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MODULAR FORMS OF DEGREE n AND REPRESENTATION BY QUADRATIC FORMS II

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Let $S^{(m)}$, $T^{(n)}$ be positive definite integral matrices and suppose that T is represented by S over each p-adic integer ring Z_p. We proved arithmetically in [3] that T is represented by S over Z provided that $m \geq 2n+3$ and the minimum of T is sufficiently large. This guarantees the existence of at least one representation but does not give any asymptotic formula for the number of representations. To get an asymptotic formula we must employ analytic methods. As a generating function of the numbers of representations we consider the theta function

$$\theta(Z) = \sum_{G \in M_{m,n}(Z)} \exp(2\pi i \sigma(S[G]\cdot Z)),$$

where $Z^{(n)} = X + iY = Z'$, $\operatorname{Im} Z = Y > 0$, and σ denotes the trace. Put $N(S, T) = \sharp\{G \in M_{m,n}(Z) \mid S[G] = T\}$; then we have

$$\theta(Z) = \sum_T N(S, T) \exp(2\pi i \sigma(TZ)).$$

$\theta(Z)$ is a modular form of degree n and we decompose $\theta(Z)$ as $\theta(Z) = E(Z) + g(Z)$, where $E(Z)$ is the Siegel's weighted sum of theta functions for quadratic forms in the genus of S. Put

$$E(Z) = \sum a(T) \exp(2\pi i \sigma(TZ)),$$
$$g(Z) = \sum b(T) \exp(2\pi i \sigma(TZ)).$$

Then $a(T)$, $T > 0$, is given by

$$\pi^{n(2m-n+1)/4} \prod_{k=0}^{n-1} \Gamma((m-k)/2)^{-1} |S|^{-n/2} |T|^{(m-n-1)/2} \prod_p \alpha_p(T, S),$$

and it is easy to see that the constant term of $g(Z)$ vanishes at every cusp. Now it may be expected that

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$$N(S, T) = a(T) + b(T)$$

gives an asymptotic formula. In fact, for $n = 1$, this is the case if $m \geq 5$, and $m = 4$ with some restrictions on T. (For $n \geq 2$, see [4, 9]). To get an asymptotic formula, it is sufficient to prove

(i) $b(T)|T|^{-(m-n-1)/2}$ tends to zero,

(ii) $\prod_p \alpha_p(T, S) > \kappa(S)(> 0)$ for every T if T is locally represented by S.

In the former part of this paper we prove (i) for $n = 2$. More precisely, we prove the following:

Let $g(Z) = \sum b(T)\exp(2\pi i\sigma(TZ))$ be a modular form of degree 2, weight $k(\in \frac{1}{2}Z)$ with level such that the constant term of $g(Z)$ vanishes at every cusp. Then we have, if $k > 3$

$$b(T) = O(m(T)^{(3-k)/2}|T|^{k-3/2}) \qquad \text{for } T > 0\,,$$

if $m(T)$(= the minimum of T) is sufficiently large.

We use the generalization of the Farey dissection due to Siegel. But his method is rather rude for our aim. It was effective for T close to scalar matrices [9, 13]. Hence we improve it although it is technical. It may be regarded as an establishment of a generalization of quite standard applications of the circle method. In the latter part we prove (ii) in case of $m \geq 2n + 3$, which is the best possible condition so that (ii) holds. Combining with the former analytic result, we have an asymptotic formula of $N(S, T)$ for $n = 2$, $m \geq 7$. Finally, we discuss some questions.

1.1. We denote by $\boldsymbol{Z}$, $\boldsymbol{R}$, and $\boldsymbol{C}$ the ring of rational integers, the field of real numbers, and the field of complex numbers. For a ring A, $M_{m,n}(A)$ is the set of $m \times n$ matrices with entries in A. If $X \in M_{m,n}(A)$, then X' is the transposed matrix. If $X \in M_{m,m}(\boldsymbol{C})$, then $\sigma(X)$ is the trace, and for $Y \in M_{m,n}(\boldsymbol{C})$ we put $X[Y] = Y'XY$.

For a positive definite matrix $P \in M_{m,m}(\boldsymbol{R})$ we put $m(P) = \min_{0 \neq a \in M_{m,1}(\boldsymbol{Z})} P[a]$. 1_n is the unit matrix of order n, $J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ and we put

$$\Gamma^{(n)} = \{X \in M_{2n,2n}(\boldsymbol{Z}) | J_n[X] = J_n\}\,,$$
$$H^{(n)} = \{Z = X + iY | X, Y \in M_{n,n}(\boldsymbol{R}),\ Z' = Z,\ \operatorname{Im} Z = Y > 0\}\,.$$

For a natural number q, we put

$$\Gamma_0^{(n)}(q) = \left\{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} | C \equiv 0 \bmod q\right\}\,.$$

$\Gamma^{(n)}$ acts discontinuously on $H^{(n)}$ by the mappings $Z \to M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We denote by $\mathscr{F}^{(n)}$ the fundamental domain $\Gamma^{(n)}\backslash H^{(n)}$ described in [11] and in Theorem on p. 169 in [6]. If $Z = X + iY \in \mathscr{F}^{(n)}$, then there exists a positive number λ_n such that $m(Y) > \lambda_n$.

A complex valued function $f(Z)$ on $H^{(n)}$ is called a modular form of degree n, level q and weight k if

(i) $f(Z)$ is an analytic function on $H^{(n)}$,

(ii) for every $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(q)$,

$$(f|M)(Z) = f(M\langle Z\rangle)|CZ+D|^{-k} = v(M)f(Z),$$

$v(M)$ being the multiplicator corresponding to M with $|v(M)| = 1$, and

(iii) for every $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)}$, $(f|M)(Z)$ has a Fourier expansion of the form

$$(f|M)(Z) = \sum_{\substack{T \geq 0 \\ T \in M_{n,n}(Z)}} a(M,T)\exp(2\pi i\sigma(TZ)/q(M)),$$

where $q(M)$ is a natural number dependent on M.

If $a(M, 0)$ in the condition (iii) vanishes for every $M \in \Gamma^{(n)}$, then we say that the constant term of $f(Z)$ vanishes at every cusp.

1.2. We give examples of modular forms which are important in this paper.

Let $S \in M_{m,m}(Z)$ be a positive definite matrix whose diagonal entries are even. Let q be a natural number such that $qS^{-1} \in M_{m,m}(Z)$ and its diagonal entries are even. We put

$$\theta_S^{(n)}(Z; X, Y) = \sum_{G \in M_{m,n}(Z)} \exp(\pi i\sigma(S[G-Y]\cdot Z) + 2\pi i\sigma(G'X) - \pi i\sigma(X'Y)),$$

where $Z \in H^{(n)}$, $X, Y \in M_{m,n}(C)$.

Then $\theta_S^{(n)}(Z; 0, 0)$ satisfies the conditions (i), (ii) for $k = m/2$ ([1]). For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)}$ with $|C| \neq 0$ it is easy to see, by using Lemma 2 in [1],

$$|CZ+D|^{-m/2}\theta_S^{(n)}(M\langle Z\rangle; 0, 0) = \sum_{\substack{G_1 \in M_{m,n}(Z) \\ G_1 \bmod 2|C|}} \exp(\pi i\sigma(S[G_1]AC^{-1}))$$
$$\cdot |S|^{-n/2}\sqrt{-1}^{-mn/2}2^{-mn}|C|^{m/2-mn}\theta_{S^{-1}}^{(n)}(4^{-1}|C|^{-2}(CZ+D)C'; -2^{-1}|C|^{-1}G_1, 0),$$

and

$$\theta_{S^{-1}}^{(n)}(4^{-1}|C|^{-2}(CZ+D)C'; -2^{-1}|C|^{-1}G_1, 0)$$
$$= \sum_{\substack{N \in M_{m,n}(Z) \\ N \bmod 8|C|^2|S|}} \exp(\pi i 4^{-1}|C|^{-2}\sigma(S^{-1}[N]DC') + 2\pi i\sigma(-2^{-1}|C|^{-1}N'G_1))$$
$$\cdot \theta_{S^{-1}}^{(n)}(16|C|^2|S|^2Z[C']; 0, (-8|C|^2|S|)^{-1}N).$$

