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THE CENTRAL LIMIT THEOREM FOR TRIGONOMETRIC SERIES

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§1. Introduction

We denote by $\boldsymbol{Z}^+$ the semi-group of positive integers. For a subset E of $\boldsymbol{Z}^+$, we denote by $|E|$ ($\leqq +\infty$) its cardinal number and by $E(n)$ the intersection of E and an interval $[1, n)$ ($n \geqq 1$). We shall identify a subset E of $\boldsymbol{Z}^+$ with a sequence, arranging elements of E according to their order. An infinite subset E of $\boldsymbol{Z}^+$ is of Gauss type (G-type), if $X_n(t; E, \mathbf{0}) = \sqrt{2/|E(n)|}\sum_{m \in E(n)} \cos mt$ converges to the normal Gaussian distribution when n tends to infinity, that is, for every two real numbers ξ, η ($\xi < \eta$),

$$(1)\qquad \lim_{n\to\infty} m(\{t \in [0, 2\pi); \xi < X_n(t; E, \mathbf{0}) < \eta\}) = \sqrt{2\pi}\int_{\xi}^{\eta} \exp(-x^2/2)dx\,,$$

where m denotes the 1-dimensional Lebesgue measure. A subset E of $\boldsymbol{Z}^+$ is Hadamard lacunary, if there exists a number $q > 1$ such that, for every $n, m \in E$ ($n > m$), $n/m \geqq q$. It is well-known that Hadamard lacunary sets are of G-type ([5] p. 264). This result was improved by P. Erdös as follows ([2]): Let $E = (n_k)_{k=1}^{\infty}$ be a sequence in $\boldsymbol{Z}^+$ such that $n_{k+1}/n_k \geq 1 + c_k/\sqrt{k}$ ($c_k \to +\infty, k \geq 1$). Then E is of G-type. On the other hand, for every $c > 0$, there exists a non-G-type sequence $E = (n_k)_{k=1}^{\infty}$ such that $n_{k+1}/n_k \geq 1 + c/\sqrt{k}$ ($k \geq 1$).

From this fact, a class $\mathscr{P}_\alpha$ was introduced, where $\mathscr{P}_\alpha$ ($0 \leq \alpha < 1$) is the totality of all sequences $E = (n_k)_{k=1}^{\infty}$ in $\boldsymbol{Z}^+$ such that

$$(2)\qquad n_{k+1}/n_k \geq 1 + c/k^\alpha \quad (k \geq 1, \text{ for some } c > 0).$$

A study of G-type sets in $\mathscr{P}_\alpha$ ($1/2 \leq \alpha < 1$) was made by I. Berkes in [1]. The purpose of this paper is to improve his results, according to the direction of S. Takahashi (Theorem 1 in [4]). Let q be an arbitrarily fixed number such that $q > 1$. For a subset E of $\boldsymbol{Z}^+$, we write $E^n = E \cap [1, q^n)$

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and $\partial E^n = E \cap [q^{n-1}, q^n)$ $(n \geq 1)$. We denote by $\mathscr{R}_\alpha$ $(0 \leq \alpha \leq 1)$ the totality of all infinite subsets E of $\boldsymbol{Z}^+$ such that

$$(3) \qquad |\partial E^n|/|E^n|^\alpha = O(1)\,.$$

We easily see that the definition of $\mathscr{R}_\alpha$ is independent of q. It is very important to note that $\mathscr{R}_\alpha$ is a class determined by a condition on the growth, different from the lacunary condition (2). Evidently $\mathscr{P}_\alpha \subset \mathscr{R}_\alpha$ and $\mathscr{R}_1$ equals the totality of all infinite subsets of $\boldsymbol{Z}^+$. We shall study G-type sets in $\mathscr{R} = \bigcup_{0 \leq \alpha < 1} \mathscr{R}_\alpha$. Different from $\mathscr{P}_\alpha$, all subset in $\mathscr{R}_\alpha$ $(0 \leq \alpha < 1/2)$ is not of G-type. A study of G-type sets in $\mathscr{R}_\alpha$ $(0 \leq \alpha < 1/2)$ was made by S. Takahashi in [4] and our results on $\mathscr{R}_\alpha$ $(0 \leq \alpha < 1/2)$ will be essentially the same as his results on G-type sets (Theorem 1 in [4]). Our theorem will give new results on $\mathscr{R}_\alpha$ $(1/2 \leq \alpha < 1)$.

To study G-type sets, we need a positive set-function $\sigma(\cdot)$, which is defined by: For every finite subset E of $\boldsymbol{Z}^+$,

$$(4) \quad \sigma(E) = \mathrm{Car}\Big(\Big\{(n_1, n_2, n_3, n_4);\ \sum_{j=1}^{4} n_j = 0,\ n_j \neq \pm n_{j'},\ (j \neq j'),\ n_j \in E \cup (-E)\Big\}\Big),$$

where $\mathrm{Car}(A)$ denotes the cardinal (number) of A. We shall show the following

THEOREM. *Let E be a set in $\mathscr{R}$ such that*

$$(5) \qquad \lim_{n\to\infty} \sigma(E(n))/|E(n)|^2 = 0\,.$$

Let U be a Borel set in $[0, 2\pi)$ having positive measure and $\Gamma = (\gamma_m)_{m=1}^{\infty}$ a sequence of numbers in $[0, 2\pi)$. Then, for every real numbers ξ, η $(\xi < \eta)$,

$$(6) \qquad \lim_{n\to\infty} m(\{t \in U;\ \xi < X_n(t; E) < \eta\}) = m(U)/\sqrt{2\pi}\int_{\xi}^{\eta} \exp(-x^2/2)\,dx\,,$$

where $X_n(t; E) = X_n(t; E, \Gamma) = \sqrt{2/|E(n)|}\sum_{m \in E(n)} \cos(mt + \gamma_m)$.

By this theorem, we know that a subset E in $\mathscr{R}$ is of G-type if (5) holds. In fact, choose $U = [0, 2\pi)$ and $\Gamma = \boldsymbol{0}$ (, i.e., $\gamma_m = 0$ for all m). It is very important to note the following fact:

$$(7) \quad \sigma(E(n))/|E(n)|^2 = o(1) \text{ if and only if } 1/2\pi \int_0^{2\pi} X_n(t; E, \boldsymbol{0})^4 dt = 3 + o(1)\,.$$

According to [2], E is of G-type if

$$(8)\qquad \frac{1}{2\pi}\int_0^{2\pi} X_n(t; E, \mathbf{0})^\ell\, dt = \begin{cases} \ell!/\{2^{\ell/2}(\ell/2)!\} + o(1) & \text{if } \ell \text{ is even} \\ o(1) & \text{if } \ell \text{ is odd}. \end{cases}$$

Our theorem signifies that the convergence of the fourth moment in (8) is sufficient in the case where $E \in \mathscr{R}$. We remark that the condition (5) is sharp in the following sense: Let E be a set in $\mathscr{R}$ such that $\int_0^{2\pi} X_n(t;\, E,\, \mathbf{0})^6 dt = O(1)$. Then it is of G-type if and only if (5) holds. Hence the characterization of G-type sets in $\mathscr{P}_{1/2}$ is given by (5), since every set in $\mathscr{P}_{1/2}$ satisfies $\int_0^{2\pi} X_n(t;\, \cdot, \mathbf{0})^6\, dt = O(1)$.

