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Nagoya Math. J.
Vol. 87 (1982), 261-271

ON THE L^p BOUND FOR DEGENERATE ELLIPTIC OPERATORS WITH TWO VARIABLES IN THE ILL POSED PROBLEM[1)]

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1. Let Ω be an open set in the upper half plane $\{y > 0\}$, whose boundary is denoted by $\partial\Omega$. Let $\partial\Omega$ contain an open segment Γ lying on the x-axis.

We consider the following system of first order degenerating on $y = 0$:

$$[\partial_y + (\mu_j + i\kappa_j)y^{k_j}\partial_x]u_j = \sum_{k=1}^{m} b_{jk}(x, y)u_k{}^{2)}\,, \qquad j = 1, \cdots, m\,, \tag{1.1}$$

where κ_j, μ_j are real constants and b_{jk} are in $L^\infty(\Omega)$, further k_j are non-negative integers. It is assumed that $\kappa_j \neq 0$, that is, (1.1) is elliptic except at $y = 0$.

In this article we shall prove

THEOREM. *There are constants C, k $(0 < k < 1)$ and a rectangle Q in Ω, whose one side lies on Γ such that if $u_j \in C^1(\Omega) \cap C^0(\bar{\Omega})$ satisfies* (1.1) *in Ω, and*

$$\|u_j\|_{L^\infty(\Omega)} \leqq M(\leqq 1)\,, \qquad \|u_j\|_{L^p(\Gamma)} \leqq \varepsilon(\leqq M)\,,$$

then it follows that

$$\|u_j\|_{L^p(Q)} \leqq C\varepsilon^{1-k} M^k\,, \tag{1.2}$$

where $1 \leqq p \lneqq \infty$ and C depends only on p, while Q, k are independent of p.

The proof is given in Section 3.

We see that our theorem holds more generally for the case of κ_j, μ_j

Received March 6, 1981.
1) This work has been supported by Grant-in-Aid for Co-operative Research A organized by the Ministry of Education, the Japanese Government.
2) We write simply $\partial/\partial x = \partial_x$ and $\partial/\partial y = \partial_y$.

being analytic in $\bar{\Omega}$. Its proof is tedious and essentially the same as in this article. Hence we treat only the case of constant coefficients for the sake of simplicity.

The inequality (1.2) is a kind of Hadamard's three circle theorem, which is required in the ill posed problem, that is, in the non-well posed Cauchy problem of partial differential equations (see e.g. [2]).

L.E. Payne and D. Sather [3] obtained a L^2-inequality of type (1.2) for Tricomi's equations arising in gas dynamics. His tool is the Jensen's inequality for convex functions. Our method is to yield Carleman's estimate with L^p-norm. We proceed along the work of T. Carleman [1] where it is treated for $p = 1$ and non-degenerate systems.

Recently, the L^p approach to unique continuation is achieved by J. C. Saut and B. Scheurer [4]. They consider Schrödinger's equations and improve Hörmander's L^2 estimates with weight.

We give an example of single equations for which our theorem is applicable. We consider the following equations with variable coefficients

$$\partial_y^2 u + a\partial_x\partial_y u + b\partial_x^2 u + Bu = 0\,, \tag{1.3}$$

where B is an operator of first order.

Let λ_1 and λ_2 are the distinct roots of the quadratic equation $\lambda^2 + a\lambda + b = 0$. We set $v_1 = u_x$ and $v_2 = u_y$. Then (1.3) becomes

$$\partial_y\begin{pmatrix} v_1 \\ v_2 \\ u \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ b & a & 0 \\ 0 & 0 & i \end{pmatrix}\partial_x\begin{pmatrix} v_1 \\ v_2 \\ u \end{pmatrix} = \mathscr{B}\,.$$