It is known (p. 205 in [6]) that every $M \in \Gamma^{(n)}$ can be written as $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 & 1_n \\ -1_n & \tilde{S} \end{pmatrix}$, $|C| \neq 0$, $\tilde{S} \in M_{n,n}(Z)$. Combining the above formula with

$$\begin{aligned} &|-Z+\tilde{S}|^{-m/2}\theta_{S^{-1}}^{(n)}(16|C|^2|S|^2(-Z+\tilde{S})^{-1}[C']; 0, (-8|C|^2|S|)^{-1}N) \\ &= \sqrt{-1}^{mn/2}\cdot 4^{-mn}|S|^{-mn+n/2}|C|^{-mn-m}\theta_S^{(n)}(-(16|C|^2|S|^2)^{-1}(-Z+\tilde{S})[C^{-1}]; \\ &\quad (-8|C|^2|S|)^{-1}N, 0), \end{aligned}$$

it is easy to see that $\theta_S^{(n)}(Z,0,0)$ satisfies the condition (iii) and the constant term of $\theta_S^{(n)}(Z,0,0)$ depends only on the genus of S, and hence the constant term of $\theta_S^{(n)}(Z,0,0) - \theta_{S_1}^{(n)}(Z,0,0)$ vanishes at every cusp if S_1 belongs to the genus of S.

1.3. LEMMA. *Let $f(Z) = \sum_{\substack{T \geq 0 \\ T \in M_{n,n}(Z)}} a(T)\exp(2\pi i\sigma(TZ))$ converge absolutely on $H^{(n)}$, and assume $a(T) = 0$ if $\operatorname{rk} T < \nu$ $(0 < \nu \leq n)$. If $Y = \operatorname{Im} Z$ runs over a fixed Siegel domain $\mathfrak{S}$ with $m(Y) > \varepsilon\,(>0)$, then we have, for some $\kappa > 0$,*

$$f(Z) = O(\exp(-\kappa\sigma(Y_\nu)))\,,$$

where Y_ν is the upper left $\nu \times \nu$ submatrix of Y.

Proof. Let $\nu \leqq h \leqq n$ and put

$$\alpha(h) = \sum_{\operatorname{rk} T = h} |a(T)|\exp(-2\pi\sigma(TY))\,.$$

If $Y \in \mathfrak{S}$, $m(Y) > \varepsilon$, then there exists $\varepsilon' > 0$ such that $Y > \varepsilon' 1_n$, and then as p.p. 184~185 in [6] we have

$$\alpha(h) < \kappa_1 \sum_{\operatorname{rk} T = h} \exp(-\pi\sigma(TY))\,,$$

where κ_1 and $\kappa_2, \cdots$ occurring hereafter are positive numbers depending only on ε, $\mathfrak{S}$ and $f(Z)$. Decompose T as

$$T = \begin{pmatrix} T_1^{(h)} & 0 \\ 0 & 0 \end{pmatrix}[U]\,, \quad |T_1| \neq 0\,, \quad U \in GL(n, Z)\,,$$

and here we assume that T_1 is any fixed representative of equivalence classes. If $T = \begin{pmatrix} T_2 & 0 \\ 0 & 0 \end{pmatrix}[V]$ is another decomposition, then we have $T_1 = T_2$ and $UV^{-1} = \begin{pmatrix} W_1^{(h)} & 0 \\ W_3 & W_4 \end{pmatrix}$, $T_1[W_1] = T_1$. Hence we have

$$\begin{aligned} \alpha(h) &< \kappa_1 \sum_{\{T_1^{(h)}\} > 0} \sum_{U \in \left\{\begin{pmatrix} 1_h & 0 \\ * & * \end{pmatrix} \in GL(n,Z)\right\}\backslash GL(n,Z)} \exp\left(-\pi\sigma\left(\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}[U]\cdot Y\right)\right) \\ &= \kappa_1 \sum_{\{T_1\}>0} \sum_F \exp(-\pi\sigma(T_1 \cdot Y[F]))\,, \end{aligned}$$

where $\{T_1\}$ means that T_1 runs over representatives in some Siegel domain of equivalence classes of positive definite integral matrices, and F runs over the set $\{F \in M_{n,h}(Z) | \text{primitive}\}$. Let $t_1, \cdots, t_h$ be diagonal entries of T_1. Then we have $T_1 \mathrel{\substack{\cup \\ \cap}} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_h \end{pmatrix}$, and the class number of positive definite integral matrices of determinant $|T_1|$ is $O(|T_1|^a)$ for some $a > 0$. Hence

$$\begin{aligned}
\alpha_h &< \kappa_2 \sum_{\substack{t_i \geq 1 \\ 1 \leq i \leq h}} (t_1 \cdots t_h)^a \sum_F \exp\left(-\kappa_3 \sigma\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_h \end{pmatrix} \cdot Y[F]\right)\right) \\
&< \kappa_4 \sum_{\substack{t_i \geq 1 \\ 1 \leq i \leq h}} \sum_F \exp\left(-\kappa_5 \sigma\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_h \end{pmatrix} \cdot Y[F]\right)\right) < \kappa_6 \sum_F \exp(-\kappa_5 \sigma(Y[F])) \\
&< \kappa_6 \sum_{\substack{1 \leq i_1 < \cdots < i_p \leq n \\ p \geq h}} \sum_G \exp(-\kappa_5 \sigma(Y[G])),
\end{aligned}$$

where G runs over the set $\{G = (g_{ij}) \in M_{n,h}(Z) | \sum_{j=1}^h g_{ij}^2 \neq 0 \text{ iff } i = i_k \text{ for } k = 1, \cdots, p\}$. If we put $Y = (y_{ij})$, $y_{ii} = y_i$, then $Y \mathrel{\substack{\cup \\ \cap}} \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{pmatrix}$ implies

$$\begin{aligned}
\sum_G \exp(-\kappa_5 \sigma(Y[G])) &< \sum_{\sum_{k=1}^h g_{ij,k}^2 \neq 0} \exp\left(-\kappa_7 \sum_{j=1}^p y_{i_j}\left(\sum_{k=1}^h g_{i_j,k}^2\right)\right) \\
&= \prod_{j=1}^p \left(\sum_{\sum_{k=1}^h g_k^2 \neq 0} \exp\left(-\kappa_7 y_{i_j} \sum_{k=1}^h g_k^2\right)\right) < \prod_{j=1}^p \left(\sum_{g=1}^\infty (2\sqrt{g} + 1)^h \exp(-\kappa_7 y_{i_j} g)\right) \\
&< \kappa_8 \prod_{j=1}^p \exp(-\kappa_9 y_{i_j}) < \kappa_8 \exp(-\kappa_{10} \sigma(Y_p)) < \kappa_8 \exp(-\kappa_{10} \sigma(Y_\nu)).
\end{aligned}$$

Q.E.D.

1.4. Put $\Gamma^{(n)}(\infty) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(n)} \middle| C = 0\right\}$ and let N_q be representatives of the right cosets of $\Gamma^{(n)}$ modulo $\Gamma^{(n)}(\infty)$, i.e.,

$$\Gamma^{(n)} = \bigcup_{q=0}^\infty \Gamma^{(n)}(\infty) N_q, \qquad N_0 = 1_{2n}.$$

We can normalize N_q, $q \geq 1$, as follows: Putting $N_q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$(C, D) = \left(\begin{pmatrix} C_1^{(h)} & 0 \\ 0 & 0 \end{pmatrix} U', \begin{pmatrix} D_1^{(h)} & 0 \\ 0 & 1_{n-h} \end{pmatrix} U^{-1}\right),$$

where $|C_1| \neq 0$, $U \in GL(n, Z)$.

If we put $U = (F^{(n,h)}, *)$, then the coset $\Gamma^{(n)}(\infty) N_q$, $q \geq 1$, corresponds bijectively to $C_1^{-1} D_1 \in$ the set of rational symmetric matrices of degree h

and $FGL(h, Z)$, $1 \leq h \leq n$. (p. 160 and p. 166 in [6]).

