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JULIA DIRECTIONS OF ENTIRE FUNCTIONS OF SMOOTH GROWTH

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§ 1. Introduction

Let $f(z)$ be entire i.e. analytic in the finite whole plane Z. The *order* of $f(z)$ is defined as

$$\rho = \varlimsup_{r\to\infty} \frac{\log^+(\log^+ M(r,f))}{\log r}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$. A ray $\chi(\theta) = \{z = r\cdot e^{i\theta} : 0 < r < +\infty\}$ is called a *Julia direction* of $f(z)$ if, in any open sector containing the ray, $f(z)$ takes all values of Z, with at most one finite exceptional value, infinitely often.

We can guess that the smoothness of growth of $M(r,f)$ causes simple boundary behaviors of $f(z)$. In this paper, we exemplify this fact, by picking up two kinds of smoothness conditions.

The following problem comes into question: Let $f(z)$ be an entire function of order less than $\frac{1}{2}$ and let $\chi(\theta)$ be any ray. Either is $\chi(\theta)$ a Julia direction of $f(z)$ or is $f(z)$ convergent to ∞ as $|z| \to +\infty$ on some sector containing $\chi(\theta)$? So, we shall prove in Theorem 2 that if we assume the smoothness of growth of $M(r,f)$: if there is a constant μ, $\mu < \frac{1}{2}$, such that

$$\text{(A)} \qquad \frac{\log M(x_0\cdot r, f)}{\log M(r,f)} \leqq x_0^{\mu} \quad (r \geqq r_0)$$

for some x_0, $x_0 > 1$, and r_0, this fact is true. Theorem 1 is the preliminary result for this theorem.

Further, we shall show in Theorem 3 that, under the assumption of the stronger smoothness condition:

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(B) $$\log M(2\cdot r, f) \sim \log M(r, f), \quad (r \to \infty)$$

a Julia direction $\chi(\theta)$ of $f(z)$ is characterized as the ray $\chi(\theta)$ for which θ is a limit point of the set

$$Z(f) = \{\arg z_n : f(z_n) = 0\}.$$

Hence, according to Hayman [9, p. 143], it follows that all Julia directions of entire functions $f(z)$ satisfying the condition

$$\log M(r, f) = O(\log^2 r) \quad (r \to \infty)$$

are the directions corresponding to the limit points of the set $Z(f)$. Hayman [9, p. 130] remarked that any entire function satisfying (B) has order 0. An example will be given to show that any entire function of order 0 has not always this property.

By using this Theorem 3, we shall give an example of an entire function $f(z)$ for which any non-empty closed set is precisely the set of Julia directions of $f(z)$. This generalizes an example of Anderson and Clunie [2].

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§ 2. The boundary behaviour of entire functions

In the following, the spherical derivative of a meromorphic function $f(z)$ is defined by

$$\rho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

We denote the set $\{z : |z - z_0| < \varepsilon |z_0|\}$ by $D(z_0, \varepsilon)$ and the sector $\{z : |\arg z - \theta| < \varepsilon\}$ by $V(\theta, \varepsilon)$.

LEMMA 1 (Clunie and Hayman [4, p. 125]). *Let $f(z)$ be regular in $|z - z_0| \leqq \delta$ and satisfy $|f(z)| \geqq 1$ there. If $|f(z_1)| = 1$ for some z_1 with $|z_1 - z_0| = \delta$, then for some z on the segment joining z_0 to z_1 we have*

$$\rho(f(z)) \geqq \frac{\log |f(z_0)|}{10\cdot\delta\cdot\log 2}.$$

LEMMA 2. *Let $f(z)$ be an entire function and let δ be a constant, $0 < \delta < 1$. If $\{z_n\}$, $|z_n| \to \infty$, is a sequence such that*

$$|f(z_n)| \to \infty$$

and $f(z)$ *does not converge to* ∞ *as* $|z| \to +\infty$ *on the set* $\bigcup_n D(z_n, \delta)$, *then there is a sequence* $\{\xi_k\}$, $|\xi_k| \to \infty$, $\xi_k \in \bigcup_n D(z_n, \delta)$, *satisfying*

$$\lim_{k\to\infty} |\xi_k| \cdot \rho(f(\xi_k)) = +\infty .$$

Proof. By the assumption, we can find a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and a sequence $\{\zeta_k\}$, $|\zeta_k| \to \infty$, $\zeta_k \in D(z_{n_k}, \delta)$, for which

$$|f(\zeta_k)| \leqq K$$

where K is a constant, $K \geqq 1$. Put $\delta_k = \operatorname{dis}(S, z_{n_k})$, where $S = \{z: |f(z)| \leqq K\}$ and $\operatorname{dis}(A, B)$ denotes the distance between A and B. Then, we have

$$(1) \qquad \delta_k \leqq |\zeta_k - z_{n_k}| \leqq \delta |z_{n_k}| \quad (k = 1, 2, 3, \cdots) .$$