Different from Berkes's proof, our proof will be elementary. In fact, we shall use the second mean-value theorem, instead of martingale tools.

§ 2. Applications

In [2], P. Erdös conjectured that $([e^{k^\beta}])_{k=1}^\infty$ $(0 < \beta \leq 1/2)$ is of G-type, where $[x]$ denotes the integral part of x. In [1], I. Berkes showed that, for every $0 < \beta \leq 1/2$, there exists a G-type sequence $(n_k)_{k=1}^\infty$ such that $|n_k - e^{k^\beta}| = O(k^3)$ and he also wrote that there is some connection between this conjecture and the theory of numbers. We shall show that our theorem is applicable to this conjecture. We shall show, in the last section, the following

COROLLARY. *Let $Q > 1$. Then $(n_k)_{k=1}^\infty$ is of G-type if $|n_k - Q^{\sqrt{k}}|$ is of polynomial order. More in detail, this is valid with $\sqrt{k}$ replaced by k^β as long as β is sufficiently near to* 1/2.

By our theorem, the conjecture seems to be completely solved, but the author cannot determine whether $([e^{k^\beta}])_{k=1}^\infty$ satisfies (5) or not when β is small. The author emphasizes that the conjecture holds with "e" replaced by any $Q > 1$ in the case where β is sufficiently near to 1/2.

§ 3. The totality of G-type sets is fat in a sense

Throughout this paper, we denote by "const." absolute constants. For every occasion, it does not generally same. For a property P on sets in $[0, 2\pi)$, we write simply (t satisfies P) $= \{t \in [0, 2\pi);\ t \text{ satisfies } P\}$.

Now we show that the totality of all set in $\mathscr{R}$ which satisfies (5) is sufficiently fat in a sense. Let $0 < \alpha < 1$ and let $(m_n)_{n=1}^\infty$ be a sequence of positive integers ≥ 5 such that, with $M_n = \sum_{k=1}^n m_k$, $m_n/M_n^\alpha = O(1)$ and $\sum_{n=1}^\infty M_n^3 2^{-n} < +\infty$. Let Ω_n denote the totality of m_n-tuples $(r_1, \cdots, r_{m_n})$

of integers such that $2^{n-1} \leq r_1 < \cdots < r_{m_n} < 2^n$. Since the cardinal of Ω_n is $N_n = 2^{n-1}(2^{n-1}-1)\cdots(2^{n-1}-m_n+1)$, we can define a probability $\mathscr{P}_n$ such that, for every $\omega_n \in \Omega_n$, $\mathscr{P}_n(\omega_n) = N_n^{-1}$. Let $(\Omega, \mathscr{P})$ denote the product space of $(\Omega_n, \mathscr{P}_n)$ $(n \geq 1)$. Then every element $\omega = (\omega_1, \cdots, \omega_n, \cdots)$ of Ω is a sequence of integers in $\mathscr{R}$. We show that

$$\mathscr{P}(\omega \text{ satisfies (5)}) = 1\,. \tag{9}$$

For every pair (K, n) of integers such that $0 < K \leq n$, we put $a_{Kn} = \mathscr{P}(\omega = (\omega_1, \cdots) \in \Omega;\ \sigma(\omega_K, \cdots, \omega_n) = 0)$ and prove

$$a_{Kn} \geq a_{K,n-1}(1 - \text{const.}\, M_n^3 2^{-n}) \ (n \geq K+1)\,. \tag{10}$$

Since the number of 4-tuples $(r_1, \cdots, r_4)$ such that $2^{n-1} \leq r_1 < \cdots < r_4 < 2^n$ and $\pm r_1 \pm \cdots \pm r_4 = 0$ (for some $\pm$) is at most $2^4 2^{n-1}(2^{n-1}-1)(2^{n-1}-2)$, the cardinal of a set $(\omega_n \in \Omega_n;\ \sigma(\omega_n) \neq 0)$ is less than

$$\{2^4 2^{n-1}(2^{n-1}-1)(2^{n-1}-2)\}\{(2^{n-1}-4)\cdots(2^{n-1}-m_n+1)\}$$

and hence its probability is less than

$$\begin{aligned}&\{2^4 2^{n-1}(2^{n-1}-1)(2^{n-1}-2)\}\{(2^{n-1}-4)\cdots(2^{n-1}-m_n+1)\}/N_n \\ &\quad = 2^4/(2^{n-1}-3) \leq \text{const.}\, 2^{-n}\,.\end{aligned}$$

Let r be a given integer. Since the number of 3-tuples (r_1, r_2, r_3) such that $2^{n-1} \leq r_1 < r_2 < r_3 < 2^n$ and $\pm r \pm r_1 \pm r_2 \pm r_3 = 0$ (for some $\pm$) is at most $2^4 2^{n-1}(2^{n-1}-1)$, the cardinal of a set $(\omega_n \in \Omega_n;\ \pm r \pm r_1 \pm r_2 \pm r_3 = 0$ for some $r_1, r_2, r_3 \in \{\omega_n\}$ and some $\pm)$ is less than

$$\{2^4 2^{n-1}(2^{n-1}-1)\}\{(2^{n-1}-3)\cdots(2^{n-1}-m_n+1)\}$$

and hence its probability is less than const. 2^{-n}. Thus, for every $\xi = (\omega_K, \cdots, \omega_{n-1})$,

$$\begin{aligned}&\mathscr{P}_n(\omega_n \in \Omega_n;\ \pm r \pm r_1 \pm r_2 \pm r_3 = 0 \text{ for some } r \in \{\xi\}, \text{ some } r_1, r_2, r_3 \in \{\omega_n\} \\ &\qquad \text{and some } \pm) \leq \text{const.}\Big(\sum_{k=K}^{n-1} m_k\Big)2^{-n} \leq \text{const.}\, M_n 2^{-n}\,.\end{aligned}$$

Analogously, for every $\xi = (\omega_K, \cdots, \omega_{n-1})$,

$$\left\{\begin{aligned}&\mathscr{P}_n(\omega_n \in \Omega_n;\ \pm r \pm r' \pm r_1 \pm r_2 = 0 \text{ for some } r, r' \in \{\xi\}, \\ &\qquad \text{some } r_1, r_2 \in \{\omega_n\} \text{ and some } \pm) \leq \text{const.}\, M_n^2 2^{-n} \\ &\mathscr{P}_n(\omega_n \in \Omega_n;\ \pm r \pm r' \pm r'' \pm r_1 = 0 \text{ for some } r, r', r'' \in \{\xi\}, \\ &\qquad \text{some } r_1 \in \{\omega_n\} \text{ and some } \pm) \leq \text{const.}\, M_n^3 2^{-n}\,.\end{aligned}\right.$$

From these four estimations, we have

$$a_{Kn} \geq a_{K,n-1}\{1 - \text{const.}\,(2^{-n} + M_n 2^{-n} + M_n^2 2^{-n} + M_n^3 2^{-n})\}$$
$$\geq a_{K,n-1}(1 - \text{const.}\, M_n^3 2^{-n}),$$

which signifies (10). By (10), we have

$$a_{Kn} \geq a_{KK} \prod_{k=K+1}^{n} (1 - \text{const.}\, M_k^3 2^{-k}) \geq a_{KK} \exp\Big(- \text{const.} \sum_{k=K+1}^{\infty} M_k^3 2^{-k}\Big).$$

Since $a_{KK} \geq 1 - \text{const.}\, 2^{-K}$ and $\sum_{n=1}^{\infty} M_n^3 2^{-n} < +\infty$, we have $\lim_{K\to\infty} \liminf_{n\to\infty} a_{Kn} = 1$, from which (9) follows.