Let $\mathscr{F}^{(n)}$ be a fundamental domain $\Gamma^{(n)} \backslash H^{(n)}$ in **1.1** and put

$$\begin{aligned} \mathscr{G}^{(n)} &= \sum_{M \in \Gamma^{(n)}(\infty)} M\langle \mathscr{F}^{(n)} \rangle \\ &= \bigcup_{U,S} (\mathscr{F}^{(n)}[U] + S) \ (U \in GL(n, Z),\ S = S' \in M_{n,n}(Z)), \\ \text{and} \quad \mathscr{G}_q &= N_q^{-1}\langle \mathscr{G}^{(n)} \rangle . \end{aligned}$$

If $X + iY \in \mathscr{G}^{(n)}$, then $m(Y) > \lambda_n$ for some positive constant λ_n. We introduce the "dissection" due to Siegel. Let $T^{(n)}$ be a positive definite matrix and put

$$\begin{aligned} &E^* = \{X + iT^{-1} | X = (x_{ij}), \quad 0 \leq x_{ij} = x_{ji} < 1\} \subset H^{(n)}, \\ &D_q = E^* \cap \mathscr{G}_q, \qquad E_q^* = D_q - (D_0 \cup \cdots \cup D_{q-1}), \\ &\text{and} \quad E_q = \{X | X + iT^{-1} \in E_q^*\}. \end{aligned}$$

Then we have $\{X = (x_{ij}) | 0 \leq x_{ij} = x_{ji} < 1\} = \bigcup_{q=0}^{\infty} E_q$ (finite and disjoint). If $m(T)$ is sufficiently large, then D_0 is empty. When N_q, $q \geq 1$, corresponds to $R^{(h)}$, $F^{(n,h)}GL(h, Z)$, we put $E(F, R) = E_q$.

1.5. Hereafter we confine ourselves to the case of $n = 2$.

Let $f(Z) = \sum_{\substack{\text{half-integral} \\ T \geq 0}} a(T)\exp(2\pi i\sigma(TZ))$ be a modular form of degree 2, some level and weight $k(\in \frac{1}{2}Z)$ and assume that the constant term of $f(Z)$ vanishes at every cusp. Our aim is to prove

THEOREM. *If $T^{(2)}$ is positive definite and $m(T)$ is sufficiently large, then we have*

$$a(T) = O(m(T)^{(3-k)/2} |T|^{k-3/2}) \qquad \textit{for } k > 3 .$$

We prove this in **1.5** and **1.6**. By definition of a modular form, we have $|a(T)| = |a(T[U])|$ for every $U \in GL(2, Z)$. Hence we may assume that

$$T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \left[\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right], \quad t_1/t_2 \leq 4/3, \quad |u| \leq 1/2 .$$

Then we have $T \underset{\cap}{\cup} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$ and $m(T) \underset{\cap}{\cup} t_1$. Moreover we assume that t_1 is sufficiently large. We fix such a T and use the dissection for T in **1.4** in this and the next section. Since t_1 is sufficiently large, we have $D_0 = \phi$. By using the dissection for T, we have

$$a(T) = \exp(4\pi)\int_{X \bmod 1} f(X + iT^{-1})\exp(-2\pi i\sigma(TX))dX$$
$$= \exp(4\pi)\sum_{q=1}^{\infty}\int_{E_q} f(X + iT^{-1})\exp(-2\pi i\sigma(TX))dX ,$$

where $dX = dx_1dx_2dx_3$, $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$.

Here E_q is empty for sufficiently large q. From the definition of $(f|M)(Z)$ in the condition (ii) follows that

$$f(Z) = |CZ + D|^{-k}(f|M^{-1})(M\langle Z\rangle) \qquad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{(2)} ,$$

and the number of functions $f|M$, $M \in \Gamma^{(2)}$, up to the constant multiples with the absolute value $= 1$, is finite. Hence from Lemma follows that for $X \in E_q$,

$$|(f|N_q^{-1})(N_q\langle X + iT^{-1}\rangle)| < \kappa_1 \exp(-\kappa_2 m(\operatorname{Im} N_q\langle X + iT^{-1}\rangle)) ,$$

where κ_1, κ_2 are positive constants independent of q, since $N_q\langle X + iT^{-1}\rangle \in \mathcal{G}^{(2)}$, and hence $m(\operatorname{Im} N_q\langle X + iT^{-1}\rangle) > \lambda_2$.

We put

$$\alpha(F, R) = \int_{E(F,R)} \|C(X + iT^{-1}) + D\|^{-k} \exp(-\kappa_2 m(\operatorname{Im} M\langle X + iT^{-1}\rangle))dX ,$$

where $N_q = M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ corresponds to R, F. It is clear that $|a(T)| < \exp(4\pi)\kappa_1 \sum_{R,F} \alpha(F, R)$.

Suppose $\operatorname{rk} F = 2$. Then we can take 1_2 as F, and as in [13] we have, for $k > 3$,

$$\sum_R \alpha(1_2, R) < \kappa_3 m(T)^{(3-k)/2}|T|^{k-3/2} .$$

1.6. Let $N_q = M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $q \geq 1$, and assume $|C| = 0$. Then we may assume $(C, D) = \left(\begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}U', \begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix}U^{-1}\right)$, $c_1 > 0$, $U \in GL(2, Z)$. Then we have

$$\operatorname{Im} M\langle X + iT^{-1}\rangle = \begin{pmatrix} (a_1 + a_1^{-1}(q_1 + r)^2)^{-1} & 0 \\ 0 & a_1^{-1}|T|^{-1} \end{pmatrix}\left[\begin{pmatrix} c_1^{-1} & q' \\ 0 & 1 \end{pmatrix}\right] ,$$

where $T^{-1}[U] = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $r = c_1^{-1}d_1$, $X[U] = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}$, $q' = a_1^{-1}a_2(q_1 + r) - q_3$. Put $P = P(q_1, q_3) = \begin{pmatrix} (a_1 + a_1^{-1}q_1^2)^{-1} & 0 \\ 0 & a_1^{-1}|T|^{-1} \end{pmatrix}\left[\begin{pmatrix} c_1^{-1} & a_1^{-1}a_2q_1 - q_3 \\ 0 & 1 \end{pmatrix}\right]$.

Suppose $m(P) > \lambda_2$. Then we show that $|u_{1,1}| < \kappa_5$ where $U = (u_{i,j})$ and κ_5 is a positive absolute constant, and that if $m(P) = P\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, then $b_2 \neq 0$. Since $3m(P)^2/4 \leq |P|$ for a positive definite matrix P of degree 2, we have

$$3\lambda_2^2/4 < (a_1 + a_1^{-1}q_1^2)^{-1}a_1^{-1}|T|^{-1}c_1^{-2} < a_1^{-2}|T|^{-1}.$$

Put $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ = the first column of U. Then $T \underset{\cap}{\cup} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}$ implies

$$a_1 = T^{-1}[F] > \kappa_3 \begin{pmatrix} t_1^{-1} & \\ & t_2^{-1} \end{pmatrix}[F] = \kappa_3(t_1^{-1}f_1^2 + t_2^{-1}f_2^2) \quad \text{and} \quad |T| > \kappa_4 t_1 t_2 .$$

Hence $3\lambda_2^2/4 < a_1^{-2}|T|^{-1}$ implies

$$3\lambda_2^2/4 < \kappa_3^{-2}(t_1^{-1}f_1^2 + t_2^{-1}f_2^2)^{-2}\kappa_4^{-1}t_1^{-1}t_2^{-1} < \kappa_3^{-2}\kappa_4^{-1}t_1t_2^{-1}f_1^{-4}$$

if $f_1 \neq 0$, and then $|f_1| < \kappa_5$ since $t_1/t_2 \leq 4/3$. Next we assume that $m(P) = P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1^{-2}(a_1 + a_1^{-1}q_1^2)^{-1}$. Then $m(P) > \lambda_2$ implies $c_1^{-2}(a_1 + a_1^{-1}q_1^2)^{-1} > \lambda_2$. On the other hand $3m(P)^2/4 \leq |P|$ implies

$$3c_1^{-4}(a_1 + a_1^{-1}q_1^2)^{-2}/4 < (a_1 + a_1^{-1}q_1^2)^{-1}a_1^{-1}|T|^{-1}c_1^{-2},$$

and then $3c_1^{-2}(a_1 + a_1^{-1}q_1^2)^{-1}/4 < a_1^{-1}|T|^{-1}$.