Here

$$\mathscr{B} = \begin{pmatrix} 0 \\ -Bu \\ v_2 + iv_1 \end{pmatrix}.$$

We write

$$U = \begin{pmatrix} v_1 \\ v_2 \\ u \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -1 & 0 \\ b & a & 0 \\ 0 & 0 & i \end{pmatrix},$$

$$D = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & i \end{pmatrix} \quad \text{and} \quad V = N^{-1}U\,.$$

It is obvious that $N^{-1}HN = D$ and

$$\partial_y V + D\partial_x V = N^{-1}\mathscr{B} - N^{-1}(\partial_y N + H\partial_x N)V.$$

Particularly, we put $\lambda_1 = ic_1$ and $\lambda_2 = ic_2 y^k$, where k is a positive integer and c_1, c_2 are non zero real numbers. We can then apply our theorem to (1.3).

2. We define

$$S(x, y) = y + x^2 - \alpha \sum_{j=1}^{m} y^{2(k_j+1)}.$$

where α is a positive number depending on $\{\kappa_j\}$, $\{\mu_j\}$ and $\{k_j\}$, which will be determined later (see (3.3)).

First we have

LEMMA 1. *There is a positive number ℓ_0 depending on α such that for any ℓ with $0 < \ell < \ell_0$, there exists a simple curve γ satisfying the properties*:

(i) *The end points of γ are $(\ell, 0)$ and $(-\ell, 0)$.*

(ii) *γ is contained in $\{y > 0\}$ except the end points.*

(iii) *$S = \ell^2$ on γ.*

(iv) *The length of γ is finite, more precisely, γ is of class C^1.*

(v) *Let G_ℓ be the domain enclosed by γ and the segment $[-\ell, \ell]$. Then G_ℓ is contained in any given neighborhood of the origin for sufficiently small ℓ.*

(vi) *$S \leqq \ell^2$ in G_ℓ.*

Proof. Since S is an even function of x, it is sufficient to consider only in $x \geqq 0$. The derivative $S_y(= \partial_y S)$ is independent of x. Hence we denote $S_y(x, y)$ simply by $S_y(y)$.

Taking ℓ_0 suitably, we see that for any ℓ with $0 < \ell < \ell_0$, there exists $y_\ell > 0$ satisfying

$$S(0, y_\ell) = \ell^2, \ S_y(y_\ell) > 0 \quad \text{and} \quad y_\ell \longrightarrow 0 \ (\ell \longrightarrow +0).$$

By the theorem of implicit functions there is a C^1-function $f_\ell(x)$ in a neighborhood of $x = 0$ such that $f_\ell(0) = y_\ell$ and $S(x, f_\ell(x)) = \ell^2$.

We show that the existence interval of f_ℓ is $[0, \infty)$. In fact, if it is not, we can find $x_0 > 0$ in such a way that the existence interval of f_ℓ is $[0, x_0)$. Since $f_\ell'(x) = -2x/S_y(f_\ell(x))$ in $[0, x_0)$, we see that $f_\ell'(x) \leqq 0$ there. This means that f_ℓ is monotone decreasing on $[0, x_0)$. Hence $S_y(f_\ell(x_0 - 0)) > 0$, that is, f_ℓ is prolonged over x_0. This is a contradiction.

We see immediately that the point $(\ell, 0)$ is on the curve $y = f_\ell(x)$. Let $\gamma = \{(x, f_\ell(x)) | 0 \leqq x \leqq \ell\} \cup \{(x, f_\ell(-x)) | -\ell \leqq x \leqq 0\}$. Then (i), (ii), (iii) and (iv) hold. Noting that $y_\ell \to 0$ $(\ell \to +0)$ and f_ℓ is monotone decreasing, we see that (v) also holds. Lastly (vi) is evident by the fact that $S_y > 0$ in a neighborhood of the origin. This completes the proof.

For any non-negative integer k we set

$$t = y^{k+1}/(k+1) \quad (y \geqq 0)\,. \tag{2.1}$$

Let D be a semidisk in the upper half plane, whose center is the origin. Let ρ be the radius of D. We denote by D' the image of D with the mapping $(x, y) \to (x, t)$.

