M. Brodmann
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REES RINGS AND FORM RINGS OF ALMOST COMPLETE INTERSECTIONS

MARKUS BRODMANN

1. Introduction

Recently different authors have studied the conormal modules I/I^2 of almost complete intersections in local Gorenstein rings (c. t. Aoyama [1], Herzog [8], Kunz [13], Matsuoka [16]). An essential tool in these papers is the theory of canonical modules and the fact that these modules are easy to handle in the case of almost complete intersections.

In this paper we rely on the idea that almost complete intersections should not behave very different from complete intersection and that therefore similar arguments must be possible to study Rees rings and form rings. The basic similarity is that almost complete intersections admit superregular sequences of length height (I). This is shown in section 2.

Our aim is to study to which extend two basic results of Rees [19] and Valla [22] extend from complete intersections to almost complete intersections. Rees' result states the well known fact that $\mathrm{Gr}_R(I)$ is a polynomial algebra if R is CM and I a complete intersection. Valla's Result claims that $R]I^n[$, $R\rangle I^n\langle$ and $\mathrm{Gr}_R(I^n)$ are CM under the same hypotheses for all $n>0$. The way Valla's result ought to be generalized is clear: By calculating the lengths of maximal regular sequences in the homogeneous maximal ideals of the above rings in terms of depth (R/I) and $\dim(R)$. This will be done in section 4.

How to generalize Rees' result is not evident. A reasonable way to do this may be a study of the geometry of the conormal cone $\mathrm{Spec}\,(\mathrm{Gr}_R(I))$. An attempt to this is given in section 6, where we consider the relations between the irreducible components of the conormal cone of I and

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Spec (R/I).

Since Hochster [10] asked a corresponding question, a lot of particular varieties have been decided to be normally torsion-free or not. But with the exception of some "relative" criteria (s. [21]) there are no general geometric properties which characterize normal torsion-freeness. For locally almost complete intersections such criteria may be given, as is shown in sections 3 and 6.

Our arguments strongly rely on the knowledge of depth (I^n/I^{n+1}) which is estimated in section 3.

If R is Gorenstein we may sharpen some of our statements in using a result of Herzog [8]. This is done in section 5, where we also treat two examples.

If not stated otherwise R is always assumed to be CM, I to be a generic complete intersection of height h which is an almost complete intersection. The first requirement on I means

(1.1) For all $P \in \mathrm{Min}\,(R/I)$ it holds $\mu_P(I) = h$,

where for an arbitrary finitely generated R-module M $\mathrm{Min}(M)$ is the set of minimal members of $\mathrm{supp}\,(M)$ and $\mu_P(M)$ stands for the minimal number of generators of the R_P-module M_P.

The second requirement on I is

(1.2) $\mu := \mu_{\mathfrak{m}}(I) \leqslant h + 1$.

By d we denote the Krull dimension of R. For simplicity we always assume that $R/\mathfrak{m}$ is infinite. Except for the results of section 2 this assumption may be dropped as usual by replacing R by $R[X]_{\mathfrak{m}[X]}$.

If $N \subseteq M$ are R-modules and if $S \subseteq R$ we write $(N : S)_M$ for the module $\{x \in M \mid xS \subseteq N\}$. The terminology generally is the same as in [15].

§ 2. Some preliminaries

Most of the proofs we shall give are by induction on h, sometimes also by induction on $\delta := \dim\,(R/I)$. Making induction on h we always will use the fact that I admits superregular sequences which generically generate I:

(2.1) LEMMA. *There are elements $x_1, \cdots, x_h \in I$ which make part of a minimal system of generators of I such that the leading forms $\bar{x}_1, \cdots, \bar{x}_h \in \mathrm{Gr}_R(I)$ constitute a regular sequence and such that $I_P = (x_1, \cdots, x_h)_P$ for all*

$P \in \mathrm{Min}\,(R/I)$.

This is a consequence of the following strengthened version of the classical "Primbasissatz" as it is found in [17] and [18].

(2.2) PROPOSITION. *Let I satisfy* (1.1). *Then I admits a system of generators $x_1, \cdots, x_\mu$ such that for each permutation σ of the elements* $\{1, \cdots, \mu\}$ *we have*:

(i) $\mathrm{ht}(x_{\sigma(1)}, \cdots, x_{\sigma(h)}) = h$.

(ii) $(x_{\sigma(1)}, \cdots, x_{\sigma(h)})_P = I_P$, *for all* $P \in \mathrm{Min}\,(R/I)$.

Proof. (Induction on h) If $h = 0$, there is nothing to prove. So let $h > 0$. For $P \in \mathrm{Min}\,(R/I)$ put $N^{(P)} = PI_P \cap I$. As $I \neq (0)$ we see by Nakayama that $I_P \neq PI_P$, hence that $N^{(P)} \neq I$. Applying again Nakayama we see that $N^{(P)} + \mathfrak{m}I \neq I$ for all $P \in \mathrm{Min}\,(R/I)$. As $R/\mathfrak{m}$ is infinite we get that $U := \bigcup_{P \in \mathrm{Min}(R/I)} (N^{(P)} + \mathfrak{m}I) \neq I$.

So, choose $y \in I - U$, and put $\mathfrak{P} = \mathrm{Min}(R) \cap V(yR)$. As $\mathrm{ht}(I) = \mathrm{ht}(I^2) > 0$ we have $I^2 \not\subseteq Q$ for all $Q \in \mathfrak{P}$. So we find an element $z \in I^2 \cap \bigcap_{P \in \mathrm{Min}(R) - \mathfrak{P}} P - \bigcup_{Q \in \mathfrak{P}} Q$.

If we set $x_1 = y + z$ we have $x_1 \in I - U$ and $ht(x_1R) = 1$. By the first fact x_1 is I-basic and I_P-basic for all $P \in \mathrm{Min}(R/I)$. So it is clear that R/x_1R and I/x_1R satisfy again our hypotheses, but with $h - 1$ and $\mu - 1$ instead of h and μ. By induction we therefore find elements $w_2, \cdots, w_\mu \in R$ whose images $\bar{x}_2, \cdots, \bar{x}_\mu$ in R/x_1R form a system of generators of I/x_1R satisfying the requirements of (2.2). Our aim is to find elements $\varepsilon_{h+1}, \cdots, \varepsilon_\mu \in R - \mathfrak{m}$ such that the elements

$$\begin{aligned} &x_1 \\ &x_i = w_i \ (i = 2, \cdots, h) \\ &x_j = \varepsilon_j x_1 + w_j \ (\max(2, h) < i \leqslant \mu) \end{aligned}$$

are a system of generators of I which satisfies our requirements. Clearly, for all choices of the elements ε_i we have $(x_i, \cdots, x_\mu) = I$. By induction it also is clear that (2.2) (i) and (ii) hold whenever $1 \in \{\sigma(1), \cdots, \sigma(h)\}$. From this we see that it suffices to construct the elements ε_k inductively on $k(> h)$ such that the following holds:

For each system of indices $\langle i_j \rangle = \langle i_1, \cdots, i_h \rangle$ with $1 \leqslant i_1 < \cdots < i_h = k$ it holds

$$(*) \qquad x_k \not\in \bigcup_{P \in \mathfrak{Q}\langle i_j \rangle} P, \qquad \text{where } \mathfrak{Q}_{\langle i_j \rangle} = \mathrm{Min}\,(R/(x_{i_1}, \cdots, x_{i_h}))$$

(**) $\qquad x_1R_P \subseteq (x_{i_1}, \cdots, x_{i_h})_P\,, \qquad \forall P \in \mathrm{Min}\,(R/I)\,.$

So let $h < k \leqslant \mu$ and assume that $\varepsilon_1, \cdots, \varepsilon_{k-1}$ are constructed. Put $\mathfrak{Q} = \bigcup \mathfrak{Q}_{\langle i_j \rangle}$, where $\langle i_j \rangle$ runs through all the systems of indices as above. For an arbitrary $Q \in \mathrm{Spec}\,(R)$ consider the canonical map $\pi_Q \colon R \to K(Q) := (R/Q)_Q$. By induction it is clear that $\pi_Q(x_1) \neq 0$ for all $Q \in \mathfrak{Q}$. We also get by induction that $I_P = (x_1, x_{i_1}, \cdots, x_{i_{h-1}})_P$ for all $P \in \mathrm{Min}\,(R/I)$ and all of the above systems $\langle i_j \rangle$. So in their rings R_P we have equations

$$w_k = \lambda_{P,\langle i_j \rangle} x_1 + u_{P,\langle i_j \rangle}\,,$$

with $u_{P,\langle i_j \rangle} \in (x_{i_1}, \cdots, x_{i_h})$, $P \in \mathrm{Min}\,(R/I)$.

Now, let $\alpha_{P,\langle i_j \rangle} \in K(P)$ be the canonical image of $\lambda_{P,\langle i_j \rangle}$. As $|R/\mathfrak{m}| = \infty$ there is an $\varepsilon_k \in R - \mathfrak{m}$ with

$$\pi_Q(\varepsilon_k x_1 + w_k) \neq 0,\ \forall Q \in \mathfrak{Q};\ \alpha_{P,\langle i_j \rangle} + \pi_P(\varepsilon_k) \neq 0,\ \forall \langle i_j \rangle,\ \forall P \in \mathrm{Min}\,(R/I)\,.$$

But from this it is clear, that ε_k satisfies (*) and (**).

To prove (2.1) let $x_1, \cdots, x_\mu$ ($\mu \leqslant h + 1$) be as in (2.2). We claim that $x_1, \cdots, x_h$ (hence each collection of h elements) has the properties requested in (2.1). To see this it suffices to prove that x_1 is $\mathrm{Gr}_R(I)$-regular, as then we have a canonical isomorphism

(2.3) $$\mathrm{Gr}_R(I)/(\bar{x}_1) \cong \mathrm{Gr}_{R/x_1R}(I/x_1R)\,,$$

which allows to make induction on h. So it remains to show that $(I^n \colon x_1)_R = I^{n-1}$ for all $n > 0$. Setting $J = (x_1, \cdots, x_\mu)$ we get $(I^n \colon x_1)_R = ((J^n + x_1I^{n-1}) \colon x_1)_R = (J^n \colon x_1)_R + I^{n-1}$ and it suffices to show that $(J^n \colon x_1)_R \subseteq J^{n-1}$. As R is CM $x_1, \cdots, x_h$ form an R-sequence and we have $\mathrm{Ass}(R/J^n) = \mathrm{Min}\,(R/J)$ for all $n > 0$. So we only have to show that $(J_Q^n \colon x_1)_{R_Q} \subseteq J_Q^{n-1}$ for all $n > 0$ and all $Q \in \mathrm{Min}(R/J)$. But this is immediately clear by the properties (2.2) (i) and (ii).

Making induction on δ we use the following result.

(2.4) LEMMA. *Let I satisfy* (1.1), *assume that $h < d - 1$ and let $P_1, \cdots, P_n \in \mathrm{Spec}\,(R) - \{m\}$. Then there is an $x \in \mathfrak{m} - P_1 \cup \cdots \cup P_n$ such that $I + xR$ is a generic complete intersection of height $h + 1$ with $\mu_{\mathfrak{m}}(I + xR) = \mu + 1$.*

Proof. Choose $x_1, \cdots, x_\mu$ according to (2.2). By (2.2) (ii) we have $I_S = (x_1, \cdots, x_h)_S$, where $S = R - \bigcup_{P \in \mathrm{Min}(R/I)} P$.

So there is an $s \in S$ such that $I_s = (x_1, \cdots, x_h)_s$. Now, put

$$\mathfrak{Q} - \bigcup_{P\in \operatorname{Min}(R/I)} \operatorname{Min}(R/(P, s)), \qquad \mathfrak{Q}^1 = \mathfrak{Q}\cup\{P_1, \cdots, P_n\}.$$

As $h < d - 1$ it is clear that $\dim(R/P) > 1$ for all $P \in \operatorname{Min}(R/I)$. Therefore we have $\mathfrak{m} \notin \mathfrak{Q}$, hence $\mathfrak{m} \notin \mathfrak{Q}'$. Now, choose $y \in \mathfrak{m} - \bigcup_{Q\in\mathfrak{Q}'} Q$, and let $n \in \boldsymbol{N}$.

As y^n is outside of all $P\in \operatorname{Min}(R/I)$ we have $\operatorname{ht}(I + y^nR) = h + 1$. Let $T \in \operatorname{Min}(R/(I + y^nR))$. Then we have $s \notin T$, as otherwise T would belong to a $Q \in \mathfrak{Q}$, which would imply that $y \in Q$. So we have $(I + y^nR)_T = (x_1, \cdots, x_h, y^nR)_T$, thus $\mu_T(I + y^nR) = h + 1$ for all $T \in \operatorname{Min}(R/(I + y^nR))$.

Finally by construction it is clear that $y^n \in (x_1, \cdots, x_\mu) = I$. Choosing n large enough we also have $x_i \notin (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_\mu, y^n)$ for $i = 1, \cdots, \mu$. For such values of n we therefore get $\mu_{\mathfrak{m}}(I + y^nR) = \mu + 1$.