§ 4. Approximation E^ε of E associated with ε

We give the proof of our theorem in sections 4 and 5. As well-known, it is sufficient to show that, for every real number x,

$$\text{(11)} \qquad \Phi_n(x; E) = \int_U \exp\{- ixX_n(t; E)\}dt = m(U)\exp(- x^2/2) + o(1).$$

For any positive number δ, there exists a finite union of open intervals U' such that $m(U \sim U') \leq \delta$. Hence, from the beginning, we may assume that U is an open interval and put $U = (a, b)$. To prove (11), we shall define, for every $0 < \varepsilon < 1$, an approximation E^ε and shall prove (11), replacing E by E^ε.

For a finite subset F of Z^+, we denote by $\mu(F)$ and $\nu(F)$ the smallest integer and the largest one in F, respectively. For the definition of E^ε, we need the following

LEMMA 1. *There exists a partition* $(\partial E_1, \partial E_1', \partial E_2, \partial E_2', \cdots)$ *of* E *such that, with* $E_k = \bigcup_{j=1}^{k} \partial E_j$,

$$\text{(12)} \qquad \partial E_k \neq \emptyset,\ |\partial E_k| = o(|E_k|)$$

$$\text{(13)} \qquad |\bigcup_{j<k} \partial E_j'| \leq \varepsilon/(1 - \varepsilon)\cdot|E_k|$$

$$\text{(14)} \qquad \nu(\partial E_{k-1})/\mu(\partial E_k) = o(\exp(- k^\beta)),$$

where $\beta = (1 - \alpha)/2$ *and* α *is a number satisfying* (3).

Proof. Let us consider a partition $(\partial E^1, \partial E^2, \cdots)$ of E. If this partition contains empty sets, we remove these empty sets. Then the resulting partition does not contain empty sets. Arranging this partition according to the order of $\nu(\cdot)$, we write $(\partial \tilde{E}_1, \partial \tilde{E}_2, \cdots)$. Put $\ell_0 = 0$, $\ell_1 = 2$ and $\ell_k =$

ℓ_{k-1} + (the integral part of ℓ_{k-1}^{γ}) ($k \geq 2$), where $\gamma = (3-\alpha)/4$. Set $\partial \bar{E}_k = \partial \tilde{E}_{\ell_{k-1}+1} \cup \cdots \cup \partial \tilde{E}_{\ell_k}$ ($k \geq 1$). Now we choose arbitrarily an integer $M = M(\varepsilon)$ such that $M\varepsilon \geq 1$. There exists a sequence of integers $(n_k)_{k=1}^{\infty}$ such that $|\partial \bar{E}_{n_k}| = \min\{|\partial \bar{E}_n|;\ kM \leq n_k < (k+1)M\}$. Then we put $\partial E'_k = \partial \bar{E}_{n_k}$ ($k \geq 1$), $\partial E_1 = \partial \bar{E}_1 \cup \cdots \cup \partial \bar{E}_{n_1-1}$ and $\partial E_k = \partial \bar{E}_{n_{k-1}+1} \cup \cdots \cup \partial \bar{E}_{n_k}$ ($k \geq 2$). Then $(\partial E_1, \partial E'_1, \cdots)$ is a required partition. In fact, (12) easily follows from (3) and $\alpha + \gamma < 1$. We have

$$|\bigcup_{j<k} \partial E'_j| = \sum_{j<k} |\partial E'_j| \leq \varepsilon \sum_{j<k} (|\partial \bar{E}_{Mj}| + \cdots + |\partial \bar{E}_{(M+1)j}|) \leq \varepsilon |E_k| + \varepsilon \sum_{j<k} |\partial E'_j|,$$

which signifies (13). By $\beta < \gamma$, we have

$$\nu(\partial E_{k-1})/\mu(\partial E_k) \leq \exp\{-(\log q)(\ell_{k-1}^{\gamma}/2)\} \leq o(\exp(-k^{\beta})),$$

which signifies (14).

Let $(\partial E_1, \partial E'_1, \cdots)$ be the partition in the above lemma. Then we define an approximation of E (associated with ε) by

$$F = E^{\varepsilon} = \bigcup_{k=1}^{\infty} \partial E_k. \tag{15}$$

From now, we write simply

$$\mu_k = \mu(\partial E_k), \qquad \nu_k = \nu(\partial E_k) \quad (k \geq 1). \tag{16}$$

Now we compare $\Phi.(x; F)$ with $\Phi.(x; E)$ in the following

LEMMA 2. *We have*

$$|\Phi_{\nu_k}(x; E) - \Phi_{\nu_k}(x; F)| \leq \{4\pi(b-a)\varepsilon\}^{1/2}|x|. \tag{17}$$

Proof. Since

$$|\exp\{-ixX_{\nu_k}(t; E)\} - \exp\{-ixX_{\nu_k}(t; F)\}| \leq |x||X_{\nu_k}(t; E) - X_{\nu_k}(t; F)|,$$

we have

$$|\Phi_{\nu_k}(x; E) - \Phi_{\nu_k}(x; F)| \leq |x| \int_a^b |X_{\nu_k}(t; E) - X_{\nu_k}(t; F)| dt$$
$$\leq \sqrt{(b-a)}|x| \left\{\int_0^{2\pi} (X_{\nu_k}(t; E) - X_{\nu_k}(t; F))^2 dt\right\}^{1/2}.$$

From (13), we have

$$\int_0^{2\pi} (X_{\nu_k}(t; E) - X_{\nu_k}(t; F))^2 dt = 4\pi(1 - \sqrt{|F(\nu_k)|/|E(\nu_k)|}) \leq 4\pi(1 - \sqrt{1-\varepsilon}) \leq 4\pi\varepsilon.$$

Thus we obtain (17).