Now, consider the function

$$g(z) = \frac{f(z)}{K} .$$

From Lemma 1 applied to $g(z)$, we see that there is a sequence $\{\xi_k\}$, $|\xi_k - z_{n_k}| \leqq \delta_k$, such that

$$(2) \qquad \rho(g(\xi_k)) \geqq \frac{\log |g(z_{n_k})|}{10 \cdot \delta_k \cdot \log 2} \quad (k = 1, 2, 3, \cdots) .$$

Since

$$\rho(f(z)) \geqq \frac{1}{K} \cdot \rho(g(z))$$

and

$$|\xi_k| \geqq (1 - \delta) \cdot |z_{n_k}| \quad (k = 1, 2, 3, \cdots) ,$$

from (1), we finally get from (1) and (2) that

$$|\xi_k| \cdot \rho(f(\xi_k)) \geqq \frac{(1 - \delta) \cdot \{\log |f(z_{n_k})| - \log K\}}{10 \cdot \delta \cdot K \cdot \log 2} \quad (k = 1, 2, 3, \cdots)$$

which gives us the conclusion.

LEMMA 3. *Let* θ, ρ_1 *and* ρ_2 *be constants satisfying* $0 \leqq \theta < 2\pi$, $0 < \rho_1 < 1$, $0 < \rho_2 < 1$ *and let* z_1, z_2 *be any numbers on* $\chi(\theta)$. *If the circles* $D(z_1, \rho_1)$

and $D(z_2, \rho_2)$ intersect, then the angle which is subtended at the origin by the chord connecting the points of intersection is dependent only on $t = z_2/z_1$, ρ_1 and ρ_2.

Proof. We can see from easy calculation that $(Y/X)^2$ is the function dependent on t, ρ_1 and ρ_2, where (X, Y) denotes the coordinate of the points of intersection of both circles.

LEMMA 4 (Lehto [11, Theorem 3]). *Let $f(z)$ be meromorphic in $R < |z| < \infty$. If, for some sequence $\{\xi_k\}$, $|\xi_k| \to \infty$*

$$\lim_{k\to\infty} |\xi_k| \cdot \rho(f(\xi_k)) = +\infty ,$$

then $f(z)$ assumes every value infinitely often with at most two exceptions of values in the extended plane on the set $\bigcup_k D(\xi_k, \varepsilon)$ for each fixed $\varepsilon > 0$.

We now state and prove

THEOREM 1. *Let $f(z)$ be an entire function and $\chi(\theta)$ $(0 \leqq \theta < 2\pi)$ be a ray on which there exist a sequence $\{z_n\}$, $|z_n| < |z_{n+1}|$, $|z_n| \to \infty$, and a constant M, satisfying*

$$\left| \frac{z_{n+1} - z_n}{z_n} \right| < M$$

and

$$\lim_{n\to\infty} |f(z_n)| = +\infty .$$

Then, $\chi(\theta)$ is a Julia direction of $f(z)$ or $f(z)$ is convergent to ∞ as $|z| \to +\infty$ on some sector containing $\chi(\theta)$.

Proof. First of all, suppose that $f(z)$ does not converge to ∞ as $|z| \to +\infty$ in the set $\bigcup_n D(z_n, \varepsilon)$ for any $\varepsilon > 0$. Then, by Lemma 2, for any ε, $0 < \varepsilon < 1$, we can find a sequence $\{\zeta_k\}$, $\zeta_k \in D(z_{n_k}, \varepsilon)$, such that

$$\lim_{k\to\infty} |\zeta_k| \cdot \rho(f(\zeta_k)) = +\infty .$$

Lemma 4 shows that $f(z)$ assumes every value of Z infinitely often with at most one exception in the set $V(\theta, \pi\varepsilon)$ and hence $\chi(\theta)$ is a Julia direction of $f(z)$.

So, suppose that $f(z)$ converges to ∞ as $|z| \to +\infty$ in the set $\bigcup_n D(z_n, \varepsilon)$ for some $\varepsilon > 0$, and denote by E_1, the set of these ε's. We put

$$\rho_1 = \sup_{\varepsilon \in E_1} \varepsilon .$$

If $\rho_1 > 1$, we have $\bigcup_n D(z_n, \varepsilon) = Z$ for some $\varepsilon \in E_1$, $\varepsilon > 1$, and hence we get evidently the conclusion. So, we suppose that $0 < \rho_1 \leqq 1$. Take the sequence $\{z_n^{(2)}\}$, $z_n^{(2)} \in \chi(\theta)$, satisfying

$$|z_n^{(2)}| = |z_n| \cdot (1 + \tfrac{1}{2} \cdot \rho_1) \quad (n = 1, 2, 3, \cdots) .$$

