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A PROPERTY OF NEW COORDINATES DEFINING AUGMENTED SCHOTTKY SPACES

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§ 0. Introduction

In the previous paper [3], we introduced new coordinates to the Schottky space, and defined the augmented Schottky spaces $\hat{\mathfrak{S}}_g^*(\Sigma)$. Here, in § 1, we will define fiber spaces over the augmented Schottky spaces. In § 2, we will consider a property of the new coordinates, namely, we will state a relation between limits of sequences of elements of the new coordinates and limits of sequences of length of loops on Riemann surfaces.

§ 1. Fiber spaces over the augmented Schottky spaces

1.1. We will use the same notations and terminologies as in the previous paper [3]. Throughout this paper, we fix a standard system of loops $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ on a compact Riemann surface S of genus g (see p. 156 in [3]).

For $I \subset \{1, 2, \cdots, g\}$ and $J = \{j_1, \cdots, j_m\} \subset \{1, 2, \cdots, 2g-3\}$ where $j_1 < \cdots < j_m$, we consider $\delta^{I,J}\mathfrak{S}_g(\Sigma)$. We see that $S \backslash \bigcup_{i=1}^m \gamma_{j_i}$ consists of $m+1$ components $[\sigma_0], [\sigma_{j_1}], \cdots, [\sigma_{j_m}]$, here each $[\sigma_j]$ represents one containing the cell $\sigma_j = \sigma(1, i_1, \cdots, i_\mu)$ when $\gamma_j = \gamma(1, i_1, \cdots, i_\mu)$ (see [3], p. 157). If $J = \phi$, we regard $S \backslash \bigcup \gamma_{j_i}$ as S itself. For arbitrary $\tau \in \delta^{I,J}\mathfrak{S}_g(\Sigma)$, we have $m+1$ Schottky groups (including the trivial group) $G_0(\tau), G_{j_1}(\tau), \cdots, G_{j_m}(\tau)$, and $m+1$ Riemann surfaces $S_0(\tau), S_{j_1}(\tau), \cdots, S_{j_m}(\tau)$ as well as the Riemann surface with nodes

$$S(\tau) = S_0(\tau) + S_{j_1}(\tau) + \cdots + S_{j_m}(\tau)$$

as in the previous paper [3]. We will introduce $2g-2$ Schottky groups (including the trivial group) $\tilde{G}_s(\tau)$ $(s = 0, 1, \cdots, 2g-3)$ as follows.

(i) $\tilde{G}_0(\tau)$ is defined by normalizing $G_0(\tau)$ as follows: $p_1(\tau) = 0$, $q_1(\tau) = \infty$ and $p_2(\tau) = 1$ if the cell σ_2 is contained in $[\sigma_0]$, or $p^+(1, 0, \cdots, 0) = 1$

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if $\sigma_2 \notin [\sigma_0]$, where $p^+(1, 0, \cdots, 0)$ is the right distinguished point with respect to the boundary loop $\gamma(1, 0, \cdots, 0)$ of $[\sigma_0]$.

(ii) $\tilde{G}_s(\tau)$ $(s = 2, 3, \cdots, g)$ is defined as follows: Let $[\sigma_{k(s)}]$ be the part which contains the cell σ_s. Let $G_{k(s)}(\tau)$ be the Schottky group representing the Riemann surface $S_{k(s)}(\tau)$ (see p. 172 in [3]). $\tilde{G}_s(\tau)$ is the group obtained from $G_{k(s)}(\tau)$ by the following normalization: $p_s(\tau) = 0$, $q_s(\tau) = \infty$, and $p_1(\tau) = 1$ if $\sigma_0 \in [\sigma_{k(s)}]$, or $p^-(1, i_1, \cdots, i_\mu)(\tau) = 1$ if $\sigma_0 \notin [\sigma_{k(s)}]$, where $p^-(1, i_1, \cdots, i_\mu)(\tau)$ is the left distinguished point with respect to the boundary loop $\gamma(1, i_1, \cdots, i_\mu)$ of $[\sigma_{k(s)}]$.