Hence we have $3\lambda_2/4 < a_1^{-1}|T|^{-1} < \kappa_3^{-1}\kappa_4^{-1}(t_1^{-1}f_1^2 + t_2^{-1}f_2^2)^{-1}t_1^{-1}t_2^{-1}$. This yields $4\kappa_3^{-1}\kappa_4^{-1}\lambda_2^{-1}/3 > t_1t_2(t_1^{-1}f_1^2 + t_2^{-1}f_2^2) > t_2$ or t_1. This is a contradiction if t_1 ($\underset{\cap}{\cup} m(T)$) is sufficiently large.

Since

$$\|C(X + iT^{-1}) + D\| = \left\| \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}(X[U] + iT^{-1}[U]) + \begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix} \right\|$$
$$= c_1|q_1 + r + ia_1|,$$

we have

$$\alpha(F, r) = c_1^{-k}\int_{E(F,r)} ((q_1 + r)^2 + a_1^2)^{-k/2}\exp(-\kappa_2 m(P(q_1 + r, q_3)))dX,$$

where F is the first column of U as above.

If $\begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 \end{pmatrix}, \begin{pmatrix} q_1 & q_2 \\ q_3 & q_4 + n \end{pmatrix} \in E(F, r)[U]$, $(n \in Z)$, then $n = 0$ follows. Since $dX = dq_1dq_3dq_4$, we have

$$\sum_{r_0 \equiv r \bmod 1} \alpha(F, r_0) < c_1^{-k}\sum_{n \in Z}\int_{E(F, r+n)} ((q_1 + r + n)^2 + a_1^2)^{-k/2}$$
$$\times \exp(-\kappa_2 m(P(q_1 + r + n, q_3)))dq_1dq_3dq_4$$
$$< c_1^{-k}\int_S (q_1^2 + a_1^2)^{-k/2}\exp(-\kappa_2 m(P(q_1, q_3)))dq_1dq_3dq_4,$$

where $S = \left\{(q_1, q_3, q_4) \in \boldsymbol{R}^2 \times [0, 1) \middle| \begin{pmatrix} q_1 - r - n & q_3 \\ q_3 & q_4 + m \end{pmatrix}[U^{-1}] \in E(F, r + n)\right.$ for some $\left. m, n \in \boldsymbol{Z}\right\}$. Note that if (q_1, q_3, q_4), $(q_1, q_3 + n, q_4) \in S$, $(n \in \boldsymbol{Z})$, then $n = 0$, and that for $(q_1, q_3, q_4) \in S$ we have $m(P(q_1, q_3)) > \lambda_2$.

For a natural number b_2, we denote by $S(b_2)$ the set

$$\left\{(q_1, q_3, q_4) \in S \middle| m(P(q_1, q_3)) = P(q_1, q_3)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ for some } b_1 \in \boldsymbol{Z}\right\}.$$

Then we have $S = \bigcup_{b_2=1}^{\infty} S(b_2)$ and

$$\sum_{r_0 \equiv r \bmod 1} \alpha(F, r_0) < c_1^{-k} \sum_{b_2=1}^{\infty} \int_{S(b_2)} (q_1^2 + a_1^2)^{-k/2} \exp\left(-\kappa_2 P(q_1, q_3)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) dq_1 dq_3 dq_4\,,$$

where b_1 is an integer such that $m(P(q_1, q_3)) = P(q_1, q_3)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. b_1 depends on q_1, q_3. Since $P\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (a_1 + a_1^{-1}q_1^2)^{-1} b_2^2 (c_1^{-1} b_1 b_2^{-1} + a_1^{-1} a_2 q_1 - q_3)^2 + a_1^{-1}|T|^{-1} b_2^2$ and b_1 is an integer such that $|c_1^{-1} b_1 b_2^{-1} + a_1^{-1} a_2 q_1 - q_3| \leq (2c_1 b_2)^{-1}$. Hence for fixed q_1, q_4 we have

$$\begin{aligned}
&\int_{(q_1, q_3, q_4) \in S(b_2)} \exp\left(-\kappa_2 P(q_1, q_3)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) dq_3 \\
&\quad \leq c_1 b_2 \int_{|q_3| \leq (2c_1 b_2)^{-1}} \exp(-\kappa_2((a_1 + a_1^{-1}q_1^2)^{-1} b_2^2 q_3^2 + a_1^{-1}|T|^{-1} b_2^2)) dq_3 \\
&\quad < c_1 b_2 \int_{\boldsymbol{R}} \exp(-\kappa_2((a_1 + a_1^{-1}q_1^2)^{-1} b_2^2 q_3^2 + a_1^{-1}|T|^{-1} b_2^2)) dq_3 \\
&\quad = c_1 b_2 \exp(-\kappa_2 a_1^{-1}|T|^{-1} b_2^2) \sqrt{\pi}\, \kappa_2^{-1/2} \sqrt{a_1 + a_1^{-1} q_1^2}\, b_2^{-1} \\
&\quad < c_1 \kappa_2^{-3/2} \sqrt{\pi}\, a_1 |T| b_2^{-2} \sqrt{a}_1^{-1} \sqrt{a_1^2 + q_1^2}\,.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_{r_0 \equiv r \bmod 1} \alpha(F, r_0) &< c_1^{1-k} \kappa_2^{-3/2} \sqrt{\pi} |T| \sum_{b_2=1}^{\infty} b_2^{-2} \sqrt{a_1} \int_{\boldsymbol{R}} (a_1^2 + q_1^2)^{(1-k)/2} dq_1 \\
&< \kappa_8 c_1^{1-k} |T| a_1^{5/2-k} < \kappa_9 c_1^{1-k} t_1 t_2 (t_1^{-1} f_1^2 + t_2^{-1} f_2^2)^{5/2-k}\,.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\sum_{F, r} \alpha(F, r) &< \kappa_9 \sum_{c_1=1}^{\infty} c_1^{2-k} t_1 t_2 \sum_{\substack{(f_1, f_2)=1 \\ |f_1| < \kappa_5}} (t_1^{-1} f_1^2 + t_2^{-1} f_2^2)^{5/2-k} \\
&< \kappa_{10} t_1 t_2 (t_1^{k-5/2} + t_2^{k-5/2} \sum_{f_2 \neq 0} |f_2|^{5-2k}) < \kappa_{11} t_1 t_2^{k-3/2} \\
&< \kappa_{12} m(T)^{5/2-k} |T|^{k-3/2} \qquad \text{if } k > 3\,.
\end{aligned}$$

Combining the estimate in **1.5**, we complete the proof.

Remark. A result in [4] suggests that there is room for improvement of the Siegel's method of the estimate of $\sum_R \alpha(1_2, R)$ in **1.5.**

2. In this part we study local densities of quadratic forms. It is necessary to show that the expected main term of the representation numbers of quadratic forms is the real one. Terminology and notation are generally those from [7].

2.1. Let p be an odd prime.

LEMMA 1. *Let M, N be quadratic spaces over $Z/(p)$ and $\dim M = m > 2$, $\dim N = n < m$, and assume that M is regular and that there is an isometry from N to M. Decompose N as $N = N_0 \perp \operatorname{rad} N$ and put $t = \dim N_0$, $\varepsilon = ((-1)^{(m-t)/2} dN_0 dM/p)$ (Legendre symbol), if $m \equiv t \bmod 2$, where we put $dN_0 = 1$ if $N_0 = \{0\}$. Then we have*

$$p^{n(n+1)/2-mn} \times \{\textit{the numbers of isometries from } N \textit{ to } M\}$$
$$= \prod_{i=1}^{a} (1 \pm p^{-r_i}) \times \begin{cases} 2 & \textit{if } m - 2n + t = 0\,, \\ 1 + \varepsilon p^{-1} & \textit{if } m - 2n + t = 2\,, \\ 1 & \textit{otherwise,} \end{cases}$$

where $2 \leq r_i \leq b$ and a, b are smaller than the number depending only on $m = \dim M$.

