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# EXISTENCE AND BIFURCATION OF SOLUTIONS FOR FREDHOLM OPERATORS WITH NONLINEAR PERTURBATIONS

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## Introduction

In this paper we shall discuss nonlinear eigenvalue problems for the equations of the form

$$(1) \qquad Lx + \lambda K(x) - M(x, \lambda) = 0\,, \quad x \in X\,, \quad \lambda \in R\,,$$

where $L$ is a linear operator on a real Banach space $X$ with non-zero kernel, $K(\cdot)$ is a linear or nonlinear operator on $X$ and $M(\cdot, \cdot)$ is an operator from $X \times R$ into $X$. Equations of the form (1) arise in various fields of physics and engineering. For example, if $L = \Delta - \mu$, $K(x) = f|x|^{k-1}x$ and $M(x, \lambda) = g|x|^{m-1}x$, then the equation (1) is the nonlinear stationary equation of the Klein-Gordon type.

A solution of (1) means a pair $(x, \lambda) \in X \times R$ satisfying the equation (1). The main purpose of this paper is to prove the existence of solutions of (1) and to investigate the local structure of the solution sets.

An important case is the one where $K(0) = 0$ and $M(x, \lambda) = o(\|x\|)$ uniformly in $\lambda \in \Lambda$, $\Lambda$ being an interval containing zero. Clearly, $(0, \lambda)$, for any $\lambda \in \Lambda$, is a solution of (1); this solution is called a trivial solution. We are interested in determining conditions for the existence of nontrivial solutions of (1).

We say that $(0, 0)$ is a bifurcation point of (1) with respect to the line of trivial solutions, if every neighbourhood of $(0, 0)$ in $X \times R$ contains non-trivial solutions. The bifurcation problems which are reduced to equations of the type (1) have been discussed by many authors. For example, Rabinowitz [7] has considered the case where $L = I + K$ with $K$ being compact and linear. Ize [2] has also treated the case where $L$ is a Fredholm operator of index zero and $K$ is the identity operator. They

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have shown that if the generalized kernel of $L$ has odd dimension, then $(0, 0)$ is a bifurcation point. When the generalized kernel of $L$ has even dimension, one needs much more information on $M(x, \lambda)$ as well as $L$ and $K(x)$ (see, Dancer [1], in which he treats the case where $L$ is a Fredholm operator of index zero and $K$ is the identity operator).

Our main interest lies in the treatment of (1) in the case where $K(x)$ as well as $M(x, \lambda)$ is a nonlinear (possibly linear) operator. For the operators $L$ and $M(x, \lambda)$ we assume that $L$ is a semi-simple Fredholm operator of index zero and $M(x, \lambda) = M(x)$. First, we assume that $K$ and $M$ are homogeneous operators with degree $k$ and $m$, respectively, where $0 < k < m$, $1 < m$. Let $P$ be the projection from $X$ onto $N(L)$ (see § 1). We assume that $PK$ is non-degenerate (which is introduced by Dancer [1]), i.e.,

$$PK(x) = 0 \quad \text{for } x \in N(L) \text{ implies } x = 0 .$$

Under this assumption, it is possible to define a map $K_S$ from the unit sphere $S$ of $N(L)$ to $S$ itself by $K_S(x) = PK(x)/\|PK(x)\|$ $(x \in S)$. Denote the degree of mapping $f\colon S \to S$ by $\deg f$.

We can show that $(0, 0)$ is a bifurcation point of (1) if one of the following conditions holds:

( i ) $d = \dim N(L)$ is odd and $\deg K_S \neq 0$.*)

(ii) $d$ is odd, $PM$ is non-degenerate and $\deg M_S \neq 0$.

(iii) $d$ is even, $PM$ is non-degenerate and $\deg K_S \neq \deg M_S$.

(See Theorem 1.1 in § 1.)

Next, instead of the homogeneity condition for $K$ and $M$, we assume that

$$\|K(x)\| = O(|||x|||) \quad \text{and} \quad \|M(x)\| = o(|||x|||) , \qquad \text{as } |||x||| \longrightarrow 0 ,$$

where $|||\cdot|||$ denotes the graph norm of $D(L)$. In this case, the existence of bifurcation can be derived similarly. Furthermore, our methods developped in this paper can be applied to more general equations of the form

$$(2) \qquad Lx + \lambda K(x) - M(x) + R(x, \lambda) = 0 ,$$

where $R(x, \lambda)$ is, in a sence, a 'small' perturbation of $M(x)$.

The contents of this paper are summarized as follows. In Section 1, we shall give some preliminaries and an existence result (Theorem 1.1)

*) Throughout this paper, we drop conditions on $\deg K_S$ and $\deg M_S$ if $d=1$.

of solution sets for (1) with homogeneous nonlinearity. Section 2 is devoted to the proof of Theorem 1.1. The main tools used in the proof are the implicit function theorem in a Banach space, the Lefschetz coincidence formula and some theorems on degree of mappings on spheres. In Section 3, using the technics developped in Section 2, we can show that there exists the bifurcation for (1). Section 4 treats more general equations of the form (2). Finally, we shall apply our results to nonlinear elliptic partial differential equations in Section 5.

