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A NOTE ON UNIVERSALLY CATENARY RINGS

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Introduction

The following two related problems in Commutative Algebra are certainly very popular:

1. Study the *permanence* of properties from a ring A to an idealadic completion of A; and
2. Study the *lifting* of properties from A/I to A where A is a ring which is complete and separated in the I-adic topology.

In recent years much effort has been devoted to the study of the permanence and lifting of many properties related to the theory of excellent rings, with special reference to the formal fibers and the openness of loci (see [6] for informations and bibliography).

As for the property UC (universally catenary) the permanence has been proved long since by Seydi (see [11], 1.2), while the only result we know related to the lifting is in [12], 1.1.2, (see 1.7(ii) below for the statement).

The present note contains some contributions to the problem of lifting for the UC property. The main fact is a counterexample (see 1.1), obtained by employing a technique used by Nagata to construct a non UC ring (see either [7], p. 203, ex. 2, or [6], p. 87, (14.E)).

This answers a question posed to the author by H. Matsumura, and shows that the main result of Rotthaus [10] on the lifting of "quasi-excellent" for semilocal rings does not hold for "excellent".

The paper ends with a positive result (see 2.3), namely: the lifting of UC holds if A is assumed to be catenary. This is proved by elementary techniques as a consequence of a theorem of Ratliff [9].

CONVENTIONS. All rings are assumed to be commutative and noetherian.

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We shall use the terminology of Matsumura [6], and most of the standard results contained in this book will be used without a direct reference.

1. A counterexample to the lifting of the property UC

In this section we shall prove the following

PROPOSITION 1.1. *There are a ring A and an ideal I of A such that*:
(i) *A is a semilocal domain of dimension* 3;
(ii) *A is I-complete and separated*;
(iii) *A/I is excellent*;
(iv) *A is not catenary.*
(note that A is QE (quasi-excellent) by [10])

Our ring A will be obtained from a certain ring B, by "gluing" two maximal ideals. We begin by constructing B and by giving some of its properties (Lemma 1.2); the required properties for A will be proved in 1.4, 1.5, 1.6.

Let $R = k[X, Y, Z]$ be the polynomial ring in three variables over the field k, and let C be the semilocalization of R at the prime ideals $M_1 = (X, Y, Z)$, $M_2 = (X, 1 - YZ)$, $N = (X + 1, Y, Z)$. Put $P = (Y, Z)C$, $Q = (X + 1 - XZ)C$, $J = P \cap Q$, and let B be the J-completion of C. Put $\mathfrak{M}_i = M_iB$, $\mathfrak{N} = NB$, $\mathfrak{P} = PB$, $\mathfrak{Q} = QB$ and $I = JB$.

LEMMA 1.2. (i) *B is a semilocal regular domain with maximal ideals $\mathfrak{M}_1$, $\mathfrak{M}_2$, $\mathfrak{N}$.*
(ii) $B/\mathfrak{M}_1 = k$; $B/\mathfrak{M}_2 = k(Y)$.
(iii) $\mathrm{ht}(\mathfrak{M}_1) = \mathrm{ht}(\mathfrak{N}) = 3$; $\mathrm{ht}(\mathfrak{M}_2) = \mathrm{ht}(\mathfrak{P}) = 2$; $\mathrm{ht}(\mathfrak{Q}) = 1$.
(iv) $\dim(B_{\mathfrak{M}_i}/IB_{\mathfrak{M}_i}) = 1$ *for* $i = 1, 2$; $\dim(B/\mathfrak{Q}) = 2$.

Proof. Since $\mathfrak{N} \in V(\mathfrak{P}) \cap V(\mathfrak{Q})$ we have that $\mathrm{spec}\,(B/I) = V(I)$ is connected. It follows that B is a regular domain (see e.g. [2], 10.11). This proves (i). The proof of (ii) and (iii) is easy; and (iv) follows immediately from the fact that $P \subset M_1 \cap N$, $P \not\subset M_2$ and $Q \subset M_2 \cap N$, $Q \not\subset M_1$.

Now we can construct A. For this let k_0 be any field and let Y_1, Y_2, $\cdots$ be infinitely many indeterminates over k_0. Put $k = k_0(Y_1, Y_2, \cdots)$, and let B be the k-algebra constructed above.

As $k \simeq k(Y)$ we have by 1.2 (ii) that there is an isomorphism ϕ: $B/\mathfrak{M}_1 \to B/\mathfrak{M}_2$. Let A be the ring obtained from B by gluing $\mathfrak{M}_1$ and $\mathfrak{M}_2$ via ϕ, i.e. $A = \{b \in B \mid \pi_2(b) = (\phi \circ \pi_1)(b)\}$, where $\pi_i : B \to B/\mathfrak{M}_i$ are the canonical maps.

Put $\mathfrak{m} = \mathfrak{M}_1 \cap A = \mathfrak{M}_2 \cap A$, $\mathfrak{n} = \mathfrak{N} \cap A$, $\mathfrak{p} = \mathfrak{P} \cap A$, $\mathfrak{q} = \mathfrak{Q} \cap A$, and note that $I = \mathfrak{p} \cap \mathfrak{q} \subset A$.