Proof. For quadratic spaces K, L over $Z/(p)$, we denote by $A(K, L)$ the number of isometries from K to L. By the assumption there is an isometry from N_0 to M. Hence we have $M \cong N_0 \perp M_1$ for some regular quadratic space M_1, and $\operatorname{rad} N$ is represented by M_1. It is easy to see $A(N, M) = A(N_0, M) A(\operatorname{rad} N, M_1)$. Put $\delta = ((-1)^{m/2} dM/p)$ if m is even. Then it is known ([10])

$$p^{t(t+1)/2-mt} A(N_0, M) = \begin{cases} (1 - \delta p^{-m/2})(1 + \varepsilon p^{(t-m)/2}) \prod_{k=1}^{t/2-1} (1 - p^{-(m-2k)}) \\ \qquad\qquad m \equiv t \equiv 0 \bmod 2\,, \\ (1 - \delta p^{-m/2}) \prod_{k=1}^{(t-1)/2} (1 - p^{-(m-2k)}) \\ \qquad\qquad m \equiv t + 1 \equiv 0 \bmod 2\,, \\ (1 + \varepsilon p^{(t-m)/2}) \prod_{k=1}^{(t-1)/2} (1 - p^{-(m+1-2k)}) \\ \qquad\qquad m \equiv t \equiv 1 \bmod 2\,, \\ \prod_{k=1}^{t/2} (1 - p^{-(m+1-2k)}) \qquad m \equiv t + 1 \equiv 1 \bmod 2\,. \end{cases}$$

As on p.p. 119~120 we have

$$p^{-(m-t)(n-t)+(n-t)(n-t+1)/2}A(\operatorname{rad} N, M_1)$$
$$=\begin{cases}\prod_{i=0}^{n-t-1}(1-p^{-(2i+m-2n+t+1)}) & m-t\equiv 1 \bmod 2\,,\\ \prod_{i=0}^{n-t-1}\{(1-\varepsilon p^{-(m-2n+t)/2-i-1})(1+\varepsilon p^{-(m-2n+t)/2-i})\} & m-t\equiv 0 \bmod 2\,,\end{cases}$$

Since $\dim M_1 \geq 2\dim\operatorname{rad} N$ implies $m-2n+t\geq 0$, $p^{t(t+1)/2-mt}A(N_0, M)$ is equal, up to the factors $1\pm p^{-r}$ ($r \geqq 2$), to

$$A_1=\begin{cases}1+\varepsilon p^{(t-m)/2} & m\equiv t \bmod 2\,,\\ 1 & m\not\equiv t \bmod 2\,,\end{cases}$$

and $p^{-(m-t)(n-t)+(n-t)(n-t+1)/2}A(\operatorname{rad} N, M_1)$ is equal to

$$A_2=\begin{cases}1-p^{-(m+t-2n+1)} & m\not\equiv t \bmod 2\,,\\ (1-\varepsilon p^{-(m-t)/2})(1+\varepsilon p^{-(m-t)/2+n-t}) & m\equiv t \bmod 2\,.\end{cases}$$

If $m-2n+t=0$, then M_1 is a hyperbolic space and $\varepsilon=1$ and $A_1=1+p^{n-m}$, $A_2=2(1-p^{n-m})$. If $m-2n+t=2$, then $A_1=1+\varepsilon p^{1+n-m}$ and $A_2=(1-\varepsilon p^{1+n-m})(1+\varepsilon p^{-1})$ and $m-n-1=n+1-t\geq 1$. If $m-2n+t\equiv 0 \bmod 2$ and $m-2n+t\neq 0, 2$, then $m-2n+t\geq 4$ and $(m-t)/2\geq 2$, $(m-t)/2-n+t\geq 2$. If $m-2n+t\equiv 1 \bmod 2$, then $A_1=1$ and $m+t-2n+1\geq 2$. These complete the proof.

Remark. If $m\geq 2n+3$, then $m-2n+t\neq 0, 2$.

2.2. Let p be a prime and M, N regular quadratic lattices over Z_p with $\operatorname{rk} M=m$, $\operatorname{rk} N=n$ and $\mathfrak{n}M$, $\mathfrak{n}N\subset 2Z_p$. For any quadratic lattice the letters Q, B denote the quadratic form and the bilinear form ($Q(x)=B(x,x)$).

Put

$$A_{p^t}(N,M)=\{u: N\to M/p^tM \mid B(ux,uy)\equiv B(x,y) \bmod p^t \quad \text{for } x,y\in N\}\,,$$
$$B_{p^t}(N,M)=\{u: N\to M/p^tM \mid Q(ux)\equiv Q(x) \bmod 2p^t \quad \text{for } x\in N \text{ and } u \text{ induces an injective mapping from } N/pN \text{ to } M/pM\}\,,$$
$$C_{p^t}(N,M)=\{u: N\to M/p^tM^{\#} \mid Q(ux)\equiv Q(x) \bmod 2p^t \quad \text{for } x\in N \text{ and } u \text{ induces an injective mapping from } N/pN \text{ to } M/pM\}\,.$$

It is known ([10]) that $2^{-\delta_{m,n}}(p^t)^{n(n+1)/2-mn}\#A_{p^t}(N,M)$ is independent of t if t is sufficiently large, and we denote the value by $\alpha_p(N,M)$

LEMMA 2. $\lim_{t\to\infty}(p^t)^{n(n+1)/2-mn}\#B_{p^t}(N, M)$
$$= p^{n\,\mathrm{ord}_p dM}(p^T)^{n(n+1)/2-mn}\#C_{p^T}(N, M),$$

where T is a natural number such that $p^{T-1}\mathfrak{n}M^\# \subset 2Z_p$.

Proof. Since $\mathfrak{n}M \subset 2Z_p$ implies $M^\# \supset M$, there is a canonical mapping φ from $B_{p^t}(N, M)$ to $C_{p^t}(N, M)$. If $\varphi(u_1) = \varphi(u_2)$ for $u_1, u_2 \in B_{p^t}(N, M)$, then $(u_1 - u_2)(x) \in p^t M^\#$ for $x \in N$ and $p^{-t}(u_1 - u_2) \in \mathrm{Hom}(N, M^\#/M)$. Conversely for $v \in \mathrm{Hom}(N, M^\#/M)$, $u \in B_{p^t}(N, M)$ we put $\tilde{u} = u + p^t v$. Suppose $t \geq T$; then $p^{t-1}\mathfrak{n}M^\# \subset 2Z_p$ and it implies $p^{t-1}M^\# \subset M$, and then $\tilde{u} \in C_{p^t}(N, M)$. It is easy to see that φ is surjective since we may assume that $u \in C_{p^t}(N, M)$ is isometry for $t \geq T$ by virtue of Satz in § 14 in [5]. Thus we have

$$\#C_{p^t}(N, M) = \#B_{p^t}(N, M)/[M^\# : M]^n = \#B_{p^t}(N, M)p^{-n\,\mathrm{ord}_p dM}$$

By the same "Satz", $(p^t)^{n(n+1)/2-mn}\#C_{p^t}(N, M)$ is independent of t if $t \geq T$. Q.E.D.

We put

$$d_p(N, M) = 2^{-\delta_{n,m}}\lim_{t\to\infty}(p^t)^{n(n+1)/2-mn}\#B_{p^t}(N, M)$$
$$= 2^{-\delta_{n,m}}p^{n\,\mathrm{ord}_p dM}(p^T)^{n(n+1)/2-mn}\#C_{p^T}(N, M),$$

where T is a natural number such that $p^{T-1}\mathfrak{n}M^\# \subset 2Z_p$. The set of values $d_p(N, M)$ for any fixed lattice M is a finite set. If M is unimodular and $p \neq 2$, then we can take 1 as T and $\#C_p(N, M) =$ the number of isometries from N/pN to M/pM over $Z_p/(p)$.