(iii) $\tilde{G}_1(\tau)$ is defined as follows. Let $[\sigma_{k(1)}]$ be the part of S which contains the cell σ_1. Let $G_{k(1)}(\tau)$ be the Schottky group representing $S_{k(1)}(\tau)$. $\tilde{G}_1(\tau)$ is the group obtained from $G_{k(1)}(\tau)$ by the following normalization: (1) $p_1(\tau) = 0$ if $\sigma_0 \in [\sigma_{k(1)}]$, or $p^-(1)(\tau) = 0$ if $\sigma_0 \notin [\sigma_{k(1)}]$; (2) $p_2(\tau) = \infty$ if $\sigma_2 \in [\sigma_{k(1)}]$, or $p^+(1, 0, \cdots, 0)(\tau) = \infty$ if $\sigma_2 \notin [\sigma_{k(1)}]$, where $p^+(1, 0, \cdots, 0)$ is the right distinguished point with respect to the boundary loop $\gamma(1, 0, \cdots, 0)$ of $[\sigma_{k(1)}]$; (3) $p_l(\tau) = 1$ if the terminal cell σ_l with $l = (1, 1, \underbrace{0, \cdots, 0}_{m})$ belongs to $[\sigma_{k(1)}]$, or $p^+(1, 1, \underbrace{0, \cdots, 0}_{m'})(\tau) = 1$ if $\sigma_l \notin [\sigma_{k(1)}]$, where $p^+(1, 1, \underbrace{0, \cdots, 0}_{m'})(\tau)$ is the right distinguished point with respect to the boundary loop $\gamma(1, 1, \underbrace{0, \cdots, 0}_{m'})$ of $[\sigma_{k(1)}]$.

(iv) $\tilde{G}_s(\tau)$ $(s = g+1, g+2, \cdots, 2g-3)$ are defined as follows. Let $[\sigma_{k(s)}]$ be the part of S which contains the cell σ_s. Let $G_{k(s)}(\tau)$ be the Schottky group representing $S_{k(s)}(\tau)$. Let $\gamma_s = \gamma(1, i_1, \cdots, i_\mu)$. We define $\tilde{G}_s(\tau)$ as the group obtained from $G_{k(s)}(\tau)$ by the following normalization: (1) $p_1(\tau) = 0$ if $\sigma_0 \in [\sigma_{k(s)}]$, or $p^-(1, i_1, \cdots, i_\nu)(\tau) = 0$ if $\sigma_0 \notin [\sigma_{k(s)}]$, where $p^-(1, i_1, \cdots, i_\nu)(\tau)$ is the left distinguished point with respect to the boundary loop $\gamma(1, i_1, \cdots, i_\nu)(\tau)$ of $[\sigma_{k(s)}]$; (2) $p_l(\tau) = \infty$ if the terminal cell σ_l with $l = (1, i_1, \cdots, i_\mu, \underbrace{0, \cdots, 0}_{m})$ belongs to $[\sigma_{k(s)}]$, or $p^+(1, i_1, \cdots, i_\mu, \underbrace{0, \cdots, 0}_{m'})(\tau) = \infty$ if $\sigma_l \notin [\sigma_{k(s)}]$, where $p^+(1, i_1, \cdots, i_\mu, \underbrace{0, \cdots, 0}_{m'})(\tau)$ is the right distinguished point with respect to the boundary loop $\gamma(1, i_1, \cdots, i_\mu, \underbrace{0, \cdots, 0}_{m'})$ of $[\sigma_{k(s)}]$; (3) $p_{l'}(\tau) = 1$ if the terminal cell $\sigma_{l'}$ with $l' = (1, i_1, \cdots, i_\mu, 1, \underbrace{0, \cdots, 0}_{n})$ belongs to $[\sigma_{k(s)}]$, or $p^+(1, i_1, \cdots, i_\mu, 1, \underbrace{0, \cdots, 0}_{n'})(\tau) = 1$ if $\sigma_{l'} \notin [\sigma_{k(s)}]$, where $p^+(1, i_1, \cdots, i_\mu, 1, \underbrace{0, \cdots, 0}_{n'})(\tau)$ is the right distinguished point with respect to the boundary loop $\gamma(1, i_1, \cdots, i_\mu, 1, \underbrace{0, \cdots, 0}_{n'})$ of $[\sigma_{k(s)}]$.

Remark. Each $[\sigma_j]$ is the union of all cells σ_s such that

$$k(s) = j\ .$$

Corresponding groups $\tilde{G}_s(\tau)$ are equivalent to $\tilde{G}_j(\tau)$, namely, there exist $T_s \in$ Möb with $\tilde{G}_s(\tau) = T_s\tilde{G}_j(\tau)T_s^{-1}$. The Riemann surfaces

$$\tilde{S}_s(\tau) = \Omega(\tilde{G}_s(\tau))/\tilde{G}_s(\tau)$$

are conformally equivalent to $S_j(\tau)$. Accordingly

$$S(\tau) = S_0(\tau) + S_{j_1}(\tau) + \cdots + S_{j_m}(\tau)$$

can be written as

$$S(\tau) = \tilde{S}_0(\tau) + \tilde{S}_{s_1}(\tau) + \cdots + \tilde{S}_{s_m}(\tau)$$

with some s_i with $j_i = k(s_i)$, $i = 1, 2, \cdots, m$, and $\tilde{S}_0(\tau) = S_0(\tau)$.