Using (12), we have also

LEMMA 3. *For every x,*

$$\max_{\nu_{k-1}\leq n\leq\nu_k} |\Phi_n(x;E) - \Phi_{\nu_k}(x;E)| = o(1) \quad (k\to+\infty)\,. \tag{18}$$

According to these two lemmas, it is sufficient to show that, for every $0<\varepsilon<1$ and every real number x,

$$\Phi_{\nu_k}(x;F) = \Phi_{\nu_k}(x;E^\varepsilon) = (b-a)\exp(-x^2/2) + o(1)\,. \tag{19}$$

The purpose of the rest of this section and of the following section is to prove (19). For the sake of simplicity, we write

$$\begin{cases} \Lambda_k = |E_k| = |F(\nu_k)|\,, \qquad \lambda_k = |\partial E_k| \\ \Delta_k(t) = \sum_{m\in\partial E_k}\cos(mt+\gamma_m) \\ \bar{X}_k(t) = X_{\nu_k}(t,F), \qquad \bar{\Phi}_k(x) = \Phi_{\nu_k}(x;F) \quad (k\geq 1)\,. \end{cases} \tag{20}$$

Then

$$\begin{cases} \Lambda_k = \lambda_1 + \cdots + \lambda_k\,, \qquad \lambda_k/\Lambda_k = o(1) \\ \Lambda_k/|E(\nu_k)| = |F(\nu_k)|/|E(\nu_k)| \geq 1-\varepsilon \\ \bar{X}_k(t) = \sqrt{2/\Lambda_k}\sum_{j=1}^{k}\Delta_j(t)\,. \end{cases} \tag{21}$$

In Lemma 5, we shall discuss moments of $\Delta_k(t)$'s. Preparatory to Lemma 5, we need the following one. Since this is easily seen, we omit the proof.

LEMMA 4. *Let $(m_j)_{j=1}^{\ell}$ be a finite sequence of integers and $(\theta_j)_{j=1}^{\ell}\subset[0,2\pi)$. Then*

$$\int_0^{2\pi}\prod_{j=1}^{\ell}\cos(m_jt+\theta_j)\,dt \leq \int_0^{2\pi}\prod_{j=1}^{\ell}\cos m_jt\,dt\,. \tag{22}$$

LEMMA 5. *Put*

$$\begin{cases} H_k(t) = 2/\Lambda_k\sum_{j=1}^{k}\Delta_j(t)^2 - 1 \\ h_k(t) = (2/\Lambda_k)^{3/2}\sum_{j=1}^{k}|\Delta_j(t)|^3\,. \end{cases} \tag{23}$$

Then we have $\int_0^{2\pi}|H_k(t)|\,dt = o(1)$ *and* $\int_0^{2\pi}h_k(t)\,dt = o(1)$.

Proof. First we prove the latter. Taking account of

$$\int_0^{2\pi} h_k(t)\,dt \leq (2/\Lambda_k)^{3/2} \sum_{j=1}^{k} \left\{\int_0^{2\pi} \Delta_j(t)^2 dt\right\}^{1/2} \left\{\int_0^{2\pi} \Delta_j(t)^4 dt\right\}^{1/2}$$
$$= (2/\Lambda_k)^{3/2} \sum_{j=1}^{k} \sqrt{\pi\lambda_j} \left\{\int_0^{2\pi} \Delta_j(t)^4 dt\right\}^{1/2}$$
$$\leq \sqrt{2\pi} \left\{(2/\Lambda_k)^2 \sum_{j=1}^{k} \int_0^{2\pi} \Delta_j(t)^4 dt\right\}^{1/2} = \sqrt{2\pi}\,\rho_k^{1/2},$$

we prove $\rho_k = o(1)$. From Lemma 4, we may assume that $\Gamma = \mathbf{0}$. We have, for every $j \leq k$,

$$\Delta_j(t)^4 = \sum_{m\in\partial E_j} \cos^4 mt + 4 \sum_{m\in\partial E_j} \cos^3 mt \sum_{m'\in\partial E_j, m'\neq m} \cos m't$$
$$+ 6 \sum_{m\in\partial E_j} \cos^2 mt \sum_{m'\in\partial E_j, m'\neq m} \cos^2 m't$$
$$+ 6 \sum_{m\in\partial E_j} \cos^2 mt \sum_{m', m''\in\partial E_j, m'\neq m, m''\neq m, m'\neq m''} \cos m't \cos m''t$$
$$+ \sum_{m_1, m_2, m_3, m_4\in\partial E_j, m_\sigma\neq m_{\sigma'}(\sigma\neq\sigma')} \cos m_1 t \cos m_2 t \cos m_3 t \cos m_4 t\,.$$

The mean of the first term (in the right side) is dominated by const. λ_j. That of the second term is dominated by const. λ_j. That of the third term is dominated by const. λ_j^2. That of the last term is dominated by const. $\sigma(\partial E_j)$. That of the fourth term is dominated by a constant multiple of

$$\int_0^{2\pi} \Big\{ \sum_{m\in\partial E_j} \cos 2mt \sum_{m', m''\in\partial E_j, m'\neq m, m''\neq m, m'\neq m''} \cos m't \cos m''t \Big\} dt$$
$$\leq \int_0^{2\pi} \Big(\sum_{m\in\partial E_j} \cos 2mt \Big) \Delta_j(t)^2 dt \leq \sqrt{\pi\lambda_j} \left\{\int_0^{2\pi} \Delta_j(t)^4 dt\right\}^{1/2}$$

From these five estimations,

$$\rho_k \leq \text{const. } 1/\Lambda_k^2 \left\{\sum_{j=1}^{k} \lambda_j + \sum_{j=1}^{k} \lambda_j^2 + \sum_{j=1}^{k} \sqrt{\lambda_j} \left\{\int_0^{2\pi} \Delta_j(t)^4 dt\right\}^{1/2} + \sum_{j=1}^{k} \sigma(\partial E_j)\right\}$$
$$\leq \text{const.}\{1/\Lambda_k + \max_{1\leq j\leq k} \lambda_j/\Lambda_k + \sqrt{\rho_k/\Lambda_k} + \sigma(E(\nu_k))/\Lambda_k^2\}$$
$$= \text{const.} \sqrt{\rho_k/\Lambda_k} + o(1)\,,$$

which signifies $\rho_k = o(1)$. Next we prove the former. It is sufficient to show that $\int_0^{2\pi} H_k(t)^2 dt = o(1)$. From Lemma 4, we may assume $\Gamma = \mathbf{0}$. We have

$$\int_0^{2\pi} H_k(t)^2 dt = 4/\Lambda_k^2 \int_0^{2\pi} \left\{\sum_{j=1}^{k} \sum_{m, m'\in\partial E_j, m\neq m'} \cos mt \cos m't + 1/2 \sum_{m\in F(\nu_k)} \cos 2mt\right\}^2 dt$$
$$\leq 8/\Lambda_k^2 \int_0^{2\pi} \left\{\sum_{j=1}^{k} \sum_{m, m'\in\partial E_j, m\neq m'} \cos mt \cos m't\right\}^2 dt$$

$$+ 2/A_k^2 \int_0^{2\pi} \{ \sum_{m \in F(\nu_k)} \cos 2mt \}^2 dt$$
$$\leq \text{const.} \{\rho_k + \sigma(E(\nu_k))/A_k^2 + 1/A_k\} = o(1) .$$

§ 5. Proof of (19)

