}$, we obtain the desired inequality

$$\begin{aligned} |f(\Gamma_{n,k})| &< A \exp\{-\tfrac{1}{2}(\mu_n - 2\nu_0)\} \\ &= Ae^{\nu_0} \exp(-\mu_n/2) = M \exp(-\mu_n/2) , \end{aligned}$$

where $M = Ae^{\nu_0}$.

§ 5. Proof of Theorem

5.1. Assuming that, for a Cantor set E satisfying our condition (1.2), there is an exceptionally ramified meromorphic function f in E^c with an essential singularity at each point of E, we shall arrive at a contradiction. By our previous result [5], f must have just three totally ramified values $\{w_i\}_{i=1,2,3}$.

Set

$$\delta = \tfrac{1}{72} \min_{k \neq m} \chi(w_k, w_m) \tag{5.1}$$

and

$$\delta_n = M \exp(-\mu_n/2) . \tag{5.2}$$

By our condition (1.2), there exists a positive integer L_3 such that, for $n \geqq L_3$,

$$\delta_n < \delta \tag{5.3}$$

and

$$\delta_{n+1} < \tfrac{1}{2}\delta_n . \tag{5.4}$$

Further, by Lemma 4, we can choose, for any $n \geqq L_4 = \max(L_1, L_2, L_3)$, discs $D_{n,k}$ with $|D_{n,k}| < 2\delta_n$ containing $f(\Gamma_{n,k})$. The union $\tilde{D} \equiv D_{n,k} \cup D_{n+1,2k-1} \cup D_{n+1,2k}$ consists of at most three, say α $(1 \leqq \alpha \leqq 3)$, components, which are covered by discs $\{D_{n,k}^{(j)}\}_{j=1,\dots,\alpha}$ with $D_{n,k}^{(1)} \supset D_{n,k}$, $|D_{n,k}^{(1)}| = 12\delta_n$ and $|D_{n,k}^{(j)}| = 12\delta_{n+1}$ for $j \neq 1$. Here we may assume that there are no ramified values of f on $\partial D_{n,k}^{(j)}$. Denote by $\Delta_{n,k}$ the triply connected domain bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. When the restriction of f to $\Delta_{n,k}$ takes no values outside $\tilde{D}$, then $\alpha = 1$ and the image of $\Delta_{n,k}$ is contained in $D_{n,k}^{(1)}$. In this case, we say that $\Delta_{n,k}$ is degenerate (f).

Suppose that the restriction of f to $\Delta_{n,k}$ takes values outside $\tilde{D}$. Then we see from 4°) of Lemma 3 that each component of $\tilde{D}$ contains just one of the $\{w_i\}_{i=1,2,3}$, so that the center of $D_{n,k}^{(j)}$ can be taken at the point $w_{\lambda_j} \in \{w_i\}_{i=1,2,3}$, the totally ramified value contained in the corresponding component of $\tilde{D}$. This show that $\{D_{n,k}^{(j)}\}_{j=1,\dots,\alpha}$ are mutually disjoint. We choose a triply connected subdomain $\Delta'_{n,k}$ of $\Delta_{n,k}$ corresponding to Δ' of Lemma 3. It is always known that the covering surface generated by f on $\Delta'_{n,k}$ belongs to one of the 25 classes. Each component of $\Delta_{n,k} - \Delta'_{n,k}$ is doubly connected and its image is contained in one of the $\{D_{n,k}^{(j)}\}_{j=1,\dots,\alpha}$. If the covering surface is of class m, $\Delta_{n,k}$ and $\Delta'_{n,k}$ are said to be of class m. Generically these $\Delta_{n,k}$ are said to be non-degenerate (f).

Let $\Delta_{n,k}$ be non-degenerate (f). The boundary curves of $\Delta'_{n,k}$ are denoted by $\check{\gamma}_{n,k}$, $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$, homotopic to $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, respectively. Each of them is a component of the inverse image of some $\partial D_{n,k}^{(j)}$ under f and said to be of w_{λ_j}-type (f). Assuming that $\check{\gamma}_{n,k}$ $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$ are positively oriented, we set for $\zeta_0 \in \hat{C} - \bigcup_{j=1}^{\alpha} D_{n,k}^{(j)}$, $\zeta_j \in D_{n,k}^{(j)}$ $(j = 1$ to $\alpha)$,

$$\check{s}_{n,k} = \sum_{j=1}^{\alpha} O(\check{\gamma}_{n,k};\, \zeta_0, \zeta_j)\ ;$$

$$\hat{s}_{n+1,2k-i} = \sum_{j=1}^{\alpha} O(\hat{\gamma}_{n+1,2k-i};\, \zeta_0, \zeta_j) \qquad (i = 0, 1)\ .$$