1.2. Fiber spaces. Here we will define fiber spaces $\mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)$ $(s = 0, 1, \cdots, 2g-3)$ over the augmented Schottky spaces $\hat{\mathfrak{S}}_g^*(\Sigma)$.

DEFINITION. The s-th *fiber space* $\mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)$ $(s = 0, 1, \cdots, 2g-3)$ over the augmented Schottky space $\hat{\mathfrak{S}}_g^*(\Sigma)$ is the set of all the points $(\tau, z) \in \boldsymbol{C}^{3g-2}$ with $\tau \in \hat{\mathfrak{S}}_g^*(\Sigma)$ and $z \in \Omega'(\tilde{G}_s(\tau))$, where $\Omega'(\tilde{G}_s(\tau)) = \Omega(\tilde{G}_s(\tau)) \backslash \bigcup_{A \in \tilde{G}_s(\tau)} A$ (distinguished points).

We define the following sets by using $\mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)$:

$$\begin{aligned}
\mathfrak{F}_s\mathfrak{S}_g^{I,J}(\Sigma) &= \mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)\,|\,\mathfrak{S}_g^{I,J}(\Sigma)\ ,\\
\mathfrak{F}_s\delta^{I,J}\mathfrak{S}_g(\Sigma) &= \mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)\,|\,\delta^{I,J}\mathfrak{S}_g(\Sigma)\ ,\\
\mathfrak{F}_s\hat{\delta}^J\mathfrak{S}_g^I(\Sigma) &= \mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)\,|\,\hat{\delta}^J\mathfrak{S}_g^I(\Sigma)\ ,
\end{aligned}$$

and

$$\mathfrak{F}_s\hat{\delta}^J\mathfrak{S}_g^*(\Sigma) = \mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)\,|\,\hat{\delta}^J\mathfrak{S}_g^*(\Sigma)\ .$$

Here the vertical segment $|$ represents a restriction.

1.3. PROPOSITION 1. (1) *For each* $s = 0, 1, \cdots, 2g-3$, *the fiber space* $\mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)$ *is a domain in* $\boldsymbol{C}^{3g-2}$. (2) *For each* $I \subset \{1, 2, \cdots, g\}$ *and* $J \subset \{1, 2, \cdots, 2g-3\}$, $\mathfrak{F}_s\mathfrak{S}_g^{I,J}(\Sigma)$ *is a subdomain of* $\mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)$ *and a domain in* $\boldsymbol{C}^{3g-2-|I|-|J|}$.

We can prove the above proposition by a similar way to the proof of Proposition 5 in [3], and here we omit it.

1.4. Poincaré metric. It is assumed that each component of $S\backslash\{\text{nodes}\}$ has hyperbolic universal covering surface (see [3]). For each $\tau \in \hat{\mathfrak{S}}_g^*(\Sigma)$, we denote by $\lambda_s(\tau, z)$ the Poincaré metric on $\Omega'(\tilde{G}_s(\tau))$. In this case, the Poincaré metric $\lambda_s(\tau, z)$ means the unique conformal complete Riemannian metric of Gaussian curvature -1. Then by a similar method to Bers [2], we have the following.

PROPOSITION 2. *The number $\lambda_s(\tau, z)$ is a continuous function* of (τ, z) $\in \mathfrak{F}_s\hat{\mathfrak{S}}_g^*(\Sigma)$ *for each* $s = 0, 1, \cdots, 2g-3$.

We project this $\lambda_s(\tau, z)$ to $\Omega'(\tilde{G}_s(\tau))/\tilde{G}_s(\tau)$ and we call it the Poincaré metric as well.

§ 2. A property of the new coordinates

2.1. In this section we will consider a relation between the new coordinates and the non-Euclidean length of loops on Riemann surfaces.