For a set V in $[0, 2\pi)$, we denote by $\chi(t; V)$ its indicator function, that is, $\chi(t; V) = 1$ $(t \in V)$ and $\chi(t; V) = 0$ $(t \in V^c)$. Set $V_k = (\Delta_1(t)^2 + \cdots + \Delta_k(t)^2 \leq A_k)$ $(k \geq 1)$. Then $m(V_k) = 2\pi + o(1)$, according to Lemma 5. Hence $\bar{\Phi}_k(x) = \int_a^b \chi(t; V_k) \exp\{- ix\bar{X}_k(t)\} dt + o(1)$. Since $\exp(- iy) = (1 - iy) \exp\{- y^2/2 + O(y^3)\}$ $(y > 0)$ ([5] p. 265), we have, writing simply $\phi_k(t, x) = \chi(t; V_k) \prod_{j=1}^k \{1 - ix\sqrt{2/A_k} \cdot \Delta_j(t)\}$,

$$\int_a^b \chi(t; V_k) \exp\{- ix\bar{X}_k(t)\} dt$$
$$= \int_a^b \phi_k(t, x) \exp\left\{- x^2/2 \cdot \left(2/A_k \sum_{j=1}^k \Delta_j(t)^2\right) + O\left(|x|^3 (2/A_k)^{3/2} \sum_{j=1}^k |\Delta_j(t)|^3\right)\right\} dt .$$

Since

$$|\phi_k(t, x)| \leq \chi(t; V_k) \prod_{j=1}^k \{1 + x^2 2/A_k \cdot \Delta_j(t)^2\}^{1/2}$$
$$\leq \chi(t; V_k) \exp\left\{x^2 \sum_{j=1}^k \Delta_j(t)^2\right\} \leq \exp x^2 \quad (= \tau_x) ,$$

we have

$$\int_a^b \chi(t; V_k) \exp\{- ix\bar{X}_k(t)\} dt = \exp(- x^2/2) \int_a^b \phi_k(t, x) dt + o(1) ,$$

according to Lemma 5. Thus $\bar{\Phi}_k(x) = \exp(- x^2/2) \int_a^b \phi_k(t, x) dt + o(1)$, and hence it is sufficient to show

$$(24) \qquad \Psi_k(x) = \int_a^b \chi(t; V_k) \prod_{j=1}^k \{1 - ix\sqrt{2/A_k} \cdot \Delta_j(t)\} dt = (b - a) + o(1) ,$$

whose proof is essential in this paper. Elementary calculus gives the following

LEMMA 6. *Let $Q(t)$ be a real (trigonometric) polynomial of degree $\leq N$. Then the cardinal of a set $(Q'(t) = 0)$ is at most* const. N. *For every $y > 0$, a set $(|Q(t)| \leq y)$ is a union of at most* const. N *intervals.*

For every pair (j, k) of positive integers satisfying $j \leq k$, we consider a set $V_{jk} = (\Delta_1(t)^2 + \cdots + \Delta_j(t)^2 \leq A_k)$. Since $V_{1k} \supset \cdots \supset V_{kk} (= V_k)$, we have

$$\chi(t;\, V_{jk}) = \prod_{\ell=1}^{j} \chi(t;\, V_{\ell k}) \quad (1 \le j \le k)\,. \tag{25}$$

We need also $W_{jk} = V_{j-1,k} \cap V_{jk}^c$ $(1 \le j \le k,\ V_{0k} = [0, 2\pi))$. We have $W_{jk} \cap W_{j'k} = \emptyset$ $(j \ne j')$, $V_{kk}^c = \bigcup_{j=1}^k W_{jk}$ and

$$\chi(t;\, V_{jk}) = \chi(t;\, V_{j-1,k})\{1 - \chi(t;\, W_{jk})\}\,. \tag{26}$$

To simplify the notation, we write

$$\begin{cases} \phi_{jk}(t) = \phi_{jk}(t, x) = \chi(t;\, V_{jk})\{1 - ix\sqrt{2/\Lambda_k}\cdot \Delta_j(t)\} \\ \Psi_{jk} = \int_a^b \prod_{\ell=1}^{j} \phi_{\ell k}(t)dt \quad (1 \le j \le k)\,. \end{cases} \tag{27}$$

By (25), $\Psi_{kk} = \Psi_k(x)$. Taking account of $\lim_{k\to\infty} \Psi_{p-1,k} = b - a$ (p: a given integer ≥ 2) and

$$|\Psi_{kk} - \Psi_{p-1,k}| \le \sum_{j=p}^{k} |\Psi_{j-1,k} - \Psi_{jk}|\,, \tag{28}$$

we shall estimate $|\Psi_{j-1,k} - \Psi_{jk}|$ $(2 \le j \le k)$. Using (26), we have

$$\begin{aligned} \Psi_{jk} &= \int_a^b \prod_{\ell=1}^{j} \phi_{\ell k}(t)dt = \int_a^b \prod_{\ell=1}^{j-1} \phi_{\ell k}(t)\{1 - \chi(t;\, W_{jk})\}\{1 - ix\sqrt{2/\Lambda_k}\cdot\Delta_j(t)\}dt \\ &= \Psi_{j-1,k} - \int_a^b \prod_{\ell=1}^{j-1} \phi_{\ell k}(t)\chi(t;\, W_{jk})dt + ix\sqrt{2/\Lambda_k}\int_a^b \prod_{\ell=1}^{j-1} \phi_{\ell k}(t)\chi(t;\, W_{jk})\Delta_j(t)dt \\ &\quad - ix\sqrt{2/\Lambda_k}\int_a^b \prod_{\ell=1}^{j-1} \phi_{\ell k}(t)\Delta_j(t)dt = \Psi_{j-1,k} - \psi_1 + ix\psi_2 - ix\psi_3\,. \end{aligned}$$

Since $|\prod_{\ell=1}^{j-1} \phi_{\ell k}(t)| \le \exp x^2 = \tau_x$, we have $|\psi_1| \le \tau_x m(W_{jk})$. We have

$$\begin{aligned} |\psi_2| &\le \tau_x\sqrt{2/\Lambda_k}\int_a^b \chi(t;\, W_{jk})|\Delta_j(t)|dt \le \tau_x\sqrt{2/\Lambda_k}\left\{\int_0^{2\pi} \Delta_j(t)^2 dt\right\}^{1/2} m(W_{jk})^{1/2} \\ &= \sqrt{2\pi}\,\tau_x\sqrt{\lambda_j/\Lambda_k}\,\sqrt{m(W_{jk})}\,. \end{aligned}$$