Hilfssats 17 in [10] implies immediately the following

LEMMA 3. $\alpha_p(N, M) = 2^{n\delta_{2,p}}\sum_{Q_pN\supset N_0\supset N}[N_0 : N]^{n-m+1}d_p(N_0, M)$,

where M, N are regular quadratic lattices over Z_p and $\mathrm{rk}\, M = m$, $\mathrm{rk}\, N = n$.

2.3. Let N be a free lattice over Z_p with $\mathrm{rk}\, N = n$. Then the number $A(n, s)$ of the lattices containing N with index p^s is equal to $\sum p^{\sum_{i=2}^n (i-1)e_i}$, where the summation with respect to e_i is over all n-tuples $(e_1, \cdots, e_n)$ of non-negative integers which satisfy $\sum_{i=1}^n e_i = s$ and it is easy to see $p^{(n-1)s} \leq A(n, s) \leq (1 - p^{-1})^{1-n}p^{(n-1)s}$.

PROPOSITION 4. *Let M be a regular quadratic lattice over Z_p with $\mathfrak{n}M \subset 2Z_p$. Then there is a positive constant $\kappa(M)$ such that*

$$\alpha_p(N, M) < \kappa(M)$$

for any regular quadratic lattice N *over* Z_p *with* $\operatorname{rk} M > 2 \operatorname{rk} N$.

Proof. From Lemma 2 follows that

$$\sup_N 2^{\operatorname{rk} N \cdot \delta_{2,p}} d_p(N, M) = \kappa_1(M) < \infty ,$$

where N runs over regular quadratic lattices. Put $n = \operatorname{rk} N$, $m = \operatorname{rk} M$ and assume $m > 2n$; then from Lemma 3 follows

$$\begin{aligned}\alpha_p(N, M) &< \kappa_1(M)\Big(\sum_{s=0}^{\infty} A(n, s)p^{s(n-m+1)}\Big)\\ &< \kappa_1(M)\Big(1 + (1 - p^{-1})^{1-n} \sum_{s=1}^{\infty} p^{s(2n-m)}\Big)\\ &= \kappa_1(M)(1 + (1 - p^{-1})^{1-n}(1 - p^{2n-m})^{-1}p^{2n-m}) = \kappa(M) . \quad \text{Q.E.D.}\end{aligned}$$

Remark. Let $n < m \leq 2n$. There exist regular quadratic lattices N_t, M with $\operatorname{rk} N_t = n$, $\operatorname{rk} M = m$ such that

$$\alpha_p(N_t, M) \to \infty \qquad \text{as } t \to \infty .$$

PROPOSITION 5. *Let* M *be a regular quadratic lattice over* Z *with* $\mathfrak{n}M \subset 2Z$. *Then there is a positive constant* $\kappa(M)$ *such that*

$$\prod_p \alpha_p(Z_pN, Z_pM) < \kappa(M)$$

for any regular quadratic lattice N *over* Z *with* $\operatorname{rk} M \geq 2 \operatorname{rk} N + 3$.

Proof. Put $\operatorname{rk} M = m$ and $\operatorname{rk} N = n$ and let p be an odd prime such that Z_pM is unimodular. Then for each regular lattice K over Z_p with $\operatorname{rk} K = n$ we have

$$d_p(K, Z_pM) = p^{n(n+1)/2 - mn} \sharp C_p(K, Z_pM) .$$

On the other hand,

$$\begin{aligned}&1 + (1 - p^{-1})^{1-n}(1 - p^{2n-m})^{-1}p^{2n-m}\\ &\quad = (1 - p^{2n-m})^{-1}\{1 + p^{2n-m}((1 - p^{-1})^{1-n} - 1)\}\\ &\quad < (1 - p^{2n-m})^{-1}(1 + p^{2n-m}) ,\end{aligned}$$

if p is sufficiently large. Hence, by virtue of Lemma 1, the product of the constants $\kappa(Z_pM)$ in Prop. 4 over $\{p \neq 2 | Z_pM : \text{unimodular}\}$ converges if $m \geq 2n + 3$. By virtue of Prop. 4 we have $\prod_{p | 2dM} \alpha_p(Z_pN, Z_pM) < \prod_{p | 2dM} \kappa(Z_pM)$.

Remark. Let M be a regular quadratic lattice over Z. If $\operatorname{rk} M = 2n$

$+2$, $n \in Z$, then there exist regular lattices N_t over Z with $\mathrm{rk}\, N_t = n$ such that

$$\prod_p \alpha_p(Z_pN_t, Z_pM) \to \infty \qquad \text{as } t \to \infty .$$

From this follows that for a modular form $f(Z) = \sum a(T)\exp(2\pi i\sigma(TZ))$ of degree n, weight $k = n+1$, $a(T) = O(|T|^{k-(n+1)/2})$ does not hold in general (c.f. [4]).

2.4. LEMMA 6. *Let M be a maximal quadratic lattice over Z_p with $\mathrm{rk}\, M = m$. If N is a regular quadratic lattice over Z_p with $\mathrm{rk}\, N = n$, $\mathfrak{n}N \subset \mathfrak{n}M$, and $m \geq 2n+3$, then N is primitively represented by M.*

Proof. We may assume $\mathfrak{n}M = Z_p$ and let $M \cong \perp_k \left\langle 2^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle \perp M_0$, where M_0 is an anisotropic Z_p-maximal lattice. From $m = 2k + \mathrm{rk}\, M_0 \geq 2n+3$ and $\mathrm{rk}\, M_0 \leq 4$ follows that $2(n-k) \leq \mathrm{rk}\, M_0 - 3 \leq 1$ and then $n \leq k$. Any element in Z_p is primitively represented by $\left\langle 2^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle$. Hence we have only to show that $\left\langle 2^{a-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle$, $\left\langle 2^{a-1}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right\rangle$ are primitively represented by $\perp_2 \left\langle 2^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle$. Let $v_1, \cdots, v_4$ be a basis of $\perp_2 \left\langle 2^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle$ such that $Q(\sum x_i v_i) = x_1x_2 + x_3x_4$. Put $z_1 = v_1$, $z_2 = v_1 + 2^a v_2 - 2^a v_3 + v_4$, and $w_1 = v_1 + 2^a v_2$, $w_2 = 2^a v_2 + v_3 + 2^a v_4$, then $(B(z_i, z_j)) = 2^{a-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $(B(w_i, w_j)) = 2^{a-1}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Thus N is primitively represented by $\perp_n \left\langle 2^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\rangle$ and hence by M. Q.E.D.

LEMMA 7. *Let M, N be regular lattices over Z_p and assume that $\mathfrak{n}M \subset 2Z_p$ and N is represented by M. Let E be an orthogonal summand of N, that is, $N = E \perp N_1$ for some sublattice N_1 of N, and $E_1, \cdots, E_k$ sublattices of M which are representatives of sublattices of M isometric to E modulo the orthogonal group of M. Denote by K_i the orthogonal complement of E_i in M. Then we have*

$$\alpha_p(N, M) \geqq \sum_{i=1}^{k} \alpha_p(E, M; E_i)\alpha_p(N_1, K_i) ,$$

where $\alpha_p(E, M; E_i) = \lim\limits_{t\to\infty} (p^t)^{e(e+1)/2 - em} \sharp \left\{ u : E \to M/p^tM \;\; \begin{array}{l} u : E \to M: \textit{isometry} \\ vuE = E_i \textit{ for some} \\ v \in O(M) \end{array} \right\}$

and here $e = \mathrm{rk}\, E$, $m = \mathrm{rk}\, M$.