As for the case $h = 0$ we note the following lemma whose proof is easy:

(2.5) LEMMA. *Let $I = xR \neq 0$ be a generic complete intersection of height* 0. *Then*

(i) $(I^n : x)_I = I^{n-1}$, $(n \geqslant 2)$,

(ii) *x is regular with respect to $R/(0: x)_R$*,

(iii) $I^n/I^{n+1} = R/(xR + (0: x)_R)$, $(n \geqslant 1)$.

§ 3. Normal torsion and normal depth

In this section we consider the maps

$$(3.1)\qquad \begin{aligned} A(n) &:= \operatorname{Ass}(R/I^n), & B(n) &:= \operatorname{Ass}(I^{n-1}/I^n), \\ t(n) &:= \operatorname{depth}(R/I^n), & \bar{t}(n) &:= \operatorname{depth}(I^{n-1}/I^n). \end{aligned}$$

By [2] and [3] we know that these maps take constant values A^*, B^*, t^*, $\bar{t}^*$ respectively for large n. In our particular situation, we get much more precise statements. Note that in case $\mu = h$ we have by Rees' result that $A(n) = B(n) = \operatorname{Min}(R/I)$ and $t(n) = \bar{t}(n) = \delta = d - h$. So we only need to consider the case $\mu = h + 1$. Let us first do this for $h = 0$.

(3.2) LEMMA. *Let $h = 0$, $I = xR \neq 0$. Then for all $n > 1$ we have*

(i) $B(n) = \operatorname{Ass}(R/((0: x)_R + xR))$,

$A(n) = \operatorname{Ass}(R/((0: x)_R + xR)) \cup \operatorname{Min}(R/xR)$,

(ii) $t(n) = \bar{t}(n) = \operatorname{depth}(R/(0: x)_R) - 1 = \min(d - 1, \operatorname{depth}(R/I))$.

Proof. The first statement of (i) is clear by (2.5) (iii). As for the second statement note that by (2.5) (ii) we have $xR \cap (0: x)_R = x(0: x)_R$

$= 0$, which gives rise to a sequence

$$(3.3) \qquad 0 \longrightarrow (0\colon x)_R \longrightarrow R/x^nR \longrightarrow R/((0\colon x)_R + x^nR) \longrightarrow 0\,.$$

By (1.1) it is implied that $\mathrm{Ass}((0\colon x)_R) = \mathrm{Min}\,(R/xR)$. So (3.3) induces $\mathrm{Min}\,(R/xR) \subseteq A(n) \subseteq \mathrm{Min}\,(R/xR) \cup \mathrm{Ass}\,(R/((0\colon x)_R + x^nR) = \mathrm{Min}\,(R/xR) \cup B(n)$, and $B(n) \subseteq A(n)$ shows the result.

"(ii)": The equality $\bar{t}(n) = \mathrm{depth}(R/(0\colon x)_R) - 1$ is clear by (2.5) (ii) and (iii). The sequence

$$0 \longrightarrow (0\colon x)_R \longrightarrow R \longrightarrow R/(0\colon x)_R \longrightarrow 0$$

shows that $\mathrm{depth}_R\,((0\colon x)_R) = \min(d, \mathrm{depth}(R/(0\colon x)_R) + 1)$. So by (3.3) we get that $t(n) \geqslant \mathrm{depth}(R/(0\colon x)) - 1 = \bar{t}(n)$, with equality if $\mathrm{depth}(R/(0\colon x)_R) < d$. To get the equality sign in the case where $\mathrm{depth}(R/(0\colon x)_R) = d$ note that—even if $n = 1$—we have $t(n) \geqslant d - 1$ as is shown by the sequence

$$(3.4) \qquad 0 \longrightarrow R \longrightarrow R/x^nR \oplus R/(0\colon x)_R \longrightarrow R/((0\colon x)_R + x^nR) \longrightarrow 0$$

which results from $x^nR \cap (0\colon x)_R = 0$. By (2.5) (iii) we also get a sequence

$$0 \longrightarrow R/((0\colon x)_R + xR) \longrightarrow R/x^nR \longrightarrow R/x^{n-1}R \longrightarrow 0$$

which shows that indeed $t(n) = d - 1 = \bar{t}(n)$. So it remains to prove

$$(3.5) \qquad \mathrm{depth}(R/(0\colon x)_R) = \min(d, \mathrm{depth}\,(R/I) + 1)\,.$$

But this is clear by the sequence

$$(3.6) \qquad 0 \longrightarrow R/(0\colon x)_R \xrightarrow{x} R \longrightarrow R/I \longrightarrow 0\,.$$

To treat the maps (3.1) in the case $h > 0$ we consider the 1-codimensional points of $V(I)$ in which I is not a complete intersection:

$$(3.7) \qquad U(I) := \{P \in V(I) \mid \mathrm{ht}\,(P/I) = 1,\ \mu_P(I) > h\}\,.$$

Another description of $U(I)$ is given by:

(3.8) Lemma. *Let* $h > 0$. *Then*

$$U(I) = \{P \in \bigcup_{n \in 0} A(n) \mid \mathrm{ht}(P/I) = 1\}\,.$$

Proof. "$\supseteq$" is clear by the fact that $\mu = h$ implies $A(n) = \mathrm{Min}(R/I)$.

"$\subseteq$": By a result of [5] (in a modified version found in [6]) we know that $\mu = h$ if $t(n) = \delta$ for all n. This easily implies the requested relation after eventual localization at $P \in U(I)$.

(3.9) PROPOSITION. *If $h > 0$ we have*
(i) $A(n) = B(n)$, $A(n) \subseteq A(n+1)$, $(n > 0)$;
$A^* = B^* = U(I) \cup A(1)$.
(ii) $t(n) = \bar{t}(n)$, $t(n) \geqslant t(n+1)$, $(n > 0)$;
$t^* = \bar{t}^* = \min(\delta - 1, \operatorname{depth}(R/I))$.

Proof. "(i)": By (2.1) I contains a superregular element x such that $\bar{R} := R/xR$ and $\bar{I} := I\bar{R}$ satisfy again (1.1) and (1.2) with $h-1$ instead of h. In particular we get exact sequences

$$(3.9)_n \qquad 0 \longrightarrow I^{n-1}/I^n \xrightarrow{x} I^n/I^{n+1} \longrightarrow \bar{I}^n/\bar{I}^{n+1} \longrightarrow 0$$

which show that $B(n) \subseteq B(n+1)$. Now with the sequences

$$0 \longrightarrow I^{n-1}/I^n \longrightarrow R/I^n \longrightarrow R/I^{n-1} \longrightarrow 0$$

the first two statements of (i) are immediately clear by induction on n. Consequently we also have $A^* = B^*$. So it remains to show that $B = U(I) \cup A(1)$. By (3.8) this comes up to prove

$$(3.10) \qquad P \in B(n) - A(1) \Longrightarrow \operatorname{ht}(P/I) = 1\,.$$

We prove this statement by induction on h. If $h = 0$, $P \in B(n) - A(1)$ implies $P \in \operatorname{Ass}(R/((0\colon x)_R + xR))$ (c.f. (3.2) (i)). Localizing at P we may assume that $P = \mathfrak{m}$. Then $0 = \bar{t}(n) = \min(d-1, t(1))$, $t(1) \neq 0$ show that $d = 1$. If $h > 0$, choose the least integer n with $P \in B(n) - A(1)$. Then $(3.9)_{n-1}$ shows that $P \in \operatorname{Ass}(\bar{I}^n/\bar{I}^{n+1}) - \operatorname{Ass}(\bar{R}/\bar{I})$ and we conclude by induction on h.