By using the fact that

$$|f(z_n^{(2)})| \to \infty \quad (n \to \infty) ,$$

we repeat the same argument. If $f(z)$ does not converge to ∞ as $|z| \to +\infty$ in the set $\bigcup_n D(z_n^{(2)}, \varepsilon)$ for any $\varepsilon > 0$, we can also conclude that $\chi(\theta)$ is a Julia direction of $f(z)$. In the case that $f(z)$ converges to ∞ as $|z| \to +\infty$ in the set $\bigcup_n D(z_n^{(2)}, \varepsilon)$ for some $\varepsilon > 0$, denote by E_2 the set of these ε's and put

$$\rho_2 = \sup_{\varepsilon \in E_2} \varepsilon .$$

Then we can suppose that $0 < \rho_2 \leqq 1$. Again, take the sequence $\{z_n^{(3)}\}$, $z_n^{(3)} \in \chi(\theta)$, satisfying

$$|z_n^{(3)}| = |z_n^{(2)}| \cdot (1 + \tfrac{1}{2} \cdot \rho_2) = |z_n| \cdot (1 + \tfrac{1}{2} \cdot \rho_1) \cdot (1 + \tfrac{1}{2} \cdot \rho_2) \quad (n = 1, 2, 3, \cdots) .$$

Repeat this process over and over until we get either the conclusion that $\chi(\theta)$ is a Julia direction of $f(z)$ or the conclusion

$$(3) \qquad \prod_{i=1}^{N} (1 + \tfrac{1}{2} \cdot \rho_i) > M + 1$$

at some step N. In the case that (3) happens, we can easily show from Lemma 3 that $f(z)$ converges to ∞ as $|z| \to +\infty$ on the set $V(\theta, \alpha)$ for some $\alpha > 0$.

Now, suppose that these processes are continued infinitely. Then, we have

$$\prod_{i=1}^{\infty} (1 + \tfrac{1}{2} \cdot \rho_i) \leqq M + 1 .$$

Since $f(z)$ does not converge to ∞ as $|z| \to +\infty$ on the set $\bigcup_n D(z_n^{(i)}, 2 \cdot \rho_i)$ for each i satisfying $\rho_i < \frac{1}{2}$, Lemma 2 gives a sequence $\{\xi_k^{(i)}\}$, $|\xi_k^{(i)}| \to \infty$ $(k \to \infty)$, $\xi_k^{(i)} \in \bigcup_n D(z_n^{(i)}, 2 \cdot \rho_i)$, such that

$$\lim_{k \to \infty} |\xi_k^{(i)}| \cdot \rho(f(\xi_k^{(i)})) = +\infty .$$

From the fact $\rho_i \to 0$ and Lemma 4, we can conclude that $\chi(\theta)$ is a Julia direction of $f(z)$. Thus, we complete the proof.

To prove Theorem 2, we need the following property (Lemma 7) of entire functions $f(z)$ for which $\log M(r, f)$ satisfies the smoothness condition (A).

LEMMA 5. *Let* x_0, $x_0 > 1$, μ, $\mu \geqq 0$, r_0 *and* R, $r_0 > R$, *be constants. If* $h(r)$ *is a positive, non-decreasing function defined on the interval* $R < r < +\infty$ *and satisfies the condition*:

$$\frac{h(x_0 \cdot r)}{h(r)} \leqq x_0^{\mu} \quad (r \geqq r_0),$$

then

(i)
$$\frac{h(x \cdot r)}{h(r)} \leqq x_0^{\mu} \cdot x^{\mu} \quad (r \geqq r_0)$$

for any x, $x \geqq x_0$, *and*

(ii) *for any* α, $\alpha > \mu$,

$$\int_r^{\infty} \frac{h(t)}{t^{1+\alpha}}\, dt \leqq S(x_0 \colon \alpha, \mu) \cdot \frac{h(r)}{r^{\alpha}} \quad (r \geqq r_0),$$

where

$$S(x_0 \colon \alpha, \mu) = \frac{x_0^{\alpha} - 1}{\alpha(x_0^{\alpha} - x_0^{\mu})} \cdot x_0^{\mu}.$$

Proof. Take any $x \geqq x_0$ and choose an integer p such that $x_0^p \leqq x < x_0^{p+1}$. Then,

$$h(x \cdot r) \leqq h(x_0^{p+1} \cdot r) \leqq (x_0^{p+1})^{\mu} \cdot h(r) \leqq x_0^{\mu} \cdot x^{\mu} \cdot h(r) \qquad (r \geqq r_0).$$

This gives (i).

Since

$$h(x_0^{i+1} \cdot r) \leqq (x_0^{\mu})^{i+1} \cdot h(r), \quad (r \geqq r_0) \quad (i = 0, 1, 2, \cdots)$$

we have

$$\int_r^{\infty} \frac{h(t)}{t^{1+\alpha}}\, dt \leqq \sum_{i=0}^{\infty} h(x_0^{i+1} \cdot r) \cdot \int_{x_0^i \cdot r}^{x_0^{i+1} \cdot r} \frac{1}{t^{1+\alpha}}\, dt$$
$$\leqq \frac{x_0^{\mu}}{\alpha} \cdot \left[1 - \frac{1}{x_0^{\alpha}}\right] \cdot \frac{h(r)}{r^{\alpha}} \cdot \sum_{i=0}^{\infty} (x_0^{\mu-\alpha})^i = S(x_0 \colon \alpha, \mu) \cdot \frac{h(r)}{r^{\alpha}} \quad (r \geqq r_0).$$

Thus, (ii) is obtained.