LEMMA 2. *There is a constant $C(\rho)$ such that for any $(x', t') \in D'$, it holds*

$$\iint_D ((x - x')^2 + (t - t')^2)^{-1/2}\, dxdy \leqq C(\rho)\,, \tag{2.2}$$

where $C(\rho)$ depends only on ρ and $C(\rho) \to 0$ $(\rho \to 0)$.

Proof. We may assume that $\rho < 1/2$. Let us replace the integral domain D in (2.2) by the semidisk D_1 with radius 2ρ and with center O. Then the proof is reduced to the case of $x' = 0$ without loss of generality. From (2.1) we have

$$dy = (k+1)^{-\nu} t^{-\nu} dt \quad (\nu = k/(k+1))\,.$$

Hence (2.2) is equivalent to

$$\iint_{D_1'} t^{-\nu}(x^2 + (t - t')^2)^{-1/2}\, dxdt \leqq C'(\rho) \tag{2.3}$$

for any $(0, t') \in D_1'$, where D_1' is the image of D_1 by (2.1).

Evidently, D_1' is contained in a semidisk with radius 2ρ and with the same center. And it is easily seen that $(x^2 + t^2)^{1/2} \leqq (x^2 + (t - t')^2)^{1/2}$ for $t \leqq t'/2$, and $|t - t'| \leqq t$ for $t \geqq t'/2$. Thus in order to prove (2.3), it is sufficient to show that

$$\iint_{D_2} |t|^{-\nu}(x^2 + t^2)^{-1/2}\, dxdt \leqq C''(\rho)\,,$$

where $C''(\rho) \to 0$ $(\rho \to 0)$ and D_2 is a entire disk with radius 4ρ and with center O. However this is obvious by virtue of $0 < \nu < 1$ and by the polar coordinates transformation. The proof is complete.

We fix an integer q with $1 \leqq q \leqq m$ and we put

$$t = y^{k_q+1}/(k_q + 1) \tag{2.4}$$

in place of (2.1). Let $c_q = (k_q + 1)^{1/(k_q+1)}$. Then S is written by

$$S(x, y) = c_q t^{1/(k_q+1)} + x^2 - \alpha c_q^{2(k_q+1)} t^2 - \alpha \sum_{j \neq q} c_q^{2(k_j+1)} t^{2(k_j+1)/(k_q+1)} .$$

For simplicity we rewrite

$$S(x, y) = c_q t^{1/(k_q+1)} + x^2 - \alpha c'_q t^2 - \alpha \sum_{j \neq q} d_q^{(j)} t^{2(k_j+1)/(k_q+1)} . \tag{2.5}$$

Here we note that the coefficients c_q, c'_q and $d_q^{(j)}$ are positive.

Let $\alpha > 0$, $\beta > \gamma$ and $0 < \gamma \leqq 1$. We set

$$h(t) = t^\gamma - \alpha t^\beta .$$

Then it holds

LEMMA 3. *There is a positive number δ depending on α, β and γ such that $h''(t) \leqq 0$ if $0 < t < \delta$.*

Proof. The proof is immediate from the equality

$$h''(t) = \begin{cases} \gamma(\gamma - 1)t^{\gamma-2}\{1 - \alpha\beta\gamma^{-1}(\beta - 1)(\gamma - 1)^{-1}t^{\beta-\gamma}\} & (\gamma \neq 1) \\ -\alpha\beta(\beta - 1)t^{\beta-2} \quad (\gamma = 1) . \end{cases}$$

Now we define

$$S_1(t) = c_q t^{1/(k_q+1)} - \alpha \sum_{j \neq q} d_q^{(j)} t^{2(k_j+1)/(k_q+1)} .$$