"(ii)": If $\delta \leqslant 1$, we conclude by (i). So let $\delta > 1$. Then by (i) we have either $\operatorname{depth}(R/I) = 0$ or $t^* > 0$. In the first case we conclude by (i). If $t^* > 0$ (2.4) implies the existence of a regular element $x \in \mathfrak{m}$ which is regular with respect to all R-modules R/I^n and such that (I, x) satisfies our hypotheses with $h+1$ instead of h. Put $\bar{R} = R/xR$ and $\bar{I} = I\bar{R}$. Then $\bar{R}$ and $\bar{I}$ satisfy again (1.1) and (1.2) and we have $\dim(\bar{R}/\bar{I}) = \delta - 1$. The R/I^n-regularity of x gives rise to canonical isomorphisms $\bar{I}^{n-1}/\bar{I}^n = (I^{n-1}/I^n)/x(I^{n-1}/I^n)$, $\bar{R}/\bar{I}^n = (R/I^n)/x(R/I^n)$, which allow to conclude inductively.

As an application we get:

(3.11) COROLLARY. *Let $h > 0$. Then the following are equivalent*

(i) *I is normally torsion-free*

(ii) $\bigcup_{P \in U(I)} P \subseteq \bigcup_{Q \in A(1)} Q$.

(3.12) *Remark.* Let V be an algebraic variety which is a local almost complete intersection. Then according to (3.11) we have the following geometric criterion for the normal torsion-freeness of V: V is normally torsionfree iff it is a complete intersection in codimension 1.

So if V is normal it is normally torsion-free. This behaviour is rather typical for almost complete intersections. Note for example that according to [21] there are smooth projective varieties which are arithmetically Gorenstein but not arithmetically normally torsion-free.

§ 4. Depth of Rees rings and form rings

Let $\mathfrak{m}]I[$ be the homogeneous maximal ideal of $R]I[$. In this section we calculate the depths of the local rings

$$\mathcal{O}_R]I[:= R]I[_{\mathfrak{m}]I[}\,, \qquad \mathcal{O}_R(I):= \mathrm{Gr}_R(I)_{\mathfrak{m}]I[}\,. \tag{4.1}$$

It comes up to the same to determine $g(\mathfrak{m}]I[; R]I[)$ and $g(\mathfrak{m}]I[; \mathrm{Gr}_R(I))$, where for an arbitrary ideal T of a noetherian ring A and any finitely generated A-module M, $g(T, M)$ stands for the maximal length of M-sequences in T. With our arguments we also may reprove the result of [22] we quoted previously, but for simplicity of the statements we only consider the case $\mu = h + 1$.

(4.2) PROPOSITION. *Let* $\mu = h + 1$. *Then*

(i) $\mathrm{depth}(\mathcal{O}_R]I[) = \begin{cases} \min(d, \mathrm{depth}\,(R/I) + 2), & \text{if } h = 0 \\ \min(d + 1, \mathrm{depth}(R/I) + h + 2), & \text{if } h > 0 \end{cases}$

(ii) $\mathrm{depth}(\mathcal{O}_R(I)) = \min\,(d, \mathrm{depth}(R/I) + h + 1)$.

Proof. "(i)": Let $h = 0$, $I = xR$. Let π be the canonical map $R[X] \to R]I[$ which sends X to the 1-form $x^* := (0, x, 0, \cdots)$ of $R]I[$. By (2.5) (ii) we have $(0{:}\, x)_R = (0{:}\, x^n)_R$ for all $n > 0$, which gives rise to the exact sequence

$$0 \longrightarrow C := (0{:}\, x)_R XR[X] \longrightarrow R[X] \xrightarrow{\ \pi\ } R]I[\longrightarrow 0\,. \tag{4.3}$$

In the proof of (3.2) (ii) we have seen that $\mathrm{depth}((0{:}\, x)_R) = \min\,(d, \mathrm{depth}\,(R/(0{:}\, x)_R) + 1)$. So (4.3) implies $\mathrm{depth}\,(\mathcal{O}_R]I[) = g((\mathfrak{m}, X); R]I[) \geqslant g((\mathfrak{m}, X);\ (0{:}\, x)_R XR[X]) - 1 = g(\mathfrak{m}, (0{:}\, x)_R) = \min\,(d, \mathrm{depth}\,(R/(0{:}\, x)_R) + 1) = \min\,(d, \mathrm{depth}\,(R/I) + 2)$, where the last equality follows from (3.5).

On the other hand (4.3) shows that $R]I[= R[X]/(X) \cap (0\colon x)_R R[X]$ hence that depth$(\mathcal{O}_R]I[) \leqslant d$. Using again (4.3) we therefore get the equality sign in the above estimate.

So let $h > 0$. If $h > 1$ choose $x_1, \cdots, x_h$ as in (2.1) and let x_1^* be the 1-form $(0, x_1, 0, \cdots)$ of $R]I[$.

Put $\bar{R} = R/x_1R$, $\bar{I} = I\bar{R}$ and consider the canonical projection $R]I[/(x_1^*) \xrightarrow{\alpha} \bar{R}]\bar{I}[$. In degree 0 the kernel of this map obviously is given by x_1R. In degree $n > 0$ the above map is given by the canonical map $I^n/x_1I^{n-1} \xrightarrow{\alpha_n} (I^n + x_1R)/x_1R \cong I^n/x_1R \cap I^n = I^n/x_1(I^n\colon x_1)_R$. As x_1 is superregular with respect to I we have $(I^n\colon x_1)_R = I^{n-1}$. So α_n is an isomorphism. So α gives rise to the following sequence of graded $R]I[$-modules.

$$(4.4) \qquad 0 \longrightarrow \mathfrak{R} := [x_1R \oplus (0) \oplus (0) \cdots] \longrightarrow R]I[/(x_1^*) \longrightarrow \bar{R}]\bar{I}[\longrightarrow 0\,,$$

We have to show that $g(\mathfrak{m}]I[, R]I[/(x_1^*)) = \min(d, \operatorname{depth}(R/I) + h + 1)$. Assume for the moment, that this holds for ht$(I) = h - 1$. Then we have $g(\mathfrak{m}]I[, \bar{R}]\bar{I}[) = \min(d, \operatorname{depth}(R/I) + h + 1)$. As $g(\mathfrak{m}]I[, \mathfrak{R}) = g(\mathfrak{m}, x_1R) = d$ (4.4) shows that the required equality holds. By this argument we may reduce inductively the problem to the case $h = 1$.