LEMMA 6. (Denjoy [5] and Kjellberg [10, p. 17–18].) *Let $f(z)$ be an entire function of order μ, $0 \leqq \mu < \frac{1}{2}$, and $f(0) = 1$. Then, for any α, $\mu < \alpha < \frac{1}{2}$,*

$$r^{\alpha}\cdot\int_r^{\infty}[\log m(t,f) - (\cos\pi\alpha)\cdot\log M(t,f)]\cdot\frac{dt}{t^{1+\alpha}} > \frac{1-\cos\pi\alpha}{\alpha}\cdot\log M(r,f) \qquad (0 < r < +\infty)\,,$$

where $m(t,f) = \min_{|z|=t}|f(z)|$.

LEMMA 7. *Let $f(z)$ be an entire function for which $f(0) = 1$ and $\log M(r,f)$ satisfies the condition* (A): *there is constant μ, $\mu < \frac{1}{2}$, such that*

$$\frac{\log M(x_0\cdot r,f)}{\log M(r,f)} \leqq x_0^{\mu} \qquad (r \geqq r_0)$$

for some x_0, $x_0 > 1$, and r_0. Then, for any α, $\mu < \alpha < \frac{1}{2}$, there exists a constant k such that for some t in any interval $(r, k\cdot r)$ $(r \geqq r_0)$

$$\log m(t,f) > \cos\pi\alpha\cdot\log M(t,f)\,.$$

Proof. First of all, we have

$$\begin{aligned} &r^{\alpha}\cdot\int_{x\cdot r}^{\infty}[\log m(t,f) - (\cos\pi\alpha)\cdot\log M(t,f)]\cdot\frac{dt}{t^{1+\alpha}} \\ &\leqq r^{\alpha}(1-\cos\pi\alpha)\cdot\int_{x\cdot r}^{\infty}\frac{\log M(t,f)}{t^{1+\alpha}}\,dt \\ &\leqq r^{\alpha}\cdot(1-\cos\pi\alpha)\cdot S(x_0\colon \alpha,\mu)\cdot x_0^{\mu}\cdot x^{\mu-\alpha}\cdot\log M(r,f) \qquad (x \geqq x_0,\ r \geqq r_0) \end{aligned}$$

from Lemma 5 in which $h(r) = \log M(r,f)$. Thus, since we see from (i) of Lemma 5 that $f(z)$ has at most order μ, we get

$$\begin{aligned} &r_{\alpha}\cdot\int_r^{x\cdot r}[\log m(t,f) - (\cos\pi\alpha)\cdot\log M(t,f)]\cdot\frac{dt}{t^{1+\alpha}} \\ &> (1-\cos\pi\alpha)\cdot\left[\frac{1}{\alpha} - S(x_0\colon \alpha,\mu)\cdot x_0^{\mu}x^{\mu-\alpha}\right]\cdot\log M(r,f) \qquad (x \geqq x_0,\ r \geqq r_0) \end{aligned}$$

from Lemma 6. Here, if we take a k, $k \geqq x_0$, such that

$$\frac{1}{\alpha} - S(x_0\colon \alpha,\mu)\cdot x_0^{\mu}\cdot k^{\mu-\alpha} > 0\,,$$

the right-hand side of the inequality in which x is replaced with k is always positive for all $r \geqq r_0$ and hence the left-hand side is positive. Thus, we obtain the conclusion.

Now, we have

THEOREM 2. *Let $f(z)$ be an entire function for which $\log M(r,f)$ satisfies the smoothness condition (A) for some μ, x_0 and r_0, where $\mu < \frac{1}{2}$ and $x_0 > 1$. Then, for any ray $\chi(\theta)$ $(0 \leqq \theta < 2\pi)$, $\chi(\theta)$ is a Julia direction of $f(z)$ or $f(z)$ is convergent to ∞ as $|z| \to +\infty$ on some open sector containing $\chi(\theta)$.*

Proof. It is evident that we can confine ourselves to the case $f(0) = 1$. If we denote by t_n such a t of the interval $(k^n \cdot r_0, k^{n+1} \cdot r_0)$ $(n = 0, 1, 2, \cdots)$ in Lemma 7, we have

$$\left|\frac{t_{n+1} - t_n}{t_n}\right| \leqq \frac{k^{n+2} \cdot r_0 - k^n \cdot r_0}{k^n \cdot r_0} = k^2 - 1 .$$

Thus, we see that the sequence $\{t_n \cdot e^{i\theta}\}$ for any fixed θ $(0 \leqq \theta < 2\pi)$ is a sequence satisfying the condition of Theorem 1. Theorem 1 gives the conclusion of Theorem 2.

QUESTION 1. Is Theorem 2 true for every entire function of order less than $\frac{1}{2}$ without any kind of smoothness condition?