Proof. The existence of $\alpha_p(E, M; E_i)$ is proved as usual, noting that regular sublattices $L_i = Z_p[v_{i,1}, \cdots, v_{i,n}]$ of a regular quadratic lattice L over Z_p are transformed by $O(L)$ if $(B(v_{1,h}, v_{1,j})) = (B(v_{2,h}, v_{2,j}))$ and $v_{1,h}$, $v_{2,h}$ are sufficiently close for every h. Suppose that t is sufficiently large. If u_1, u_1' are isometries from E to M such that $vu_1E = v'u_1'E = E_i$ for some $v, v' \in O(M)$ and u_2, u_2' are isometries from N_1 to K_i, then $u = u_1 \perp v^{-1}u_2$, $u' = u_1' \perp v'^{-1}u_2'$ are isometries from N to M. Suppose that $u \equiv u' \bmod p^tM$ and u_1, u_1' (resp. u_2, u_2') are representatives of $A_{p^t}(E, M)$ (resp. $A_{p^t}(N_1, K_i)$). Then we have $u_1 \equiv u_1' \bmod p^tM$ and so $u_1 = u_1'$, $v = v'$. Hence $u_2 \equiv u_2' \bmod p^tM$. Since K_i is a direct summand of M, we have $u_2 \equiv u_2' \bmod p^tK_i$ and so $u_2 = u_2'$. Hence we complete the proof. Q.E.D.

PROPOSITION 8. *Let M, N be regular quadratic lattices over Z_p with $\mathfrak{n}M \subset 2Z_p$ and assume that N is represented by M. Then there exists a positive constant $\kappa(M)$ such that*

$$\alpha_p(N, M) > \kappa(M) \qquad \textit{if } \operatorname{rk} M \geq 2 \operatorname{rk} N + 3 .$$

Proof. Let M_0 be a maximal lattice in M with $\mathfrak{n}M_0 = (p^a)$. Suppose $\mathfrak{n}N \subset \mathfrak{n}M_0$. Then N is primitively represented by M_0 by virtue of Lemma 6. Hence we have $\alpha_p(N, M_0) \geq d_p(N, M_0) \geq \kappa(M_0) > 0$ by Lemmas 2, 3. Denote by φ the canonical mapping from $M_0/p^tM_0 \to M/p^tM$. Since $\varphi u_1 = \varphi u_2$ for $u_i \in A_{p^t}(N, M_0)$ implies $(u_1 - u_2)(x) \in p^tM$ for $x \in N$, we get

$$\sharp A_{p^t}(N, M_0) \leq \sharp A_{p^t}(N, M) \sharp\{u \colon N \to p^tM/p^tM_0\} .$$

Thus we have $\alpha_p(N, M_0) \leq \alpha_p(N, M)[M : M_0]^n$ ($n = \operatorname{rk} N$), and then $\alpha_p(N, M) \geq \alpha_p(N, M_0)\ [M : M_0]^{-n} \geq \kappa(M_0)\ [M : M_0]^{-n}$. Now we come back to the general case and assume that M has the minimal rank so that the proposition is false. Suppose that N_i is represented by M and $\operatorname{rk} N_i = \operatorname{rk} N$ and $\alpha_p(N_i, M) \to 0$ as $i \to \infty$. By the former part we may assume $\mathfrak{n}N_i \not\subset \mathfrak{n}M_0$. Let $N_i = N_1^{(i)} \perp \cdots \perp N_{t_i}^{(i)}$ be the Jordan splitting such that $N_j^{(i)}$ is $p^{a_j^{(i)}}$-modular and $0 \leq a_1^{(i)} \leq \cdots \leq a_{t_i}^{(i)}$. $\mathfrak{n}N_i \not\subset \mathfrak{n}M_0$ implies $a_1^{(i)} < a$. Since the number of p^c-modular lattices K such that $0 \leq c < a$ and $\operatorname{rk} K \leq \operatorname{rk} N$ is finite up to isometry, we may assume that $N_1^{(i)} \cong L$ for every i and $\operatorname{rk} L < \operatorname{rk} N$, taking a subsequence. Applying Lemma 7, there exist sublattices L_i, K_i of M with $\operatorname{rk} K_i = \operatorname{rk} M - \operatorname{rk} L$ such that

$$\alpha_p(N_i, M) \geqq \sum_h \alpha_p(L, M; L_h)\alpha_p(N_i', K_h) ,$$

where N_i' is the orthogonal complement of $N_1^{(i)}$ in N_i. Since $\operatorname{rk} K_h - (2\operatorname{rk} N_i' + 3) \geq 0$ and $\alpha_p(N_i', K_h) > \kappa(K_h)(> 0)$ if N_i' is represented by K_h, we have a contradiction. Q.E.D.

Remark. If $n < m \leq 2n + 2$, then there exist regular quadratic lattices M, N_i over Z_p with $\mathrm{rk}\, M = m$, $\mathrm{rk}\, N_i = n$ such that

$$0 < \alpha_p(N_i, M) \to 0 \qquad \text{as } i \to \infty .$$

PROPOSITION 9. *Let M be a regular quadratic lattice over Z with $\mathfrak{n}M \subset 2Z$. Then there exist a positive constant $\kappa(M)$ such that*

$$\prod_p \alpha_p(Z_pN, Z_pM) > \kappa(M)$$

for any regular quadratic lattice N over Z if $\mathrm{rk}\, M \geq 2\mathrm{rk}\, N + 3$ and Z_pN is represented by Z_pM for every prime p.

Proof. Let p be an odd prime such that Z_pM is unimodular. Then from $\alpha_p(Z_pN, Z_pM) \geq d_p(Z_pN, Z_pM) = p^{n(n+1)/2-mn} \sharp C_p(N, M)$ ($n = \mathrm{rk}\, N$, $m = \mathrm{rk}\, M$) and Lemma 1 follows that there is a positive constant κ_1 such that

$$\prod_p \alpha_p(Z_pN, Z_pM) > \kappa_1 ,$$

where p runs over the set $\{p \neq 2 | Z_pM\text{: unimodular}\}$. Prop. 8 completes the proof. Q.E.D.

Remark. Let $m > n$ be natural numbers. Let M be a regular quadratic lattice over Z with $\mathrm{rk}\, M = m$ and denote by P the set of primes p such that $p \neq 2$ and Z_pM is unimodular. Then there exists a positive constant $\kappa(M)$ such that

$$\prod_{p \in P} \alpha_p(Z_pN, Z_pM) > \kappa(M) \prod_{p \in P(N)} (1 + \varepsilon_p p^{-1}) ,$$

if N is a regular quadratic lattice over Z with $\mathrm{rk}\, N = n$ such that

$$Z_pN \text{ is primitively represented by } Z_pM \text{ for each } p \in P .$$

Here ε_p, $P(N)$ are defined as follows:

For $p \in P$, Z_pN/pZ_pN becomes a quadratic space over $Z/(p)$. Decompose Z_pN/pZ_pN as $Z_pN/pZ_pN = (Z_pN/pZ_pN)_0 \perp \mathrm{rad}\, Z_pN/pZ_pN$ and put $t_p = \dim (Z_pN/pZ_pN)_0$. Then, by definition, $p \in P(N)$ iff $m - 2n + t_p = 2$

$$\varepsilon_p = \left(\frac{(-1)^{m-n-1} d(Z_pN/pZ_pN)_0 dM}{p}\right).$$

Since $t_p = n$ if Z_pN is unimodular, $P(N)$ is a finite set if $m > n + 2$.

Assume $m = 2n + 2$. If $\mathfrak{n}Z_pN \subset Z_p$, then Z_pN is primitively represented by Z_pM for $p \in P$ as in the proof of Lemma 6 since the Witt index of $Q_pM \geq n$. Since $p \in P(N)$ implies $t_p = 0$ and $\mathfrak{n}Z_pN \subset pZ_p$, we have $\prod_{p \in P(N)}(1 + \varepsilon_p p^{-1}) \geq \prod_{p \mid \mathfrak{n}N}(1 - p^{-1}) \gg (\mathfrak{n}N)^{-\varepsilon}$ for any $\varepsilon > 0$ if $\mathfrak{n}N \subset Z$.