So let x, y be a system of generators of I which satisfies the conditions in (2.2). We have to show that

$$(4.5) \qquad g(\mathfrak{m}]I[; R]I[/(x^*)) = \min(d, \operatorname{depth}(R/I) + 2)\,.$$

Applying (4.4) (which also holds for $h = 1$) we get the following isomorphism of graded $R]I[$-algebras

$$R]I[/(x^*) = R \oplus (y, x)/(x) \oplus (y^2, x)/(x) \oplus \cdots\,.$$

So we may define a canonical homomorphism $\rho\colon R[Y] \to R]I[/(x^*)$ which sends Y to the 1-form $(0, y/(x), 0, 0, \cdots)$. Applying (2.5) (ii) to the $R/(x)$-ideal $IR/(x)$ we see again that $(xR\colon y^n)_R = (xR\colon y)_R$, hence that Ker$(\rho) = (xR\colon y)_R YR[Y] = (xR\colon y)_R R[Y] \cap (Y)$. This gives rise to the following exact sequence of $R[Y]$-modules

$$0 \longrightarrow R]I[/(x^*) \longrightarrow R/(xR\colon y)[Y] \oplus R \longrightarrow R/(xR\colon y)_R \longrightarrow 0\,.$$

By (3.5) we get $g((\mathfrak{m}, Y); R/(xR\colon y)_R) = \min(d - 1, \operatorname{depth}(R/I) + 1)$ and $g((\mathfrak{m}, Y); R(xR\colon y)_R[Y]) = \min(d, \operatorname{depth}(R/I) + 2)$. So (4.5) is clear.

"(ii)": Let $h = 0$, $I = xR$. By (2.5) (iii) there is a canonical map $\eta\colon \operatorname{Gr}_R(I) \to R/((0\colon x)_R + xR)[X]$ which sends $\bar{x}$ to X, and which is an isomorphism in positive degrees. As $xR \cap (0\colon x)_R = 0$ we get an exact sequence

$$(4.7) \qquad 0 \longrightarrow N := [(0\colon x)_R \oplus (0) \oplus (0) \cdots] \longrightarrow \mathrm{Gr}_R(I) \longrightarrow R/((0\colon x)_R + xR[X]) \longrightarrow 0\,.$$

Using $g(\mathfrak{m}]I[;N) = g(\mathfrak{m}, (0\colon x)_R) = \min(d, \operatorname{depth}(R/I) + 2)$ and $g(\mathfrak{m}]I[; R/((0\colon x)_R + xR)[X]) = \min(d, \operatorname{depth}(R/I) + 1)$ we see that $\operatorname{depth}(\mathcal{O}_R(I)) \geqslant \min(d, \operatorname{depth}(R/I) + 1)$. On the other side using $\mathrm{Gr}_R(I) = R]I[/(x)$, we get by (4.3) a sequence

$$(4.8) \qquad 0 \longrightarrow W := X(0\colon x)_R R/I[X] \longrightarrow R/I[X] \longrightarrow \mathrm{Gr}_R(I) \longrightarrow 0\,.$$

By $g((\mathfrak{m}, X); W) = g(\mathfrak{m}, (0\colon x)_R R/I) + 1 = \operatorname{depth}(((0\colon x)_R + xR)/xR) + 1 = \operatorname{depth}((0\colon x)_R) + 1 = \min(d+1, \operatorname{depth}(R/I) + 2)$ $(g > ((\mathfrak{m}, X); R/I[X])$ = if $\operatorname{depth}(R/I) < d)$ we conclude that $\operatorname{depth}(\mathcal{O}_R(I)) = g((\mathfrak{m}, X); \mathrm{Gr}_R(I)) = g((\mathfrak{m}, X); R/I[X]) = \operatorname{depth}(R/I) + 1$, which gives the requested result.

So let $h > 0$. Then we choose again a superregular element $x_1 \in I$ and make induction by the isomorphism (2.3).

Let us introduce the homogeneous maximal ideal $\mathfrak{m}\rangle I\langle$ of $R\rangle I\langle$ and the local ring $\mathcal{O}_R\rangle I\langle := R\rangle I\langle_{\mathfrak{m}\rangle I\langle}$. (4.2) and the canonical isomorphism

$$(4.9) \qquad \mathrm{Gr}_R(I) \cong R\rangle I\langle / X^{-1} R\rangle I\langle$$

show that the following holds.

(4.10) Corollary. *Let* $\mu = h + 1$. *Then*

$$\operatorname{depth}(\mathcal{O}_R\rangle I\langle) = \min(d+1, \operatorname{depth}(R/I) + h + 2).$$

(4.11) *Remark.* If we replace I by a power I^n, the statements (4.2) and (4.10) are not affected by [7]. So we get the following generalization of Vallas' result:

(4.12) Corollary. *Let* I *satisfy* (1.1) *and* (1.2) *and assume that* $\operatorname{depth}(R/I) \geqslant \dim(R/I) - 1$. *Then for all* $n > 0$ *the rings* $\mathrm{Gr}_R(I^n)$ *and* $R\rangle I^n\langle$ are CM. *If* $h > 0$, *the same is true for* $R]I^n[$.

Proof. By (4.11) we may assume that $n = 1$. By [12] it suffices to see that the local rings $\mathcal{O}_R(I)$, $\mathcal{O}_R\rangle I\langle$ and $\mathcal{O}_R]I[$ are CM. This is clear by (4.2), (4.12) and the well known fact that the above local rings are respectively of dimension d, $d+1$ and $d+1$.

§ 5. The Gorenstein case

In this section we consider the case where R is Gorenstein. The sharpening of some of the previous results that we get in this case is a consequence that we dispose on canonical modules [9]. One of the resulting facts we use is the following result of Herzog [8] (the result found in [8] requests slightly stronger hypotheses but (3.2) and (3.9) show that these may be weakened as below).

(5.1) PROPOSITION. *Let R be Gorenstein and suppose that I satisfies* (1.1) *and* (1.2). *Then $\mu = h$ iff $\bar{t}(2) = \dim(R/I)$.*

We call I unmixed, if $\operatorname{Ass}(R/I) = \operatorname{Min}(R/I)$. Then as a consequence of (5.1) we get:

(5.2) PROPOSITION. *Let R be Gorenstein $\mu = h + 1$ and assume that I is unmixed. Then we have*:

(i) $B(2) = B(3) = \cdots = B^*$.
$\bar{t}(2) = \bar{t}(3) = \cdots = \bar{t}^*$.