Remark 1. We note that (A) is implied by the following smooth condition: there exist a proximate order $\rho(r)$, $\rho(r) \to \rho$ $(r \to \infty)$ for some ρ, $0 \leqq \rho < \frac{1}{2}$, and two constants a, b such that

$$0 < a \leqq \varliminf_{r \to \infty} \frac{\log M(r,f)}{r^{\rho(r)}} \leqq \varlimsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho(r)}} \leqq b < +\infty$$

(see Cartwright [3] for the definition and the properties of proximate order). Hence, for example, Theorem 2 is true for entire functions $f(z)$ which satisfy the condition

$$\log M(r,f) \sim r^{\rho} \cdot \log^{\rho_1} r \cdot \log_2^{\rho_2} r \cdots \log_p^{\rho_p} r \quad (r \to \infty)$$

where $\log_j r = \log(\log_{j-1} r)$ and ρ $(0 \leqq \rho < \frac{1}{2})$, $\rho_1, \rho_2, \cdots, \rho_p$ are real numbers.

Next, we shall consider Julia directions of entire functions satisfying the smoothness condition (B).

A countable set of circles C_ν in Z is said to form a *slim set* S, $S = \bigcup_\nu C_\nu$, if the sum $\sum_\nu r_{\nu,k}$ of the radii $r_{\nu,k}$ of those circles $C_{\nu,k}$ intersecting the annulus $\{z: 2^k \leqq |z| < 2^{k+1}\}$ is $o(2^k)$ $(k \to \infty)$ i.e.,

$$\varepsilon_k \to 0 \quad (k \to \infty) \quad \text{for} \quad \sum_\nu r_{\nu,k} = \varepsilon_k \cdot 2^k$$

(see Anderson [1]).

LEMMA 8. *A slim set S has the following properties:*

(i) *Each component of S that intersect the set $\{z: |z| > N\}$ for a sufficiently large number N is contained in some annulus $R_k = \{z: 2^{k-1} < |z| < 2^{k+1}\}$,*

(ii) *Let G_k be a component of S contained in R_k. If we denote by θ_k the angle which G_k subtends at the origin, then*

$$\theta_k \to 0 \quad (k \to \infty)\,.$$

Proof. Evidently, (i) is true. If we denote $\theta_{\nu,k}$ the angle subtended at the origin by the circle $C_{\nu,k}$, we have

$$\theta_k \leqq \sum_\nu \theta_{\nu,k-1} + \sum_\nu \theta_{\nu,k} \leqq \pi(\varepsilon_{k-1} + 2\cdot\varepsilon_k)\,.$$

Since $\varepsilon_k \to 0$ $(k \to \infty)$, (ii) follows.

LEMMA 9 (Anderson [1, Theorem 2]). *Let $f(z)$ be an entire function for which* $\log M(r, f)$ *satisfies the condition* (B). *Then,*

$$\log|f(z)| \sim \log M(r, f) \qquad (|z| = r \to \infty)$$

outside a slim set S_f.

We deduce

THEOREM 3. *Let $f(z)$ be an entire function for which* $\log M(r, f)$ *satisfies the condition* (B). *Then, the set of ray $\chi(\theta)$ for which θ is a limit point of the set*

$$E(f) = \{\arg z_n : f(z_n) = 0\}$$

is precisely the set of Julia directions of $f(z)$. In fact, if $\theta \in E(f)$, $f(z)$ assumes every value without exception infinitely often in any sector containing $\chi(\theta)$. Otherwise $f(z)$ converges to ∞ as $|z| \to +\infty$ in some such sector and so assumes no value more than a finite number of times in this sector.

Proof. It is evident from Lemma 9 that $f(z)$ converges to ∞ as $|z| \to +\infty$ in the sector which intersects a finite number of components of the slim set S_f.

Now, suppose that any sector containing $\chi(\theta)$ intersects an infinite number of components of S_f. Then, Lemma 8 shows that such sector contains an infinite number of components of S_f completely. Here, we can easily see from Lemma 9 that for any fixed $M > 0$, any component contained inside R_k, where k is sufficiently large, contains at least one component of F_M^c, where F_M^c denotes the complement of the set $\{z: |f(z)| \geqq M\}$. Thus, since such sector contains an infinite number of components of F_M^c, Rouche's theorem gives us the conclusion of Theorem 3.

QUESTION 2. A function satisfying (B) has order 0 (see Hayman [9, p. 130.]). As a natural generalization, we can consider the class of entire functions of order ρ, $0 \leqq \rho < \frac{1}{2}$, satisfying the condition:

$$\varlimsup_{r\to\infty} \frac{\log M(x\cdot r, f)}{x^\rho \cdot \log M(r, f)} \leqq 1$$

for any x, $1 < x$.

Is the analogie of Theorem 3 true for this wider class, or for the still more general class satisfying the condition (A)?

The following example shows that Theorem 3 depends on the smoothness of growth of $M(r, f)$.