3. THEOREM. *Let M be a positive definite quadratic lattice over Z with $\mathfrak{n}M \subset 2Z$ with $\operatorname{rk} M = m \geq 7$. Let N be a positive definite quadratic lattice with $\operatorname{rk} N = 2$ and suppose that Z_pN is represented by Z_pM for every prime p. Then we have:*

The number of isometries from N to M

$$= \frac{\pi^{m-1/2}}{\Gamma(m/2)\Gamma((m-1)/2)} \cdot \frac{(dN)^{(m-3)/2}}{dM} \cdot \prod_p \alpha_p(Z_pN, Z_pM)$$
$$+ O(m(N)^{(3-m/2)/2} \cdot dN^{(m-3)/2}) \qquad \text{for } m \geq 7,$$

when $m(N) = \min_{0 \neq x \in N} Q(x)$ is sufficiently large.

Proof. Let M_i be representatives of classes in $\operatorname{gen} M$ and S_i the corresponding matrix to M_i. Put $\theta_i(Z) = \theta_{S_i}^{(2)}(Z, 0, 0)$ (in **1.2.**). Then the constant term of $\theta_i(Z)$-$\theta_1(Z)$ vanishes at every cusp. Put $E(Z) = M(S_1)^{-1} \sum |O(S_i)|^{-1}\theta_i(Z)$, where $M(S_1)^{-1} = \sum |O(S_i)|^{-1}$. Then the constant term of $\theta_1(Z)$-$E(Z)$ vanishes at every cusp. From the Siegel formula and Theorem in **1.5.** follows our theorem. Q.E.D.

Remark. The formula in Theorem gives an asymptotic one when $m(N)$ tends to infinity by virtue of Prop. 9.

4. We discuss here questions about local densities and representations of quadratic forms.

The most fundamental one is

(a) to evaluate the density $\alpha_p(N, M)$.

(b) Let M be a regular quadratic lattice over Z_p.

When does the set of accumulation points of $\{\alpha_p(N, M) \mid N$: regular sublattice of M with a fixed rank$\}$ contain 0 and/or ∞?

(c) We proved in [3];

Let M, N be positive definite quadratic lattices over Z, and assume that Z_pN is represented by Z_pM for every p. Then N is represented by M if $m(N)$ is sufficiently large and $\operatorname{rk} M \geq 2\operatorname{rk} N + 3$. Our results here seem to suggest that $\operatorname{rk} M \geq 2\operatorname{rk} N + 3$ is the best possible condition. The counter-example may be found in the sequence $\{N_t\}$ such that $\prod \alpha_p(Z_pN_t,$

$Z_pM) \to 0$. We can only give the following example in case of $\operatorname{rk} M = \operatorname{rk} N + 3$.

Let $p_1 < \cdots < p_{n+1}$ be primes $\equiv 1 \bmod 24$ and $M = \langle p_1 \rangle \perp \cdots \perp \langle p_{n+1} \rangle \perp \langle 3 \rangle \perp \langle 3 \rangle$ and $N_t = \langle p_1^t \rangle \perp \cdots \perp \langle p_{n-1}^t \rangle \perp \langle 3^{2t} \rangle$ be positive definite quadratic lattices over Z. Then Z_pN_t is represented by Z_pM for every prime p and $m(N_t) = 3^{2t} \to \infty$, but N_t is not represented by M over Z.

(*Proof.* It is easy to see that N_t is represented by M over Z_p. Suppose that there is an isometry u from N_t to M. Since $\langle p_1^t \rangle \perp \cdots \perp \langle p_{n-1}^t \rangle \cong \perp_{n-1} \langle 1 \rangle$ over Z_3, and any sublattice of Z_3M which is isometric to $\perp_{n-1}\langle 1 \rangle$ is mapped to $Z_3(\langle p_1 \rangle \perp \cdots \perp \langle p_{n-1} \rangle)$ by an isometry of Z_3M, the orthogonal complement of $u(\langle p_1^t \rangle \perp \cdots \perp \langle p_{n-1}^t \rangle)$ in M is isometric to $\langle 1 \rangle \perp \langle 1 \rangle \perp \langle 3 \rangle \perp \langle 3 \rangle$ over Z_3. Hence we have $u\langle 3^{2t} \rangle = Z \cdot 3^t x$ for $x \in M$, and then $Q(x) = 1$. This is a contradiction.)

(d) Let m, n be natural numbers with $m \geq n + 2$, M a positive definite quadratic lattice over Z with $\operatorname{rk} M = m$, $\mathfrak{n}M \subset 2Z$ and N_p^0 a regular quadratic sublattice of Z_pM with $\operatorname{rk} N_p^0 = n$ for $p | 2dM$. If a positive definite quadratic lattice N over Z with $\operatorname{rk} N = n$ satisfies the following conditions 1)$\sim$5), then is N represented by M?

1) $Z_pN \cong N_p^0$ for $p | 2dM$,

2) Z_pN is represented by Z_pM for every prime p,

3) the corresponding matrix to N is sufficiently large in an appropriate sense,

4) $\prod \alpha_p(Z_pN, Z_pM) > \kappa$ for any fixed positive constant κ,

5) N is not a spinor exceptional lattice for M in case of $m = n + 2$.

Analytically it is (almost in case of $m = n + 2$) sufficient to show the following

(d′) Let $f(Z) = \sum a(T)\exp(2\pi i\sigma(TZ))$ be a modular form of degree n and weight $k(\in \frac{1}{2}Z)$, and assume that $k \geq n/2 + 1$, and the constant term of $f(Z)$ vanishes at every cusp. Then does $a(T)|T|^{(n+1)/2-k} = o(1)$ hold for $T > 0$? In case of $k = n/2 + 1$ we restrict T by the condition that $|2T|$ is not numbers of form ab^2 where a, b are integers and a divides $2\times$(the level of $f(Z)$).

When k is sufficiently large and even, it is known ([4] and a letter from S. Raghavan) that $a(T)|T|^{(n+1)/2-k} = O(m(T)^{-\varepsilon})$ $(\varepsilon > 0)$.

The condition 4) may be weakened:
Suppose $n = 1$, and consider the following condition 1′) weaker than 1)

1′) $Z_pN \cong N_p^0$ for p such that Z_pM is anisotropic.

Then for $m \geq 3$ 1′), 2) imply

4′) $\prod_p \alpha_p(Z_pN, Z_pM) > \kappa(M, \varepsilon)(dN)^{-\varepsilon}$ for a positive constant $\kappa(M, \varepsilon)$ and any small number $\varepsilon > 0$.

When $m = 4$, the condition 1′), 2), 3) imply the representation of N by M since for each cusp form $f(z) = \sum c_k \exp(2\pi ikz)$ we know $c_k = O(k^{1/2+\varepsilon})$, $\varepsilon > 0$. When $m = 3$, via an arithmetic approach of Linnik, Malyshev, Peters [8] it is shown under generalized Riemann hypotheses that 1′), 2), 3), 5′) imply the representation of N by M. Here the condition 5′), which is stronger than 5), is as follows:

5′) $dN \neq ab^2$ $(a, b \in \boldsymbol{Z}, a|2dM)$.

Let A be a matrix corresponding to a positive definite ternary lattice M. Put $\theta(z) = \theta_{2A}^{(1)}(z, 0, 0) = \sum a(n)\exp(2\pi inz)$ and decompose it as $a(n) = b(n) + c(n)$ where $b(n)$ (resp. $c(n)$) is a Fourier coefficient of Eisenstein series (resp. a cusp form) as usual. It is known, by the Siegel formula, that $b(n) \geq \kappa(M)h(-4n|A|)$ where $\kappa(M)$ is a positive constant and $h(-4n|A|)$ is the class number of primitive positive definite binary quadratic forms with discriminant $-4n|A|$ if we assume 1′), 2). If, hence, $c(n) = O(n^{1/2-\varepsilon})$ $(\varepsilon > 0)$ for $n \neq ab^2$ $(a, b \in \boldsymbol{Z}, a|$the level of $A)$, then the conditions 1′), 2), 3), 5′) imply the representation of N by M. Recent developments of the theory of modular forms of weight 3/2 show: for a fixed square-free t such that $t \nmid$ the level of A, $c(tn^2) = O(n^{1/2+\varepsilon})$ $(\varepsilon > 0)$ holds. Hence $a(tn^2) = b(tn^2) + c(tn^2)$ gives an asymptotic formula as $n \to \infty$ if we assume the conditions 1′), 2), 3).

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