At last we estimate ψ_3. Putting $Q(t) = \mathrm{Re}\prod_{\ell=1}^{j-1} \phi_{\ell k}(t)$, we have $\mathrm{Re}\,\psi_3 = \sqrt{2/\Lambda_k}\int_a^b Q(t)\Delta_j(t)dt$. Since $\mathrm{Re}\prod_{\ell=1}^{j-1}\{1 - ix\sqrt{2/\Lambda_k}\cdot\Delta_\ell(t)\}$ is a real polynomial of degree $\le \sum_{\ell=1}^{j-1}\nu_\ell$, the cardinal of the set of all point where its derivative vanishes is less than const. $\sum_{\ell=1}^{j-1}\nu_\ell$, according to Lemma 6. Since $\Delta_1(t)^2 + \cdots + \Delta_{j-1}(t)^2$ is a real polynomial of degree $2\nu_{j-1}$, $V_{j-1,k}$ is a finite union of at most const. ν_{j-1} intervals, according to Lemma 6. Thus there exist mutually disjoint intervals (a_θ, b_θ) $(1 \le \theta \le \sigma)$ such that $\sum_{\theta=1}^{\sigma}(b_\theta - a_\theta) = b - a$, $\sigma \le$ const. $(\sum_{\ell=1}^{j-1}\nu_\ell + \nu_{j-1}) \le$ const. $\sum_{\ell=1}^{j-1}\nu_\ell$ and $Q(t)$ is monotone in every (a_θ, b_θ). For every θ, we have, from the second mean value theorem,

$$\int_{a_\theta}^{b_\theta} Q(t)\Delta_j(t)\,dt = Q(a_\theta)\int_{a_\theta}^{\xi_\theta} \Delta_j(t)\,dt + Q(b_\theta)\int_{\xi_\theta}^{b_\theta} \Delta_j(t)\,dt$$
$$\text{(for some } \xi_\theta \ (a_\theta \le \xi_\theta \le b_\theta)) .$$

Hence

$$|\mathrm{Re}\,\psi_3| \le \sqrt{2/\Lambda_k}\sum_{\theta=1}^{\sigma}\left|Q(a_\theta)\int_{a_\theta}^{\xi_\theta}\Delta_j(t)\,dt + Q(b_\theta)\int_{\xi_\theta}^{b_\theta}\Delta_j(t)\,dt\right|$$
$$\le \tau_x\sqrt{2/\Lambda_k}\sum_{\theta=1}^{\sigma}\left(\left|\int_{a_\theta}^{\xi_\theta}\Delta_j(t)\,dt\right| + \left|\int_{\xi_\theta}^{b_\theta}\Delta_j(t)\,dt\right|\right).$$

To estimate $\sqrt{2/\Lambda_k}\int_{a_\theta}^{\xi_\theta}\Delta_j(t)\,dt$ $(1 \le \theta \le \sigma)$, we use a set $A_{jk} = (|V_{jk}(t)| \ge j^2)$, where $V_{jk}(t) = \sqrt{2/\Lambda_k}\sum_{m\in\partial E_j}(\mu_j/m)\sin(mt+\gamma_m)$. Since $\int_0^{2\pi} V_{jk}(t)^2 dt = 2\pi/\Lambda_k\sum_{m\in\partial E_j}(\mu_j/m)^2 \le 2\pi\lambda_j/\Lambda_k \le 2\pi$, we have $m(A_{jk}) \le \text{const.}\, j^{-4}$. If $(a_\theta, \xi_\theta) \not\subset A_{jk}$, we put $a'_\theta = \min\{y;\ a_\theta \le y,\ y \in A^c_{jk}\}$ and $\xi'_\theta = \max\{y;\ y \le \xi_\theta,\ y \in A^c_{jk}\}$. Then $|V_{jk}(a'_\theta)|$, $|V_{jk}(\xi'_\theta)|$ are less than j^2 and $(a_\theta, a'_\theta) \cup (\xi'_\theta, \xi_\theta) \subset A_{jk}$. Hence

$$\sqrt{2/\Lambda_k}\left|\int_{a_\theta}^{\xi_\theta}\Delta_j(t)\,dt\right| = \left|\sqrt{2/\Lambda_k}\left\{\int_{a_\theta}^{a_\theta'} + \int_{\xi'_\theta}^{\xi_\theta}\right\}\Delta_j(t)\,dt + \{V_{jk}(\xi'_\theta) - V_{jk}(a'_\theta)\}/\mu_j\right|$$
$$\le \sqrt{2/\Lambda_k}\int_{a_\theta}^{\xi_\theta}\chi(t;A_{jk})|\Delta_j(t)|\,dt + 2j^2/\mu_j .$$

This estimation is valid in the case where $(a_\theta, \xi_\theta) \subset A_{jk}$. The same estimation exists with $\int_{a_\theta}^{\xi_\theta}$ replaced by $\int_{\xi_\theta}^{b_\theta}$. Thus

$$(29)\qquad |\mathrm{Re}\,\psi_3| \le \tau_x\left\{\sqrt{2/\Lambda_k}\int_0^{2\pi}\chi(t;A_{jk})|\Delta_j(t)|\,dt + 4\sigma j^2/\mu_j\right\}$$
$$\le \tau_x\left\{\sqrt{2\pi\lambda_j/\Lambda_k}\sqrt{m(A_{jk})} + \text{const.}\left(\sum_{\ell=1}^{j-1}\nu_\ell\right)j^2/\mu_j\right\}$$
$$\le \text{const.}\ \tau_x\left\{j^{-2} + \left(\sum_{\ell=1}^{j-1}\nu_\ell\right)j^2/\mu_j\right\}.$$

Analogously, (29) is valid with $\mathrm{Re}\,\psi_3$ replaced by $\mathrm{Im}\,\psi_3$ and hence this is valid with $\mathrm{Re}\,\psi_3$ replaced by ψ_3 if we neglect midle two parts. From these estimations, we have

$$|\Psi_{jk} - \Psi_{j-1,k}| \le \text{const.}\ \tau_x\left\{m(W_{jk}) + \sqrt{\lambda_j/\Lambda_k}\sqrt{m(W_{jk})} + j^{-2} + \left(\sum_{\ell=1}^{j-1}\nu_\ell\right)j^2/\mu_j\right\}.$$

By (28), we have

$$|\Psi_{kk} - \Psi_{p-1,k}| \le \text{const.}\ \tau_x\sum_{j=p}^{k}\left\{m(W_{jk}) + \sqrt{\lambda_j/\Lambda_k}\sqrt{m(W_{jk})} + j^{-2} + \left(\sum_{\ell=1}^{j-1}\nu_\ell\right)j^2/\mu_j\right\}$$
$$\le \text{const.}\ \tau_x\left\{m(V^c_{kk}) + \sqrt{m(V^c_{kk})} + \sum_{j=p}^{\infty}j^{-2} + \sum_{j=p}^{\infty}\left(\sum_{\ell=1}^{j-1}\nu_\ell\right)j^2/\mu_j\right\}$$

$$= \text{const.}\ \tau_x\left\{\sum_{j=p}^{\infty} j^{-2} + \sum_{j=p}^{\infty}\left(\sum_{\ell=1}^{j-1}\nu_\ell\right)j^2/\mu_j\right\} + o(1) = \delta(p, x) + o(1)\,.$$

Hence $\limsup_{k\to\infty}|\Psi_{kk} - (b-a)| \leq \delta(p, x)$. By (14), we have $\sum_{j=2}^{\infty}(\sum_{\ell=1}^{j-1}\nu_\ell)j^2/\mu_j < +\infty$. Since p is arbitrarily given, we have (19). This completes the proof of our theorem.