EXAMPLE. Let ρ be any positive number. Take two sequences $\{a_n\}$, $\{b_n\}$ $(n = 1, 2, 3, \cdots)$ defined by

$$a_n = c^{c^n}$$

where $c = [1 + 1/\rho] + 1$, $[x]$ is the integral part of x, and

$$\log^{1+\rho} b_n = a_n .$$

We define the entire function $f(z)$ by

$$f(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{b_k}\right)^{a_k} . \tag{4}$$

This $f(z)$ has the following properties:

(a) Any $\chi(\theta)$, $|\theta| \leqq \pi/2$, is a Julia direction of $f(z)$, in spite of the fact that only $\theta = 0$ is the limit point of the set $\{\arg z_n : f(z_n) = 0\}$;

(b) $\log M(r, f) = O(\log^{2+\rho} r)$.

First of all, we shall show that

(5) $f(z)$ converges to 0 as $|z| \to +\infty$ on the set $\bigcup_n \{z: |z - b_n| < c_1 \cdot b_n\}$ for any fixed c_1, $0 < c_1 < 1$.

Decompose the product (4) into four subproducts $I_i(z)$ $(i = 1, 2, 3, 4)$:

$$I_1(z) = \prod_{k=1}^{n-1}\left(\frac{z}{b_k}\right)^{a_k}, \qquad I_2(z) = \prod_{k=1}^{n-1}\left(\frac{b_k}{z} - 1\right)^{a_k},$$

$$I_3(z) = \left(1 - \frac{z}{b_n}\right)^{a_n}, \qquad I_4(z) = \prod_{k=n+1}^{\infty}\left(1 - \frac{z}{b_k}\right)^{a_k}.$$

We have to determine a upper bound of $I_i(z)$ $(i = 1, 2, 3, 4)$ for any z, $|z - b_n| < c_1 \cdot b_n$. First, we have

$$\begin{aligned}|I_1(z)| &\leqq \prod_{k=1}^{n-1}[(1 + c_1)b_n]^{a_k} = (1 + c_1)^{o(1)\cdot a_n} b_n^{(1+o(1))\cdot a_{n-1}} \\ &= (1 + c_1)^{o(1)\cdot a_n}(b_n^{a_{n-1}/a_n})^{(1+o(1))\cdot a_n} = (1 + o(1))^{a_n} \quad (n \to \infty),\end{aligned}$$

because of the fact

$$\sum_{k=1}^{n-1} a_k = o(1)\cdot a_n \quad (n \to \infty), \tag{6}$$

and, since $c > 1 + 1/\rho$, we deduce

$$b_n^{a_{n-1}/a_n} \to 1 \qquad (n \to \infty).$$

Next, we have

$$\begin{aligned}|I_2(z)| &\leqq \prod_{k=1}^{n-1}\left(\left|\frac{b_k}{z}\right| + 1\right)^{a_k} \leqq \prod_{k=1}^{n-1}\left(\frac{2 - c_1}{1 - c_1}\right)^{a_k} = \left(\frac{2 - c_1}{1 - c_1}\right)^{o(1)\cdot a_n} \\ &= (1 + o(1))^{a_n}, \qquad (n \to \infty)\end{aligned}$$

since

$$\left|\frac{b_k}{z}\right| \leqq \frac{b^k}{(1 - c_1)b_n} < \frac{1}{1 - c_1} \quad (k = 1, 2, 3, \cdots, n - 1).$$

For $I_4(z)$, we have

$$\begin{aligned}|I_4(z)| &\leqq \prod_{k=n+1}^{\infty}\left(1 + \frac{(1 + c_1)b_n}{b_k}\right)^{a_k} \leqq \exp\left[(1 + c_1)b_n \cdot \prod_{k=n+1}^{\infty}\frac{a_k}{b_k}\right] \\ &= 1 + o(1) \qquad (n \to \infty)\end{aligned}$$

by using the inequality

$$1 + x < e^x \qquad (x > 0)$$

and

$$b_n \cdot \prod_{k=n+1}^{\infty}\frac{a_k}{b_k} \to 0 \qquad (n \to \infty). \tag{7}$$

Thus, we get

$$|f(z)| \leqq [1 + o(1)]\cdot[(1 + o(1))\cdot c_1]^{a_n} \qquad (n \to \infty)\,,$$

which shows (5).

Next, we shall show that

(8) $f(z)$ converges to ∞ as $|z| \to +\infty$ on the sequence of circles $\{z: |z - b_n| = c_2\cdot b_n\}$ for any fixed c_2, $c_2 > 1$.

Decompose the product (4) into three subproducts $J_j(z)$ $(j = 1, 2, 3)$:

$$J_1(z) = \prod_{k=1}^{n-1}\left(1 - \frac{z}{b_k}\right)^{a_k}, \quad J_2(z) = \left(1 - \frac{z}{b_n}\right)^{a_n}, \quad J_3(z) = \prod_{k=n+1}^{\infty}\left(1 - \frac{z}{b_k}\right)^{a_k}.$$