(ii) *The following statements are equivalent*:

(a) $t(1) \geqslant \dim(R/I) - 1$
(b) *R/I is C.M.*
(c) $\bar{t}(2) = \bar{t}(3) = \cdots = \dim(R/I) - 1$
(d) *One of the rings* $\operatorname{Gr}(I)$, $R\rangle I\langle$ *is C.M.*
(e) *Both of the above rings are C.M.*

Proof. "(i)": Let $P \in B(n)$, $n > 2$. We have to show that $P \in B(2)$. To do this we may assume after localizing that $P = \mathfrak{m}$. We first want to show that $\delta \leqslant 1$. If $h > 0$, this is indeed clear by (3.2) (i) and the unmixedness of I. If $h = 0$ we have by (3.2) (ii) that either $d - 1 = 0$ or $\operatorname{depth}(R/I) = 0$. Then by the unmixedness of I we conclude again that $d \leqslant 1$. But now $P = \mathfrak{m} \notin B(2)$ would imply $\delta = \bar{t}(2)$, hence a contradiction to (5.1).

To prove the second statement we proceed by induction on $t^* = \min(\delta - 1, \operatorname{depth}(R/I))$. By (3.2) the case $h = 0$ is clear. So let $h > 0$. Let $t^* = 0$. Then $\mathfrak{m} \in A^* = B^*$ implies by the first statement that $\mathfrak{m} \in B(n) = A(n)$ for all $n > 1$. So we have $t(n) = 0$, $\forall n > 1$. Now, let $t^* = 1$. Then we have either $\operatorname{depth}(R/I) = 1$ or $\operatorname{depth}(R/I) = 2$ and $\delta = 2$. Let us treat the first case; noticing that $t(1) = \operatorname{depth}(R/I)$ we get by (3.9) that $1 = t(1) \geqslant t(2) \geqslant \cdots \geqslant t^* = 1$. In the second case R/I is CM. By (3.9) (ii) it suffices to show that $\bar{t}(2) \leqslant 1$. Assuming the opposite, we would get by (5.1) that I is a complete intersection. But then we have the contradiction $2 = \bar{t}(2) = t^*$.

So let $t^* > 1$. Then we see that $\operatorname{depth}(R/I) > 1$. As I is unmixed and as R is CM, all $\hat{P} \in \operatorname{Ass}(\hat{R}/I\hat{R})$ satisfy $\dim(\hat{R}/\hat{P}) = \delta$. Now, let $\mathscr{F}$ be the (multiplicatively closed) set of all those ideals $\mathfrak{a} \subseteq R/I$ for which $\operatorname{ht}(\mathfrak{a}) > 1$. Then by [4, (3.13)]

$$D_{\mathscr{F}}(R/I) = \bigcup_{\mathfrak{a} \in \mathscr{F}} (R/I\colon \mathfrak{a})_{Q(R/I)}$$

is a finitely generated R-module, contained in the total ring of fractions $Q(R/I)$ of R/I. So by [4, (3.2)] there are only finitely many $P \in V(I)$ satisfying simultanously

(α) $\operatorname{ht}(P/I) > 1$

(β) there is a $s \in \operatorname{reg}(P/I)$ with $P \in \operatorname{Ass}(R/(I, s))$.

We denote these primes by $P_1, \cdots, P_r$. By condition (β) we have then $\mathfrak{m} \neq P_j$ $(j = 1, \cdots, r)$. Moreover put $\bigcup_n A(n) = \{P_{r+1}, \cdots, P_s\}$. Then, according to (2.4) there is a $x \in \mathfrak{m} - P_1 \cup \cdots \cup P_s$ such that $I + xR$ is a generic complete intersection of height $h + 1$ and an almost complete intersection. Moreover, as $x \notin P_1 \cup \cdots \cup P_r$ $I + xR$ is unmixed. Put $R' = R/xR$, $I' = IR'$. Then I' satisfies (1.1) and (1.2). As $x \notin P_{r+1} \cup \cdots \cup P_s$ it follows $t(R'/I'^n) = t(R/(I^n + xR)) = t(n) - 1$. This allows to conclude inductively

"(ii)": (a) $\leftrightarrow$ (d) $\leftrightarrow$ (e) are clear by (4.2). (b) $\leftrightarrow$ (a) is obvious. (c) $\leftrightarrow$ (a) is clear by (3.2) and (3.9). So it remains to show (a) $\leftrightarrow$ (c) and (a) $\leftrightarrow$ (b).

"(a) $\leftrightarrow$ (c)": As $t^* = \bar{t}^* = \min(t(1), \dim(R/I) - 1)$, we conclude by the second statement of (i).

"(a) $\leftrightarrow$ (b)": By the isomorphism (2.3) we inductively may reduce the situation to the case $h = 0$, $\mu = 1$. We still know that (a) implies (c). By (3.2) (ii) we conclude that $\operatorname{depth}(R/(0\colon I)_R) = \dim(R)$. As I is unmixed, this implies that R/I is C.M. by [18, 1.3].

(5.3) *Remark.* Let $V \subseteq P^n$ be an algebraic variety of pure dimension δ which is locally an almost complete intersection. Then (a) $\leftrightarrow$ (b) of (5.2) (ii) shows that V is C.M. if $\delta = 2$ or of $\delta = 3$ and V is normal.

In [8] resp. [13] it is shown that under the hypotheses of (5.2) the torsion-freeness of the conormal module I/I^2 is equivalent to the vanishing of $U(I)$. We may sharpen this in showing that the vanishing of $U(I)$ also implies the normal torsion-freeness of I.

Indeed by (3.11) and (5.2) it follows

(5.4) COROLLARY. *Let R and I be as in* (5.2). *Then the following are equivalent*:

(i) $U(I) = \emptyset$,

(ii) *I is normally torsion-free,*

(iii) *I/I^2 is R/I-torsion-free.*

(5.5) EXAMPLES. We look at two examples which with different methods have been studied by Robbiano [20].

(a) Let $R = k[X_1, \cdots, X_6]_{(X_1,\ldots,X_6)}$ and let I be the ideal generated by the 2×2-minors of the matrix $\begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{pmatrix}$.

I is prime and perfect [11]. Each pair of minors generates a complete intersection which coincides with I on the punctured spectrum of R. So we have $U(I) = \emptyset$ and the unmixedness of I. By (5.4) I is normally torsion-free (this holds for a larger class of determinantal ideals [10]). By (4.2) $\mathrm{Gr}_R(I)$ is CM and we have $t(2) = t(3) = \cdots = \bar{t}(2) = \bar{t}(3) = \cdots = 3$.