Remark 7. More in detail, our theorem holds with the condition "$E \in \mathscr{R}$" replaced by

$$|\partial E^n|/|E^n| = o(1/\log|E^n|)\,. \tag{30}$$

In fact, by (30), Lemma 1 holds with (14) replaced by $\nu(\partial E_{k-1})/\mu(\partial E_k) = O(k^{-4})$. In the proof of (19), the convergence of $\sum_{j=2}^{\infty}(\sum_{\ell=1}^{j-1}\nu_\ell)j^2/\mu_j$ is essential and this is valid in this case.

§ 6. Proof of Corollary

For the sake of simplicity, we give the proof in the case of $n_k = [Q^{\sqrt{k}}]$, where $[x]$ denotes the integral part of x. Throughout the proof, $O(1)$ and $o(1)$ will depend only on Q.

Set $E = (n_k)_{k=1}^{\infty}$ and $W_M = E \cap [Q^{(M-1)}, Q^M)$ $(M \geq 1)$. We note that $|W_M| = 2M(1+o(1))$ and $\sup\{|m/M^2 - 1|;\, n_m \in W_M\} = o(1)$. For every integer A and every positive integer M, we put $Z_\pm(A, M) = \{(n, n');\, n \pm n' = A,\ n > n',\ n \in W_M,\ n' \in E\}$ and $Z(A, M) = Z_-(A, M) \cup Z_+(A, M)$. We need two lemmas in which the first is essential.

LEMMA 8. *There exists M_1 depending only on Q such that, for any A and any $M \geq M_1$,* Car. $Z(A, M) \leq$ const. $M^{9/10}$.

Proof. Put $\gamma = 1/10$ and $\varepsilon = 1/20$. We shall show that Car. $Z_-(A, M) \leq$ const. $M^{1-\gamma}$ $(M \geq M_1)$. Since the same estimate is valid for Car. $Z_+(A, M)$, we obtain the required inequality. Assuming $Z_-(A, M) \neq \emptyset$, we denote by $(n_{m_0}, n_{m_0'})$ the pair in $Z_-(A, M)$ such that n_{m_0} is minimum in W_M. Now let us consider $Z'_-(A, M) = \{(n_m, n_{m'}) \in Z_-(A, M);\, m - m_0 \geq M^{1-\gamma}\}$. Since Car$(Z_-(A, M) - Z'_-(A, M)) \leq M^{1-\gamma}$, the estimate of Car. $Z'_-(A, M)$ is necessary. To do this, we define two partitions of W_M and $Z'_-(A, M)$ as follows:

$$W_{MJ} = W_M \cap [Q^{\sqrt{(M-1)^2+JM^\gamma}}, Q^{\sqrt{(M-1)^2+(J+1)M^\gamma}})\,,$$
$$Z_J(A, M) = \{(n, n') \in Z'_-(A, M);\, n \in W_{MJ}\} \qquad (J = 0, 1, \cdots, J_M)\,.$$

It is sufficient to show that

(31) There exists M_1 depending only on Q such that, for any A, $M \geq M_1$ and any $J = 0, 1, \cdots, J_M$, Car. $Z_J(A, M) \leq 2$.

In fact, if this is known, we have Car. $Z'_-(A, M) \leq$ const. $M^{1-\gamma}$, since $J_M \leq$ const. $M^{1-\gamma}$. Then the required inequality immediately follows. To prove (31), we begin with showing two properties:

(32) There exists M_2 such that, for $M \geq M_2$,

$$(n_m, n_{m'}) \in Z'_-(A, M) \Rightarrow m - m' \leq (\gamma + \varepsilon)/\log Q \cdot \sqrt{m} \log m .$$

(33) There exists M_3 such that, for $M \geq M_3$,

$$(n_m, n_{m'}), (n_{m+r}, n_{m'+r'}) \in Z_J(A, M) \quad (r > 0) \Rightarrow r' \leq rm^{(\gamma+2\varepsilon)/2} .$$

(32): We have $[Q^{\sqrt{m}}] - [Q^{\sqrt{m_0}}] = [Q^{\sqrt{m'}}] - [Q^{\sqrt{m_0'}}]$, and hence

$$Q^{\sqrt{m'}} \geq Q^{\sqrt{m}} - Q^{\sqrt{m_0}} - 3 \geq Q^{\sqrt{m}} \frac{\log Q}{2Q} \frac{m - m_0}{\sqrt{m}} (1 + o(1))$$

$$\geq Q^{\sqrt{m}} \frac{\log Q}{2Q} m^{-\gamma/2}(1 + o(1)) .$$

Putting $r = m - m'$, we have

$$Q^{\sqrt{m'}} \leq Q^{\sqrt{m} - r/(2\sqrt{m})} = Q^{\sqrt{m}} \exp\left(- \frac{\log Q}{2} \frac{r}{\sqrt{m}}\right) .$$

Thus

$$\exp\left(- \frac{\log Q}{2} \frac{r}{\sqrt{m}}\right) \geq \frac{\log Q}{2Q} m^{-\gamma/2}(1 + o(1)) ,$$

which signifies $r \leq \gamma(1 + o(1))/\log Q \cdot \sqrt{m} \log m$. Hence choosing M_2 sufficiently large, we have the required implication.

(33): Let $M \geq M_2$. We have

$$\begin{cases} [Q^{\sqrt{m+r}}] - [Q^{\sqrt{m}}] = [Q^{\sqrt{m'+r'}}] - [Q^{\sqrt{m'}}] \\ [Q^{\sqrt{m+r}}] - [Q^{\sqrt{m}}] \leq Q^{\sqrt{m}} \dfrac{\log Q}{2} (1 + o(1)) \dfrac{r}{\sqrt{m}} \\ [Q^{\sqrt{m'+r'}}] - [Q^{\sqrt{m'}}] \geq Q^{\sqrt{m'}} \dfrac{\log Q}{2} (1 + o(1)) \dfrac{r'}{\sqrt{m'}} , \end{cases}$$

and hence

$$r' \leq \sqrt{m'}\, r/\sqrt{m} \cdot Q^{\sqrt{m} - \sqrt{m'}}(1 + o(1)) = (1 + o(1)) r \exp\left\{\frac{\log Q}{2} (1 + o(1)) \frac{m - m'}{\sqrt{m}}\right\}$$

$$\leq (1 + o(1)) r \exp\{(\gamma + \varepsilon)(1 + o(1))/2 \cdot \log m\} .$$

Choosing M_3 ($\geq M_2$) sufficiently large, we have the required implication.

Now we prove (31). Suppose that $Z_J(A, M)$ ($M \geq M_3$) contains at least three elements, saying $(n_m, n_{m'})$, $(n_{m+r}, n_{m'+r'})$, $(n_{m+s}, n_{m'+s'})$ $(0 < r < s)$.