First of all, we have

$$|J_1(z)| \geqq \prod_{k=1}^{n-1}\left(\left|\frac{z}{b_k}\right| - 1\right)^{a_k} \geqq 1$$

since

$$\frac{|z|}{b_k} \geqq (c_2 - 1)\cdot\frac{b_n}{b_k} \geqq 2 \qquad (k = 1, 2, 3, \cdots, n-1)$$

for sufficiently large n. Secondly, we have

$$\begin{aligned}\log|J_3(z)| &\geqq \sum_{k=n+1}^{\infty} a_k\cdot\log\left(1 - \frac{(1 + c_2)b_n}{b_k}\right)\\ &\geqq -2\cdot\log 2\cdot(1 + c_2)\cdot b_n\cdot\sum_{k=n+1}^{\infty}\frac{a_k}{b_k} = o(1) \qquad (n \to \infty)\,,\end{aligned}$$

by using the inequality

$$\log(1 - x) \geqq -2\cdot(\log 2)\cdot x \qquad (0 \leqq x \leqq 1/2)$$

and (7). Thus, we get

$$|f(z)| \geqq (1 - o(1))\cdot c_2^{a_n}\,, \qquad (n \to \infty)$$

which shows (8).

Now, we can prove (a). Let θ be any fixed number satisfying $|\theta| < \pi/2$ and denote by $\{z_n\}$ the point, other than the origin, where the ray $\chi(\theta)$ meets the circle $\{z: |z - b_n| = b_n\}$. Consider the sequence of functions

$$f_n(z) = f(|z_n|\cdot z + z_n)$$

and suppose that $\{f_n(z)\}$ is normal at $z = 0$. Then, there is a δ, $\delta > 0$, such that $f(z)$ converges uniformly to some function $g(z)$ on the sequence of discs $D(z_n, \delta)$. If we take a c_1 in (5) and a c_2 in (8) such that

$$1 > c_1 > 1 - 2\delta\cdot\cos\theta\,, \qquad 1 < c_2 < 1 + 2\delta\cdot\cos\theta\,,$$

then (5) and (8) show that $g(z) \equiv 0$ and $g(z) \equiv \infty$, respectively, which is a contradiction. Hence, we see that $\{f_n(z)\}$ is not normal at $z = 0$. Now, Ostrowski [13, Satz 1 and p. 234] gives that $\chi(\theta)$, $|\theta|<\pi/2$, is Julia direction of $f(z)$. It is easy to see that $(\pm\pi/2)$ is also a Julia direction of $f(z)$.

Next, we shall prove (b). For any r, $r \geqq b_1$, take an n such that $b_n \leqq r < b_{n+1}$. Then, for the number $n(r, 1/f)$ of zeros of $f(z)$ inside the circle $\{z: |z| \leqq r\}$, we have

$$n\left(r, \frac{1}{f}\right) = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n-1} a_k + a_n = (1 + o(1))\cdot a_n \leqq (1 + o(1))\cdot\log^{1+\rho} r\,,$$

from (6). Thus,

$$r\cdot\int_r^\infty \frac{n(t, 1/f)}{t^2}\,dt \leqq (1 + o(1))\cdot(\log^{1+\rho} r)\,. \qquad (r\to\infty)$$

So we get

$$\begin{aligned}\log M(r, f) &= \log f(-r) = \int_0^\infty \log\left(1 + \frac{r}{t}\right) dn\left(t, \frac{1}{f}\right)\\ &= r\cdot\int_0^\infty \frac{n(t, 1/f)}{t(t+r)}\,dt \leqq \int_0^r \frac{n(t, 1/f)}{t}\,dt + r\int_r^\infty \frac{n(t, 1/f)}{t^2}\,dt\\ &= O(\log^{2+\rho} r)\,.\end{aligned}$$

Remark 2. The property (5) shows that Lemma 9 holds only for the functions having some smoothness of growth of $M(r, f)$. From this fact, we can see that this example also satisfies

$$\varlimsup_{r\to\infty} \frac{\log M(r, f)}{\log^2 r} = +\infty$$

by the fact of Hayman [9, p. 143].

§ 3. The set of Julia direction and growth of $M(r, f)$

It is easily observed that the set of Julia directions of a transcendental entire function is a non-empty closed set. Polya [14] showed that for any given non-empty closed set E, there exists an entire function $f(z)$ of order ∞ having just E as the set of Julia directions of $f(z)$. Anderson and

Clunie [2, Theorem 1] also gave this sort of an example in the case $\rho = 0$. Drasin and Weitsman [6, Theorem 1 and p. 209–210] constructed an example in the case $0 < \rho \leqq 1/2$. But their construction depends on a general theorem of Levin [12, p. 95 and Chapter 2] and hence the condition $\rho > 0$ is essential to show that a direction is a Julia direction.

The example in the following Theorem 4 generalizes the example of Anderson and Clunie [2] in the sense not only that it has order $\rho = 0$ but also that it has an arbitrarily given growth subject to (B).

LEMMA 10 (Valiron [15, p. 130], Edrei and Fuchs [7, Theorem 1]). *Let $\Lambda(r)$ be a function*

$$\Lambda(r) = \text{constant} + \int_{r_0}^{r} \frac{\psi(t)}{t} dt, \qquad (r \geqq r_0 > 0)$$

where $\psi(t)$ is a non-negative, non-decreasing and unbounded function.