5.2. The centers of $D_{n,k}^{(j)}$ are totally ramified values $w_{\lambda_j} \in \{w_i\}_{i=1,2,3}$ for any $\Delta_{n,k}$ being non-degenerate (f), while $D_{n,k}^{(1)}$ might contain no values $\{w_i\}_{i=1,2,3}$ for $\Delta_{n,k}$ being degenerate (f). However $D_{n,k}^{(1)}$ stay always considerably near one of the $\{w_i\}_{i=1,2,3}$.

PROPOSITION. *Let $\Delta_{n,k}$ be degenerate (f). Then $D_{n,k}^{(1)}$, the disc covering $f(\Delta_{n,k})$, is contained in one of the $\{D(w_i, 24\delta_n)\}_{i=1,2,3}$.*

Proof of Proposition. Suppose that $D^{(1)}_{n,k} \not\subset \bigcup_{i=1}^{3} D(w_i, 24\delta_n)$. Since $|D^{(1)}_{n,k}| = 12\delta_n$,

$$D^{(1)}_{n,k} \subset \left\{\bigcup_{i=1}^{3} D(w_i, 12\delta_n)\right\}^c ,$$

so that

$$f(\Delta_{n,k}) \subset \left\{\bigcup_{i=1}^{3} D(w_i, 12\delta_n)\right\}^c .$$

By (5.4) and this inclusion

$$D^{(1)}_{n+1,2k-j} \subset \left\{\bigcup_{i=1}^{3} D(w_i, 12\delta_{n+1})\right\}^c \qquad (j = 0, 1) .$$

This shows that $D^{(1)}_{n+1,2k-j}$ $(j = 0, 1)$ contain no totally ramified values $\{w_i\}_{i=1,2,3}$ and $\Delta_{n+1,2k-j}$ must be degenerate (f). Therefore

$$f(\bar{\Delta}_{n,k} \cup \bar{\Delta}_{n+1,2k-1} \cup \bar{\Delta}_{n+1,2k}) \subset D^{(1)}_{n+1} \cup D^{(1)}_{n+1,2k-1} \cup D^{(1)}_{n+1,2k} ,$$

which imply

$$|f(\bar{\Delta}_{n,k} \cup \bar{\Delta}_{n+1,2k-1} \cup \bar{\Delta}_{n+1,2k})| < 12\delta_n + 24\delta_{n+1} .$$

By repeating this procedure, we have

$$\begin{aligned} |f((\Gamma_{n,k}) - E)| &\leqq 12\delta_n + 24(\delta_{n+1} + \delta_{n+2} + \cdots) \\ &< 36\delta_n < \tfrac{1}{2} \min_{k \neq m} \chi(w_k, w_m) < \sqrt{2} , \end{aligned}$$

where $(\Gamma_{n,k})$ denotes the domain bounded by $\Gamma_{n,k}$ (see (5.1), (5.3), (5.4)).

We may assume that f is bounded in $(\Gamma_{n,k})$, because if necessary, we take a certain linear transformation of f in place of f. Since E is of linear measure zero, $(\Gamma_{n,k}) \cap E$ must be removable for any bounded analytic function (cf. A. S. Besicovitch [1]). This contradicts our assumption that each point of E is an essential singularity of f.

5.3. Now assume that infinitely many of $\Delta_{n,k}$ are non-degenerate (f). Then there are $\Delta_{n,k}$'s being non-degenerate (f) with $n \geqq L_4$. We take such a fixed $\Delta_{n,k}$. Let the boundary curves $\hat{\gamma}_{n+1,2k}$ and $\hat{\gamma}_{n+1,2k-1}$ of $\Delta'_{n,k}$ be of w_λ-type (f) and of $w_{\lambda'}$-type (f), respectively. Here we may assume that $\hat{s}_{n+1,2k-1} \geqq \hat{s}_{n+1,2k}$ and that $\lambda \geqq \lambda'$ if $\hat{s}_{n+1,2k-1} = \hat{s}_{n+1,2k}$. From Table 1 we see that $\hat{s}_{n+1,2k} = 1$ or 2.