(b) Let $R = k[X_1, \cdots, X_4]_{(X_1,\ldots,X_4)}$ and let I be the ideal generated by the 2×2-minors of the matrix $\begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_4 \end{pmatrix}$.

I is again a perfect prime (example (b) follows from example (a) by factoring out two independent linear forms) and satisfies (1.1) and (1.2) with $h = 2$. In this case (5.2) gives $t(2) = t(3) = \cdots = \bar{t}(2) = \bar{t}(3) = \cdots = 1$ and the CM-property of $\mathrm{Gr}_R(I)$.

§ 6. On the structure of the conormal cone

In this section we consider the morphism

$$\Phi : \mathrm{Spec}\,(\mathrm{Gr}_R(I)) \longrightarrow \mathrm{Spec}\,(R)\,. \tag{6.1}$$

(6.2) LEMMA. *Let $P \in V(I)$. Then there is an isomorphism $K(P) \otimes_R \mathrm{Gr}_R(I) = K(P)[X_1, \cdots, X_{\mu_P(I)}]$, where $K(P)$ stands for the field $(R/P)_P$.*

Proof. Obviously we may assume that $P = \mathfrak{m}$. If $\mu = h$, the isomorphism is a consequence of the fact that $\mathrm{Gr}_R(I)$ is a polynomial ring in h indeterminates over R/I. So let $\mu = h + 1$ and let $x_1, \cdots, x_\mu$ be a system of generators of I which satisfies the conditions (2.2) (i), (ii). Consider the canonical morphism

$$\varphi_I\colon K(\mathfrak{m})[X_1, \cdots, X_\mu] \longrightarrow K(\mathfrak{m})\otimes_R \mathrm{Gr}_R(I)$$

which sends X_i to $1\otimes\bar{x}_i$. We want to show that φ_I is an isomorphism. If $h = 0$, this is clear by (2.5) (i). So let $h > 0$. We have to show that $\mathrm{Ker}(\varphi_I) = 0$. Choose $i \in \{1, \cdots, \mu\}$. According to section 2 the element x_i is superregular with respect to I. So (2.3) gives rise to diagrams

$$\begin{array}{ccccccc} 0 \to K(\mathfrak{m})[X_1, \cdots, X_\mu] & \xrightarrow{X_i} & K(\mathfrak{m})[X_1, \cdots, X_\mu] & \to & K(\mathfrak{m})[X_1, \cdots, \hat{X}_i, \cdots, X_\mu] & \to & 0 \\ & & \downarrow{\scriptstyle \varphi_I} & & \downarrow{\scriptstyle \varphi_{I/x_iR}} & & \\ & & K(\mathfrak{m})\otimes \mathrm{Gr}_R(I) & \longrightarrow & K(\mathfrak{m})\otimes \mathrm{Gr}_{R/x_iR}(I/x_iR) & \longrightarrow & 0 \end{array}$$

which show inductively that $\mathrm{Ker}(\varphi_I) \subseteq X_i K(\mathfrak{m})[X_1, \cdots, X_\mu]$.

(6.3) *Remark.* (6.2) shows in particular that for $P \in V(I)$

$$\tilde{P} := P\,\mathrm{Gr}(I)_P \cap \mathrm{Gr}_R(I)$$

is prime. $\tilde{P}$ is the generic point of $\Phi^{-1}(P)$.

(6.4) PROPOSITION. *Let* $\mathfrak{A} = \mathrm{Ass}(\mathrm{Gr}_R(I))$ *and* $\mathfrak{B} = \mathrm{Min}(\mathrm{Gr}_R(I))$. *Then we have for the map* $\sim\colon P \to \tilde{P}$:

(i) $\sim$ *maps* $A(1) \cup A^*$ (*which equals* $A^* = U(I) \cup A(1)$ *if* $h > 0$) *bijectively onto* $\mathfrak{A}$.

(ii) $\sim$ *maps* $\mathrm{Min}(R/I) \cup U(I)$ *bijectively onto* $\mathfrak{B}$.

Proof. "(i)": $\Phi(\tilde{P}) = P$ shows that $\sim$ is injective on $V(I)$. So it remains to show that $(A(1)\cup A^*)^\sim = \mathfrak{A}$. We first prove that $A(1)\cup A^* = \Phi(\mathfrak{A})$:

Using (3.2) (i) and (3.9) (i) we may write $A(1)\cup A^* = \bigcup_{n\geqslant 1} B(n)$. So $A(1)\cup A^*$ is the set of those $P \in V(I)$ for which $P\,\mathrm{Gr}_R(I)$ is annihilated by a form. This immediately proves $A(1)\cup A^* = \Phi(\mathfrak{A})$.

Having this it remains to show that $\Phi(\mathfrak{P})^\sim = \mathfrak{P}$ for all $\mathfrak{P} \in \mathfrak{A}$. After localization we may assume that $\Phi(\mathfrak{P}) = \mathfrak{m}$, once having chosen such a $\mathfrak{P}$. Now we prove the statement that $\mathfrak{m}\,\mathrm{Gr}_R(I) = \mathfrak{P}$ by induction on h.

Let $h = 0$. If $\mu = 0$ there is nothing to prove. So let $\mu = 1$.

By (6.2) the only homogeneous primes containing $\mathfrak{m}\,\mathrm{Gr}_R(I)$ are $\mathfrak{m}\,\mathrm{Gr}(I)$ and the homogeneous maximal ideal. So $\mathfrak{P}$ must equal one of these. As depth $\mathcal{O}_R(I) > 0$ (by (4.2) (ii)) $\mathfrak{P} \in \mathrm{Ass}(\mathrm{Gr}_R(I))$ implies that $\mathfrak{P} = \mathfrak{m}\,\mathrm{Gr}_R(I)$.