Assume further that

$$\Lambda(r) \leqq r^K \tag{9}$$

for some K and all sufficiently large r.

Then, there exists an entire function $g(z)$ such that

$$\log M(r, g) \sim \Lambda(r) \sim N\left(r, \frac{1}{g}\right) \qquad (r \to \infty)$$

where

$$N\left(r, \frac{1}{g}\right) = \int_0^r \frac{n(t, 1/g) - n(0, 1/g)}{t} dt + n\left(0, \frac{1}{g}\right) \cdot \log r .$$

LEMMA 11 (Hayman [9, Theorem 6]). *Let $f(z)$ be an entire function. Then, $f(z)$ satisfies*

$$T(r, f) \sim T(2r, f) \qquad (r \to \infty) \tag{10}$$

if and only if $f(z)$ has genus zero and further

$$n\left(r, \frac{1}{f}\right) = o\left(N\left(r, \frac{1}{f}\right)\right) \qquad (r \to \infty),$$

where $T(r, f)$ denotes the characteristic function of $f(z)$.

Remark 3. That (B) is equivalent to (10) is easily seen from the inequality

$$T(r, f) \leqq \log^+ M(r, f) \leqq \frac{R + r}{R - r} T(r, f) \qquad (0 \leqq r < R)$$

(see [8, p. 18]).

THEOREM 4. *Let E be any non-empty closed set on $[0, 2\pi)$ and let $\Lambda(r)$ be a function given by*

$$\Lambda(r) = \text{constant} + \int_{r_0}^{r} \frac{\psi(t)}{t} dt \qquad (r \geqq r_0 > 0)$$

where $\psi(t)$ is a non-negative, non-decreasing and unbounded function.

Further, in the case

$$\varlimsup_{r\to\infty} \frac{\Lambda(r)}{\log^2 r} = +\infty ,$$

we assume that

(11) $$\Lambda(2r) \sim \Lambda(r) \qquad (r \to \infty) .$$

Then, there exists an entire function $f(z)$ such that

$$\log M(r, f) \sim \Lambda(r) \qquad (r \to \infty)$$

and E is precisely the set of Julia directions of $f(z)$.

Proof. First of all we remark by an argument of Hayman [9, p. 130] that (9) is satisfied for any positive K if (11) holds.

Now, as in Edrei and Fuchs [7] we construct the function

$$g(z) = \prod_{j=1}^{\infty} \left\{1 + \left(\frac{z}{t_j}\right)^{q_j}\right\}$$

such that

(12) $$\log M(r, g) \sim \Lambda(r) \sim N\left(r, \frac{1}{g}\right) \qquad (r \to \infty)$$

where $\{t_j\}$ and $\{q_j\}$ are the sequences chosen in [5, p. 388]. We take a countable dense subset $\{\theta_1, \theta_2, \theta_3, \cdots\}$ of E and put

$$z_{j,k} = t_j e^{i\theta_k} \qquad (k = 1, 2, 3, \cdots, q_j ; j = 1, 2, 3, \cdots) .$$

We define the required function $f(z)$ by

$$f(z) = \prod_{j=1}^{\infty} \prod_{k=1}^{k=q_j} \left(1 - \frac{z}{z_{j,k}}\right) .$$

First, in the case

$$\overline{\lim_{r\to\infty}}\frac{\Lambda(r)}{\log^2 r}=+\infty\,,$$

we have from (11) and (12) that

$$\log M(2r,g)\sim\log M(r,g)\qquad(r\to\infty)\,.$$

Hence, by Lemma 11 and Remark 3,

$$n\left(r,\frac{1}{g}\right)=o\left(N\left(r,\frac{1}{g}\right)\right)\qquad(r\to\infty)$$

and $g(z)$ has genus zero. Again from Lemma 11, Remark 3, (12) and the fact of Hayman [9, p. 133],

$$\log M(2r,f)\sim\log M(r,f)\sim N\left(r,\frac{1}{f}\right)=N\left(r,\frac{1}{g}\right)\sim\Lambda(r)\quad(r\to\infty)\,. \tag{13}$$

Thus, this $f(z)$ satisfies

$$\log M(r,f)\sim\Lambda(r)\qquad(r\to\infty)\,.$$

In the case

$$\overline{\lim_{r\to\infty}}\frac{\Lambda(r)}{\log^2 r}<+\infty\,,$$

from (12) and Hayman [9, p. 143],

$$n\left(r,\frac{1}{g}\right)=o\left(N\left(r,\frac{1}{g}\right)\right)\qquad(r\to\infty)$$

and hence

$$n\left(r,\frac{1}{f}\right)=o\left(N\left(r,\frac{1}{f}\right)\right)\qquad(r\to\infty)\,.$$

Thus by the same argument, this $f(z)$ satisfies

$$\log M(2r,f)\sim\log M(r,f)\sim\Lambda(r)\qquad(r\to\infty)\,. \tag{14}$$

Now, it is easily observed from (13) and (14) and Theorem 3 that E is precisely the set of Julia directions of $f(z)$.

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