In the case where $s/r \neq s'/r'$: We have

$$(34) \qquad \begin{cases} [Q^{\sqrt{m+r}}] - [Q^{\sqrt{m}}] = [Q^{\sqrt{m'+r'}}] - [Q^{\sqrt{m'}}] \\ [Q^{\sqrt{m+s}}] - [Q^{\sqrt{m}}] = [Q^{\sqrt{m'+s'}}] - [Q^{\sqrt{m'}}], \end{cases}$$

and hence

$$\begin{cases} Q^{\sqrt{m}}\left\{\dfrac{\log Q}{2}\dfrac{r}{\sqrt{m}} + O\left(\dfrac{r^2}{m}\right)\right\} - Q^{\sqrt{m'}}\left\{\dfrac{\log Q}{2}\dfrac{r'}{\sqrt{m'}} + O\left(\dfrac{r'^2}{m'}\right)\right\} = O(1) \\ Q^{\sqrt{m}}\left\{\dfrac{\log Q}{2}\dfrac{s}{\sqrt{m}} + O\left(\dfrac{s^2}{m}\right)\right\} - Q^{\sqrt{m'}}\left\{\dfrac{\log Q}{2}\dfrac{s'}{\sqrt{m'}} + O\left(\dfrac{s'^2}{m'}\right)\right\} = O(1). \end{cases}$$

Eliminating $Q^{\sqrt{m'}}$-terms, we have

$$Q^{\sqrt{m}}\left\{\left(\frac{\log Q}{2}\frac{r}{\sqrt{m}} + O\left(\frac{r^2}{m}\right)\right)\left(\frac{\log Q}{2}\frac{s'}{\sqrt{m'}} + O\left(\frac{s'^2}{m'}\right)\right)\right.$$
$$\left. - \left(\frac{\log Q}{2}\frac{s}{\sqrt{m}} + O\left(\frac{s^2}{m}\right)\right)\left(\frac{\log Q}{2}\frac{r'}{\sqrt{m'}} + O\left(\frac{r'^2}{m'}\right)\right)\right\} = O(1).$$

Using

(35) $\quad r, s = O(M^{\gamma})$; $r', s' = O(M^{\gamma+(\gamma+2\varepsilon)})$, see (33); $\sqrt{m}, \sqrt{m'} = M(1 + o(1))$,

we have

$$Q^{\sqrt{m}}\left\{\frac{(\log Q)^2}{4}\frac{rs' - sr'}{\sqrt{mm'}} + O(M^{3\gamma+2(\gamma+2\varepsilon)-3})\right\} = O(1),$$

which is a contradiction for all sufficiently large M, according to $rs' \neq sr'$ and $3 - 3\gamma - 2(\gamma + 2\varepsilon) > 2$.

In the case where $s/r = s'/r'$ ($= \sigma$): Note that $\sigma \neq 1$. By (34), we have:

$$Q^{\sqrt{m}}\left\{\frac{\log Q}{2}\frac{r}{\sqrt{m}} + \frac{(\log Q)^2}{8}\frac{r^2}{m} + O\left(\frac{r^3}{m^{3/2}}\right)\right\}$$
$$- Q^{\sqrt{m'}}\left\{\frac{\log Q}{2}\frac{r'}{\sqrt{m'}} + \frac{(\log Q)^2}{8}\frac{r'^2}{m'} + O\left(\frac{r'^3}{m'^{3/2}}\right)\right\} = O(1)$$
$$Q^{\sqrt{m}}\left\{\frac{\log Q}{2}\frac{\sigma r}{\sqrt{m}} + \frac{(\log Q)^2}{8}\frac{\sigma^2 r^2}{m} + O\left(\frac{\sigma^3 r^3}{m^{3/2}}\right)\right\}$$
$$- Q^{\sqrt{m'}}\left\{\frac{\log Q}{2}\frac{\sigma r'}{\sqrt{m'}} + \frac{(\log Q)^2}{8}\frac{\sigma^2 r'^2}{m'} + O\left(\frac{\sigma^3 r'^3}{m'^{3/2}}\right)\right\} = O(1).$$

Eliminate $Q^{\sqrt{m'}}$-terms. Using (35), we can write simply

$$(36)\qquad Q^{\sqrt{m}}\frac{(\log Q)^3}{16}\frac{\sigma rr'}{\sqrt{mm'}}\left\{\frac{(\sigma-1)r'}{\sqrt{m'}}-\frac{(\sigma-1)r}{\sqrt{m}}+O(M^{2\gamma+2(\gamma+2\varepsilon)-2})\right\} = O(1)\,.$$

Since $\frac{1}{\sqrt{m'}}-\frac{1}{\sqrt{m}}=O\left(\frac{m-m'}{\sqrt{m'm}}\right)=O\left(\frac{\log M}{M^2}\right)$ (see (32)), we have

$$\left|\frac{(\sigma-1)r'}{\sqrt{m'}}-\frac{(\sigma-1)r}{\sqrt{m}}\right|=\left|\frac{(\sigma-1)(r'-r)}{\sqrt{m'}}+(\sigma-1)r\left(\frac{1}{\sqrt{m'}}-\frac{1}{\sqrt{m}}\right)\right| \geq M^{-1}(1+o(1))-O(M^{\gamma-2}\log M)=M^{-1}(1+o(1))\,.$$

Thus (36) is a contradiction for all sufficiently large M, according to $2-2\gamma-2(\gamma+2\varepsilon)>1$. (If $r'=r$, (34) does not evidently hold for all large M.)

In any case, from the first hypothese, we have a contradiction for all large M, which signifies (31), and hence the proof completes.

Elementary calculus gives the following

LEMMA 9. *For every positive integer K, we put*

$$(37)\qquad Z_K=\{(q_1,q_2,q_3,q_4);\ \pm q_1\pm\cdots\pm q_4=0 \textit{ for some } \pm\,,\quad q_1>\cdots>q_4\,,\ q_1\leq Q^K\,,\ q_j\in E\}\,.$$

Let $(q_1,q_2,q_3,q_4)\in Z_K$ and $q_1\in W_M$ $(M\leq K)$. Then

$$(38)\qquad q_2\geq Q^{M-c}\,,$$

$$(39)\qquad q_3\geq Q^{M-c'\log M}\,,$$

where $c=c_Q$ and $c'=c'_Q$ are constants depending only on Q.

Now we give the proof of Corollary. Since the order of $|E(Q^K)|^2$ is K^4, it is sufficient to show $\sigma(E(Q^K))=o(K^4)$, according to our theorem. Note that $\sigma(E(Q^K))=$ const. Car (Z_K). Choose arbitrarily a positive integer M'_1 such that $M'_1-c'_Q\log M'_1\geq M_1$, where M_1 is the integer in Lemma 8. Let $K\geq M'_1$. For any $M(M'_1\leq M\leq K)$, the cardinal of $\{(q_1,q_2);(q_1,q_2,q_3,q_4)\in Z_K$ for some $q_3,q_4;\ q_1\in W_M\}$ is less than const. $c_Q^2M^2$, according to (38). For a fixed pair (q_1,q_2) $(q_1\in W_M)$, the cardinal of $\{(q_3,q_4);(q_1,q_2,q_3,q_4)\in Z_K\}$ is less than const.$c'_Q(\log M)M^{9/10}$, according to Lemma 8 and (39). Hence the cardinal of $\{(q_1,\cdots,q_4)\in Z_K;\ q_1\in W_M\}$ is less than const. $c_Q^2c'_Q(\log M)M^{2+9/10}$. Adding from M'_1 to K, we have $\mathrm{Car}(Z_K)=O(K^{3+9/10}\log K)=o(K^4)$. This completes the proof.

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