From Lemma 3 we see immediately

LEMMA 4. *There is a positive number δ_0 such that*

$$S''_1(t) \leqq 0 , \qquad \text{if } 0 < t < \delta_0 .$$

We fix any t' with $0 < t' < \delta_0$ and we set

$$S_2(t) = S_1(t') + (t - t')S'_1(t') - S_1(t) .$$

Then we have

LEMMA 5. *$S_2(t) \geqq 0$ for $0 < t < \delta_0$ and $S_2(t') = 0$.*

Proof. It is trivial that $S_2(t') = 0$. We see that $S'_2(t) = S'_1(t') - S'_1(t)$, $S''_2(t) = - S''_1(t) \geqq 0$ by Lemma 4 and $S'_2(t') = 0$. Accordingly, $S'_2(t) \geqq 0$ for $t' \leqq t < \delta_0$ and $S'_2(t) \leqq 0$ for $0 < t \leqq t'$, which proves the lemma.

LEMMA 6. *Let $1 \leqq p < \infty$, $0 \leqq \nu < 1$ and let $A_1, A_2 > 0$. We put*

$$u(x, y) = \int_{-\infty}^{\infty} ((x - x')^2 + y^2)^{-1/2} f(x')\, dx'$$

for any $f \in L^p(R^1)$ with supp. $f \subset (-A_1, A_1)$. *Then it holds*

$$\left(\int_0^1 \int_{-A_2}^{A_2} |u(x, y)|^p y^{-\nu}\, dxdy\right)^{1/p} \leqq C\|f\|_{L^p(R^1)} ,$$

where C is independent of f.

Proof. We write $A_3 = A_1 + A_2$. The proof is obtained from the following Hausdorff-Young's inequality

$$\begin{aligned}\int_{-A_2}^{A_2} |u(x, y)|^p\, dx &\leqq \left(\int_{-A_3}^{A_3} (x^2 + y^2)^{-1/2}\, dx\right)^p (\|f\|_{L^p(R^1)})^p \\ &\leqq C y^{(\nu-1)/2} (\|f\|_{L^p(R^1)})^p .\end{aligned}$$

LEMMA 7. *Let Γ be a curve of class C^1 with finite length. Let G be a bounded domain in the upper half plane. Then, if $0 \leqq \nu < 1$ and $1 \leqq p < \infty$, we have*

$$\iint_G \left(\int_\Gamma ((x - x')^2 + (y - y')^2)^{-1/2}\, ds_{x,y}\right)^p y'^{-\nu} dx'dy' < \infty .$$

Proof. We write $P = (x', y')$, $Q = (x, y)$ and dis $(P, \Gamma) = |P - R| (R \in \Gamma)$. First we prove

$$\int_\Gamma |P - Q|^{-\alpha}\, ds_Q \leqq C \tag{2.6}$$

for $0 < \alpha < 1$. When $P \in \Gamma$, the inequality is trivial. In general, (2.6) is reduced to the case of $P \in \Gamma$, because

$$|R - Q| \leqq |R - P| + |P - Q| \leqq 2|P - Q| .$$

From (2.6) we see

$$\begin{aligned}(\text{dis}\,(P, \Gamma))^\alpha \int_\Gamma |P - Q|^{-1}\, ds_Q &= \int_\Gamma (\text{dis}(P, \Gamma)/|P - Q|)^\alpha |P - Q|^{\alpha-1}\, ds_Q \\ &\leqq \int_\Gamma |P - Q|^{\alpha-1}\, ds_Q \leqq C_{1-\alpha} .\end{aligned}$$

Thus it holds

$$\int_\Gamma |P - Q|^{-1}\, ds_Q \leqq C_{1-\alpha} (\text{dis}\,(P, \Gamma))^{-\alpha} .$$

Therefore it is sufficient to prove

$$\iint_G (\mathrm{dis}(P,\Gamma))^{-\alpha p} y'^{-\nu}\, dx'dy' < \infty . \tag{2.7}$$