The adjacent domain $\Delta_{n+1,2k}$ will be either

(A) degenerate (f)

or

(B) non-degenerate (f).

Case (A). Let $\hat{\Delta}_{n+1,2k}$ be the triply connected domain bounded by $\hat{\gamma}_{n+1,2k}$, $\Gamma_{n+2,4k-1}$ and $\Gamma_{n+2,4k}$. By virtue of the maximum principle, Proposition implies

$$f(\hat{\Delta}_{n+1,2k}) \subset D(w_\lambda, 6\delta_{n+p}) ,$$

where $p = 0$ or $p = 1$ according to $f(\hat{\gamma}_{n+1,2k}) \subset \partial D^{(1)}_{n,k}$ or $f(\hat{\gamma}_{n+1,2k}) \subset \partial D^{(j)}_{n,k}$ $(j \neq 1)$. We choose the component $J_{n+1,2k}$ of the inverse image $f^{-1}(R(w_\lambda, 24\delta_{n+2}, 6\delta_{n+p}))$ in $\hat{\Delta}_{n+1,2k}$ having $\hat{\gamma}_{n+1,2k}$ as a boundary curve, where $R(w_\lambda, 24\delta_{n+2}, 6\delta_{n+p}) = \{\zeta \mid 24\delta_{n+2} < \chi(\zeta, w_\lambda) < 6\delta_{n+p}\}$. From Lemma 2, it is easy to see that the boundary of $J_{n+1,2k}$ outside $\hat{\gamma}_{n+1,2k}$ is mapped onto $C(w_\lambda, 24\delta_{n+2})$ under f. We shall show that the boundary of $J_{n+1,2k}$ outside $\hat{\gamma}_{n+1,2k}$ consists of

(A_1) one boundary curve $\kappa_{n+1,2k}$ separating $\Gamma_{n+2,4k-1} \cup \Gamma_{n+2,4k}$ from $\hat{\gamma}_{n+1,2k}$

or

(A_2) two boundary curves $\kappa_{n+2,4k-1}$ and $\kappa_{n+2,4k}$ separating $\Gamma_{n+2,4k-1}$ and $\Gamma_{n+2,4k}$ from $\Gamma_{n+2,4k} \cup \hat{\gamma}_{n+1,2k}$ and $\Gamma_{n+2,4k-1} \cup \hat{\gamma}_{n+1,2k}$,

respectively.

In fact, we assume contrary that $J_{n+1,2k}$ has boundary curves β_i $(i = 1, \cdots, h)$ other than the above, then each β_i is homotopic to zero. Set

$$s_{i,j} = O(\kappa_{i,j}; \zeta_0, w_\lambda) \quad \text{and} \quad t_i = O(\beta_i; \zeta_0, w_\lambda)$$

for $\zeta_0 \in \hat{C} - \bar{D}(w_\lambda, 24\delta_{n+2})$, where $\kappa_{i,j}$ and β_i are positively oriented. Applying the argument principle to f in $J_{n+1,2k}$, we have

$$\hat{s}_{n+1,2k} = s_{n+1,2k} + \sum_{i=1}^{h} t_i \qquad \text{in the case } (A_1)$$

or

$$\hat{s}_{n+1,2k} = s_{n+2,4k-1} + s_{n+2,4k} + \sum_{i=1}^{h} t_i \qquad \text{in the case } (A_2) .$$

Since $\hat{s}_{n+1,2k} = 1$ or 2, $s_{i,j} \geqq 1$ and

$$t_i = O(-\beta_i; w_\lambda, \zeta_0) = \nu(w_\lambda, f, (-\beta_j)) \geqq \nu_\lambda \geqq 2 ,$$

which is a contradiction.

Case (A_1). The domain $J_{n+1,2k}$ is doubly connected. By the Hurwitz formula, f have no ramified values on $J_{n+1,2k}$. Hence $J_{n+1,2k}$ is conformally equivalent to

$$R^* = \left\{\zeta \middle| \left\{\frac{24\delta_{n+2}}{\sqrt{1-24^2\delta_{n+2}^2}}\right\}^{(\hat{s}_{n+1,2k})^{-1}} < |\zeta| < \left\{\frac{6\delta_{n+j}}{\sqrt{1-36\delta_{n+j}^2}}\right\}^{(\hat{s}_{n+1,2k})^{-1}}\right\}.$$