By general properties of gluings (see e.g. [8], Teorema 1), we have:

LEMMA 1.3. (i) *B is finite over A.*
(ii) $\mathrm{Ann}_A(B/A) = \mathfrak{m} = \mathfrak{M}_1 \cap \mathfrak{M}_2$.

LEMMA 1.4. (i) *A is a semilocal domain with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$.*
(ii) $\mathrm{ht}(\mathfrak{m}) = \mathrm{ht}(\mathfrak{n}) = 3$; $\mathrm{ht}(\mathfrak{p}) = 2$
(iii) $\dim(A_\mathfrak{m}/IA_\mathfrak{m}) = 1$.
(iv) $\mathrm{ht}(\mathfrak{q}) = 1$.
(v) *A is not catenary.*

Proof. (i) and (ii) are easy consequences of 1.2, 1.3 and of the going up.

(iii) $B_\mathfrak{m}$ is the semilocalization of B at $\mathfrak{M}_1$, $\mathfrak{M}_2$ (by 1.3), and hence $B_\mathfrak{m}/IB_\mathfrak{m}$ is a finite overring of $A_\mathfrak{m}/IA_\mathfrak{m}$. The conclusion follows by 2.2 (iv) and the going up.

(iv) As $B/\mathfrak{Q}$ is a finite overdomain of $A/\mathfrak{q}$ we have $\dim(A/\mathfrak{q}) = \dim(B/\mathfrak{Q}) = 2$ by 1.2 (iv),; and since $\dim(A) = 3$ we must have $\mathrm{ht}(\mathfrak{q}) = 1$.

(v) By (iii) and (iv) we have $\dim(A_\mathfrak{m}) = 3$, $\dim(A_\mathfrak{m}/\mathfrak{q}A_\mathfrak{m}) = 1$ and $\mathrm{ht}(\mathfrak{q}A_\mathfrak{m}) = 1$. Since A is a domain $A_\mathfrak{m}$ (and a fortiori A) is not catenary.

LEMMA 1.5. *A/I is excellent (in particular UC)*

Proof. Since $B/I = C/J$ is excellent and is finite over A/I, the latter is QE by [1], 1.3. Thus it is sufficient to prove that A/I is UC. The maximal ideals of A/I are $\mathfrak{m}' = \mathfrak{m}/I$ and $\mathfrak{n}' = \mathfrak{n}/I$. Moreover $\dim(A/I)_{\mathfrak{m}'} = 1$, and since I is a radical ideal we have that $(A/I)_{\mathfrak{m}'}$, is Cohen-Macaulay and hence UC ([6], p. 111, Theorem 33). Moreover by 1.3. (ii) we have $A_\mathfrak{n} = B_\mathfrak{N}$ and hence $(A/I)_{\mathfrak{n}'}$ is UC being a quotient of a regular ring.

To conclude the proof of 1.1 we have only to show that A is complete and separated in the I-adic topology. This is an immediate consequence of 1.3. (i) and of the following well known Lemma which we include by lack of a ready reference.

LEMMA 1.6. *Let $f: R \to S$ be a finite injective ring homomorphism. Let b be an ideal of S and put $\mathfrak{a} = f^{-1}(\mathfrak{b})$. Then if S is $\mathfrak{b}$-separated and complete we have that R is $\mathfrak{a}$-separated and complete.*

Proof. By]13], p. 275, Theorem 14 we may assume $\mathfrak{a}S = \mathfrak{b}$. Thus if

* denotes $\mathfrak{a}$-completion it is easy to see that $R \subset R^* \subset S$, and the conclusion follows by Nakayama.

Remarks 1.7. (i) The key idea of our counterexample (due to Nagata) is that by gluing two prime ideals of different height of a domain one gets a non UC ring. This explains our construction of A and makes one to believe that it is not so easy to get, by the same technique, a better counterexample (e.g. with A local and/or I prime).

(ii) If the characteristic of k_0 is $p>0$ we have $[k : k^p] = \infty$ whence $[k(\mathfrak{m}) : k(\mathfrak{m})^p] = \infty$. This fact is necessary in order to have a counterexample as in 1.1 in positive characteristic. Indeed there is the following result of Seydi ([12], 1.1.2): Let A be a ring containing a field of characteristic $p>0$ and let I be an ideal of A. Assume: (i) A is I-complete and separated, (ii) A/I is a Nagata ring, and (iii) $[k(\mathfrak{m}) : k(\mathfrak{m})^p] < \infty$ for all maximal ideals $\mathfrak{m}$ of A. Then A is excellent.

We do not know if in characteristic zero one can have a counterexample with "better" residue fields (e.g. algebraically closed).

(iii) If $\dim(A) = 2$ the lifting of UC holds (see 2.3 below), hence in 1.1 we have the least possible dimension.