We can assume that Γ is written by $y = f(x)$ $(a \leqq x \leqq b)$, without loss of generality. And it is sufficient to consider that P is close to Γ and the x coordinate of P is in $[a+\varepsilon_0, b-\varepsilon_0]$ for some $\varepsilon_0 > 0$. Let $R = (x'', y'')$. Then we easily see

$$\mathrm{dis}(P,\Gamma) = |x' - x''|(1 + (f'(x''))^{-2})^{1/2} .$$

Let S be the point where the line being parallel to y-axis through P intersects Γ (see Figure 1). Evidently $S = (x', f(x'))$ and we have

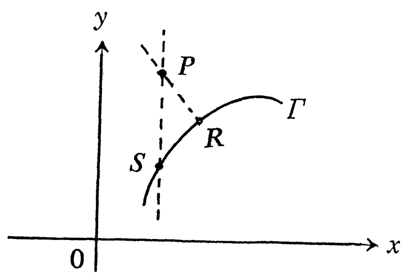

Figure 1

$$\begin{aligned}|P - S| &= |y' - f(x')| = |y' - f(x'') - (x' - x'')f'(c)| \\ &= |x' - x''||f'(x'')^{-1} + f'(c)| ,\end{aligned}$$

where c lies between x' and x''. Consequently, it holds

$$|P - S| \leqq C\, \mathrm{dis}(P,\Gamma) .$$

Hence (2.7) is equivalent to

$$\iint_G |y' - f(x')|^{-\alpha p} y'^{-\nu}\, dx'dy' < \infty .$$

This inequality is correct for sufficiently small α, because the integral

$$\int_0^1 |s - c|^{-\mu} s^{-\nu}\, ds \quad (0 \leqq c \leqq 1)$$

is finite and uniformly bounded with respect to c, if $\mu + \nu < 1$. This completes the proof.

3. In this section we give the proof of our theorem, following the method of T. Carleman [1]. And in the final part of the proof we use the idea of F. John (page 559 in [2]), where the case of analytic functions with one complex variable was treated.

We may assume that the origin is in Γ. We choose a fixed ℓ such that $[-\ell, \ell] \subset \Gamma$, $G_\ell \subset \Omega$ and $\ell < \delta_0/2$[3].

Let q be any fixed integer with $1 \leqq q \leqq m$. For simplicity we write $\kappa = \kappa_q$, $\mu = \mu_q$, $k = k_q$ and $u = u_q$. And we write

$$(3.1) \qquad [\partial_y + (\mu + i\kappa)y^k \partial_x]u = f.$$

It can be assumed that $\kappa > 0$, since the following argument is quite similar for the case of $\kappa < 0$.

We denote by G'_ℓ the image of G_ℓ with the transformation (2.4). Then (3.1) becomes

$$(3.2) \qquad [\partial_t + (\mu + i\kappa)\partial_x]u = g,$$

where $g = (k+1)^{-\nu} t^{-\nu} f$ and $\nu = k/(k+1)$.

Let (x', t') be any fixed point in G'_ℓ and let us set

$$\xi = x - x' - \mu(t - t'), \qquad \eta = \kappa(t - t').$$

Then we see

$$x^2 - \alpha c'_q t^2 = C_0 + C_1\xi + C_2\eta + C_3\xi\eta + \xi^2 + \kappa^{-2}\mu^2\eta^2 - \alpha\kappa^{-2}c'_q\eta^2,$$

where C_j are real constants depending on κ, μ, x', t', α and c'_q. We write

$$\begin{aligned}\xi^2 + \kappa^{-2}\mu^2\eta^2 - \alpha\kappa^{-2}c'_q\eta^2 = {} & \tfrac{1}{2}(1 + \kappa^{-2}(\alpha c'_q - \mu^2))(\xi^2 - \eta^2) \\ & + \tfrac{1}{2}(1 + \kappa^{-2}(\mu^2 - \alpha c'_q))(\xi^2 + \eta^2).\end{aligned}$$