We have

$$\mu(J_{n+1,2k}) = \mu(R^*)\ . \tag{5.5}$$

As well-known, $\mu(J_{n+1,2k})$ is dominated by the harmonic modulus of the extremal domain of Teichmüller, i.e.

$$\mu(J_{n+1,2k}) \leqq \log 16\Big(\frac{r_2}{r_1} + 1\Big) = \log 16\Big(\frac{2}{\xi_{n+1}} - 1\Big)\,, \tag{5.6}$$

where $r_1 = \prod_{p=1}^{n+1} \eta_p$ and $r_2 = \prod_{p=1}^{n} \eta_p(1-2\eta_{n+1})$ (cf. O. Lehto and K. I. Virtanen [6] 55–62).

Hence, by (4.1), (5.2), (5.5) and (5.6), we have

$$\log 16\Big(\frac{2}{\xi_{n+1}} - 1\Big) \geqq \log\left\{\frac{\delta_{n+1}}{8\delta_{n+2}}\right\}^{(\hat{s}_{n+1,2k})^{-1}},$$

$$\left\{16\Big(\frac{2}{\xi_{n+1}} - 1\Big)\right\}^2 \geqq \frac{1}{8}\sqrt{\frac{\xi_{n+1}}{\xi_{n+2}}}\,,$$

so

$$\xi_{n+2} \geqq \frac{\xi_{n+1}^5}{2^{22}(2-\xi_{n+1})^4}\,.$$

This inequality contradicts our assumption (1.2), for a sufficiently large n, which imply that (A_1) cannot occur.

Case (A_2). The domain $J_{n+1,2k}$ is triply connected. In this case, $\hat{s}_{n+1,2k} = 2$. From Table 1 we see that $\Delta_{n,k}$ is of classes 9, 19, 22 or 23 and $\lambda = 3$. The domain $\Delta_{n+2,4k}$ is degenerate (f). In fact, assume that $\Delta_{n+2,4k}$ is non-degenerate (f). Then f takes the value w_λ in the ring domain $R'_{n+2,4k}$ bounded by $\kappa_{n+2,4k}$ and $\check{\gamma}_{n+2,4k}$, and by virtue of the argument principle

$$7 \leqq \nu_\lambda \leqq \nu(w_\lambda, f, R'_{n+2,4k}) = s_{n+2,4k} + \check{s}_{n+2,4k} \leqq 5\,,$$

which is a contradiction. Let f be restricted to the domain $\hat{\Delta}_{n+2,4k}$ bounded by $\kappa_{n+2,4k}$, $\Gamma_{n+3,8k-1}$ and $\Gamma_{n+3,8k}$ and let $J_{n+2,4k}$ be the component of the inverse image of $R(w_\lambda, 24\delta_{n+2}, 24\delta_{n+1})$, one of whose boundary curves is $\kappa_{n+2,4k}$. Since $s_{n+2,4k} = 1$, (A_1) is only possible for $J_{n+2,4k}$, that is, $J_{n+2,4k}$ is doubly connected. In the same way as above, we conclude that (A_2) cannot occur.

In conclusion, $\Delta_{n+1,2k}$ must be non-degenerate (f), i.e., of the case (B).

5.4. *Case* (B). Suppose that both of $\Delta_{m,n}$ and $\Delta_{m+1,2n}$ are non-degenerate (f) and $\hat{\gamma}_{m+1,2n}$ is of w_λ-type (f). By the argument principle

$$\nu_\lambda \leqq \nu(w_\lambda, f, R_{m+1,2n}) = \hat{s}_{m+1,2n} + \check{s}_{m+1,2n} , \tag{5.7}$$

where $R_{m+1,2n}$ denotes the domain bounded by $\hat{\gamma}_{m+1,2n}$ and $\check{\gamma}_{m+1,2n}$. The inequality (5.7) will be useful in this paragraph.

The (B) is divided into the following four cases.

(B_1) $\Delta_{n,k}$ is of classes 1 or 2.

(B_2) $\Delta_{n,k}$ is of classes 3, 4, $\cdots$, 22 or 23.

(B_3) $\Delta_{n,k}$ is of class 24.

(B_4) $\Delta_{n,k}$ is of class 25.