(iv) Proposition 1.1 implies that the main result of Rotthaus [10] on the lifting of the QE property for semilocal rings is false if one replaces QE by "excellent".

(v) In 1.4 we have shown that a one-dimensional reduced ring is UC. A more general (and more difficult) result is given in [9], 2.6: a ring A is UC if and only if the polynomial ring in one variable $A[X]$ is catenary.

§2. A sufficient condition for the lifting of the UC property

In this section we show that the lifting of the property UC holds if the ring A is assumed to be catenary (Proposition 2.3). We begin by recalling the following Theorem.

THEOREM 2.1. *For a local ring R the following conditions are equivalent*:

(i) *R is CU*;

(ii) *for every $\mathfrak{p} \in \operatorname{Spec}(R)$ the completion of $R/\mathfrak{p}$ is equidimensional*;

(iii) *for every minimal prime ideal $\mathfrak{p}$ of R the completion of $R/\mathfrak{p}$ is equidimensional.*

Proof. By Ratliff [9], 2.6 we have that (i) implies (ii); and by EGA ([3], 7.1.11) we have that (iii) implies (i).

If $\mathfrak{P}$ is a prime ideal in a ring R we call "dimension of $\mathfrak{P}$" the dimension of the ring $R/\mathfrak{P}$.

LEMMA 2.2. *Let* $(B, \mathfrak{n})$ *be a local ring of dimension* $d>0$, *and assume that* $\mathrm{ht}(\mathfrak{P}) = d - 1$ *for all prime ideals* $\mathfrak{P}$ *of* B *having dimension* 1. *Then* B *is equidimensional.*

Proof. Let $\mathfrak{p}$ be a minimal prime of A. Then there is a one-dimensional prime $\mathfrak{P}$ which contains $\mathfrak{p}$ and no other minimal prime (see e.g. [5], Theorems 26 and 146; or [4], 10.5.9). The conclusion follows easily.

PROPOSITION 2.3. *Let* A *be a ring and let* I *be an ideal of* A. *Assume that*:

(i) *A is I-complete and separated*;

(ii) *A is catenary* (e.g. $\dim(A) = 2$);

(iii) *A/I is UC.*

Then A is UC.

Proof. We may assume that A be a domain, and to prove our claim it is sufficient to apply 2.1 to the localizations of A at its maximal ideals. Let then $\mathfrak{m}$ be a maximal ideal of A and let B be the completion of $A_{\mathfrak{m}}$ (as a local ring); we have to show that B is equidimensional, and for this we use 2.2. So let $\mathfrak{P}$ be a prime ideal of B having dimension 1: we have to show that $\mathrm{ht}(\mathfrak{P}) \geq d - 1$, where $d = \dim(B) = \mathrm{ht}(\mathfrak{m})$. Let $\mathfrak{p}$ be the contraction of $\mathfrak{P}$ to A; we have two cases:

Case 1: $\mathfrak{p} \not\supset I$. Then $IB + \mathfrak{P}$ is an ideal of definition of B, and hence it is easy to see that the canonical map

$$(A/\mathfrak{p})/I(A/\mathfrak{p}) \longrightarrow (B/\mathfrak{P})/I(B/\mathfrak{P})$$

is surjective, in particular finite. It follows then by [6], p. 212, Lemma (or by [13], p. 259, Corollary 2) that the injective homomorphism $A/\mathfrak{p} \to B/\mathfrak{P}$ is finite (actually an isomorphism), whence $\dim(A/\mathfrak{p}) = \dim(B/\mathfrak{P}) = 1$. Since A is a catenary domain this implies $d = \mathrm{ht}(\mathfrak{m}) = 1 + \mathrm{ht}(\mathfrak{p})$, whence by flatness $\mathrm{ht}(\mathfrak{P}) \geq \mathrm{ht}(\mathfrak{p}) = d - 1$.

Case 2: $I \subset \mathfrak{p}$. Let $\mathfrak{Q} \subset \mathfrak{P}$ be a minimal prime of $\mathfrak{p}B$. By flatness we have $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{Q})$, and by 2.1 we have $\dim(B/\mathfrak{Q}) = \dim(A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}})$. Moreover both $A_{\mathfrak{m}}$ and B are catenary and hence we have:

$$\begin{aligned}\mathrm{ht}(\mathfrak{P}) &\geq \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{P}/\mathfrak{Q}) = \mathrm{ht}(\mathfrak{p}) + \dim(B/\mathfrak{Q}) - \dim(B/\mathfrak{P}) \\ &= \mathrm{ht}(\mathfrak{p}) + \dim(A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}}) - 1 = d - 1.\end{aligned}$$

The conclusion follows then by 2.2 and 2.1.

Remark 2.4. By 1.1 the assumption "A catenary" in 2.3 cannot be dropped. However it should be possible to weaken it by making suitable assumption on the minimal primes of I. It is also possible that this assumption on A could be replaced by assumptions on the residue fields of A at its maximal ideals (compare with 1.7. (ii)).

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