Here let α be such that

$$(3.3) \qquad \max_j (\kappa_j^2 + \mu_j^2) < \alpha c'_q$$

for any q. Then it follows

$$\begin{aligned} S(x, y) &= x^2 - \alpha c'_q t^2 + S_1(t) \\ &= C'_0 + C_1\xi + C'_2\eta + C_3\xi\eta + C_4(\xi^2 - \eta^2) - C_5(\xi^2 + \eta^2) - S_2(t),\end{aligned}$$

where $C_5 > 0$. Hence we have

$$\begin{aligned} S(x, y) = Re[&C'_0 + (C'_2 - iC_1)(\eta + i\xi) \\ &- (C_4 + (i/2)C_3)(\eta + i\xi)^2] - C_5(\xi^2 + \eta^2) - S_2(t).\end{aligned}$$

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3) The number δ_0 is the same as in Lemma 5.

For $\tau \geqq 0$ we set

$$\Phi(\eta + i\xi) = \frac{1}{\eta + i\xi} \exp[-\tau(C_0' + (C_2' - iC_1)(\eta + i\xi) - (C_4 + (i/2)C_3)(\eta + i\xi)^2)] .$$

Then it is obvious that

$$(\partial_\eta + i\partial_\xi)\Phi = 0 .$$

We remark that the following two equations are equivalent:

$$[\partial_t + (\mu + i\kappa)\partial_x]Z = 0 , \qquad (\partial_\eta + i\partial_\xi)Z = 0 .$$

Hence if we put $\psi(x, t; x', t') = \Phi(\eta + i\xi)\exp(\tau S(x, y))$, we obtain

(3.4) $$[\partial_t + (\mu + i\kappa)\partial_x](\psi e^{-\tau S}) = 0 .$$

Since

$$(\eta + i\xi)\psi(x, t; x', t') = \exp(-\tau[C_5(\xi^2 + \eta^2) + S_2(t)])\cdot \exp(-i\tau[C_2'\xi - C_1\eta - 2C_4\xi\eta - \tfrac{1}{2}C_3(\eta^2 - \xi^2)]) ,$$

it follows from Lemma 5 that

$$|\psi| \leqq 1/|\eta + i\xi| , \qquad \lim_{\eta + i\xi \to 0} (\eta + i\xi)\psi = 1 .$$

If we set $\varphi = ue^{-\tau S}$, *(3.2) becomes*

$$(\partial_\eta + i\partial_\xi)\varphi + \tau\varphi\cdot(\partial_\eta + i\partial_\xi)S = \kappa^{-1}ge^{-\tau S} .$$

Let ω be a disk with center (x', t') and with sufficiently small radius. Multiplying the both sides of the above equality by ψ, we integrate it over $G_\varepsilon' - \omega$. By Green's formula and by (3.4) we get

$$-\int_{\partial G_\varepsilon' - \partial\omega} \varphi\psi d\xi + i\int_{\partial G_\varepsilon' - \partial\omega} \varphi\psi d\eta = \kappa^{-1}\iint_{G_\varepsilon' - \omega} g\psi e^{-\tau S}\, d\xi d\eta ,$$

where the boundaries are oriented to the positive direction. Letting the radius of $\omega \to 0$, we see

$$\int_{\partial\omega} \varphi\psi(d\xi - i d\eta) \to -2\pi\varphi(x', t') .$$

Therefore it follows that

$$\varphi(x', t') = -\frac{1}{2\pi}\left[\int_L \varphi\psi\, dx + \int_{\gamma'} \varphi\psi(dx - (\mu + i\kappa)dt) + \iint_{G_\ell'} g\psi e^{-\tau S}\, dxdt\right],$$

where $L = \{(x, 0)\,|\,|x| \leqq \ell\}$ and γ' is the image of γ by (2.4).