Let S be a fixed compact Riemann surface of genus g and $\Sigma = \{\alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3}\}$ a fixed standard system of loops on S. Let $S(\nu)$ be a compact Riemann surface of genus g and $\Sigma_\nu = \{\alpha_1(\nu), \cdots, \alpha_g(\nu); \gamma_1(\nu), \cdots, \gamma_{2g-3}(\nu)\}$ a standard system of loops on $S(\nu)$ such that $\gamma_j(\nu)$ $(j = 1, 2, \cdots, 2g-3)$ give the same partitions of the set $\{1, 2, \cdots, g\}$ as γ_j (see p. 157 and p. 171 in [3]). Then there exists a Schottky group

$$G(\nu) = \langle A_1(\nu, z), \cdots, A_g(\nu, z)\rangle$$

with $\Omega(G(\nu))/G(\nu) = S(\nu)$, where the defining curves $C_j(\nu)$ and $C_j'(\nu)$ of $A_j(\nu, z)$ have the property $\Pi_\nu(C_j(\nu)) = \alpha_j(\nu) = \Pi_\nu(C_j'(\nu))$ and Π_ν is the natural projection of $\Omega(G(\nu))$ onto $S(\nu)$. We call $A_j(\nu, z)$ the generator of $G(\nu)$ associated with $\alpha_j(\nu)$. Then we can uniquely determine $\tau_\nu \in \mathfrak{S}_g(\Sigma)$ such that $G(\tau_\nu) = G(\nu)$. Let $L(\alpha_i(\nu))$ and $L(\gamma_j(\nu))$ denote the length of geodesic loops homotopic to $\alpha_i(\nu)$ and $\gamma_j(\nu)$ on $S(\nu)$, respectively $(i = 1, 2, \cdots, g;\ j = 1, 2, \cdots, 2g-3)$.

2.2. THEOREM. *Let*

$$\tau_\nu = (t_1(\tau_\nu), \cdots, t_g(\tau_\nu),\ \rho_1(\tau_\nu), \cdots, \rho_{2g-3}(\tau_\nu))$$

$(\nu = 1, 2, \cdots)$ *be elements of* $\mathfrak{S}_g(\Sigma)$ *determined by* $S(\nu)$ *and* Σ_ν *as above.*

(1) *Suppose* $\lim_{\nu\to\infty} \tau_\nu = \tau_0 \in \delta^I\mathfrak{S}_g(\Sigma)$ $(I \neq \phi)$. *Then* $\lim_{\nu\to\infty} L(\alpha_i(\nu)) = 0$ *if and only if* $t_i(\tau_0) = 0$, *that is,* $i \in I$.

(2) *Suppose* $\lim_{\nu\to\infty} \tau_\nu = \tau_0 \in \hat{\delta}^J\mathfrak{S}_g(\Sigma)$ $(J \neq \phi)$. *Then* $\lim_{\nu\to\infty} L(\gamma_j(\nu)) = 0$ *if and only if* $\rho_j(\tau_0) = 1$, *that is,* $j \in J$.

Proof. (1) We show that if $\lim_{\nu\to\infty} L(\alpha_i(\nu)) = 0$, then $t_i(\tau_0) = 0$. Suppose $t_i(\tau_0) \neq 0$. Since $\tau_0 \in \delta'\mathfrak{S}_g(\Sigma)$, $A_i(\tau_0, z)$ is one of generators of the Schottky group $G(\tau_0)$. We may set $A_i(\tau_0, z) = (1/t_i(\tau_0))z$. Let $C_i(\tau_0)$ and $C_i'(\tau_0)$ be defining curves of $A_i(\tau_0, z)$. Let $\Lambda(z)|dz|$ be the Poincaré metric on $C\backslash\{0, 1\}$. Then noting that $\Lambda(z) \leqq \lambda(\tau_0, z)$, it is easily seen that $L(\alpha_i(\tau_0)) \neq 0$, which contradicts the assumption.

Next we will show that if $t_i(\tau_0) = 0$, then $\lim_{\nu\to\infty} L(\alpha_i(\nu)) = 0$. Let c be a circle with the center $p_i(\tau_0)$ (the extended repelling fixed point associated with $i \in I$ in a standard fundamental domain $\omega(G(\tau_0))$) (see § 5–1 in [3]) such that $l_{\tau_0}(c) < \varepsilon$ for sufficiently small ε, where $l_{\tau_0}(c)$ denotes the length of c with respect to the Poincaré metric on $\Omega'(G(\tau_0))$. For sufficiently large ν, $p_i(\tau_\nu)$ is contained in the interior to c, and all defining curves of $G(\tau_\nu)$ other than $C_i(\tau_\nu)$ can be taken to be to the exterior to c, where $C_i(\tau_\nu)$ is a defining curve of $A_i(\tau_\nu, z)$ containing $p_i(\tau_\nu)$ in the interior. Then it is easily seen that $\alpha_i(\nu)$ is homotopic to the image of c under the natural projection from $\Omega(G(\tau_\nu))$ to $S(\nu)$. By Proposition 2, $|l_{\tau_0}(c) - l_{\tau_\nu}(c)| < \varepsilon$ for sufficiently large ν. Hence $l_{\tau_\nu}(c) < 2\varepsilon$. Thus $L(\alpha_i(\nu)) \leqq l_{\tau_\nu}(c) < 2\varepsilon$. Since ε may be taken arbitrarily small, we have $\lim_{\nu\to\infty} L(\alpha_i(\nu)) = 0$.