Case (B_1). The adjacent domain $\Delta_{n+1,2k}$ must be of classes 1, 2, 24 or 25. From Table 1 we see

$$\hat{s}_{n+1,2k} = 1 , \qquad \check{s}_{n+1,2k} = 1 \text{ or } 2 .$$

These equalities and (5.7) give

$$\nu_\lambda \leqq 3 . \tag{5.8}$$

On the other hand, since $\lambda = 2$ or 3, we have $\nu_\lambda = 4$ or 5. This contradicts (5.8).

Case (B_2). By (5.7), we have

$$\nu_\lambda \leqq 2 + 4 = 6 ,$$

so that

$$\lambda = 1 \text{ or } 2 .$$

This implies that $\Delta_{n,k}$ cannot be of classes 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 21, 22 and 23.

In the case $\lambda = 1$, $\Delta_{n,k}$ is of class 20 and $\hat{s}_{n+1,2k} = 1$. Hence $\Delta_{n+1,2k}$ is of classes 4, 20 or 25.

In the case $\lambda = 2$, $\Delta_{n,k}$ is of classes 3, 5, 18 or 19 and $\hat{s}_{n+1,2k} = 1$. Hence $\Delta_{n+1,2k}$ is of classes 3, 5, 6, 12, 18, 19, 24 or 25. We see that $\Delta_{n+1,2k}$

is of class 24 in the following way. Assume that $\varDelta_{n+1,2k}$ is of classes 3, 5, 6, 12, 18, 19 or 25, then (5.7) gives $3 \leqq \nu_2$ and $\nu_2 \leqq 2$, which is impossible.

In either case, at least one of $\{\hat{\gamma}_{n+2,4k-1}, \hat{\gamma}_{n+2,4k}\}$, say $\hat{\gamma}_{n+2,4k}$, is of w_3-type and $\hat{s}_{n+2,4k} = 1$. Assuming that $\varDelta_{n+2,4k}$ is non-degenerate (f), we are led to a contradiction $7 \leqq \nu_3 \leqq 1 + 4 = 5$. However $\varDelta_{n+2,4k}$ is not degenerate (f). Both cases cannot occur.

Case (B_3). In this case, $\hat{s}_{n+1,2k} = 1$ and $\lambda = 3$. By (5.7), we have $7 \leqq \nu_3$ and $\nu_3 \leqq 5$, which is impossible.

Case (B_4). We have shown that no $\varDelta_{n,k}$'s of other classes than 25 class appear. It follows that $\varDelta_{n+1,2k}$ and $\varDelta_{n+2,4k}$ are also of class 25. By (5.7) we have

$$2 \leqq \nu_\lambda \leqq \nu(w_\lambda, f, R_{n+1,2k}) = 1 + 1 = 2$$

and

$$2 \leqq \nu_{\lambda'} \leqq \nu(w_{\lambda'}, f, R_{n+2,4k}) = 1 + 1 = 2 \,,$$

which contradict (1.1), because $\lambda' \neq \lambda$.

Thus the case (B) also cannot occur. Consequently, there exists a positive integer N ($\geqq L_4$) such that every $\varDelta_{n,k}$ ($n \geqq N, k = 1, 2, \cdots, 2^n$) is degenerate (f).

5.5. Finally, we take a fixed $n(\geqq N)$. Since $\varDelta_{n+p,q}$ is degenerate (f), we have $f(\bar{\varDelta}_{n+p,q}) \subset D^{(1)}_{n+p,q}$. For any $z \in (\Gamma_{n,k}) - E$, there is a chain of $\{\varDelta_{n+p,q}\}$ connecting $\varDelta_{n,k}$ to z. The diameter of the chain $\leqq 12$ $(\delta_n + \delta_{n+1} + \cdots + \delta_{n+m} + \cdots) \leqq 24\delta_n$, because $|D^{(1)}_{n+p,q}| = 12\delta_{n+p}$ and $\delta_{n+p+1} < (1/2)\delta_{n+p}$ (see (5.4)). Hence

$$f((\Gamma_{n,k}) - E) \subset D(w_0, 24\delta_n) \,,$$

where $w_0 \in f(\bar{\varDelta}_{n,k})$, that is,

$$|f((\Gamma_{n,k}) - E)| < 48\delta_n < 48\delta < \sqrt{2} \,.$$

We may assume that f is bounded in $(\Gamma_{n,k})$, because if necessary, we take a certain linear transformation of f in place of f. The Cantor set E is of linear measure zero, so that $(\Gamma_{n,k}) \cap E$ is removable for f. This contradicts our assumption that each point of E is an essential singularity of f.

The proof of Theorem is thus complete.

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