So let $h > 0$ and choose x as a supperregular element (which induces an isomorphism (2.3)). $\mathfrak{P} \in \mathfrak{A}$ shows that $\bar{x} \in \mathfrak{P}$. As $\mathfrak{P}$ is homogeneous and $\bar{x}$ of degree 1 we have $(\bar{x}, \mathfrak{P}) \neq \mathrm{Gr}_R(I)$. So $(\bar{x}, \mathfrak{P})$ has a minimal prime divisor Q which belongs to $\mathrm{Ass}(\mathrm{Gr}_R(I)/(\bar{x}))$ by [14]. Using (2.3) we thus get by induction that $Q = (\mathfrak{m}, x)\mathrm{Gr}(I)$. $\tilde{\mathfrak{m}} = \mathfrak{m}\,\mathrm{Gr}_R(I) \subseteq \mathfrak{P} \subseteq Q$ and $\mathrm{ht}(Q/\mathfrak{m}) = 1$ then show that $\tilde{\mathfrak{m}} = \mathfrak{P}$. "(ii)": Let $P \in A(1) \cup A^*$. We have to show that

$$\tilde{P} \in \mathfrak{B} \leftrightarrow P \in \mathrm{Min}(R/I) \cup U(I)\,.$$

To do this we may assume that $P = \mathfrak{m}$. Let $h = 0$, $\mu = 1$. Then we have $\mathfrak{m} \notin \mathrm{Min}\,(R/I)$ and it remains to prove that $\mathfrak{m} \in U(I) \leftrightarrow \tilde{\mathfrak{m}} \in \mathfrak{B}$. So let $\mathfrak{m} \in U(I)$. As $d = 1$, $\tilde{\mathfrak{m}} \neq \mathfrak{m}]I[\mathrm{Gr}\,(I)$ shows that $\tilde{\mathfrak{m}} \in \mathfrak{B}$. If $\tilde{\mathfrak{m}} \in \mathfrak{B}$ (6.2) shows that $\dim(\mathrm{Gr}_R(I)/\tilde{\mathfrak{m}}) = 1$. On the other hand we have by (4.8) an isomorphism $\mathrm{Gr}_R(I) = R[X]/(X(0\colon I)_R, I)$.

Let $\mathfrak{P} \in \mathrm{Min}\,(R[X]/(X(0\colon I)_R, I))$. Then we have $\mathfrak{P} \in \mathrm{Min}\,(R[X]/(X, I))$ or $\mathfrak{P} \in \mathrm{Min}\,(R[X]/(0\colon I), I))$.

In the first case we have $\dim(R[X]/\mathfrak{P}) = d$ as R/I is purely d-dimensional ($\dim(R/P) = d$ for all $P \in \mathrm{Min}\,(R/I)$). In the second case we have $\dim(R[X]/\mathfrak{P}) = d$ as $R/((0\colon I)_R, I)$ is purely d-1-dimensional (which follows as $R/(0\colon I)$ is purely d-dimensional and as the generator of I is $R/(0\colon I)$-regular). So $\mathrm{Gr}(I)$ is purely d-dimensional. Thus $\tilde{\mathfrak{m}} \in \mathfrak{B}$ induces that $d = 1$. But this means that $\mathfrak{m} \in U(I)$.

Now let $h > 0$. Again we may assume that $\mu = h + 1$. As R is CM, $\mathrm{Gr}_R(I)$ is purely d-dimensional (as $R\rangle I\langle$ is) and therefore we may conclude as previously by (6.2).

(6.5) gives a geometric characterization of the normal torsion-freeness. Indeed, by (6.5) (ii) and (3.11) it clearly follows

(6.6) COROLLARY. *Let I be unmixed, $h > 0$. Then I is normally torsion-free iff the conormal cone of I has the same number of components as $V(I)$.*

(6.7) *Remark.* Using [4] (6.6) says that under the above hypotheses the symbolic Rees ring $R]I[\,:= \bigoplus_{n>0} I^{(n)}$ either equals $R]I[$ or is not finite over $R]I[$.

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Remark. In the meantime Huneke [23] [24] has studied ideals generated by d-Sequences, and so got results which partly cover ours.

BIBLIOGRAPHY

[1] Aoyama, Y., A remark on almost complete intersections, Manuscripta Math., **22** (1977), 225–228.

[2] Brodmann, M., Asymptotic stability of Ass(M/I^nM), Proc. Amer. Math. Soc. **74**, no. 1 (1979), 16–18.

[3] ——, On the asymptotic nature of the analytic spread, Math. Proc. Cambridge Phil. Soc., **86** (1979), 35–39.

[4] ——, Finiteness of ideal Transforms, J. Algebra, **63**, no. 1 (1980), 162–185.

[5] Cowsik, R. C. and Nori, L., On the fibres of blowing up, J. Indian Math. Soc., **40** (1976), 217–222.

[6] Goto, S., On the Rees algebras of the powers of an ideal generated by a regular sequence, Proceedings of the Institute of Natural Sciences, College of Humanities and Sciences, Nihon University, **13** (1978), 9–11.

[7] Eisenbud, D., Herrmann, M. and Vogel, W., Remarks on regular sequences, Nagoya Math. J., **67** (1977), 177–180.

[8] Herzog, J., Ein Cohen-Macaulay Kriterium mit Anwendung auf den Konormalenmodul und den Differentialmodul, Math. Z., **163** (1978), 149–162.

[9] Herzog, J. and Kunz, E., Der kanonische Modul eines Cohen-Macaulay-Ringes, Lect. Notes in Math., **238** (1971).

[10] Hochster, M., Criteria for equality of ordinary and symbolic powers of primes, Math. Z., **133** (1973), 54–65.

[11] Hochster, M. and Eagon, J. A., Cohen-Macaulay rings invariant theory and the generic perfection of determinantal loci, Amer. J. Math., **93** (1971), 1020–1058.

[12] Hochster, M. and Ratliff, L. J. jr., Five theorems on Macaulay rings, Pacific J. Math., **44** (1973), 147–172.

[13] Kunz, E., The conormal module of an almost complete intersection, Proc. Amer. Math. Soc., **73** (1979), 15–21.

[14] Peskine, C. and Szpiro, L., Liaison des variétés algébriques, Invent. Math., **26** (1974), 271–302.

[15] Matsumura, H., Commutative Algebra, W. A. Benjamin, New York (1970).

[16] Matsuoka, T., On almost complete intersections, Manuscripta Math., **22** (1977), 329–340.

[17] Murthy, M. P., A note on the "Primbasissatz", Arch. Math., **XII** (1961), 425–428.

[18] Nastold, H.-J., Zum Primbasissatz in regulären lokalen Ringen, Arch. Math., **XII** (1961), 30–33.

[19] Rees, D., The grade of an ideal or module, Proc. Cambridge Phil. Soc., **53** (1957), 28–42.

[20] Robbiano, L., An algebraic property of $P^1 \times P^N$, comm. Algebra, **7** (1978), 641–655.

[21] Robbiano, L. and Valla, G., On normal flatness and normal torsion-freeness, J. Algebra, **53** (1976), 552–560.

[22] Valla, G., Certain graded algebras are always Cohen-Macaulay, J. Algebra, **42** (1976), 537–548.

[23] Huneke, C., On the symmetric and Rees-Algebra of an ideal generated by a d-sequence, J. Algebra, **62** (1980), 268–275.

[24] ——, The Theory of d-Sequences and powers of ideals, Adv. in Math., to appear.

Forschungsinstitut für Mathematik
ETH-Zentrum — CH-8092 Zürich