Hereafter we denote simply by C the constant independent of τ and $\{u_j\}$. Letting t' be the image of y' with (2.4), we estimate the integral

$$\iint_{G_\ell} |\varphi(x', t')|^p\, dx'dy'.$$

First we see

$$\iint_{G_\ell} \left|\int_L \varphi\psi dx\right|^p dx'dy' \leqq C\iint_{G_\ell'}\left(\int_L ((x - x')^2 + t'^2)^{-1/2}|\varphi(x, 0)|\,dx\right)^p \cdot t'^{-\nu} dx'dt'$$ [4)]

(by Lemma 6)

$$\leqq C(\|\varphi(\cdot, 0)\|_{L^p(L)})^p.$$

And in virtue of Lemma 7 we have

$$\begin{aligned}&\iint_{G_\ell}\left|\int_{\gamma'} \varphi\psi(dx - (\mu + i\kappa)dt)\right|^p dx'dy' \\ &\leqq \iint_{G_\ell'}\left(\int_{\gamma'} |\varphi|((x - x')^2 + (t - t')^2)^{-1/2} ds_{x,t}\right)^p \cdot t'^{-\nu} dx'dt' \\ &\leqq C(\|\varphi\|_{L^\infty(\gamma)})^p.\end{aligned}$$

Finally Lemma 2 and Hausdorff-Young's inequality give

$$\iint_{G_\ell}\left|\iint_{G_\ell'} g\psi e^{-\tau S}\, dxdt\right|^p dx'dy' \leqq C(\ell)^p(\|fe^{-\tau S}\|_{L^p(G_\ell)})^p.$$

Combining the above inequalities we obtain

$$\|\varphi\|_{L^p(G_\ell)} \leqq C(\|\varphi\|_{L^p(L)} + \|\varphi\|_{L^\infty(\gamma)} + C(\ell)\|fe^{-\tau S}\|_{L^p(G_\ell)}).$$

Setting $\varphi_j = u_j e^{-\tau S}$ ($\tau \geqq 0$) for each u_j of (1.1), we conclude that

$$\sum_{j=1}^m \|\varphi_j\|_{L^p(G_\ell)} \leqq C\left(\sum_{j=1}^m \|\varphi_j\|_{L^p(L)} + \sum_{j=1}^m \|\varphi_j\|_{L^\infty(\gamma)}\right) \tag{3.5}$$

for small ℓ if necessary.

If we put $\tau = \log (M/\varepsilon)^{1/\ell^2}$, it holds

$$\|\varphi_j\|_{L^\infty(\gamma)} \leqq M \exp((-\ell^2) \log (M/\varepsilon)^{1/\ell^2}) = \varepsilon,$$

because $S = \ell^2$ on γ. Since $S \geqq 0$ on $y = 0$, we see that $\|\varphi_j\|_{L^p(L)} \leqq \|u_j\|_{L^p(L)}$

4) $\nu = k/(k+1)$.

$\leqq \varepsilon$. Hence by virtue of (3.5) it follows that

$$\sum_{j=1}^{m} \|\varphi_j\|_{L^p(G_\ell)} \leqq C\varepsilon .$$

Let ℓ' be any fixed with $0 < \ell' < \ell$. It is obvious that $G_{\ell'} \subset G_\ell$ and $S \leqq \ell'^2$ in $G_{\ell'}$. Hence we have

$$\sum_{j=1}^{m} \|u_j\|_{L^p(G_{\ell'})} = \sum_{j=1}^{m} \|\varphi_j e^{\tau S}\|_{L^p(G_{\ell'})} \leqq C\varepsilon e^{\tau\ell'^2} = C\varepsilon \exp(\log(M/\varepsilon)^{\ell'^2/\ell^2}) .$$

Therefore setting $k = (\ell'/\ell)^2$ (< 1), we obtain

$$\sum_{j=1}^{m} \|u_j\|_{L^p(G_{\ell'})} \leqq C\varepsilon^{1-k} M^k .$$

This completes the proof.

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