(2) We will show that if $\lim_{\nu\to\infty} L(\gamma_j(\nu)) = 0$, then $\lim_{\nu\to\infty} \rho_j(\tau_\nu) = 1$. Let $S(\nu) = \Omega(G(\tau_\nu))/G(\tau_\nu)$ be divided into two parts $S_1(\tau_\nu)$ and $S_2(\tau_\nu)$ by the loop $\gamma_j(\nu)$. We denote by $\alpha_1^i(\nu), \cdots, \alpha_{g_i}^i(\nu)$ the "α-loops" on $S_i(\tau_\nu)$ $(i = 1, 2)$. We denote by $A_k^{(i)}(\tau_\nu, z)$ the generators of $G(\tau_\nu)$ associated with $\alpha_k^i(\nu)$. Let $G_1(\tau_\nu)$ be the group $G(\tau_\nu)$ normalized by $p_1^{(1)}(\tau_\nu) = 0$, $q_1^{(1)}(\tau_\nu) = \infty$ and $p_1^{(2)}(\tau_\nu) = 1$, where $p_1^{(l)}(\tau_\nu)$ $(l = 1, 2)$ and $q_1^{(1)}(\tau_\nu)$ are the repelling fixed points of $A_1^{(l)}(\tau_\nu, z)$ and the attracting fixed point of $A_1^{(1)}(\tau_\nu, z)$, respectively.

With the aid of a standard fundamental domain for $G(\tau_\nu)$, we can find a simple closed curve $\tilde{\gamma}_j(\nu)$, which is a lift of $\gamma_j(\nu)$, whose interior contains all the fixed points of $A_1^{(2)}(\tau_\nu, z), \cdots, A_{g_2}^{(2)}(\tau_\nu, z)$. It is easily seen that the Euclidean length of $\tilde{\gamma}_j(\nu)$ tends to 0 as $\nu \to \infty$, since $L(\gamma_j(\nu)) \to 0$ $(\nu \to \infty)$. Hence

$$\lim_{\nu\to\infty} p_1^{(2)}(\tau_\nu) = \lim_{\nu\to\infty} q_1^{(2)}(\tau_\nu) = \cdots = \lim_{\nu\to\infty} p_{g_2}^{(2)}(\tau_\nu) = \lim_{\nu\to\infty} q_{g_2}^{(2)}(\tau_\nu) = 1 .$$

Thus from the definition of $\rho_j(\tau_\nu)$ (see p. 161 in [3]) we have the desired result, $\lim_{\nu\to\infty} \rho_j(\tau_\nu) = 1$.

Conversely, we show that if $\lim_{\nu\to\infty} \rho_j(\tau_\nu) = 1$, then $\lim_{\nu\to\infty} L(\gamma_j(\nu)) = 0$. We denote by $p(\tau_0)$ the distinguished point resulted from the deformation. Then we choose a circle c with the center $p(\tau_0)$ in a standard fundamental domain $\omega(\tau_0)$ for $G(\tau_0)$ such that $l_{\tau_0}(c) < \varepsilon$ for sufficiently small ε. By a similar method to the proof of Proposition 5 in [3], we can assume that all defining curves of $A_1^{(2)}(\tau_\nu, z), \cdots, A_{g_2}^{(2)}(\tau_\nu, z)$ can be taken to be in the interior to the circle c, and all defining curves of $A_1^{(1)}(\tau_\nu, z), \cdots, A_{g_1}^{(1)}(\tau_\nu, z)$ can be taken to be to the exterior to the circle c for sufficiently large ν. Thus the image of c under the natural projection is homotopic to the loop $\gamma_j(\nu)$. By Proposition 2, we have $|l_{\tau_\nu}(c) - l_{\tau_0}(c)| < \varepsilon$. Hence $l_{\tau_\nu}(c) < 2\varepsilon$. Since $L(\gamma_j(\nu)) \leqq l_{\tau_\nu}(c)$, we have $\lim_{\nu\to\infty} L(\gamma_j(\nu)) = 0$.

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