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## §1. Existence results for homogeneous nonlinearity

Let $X$ be a real Banach space with norm $\|\cdot\|$. We consider the equations of the form

$$Lx + \lambda K(x) - M(x) = 0 , \tag{1.1}$$

where $\lambda \in R$, $x \in X$, $L$ is a linear operator and $K$, $M$ are nonlinear operators in $X$. Throughout this paper we put the following assumptions on $L$:

(a.1) $L$ is a Fredholm operator of index zero and $d = \dim N(L) = \operatorname{codim} R(L) \neq 0$, where $N(L)$ and $R(L)$ denote the kernel of $L$ and the range of $L$ respectively.

(a.2) $N(L) = N(L^n)$ and $R(L) = R(L^n)$ for $n = 1, 2, \cdots$.

Let $D(L)$ denote the domain of $L$. $D(L)$ is a Banach space equipped with the graph norm of $L$; $|||x||| = \|x\| + \|Lx\|$ for $x \in D(L)$. Nonlinear operators $K$ and $M$ satisfy the following assumptions:

(a.3) $K$ and $M$ are defined on an open set U of $D(L)$ containing the unit sphere $S$ of $N(L)$. Moreover, $K$ and $M: U \to X$ are continuously Fréchet differentiable (which is denoted by $K, M \in C^1(U \to X)$).

(a.4) If $x \in U$ and $\alpha > 0$, then $\alpha x \in U$, $K(\alpha x) = \alpha^k K(x)$ and $M(\alpha x) = \alpha^m M(x)$, where $k$ and $m$ are real numbers such that $m \neq 0, 1, k$.

By the assumptions (a.1) and (a.2), $X$ can be decomposed as

$$X = N(L) + R(L)$$

(see Ize [2] and Kato [3]). The projection $P: X \to N(L)$ is given by

$$P = \frac{1}{2\pi i} \int_c (\lambda - L)^{-1} d\lambda ,$$

where $c$ is a small circle around the origin in $C$ (see Dancer [1] and Kato [3]). Clearly, $I - P$ is the projection from $X$ onto $R(L)$. The expression given above proves that $P$ commutes with $L$.

By the assumptions (a.3) and (a.4), we have $N(L) - \{0\} \subset D(K) \cap D(M)$. If

(c.1) $$PK(x) \neq 0 \qquad \text{for } x \in N(L) - \{0\} ,$$

we can define a map $K_S: S \to S$ by

$$K_S(x) = \frac{PK(x)}{\|PK(x)\|} \qquad \text{for } x \in S .$$

Similarly, we can define $M_S: S \to S$ if

(c.2) $$PM(x) \neq 0 \qquad \text{for } X \in N(L) - \{0\} .$$

For a continuous map $f: S \to S$, $\deg f$ denotes the degree of $f$. For the definition of the degree, we refer to Schwartz [5] and Nirenberg [4]. We shall summarize several properties of the degree in § 2.3.

We are now ready to state

THEOREM 1.1. *Suppose that one of the following assumptions is satisfied:*
(i) $d = \dim N(L)$ *is odd,* (c.1) *holds and* $\deg K_S \neq 0$.
(ii) $d$ *is odd,* (c.1) *and* (c.2) *hold and* $\deg K_S \neq 0$ *or* $\deg M_S \neq 0$.
(iii) $d$ *is even,* (c.1) *and* (c.2) *hold and* $\deg K_S \neq \deg M_S$.

*Then, there exists a continuum* $\{(x(e), \lambda(e)) \mid 0 \leqq e \leqq \rho\}$ *of solutions of* (1.1) *of the form*

$$\begin{cases} x(e) = e^{1/(m-1)}\{y(e) + ez(e)\} , & y(e) \subset N(L),\ z(e) \subset R(L) , \\ \lambda(e) = e^{(m-k)/(m-1)}a(e) , & a(e) \subset R , \end{cases}$$

*where* $\rho > 0$, $\|y(e)\| = 1$ *and* $\|z(e)\|$ *and* $|a(e)|$ *are bounded. In particular, under the assumption* (ii) *or* (iii), $|a(e)|$ *is bounded also from below.*

*Remark* 1.1. The correspondence $e \to (x(e), \lambda(e))$ is set-valued. In other words, $(x(e), \lambda(e))$ is a subset of $X \times R$ for each $0 \leqq e \leqq \rho$.

COROLLARY 1.2. *In the case of* (ii) *or* (iii) *in Theorem* 1.1, *the following holds: Along the solution set obtained in Theorem* 1.1,

(i) *if* $m > 1$ *and* $m > k$, *then* $x \to 0$ *as* $\lambda \to 0$.
(ii) *if* $m > 1$ *and* $m < k$, *then* $x \to 0$ *as* $\lambda \to \infty$.
(iii) *if* $m < 1$ *and* $m > k$, *then* $x \to \infty$ *as* $\lambda \to \infty$.
(iv) *if* $m < 1$ *and* $m < k$, *then* $x \to \infty$ *as* $\lambda \to 0$.
*In the case of* (*i*) *in Theorem 1.1, both* (*i*) *and* (*iv*) *hold.*

*Remark* 1.2. Let $m > 1$, $m > k > 0$ and $K(0) = M(0) = 0$. Since the curve $\{(0, \lambda) \mid \lambda \in R\}$ is the line of trivial solutions of (1.1), it follows from Corollary 1.2 (i) that $(0, 0)$ is a bifurcation point of (1.1).

## § 2. Proof of Theorem 1.1

**2.1.** Reduction to finite dimension.

Since $X$ is decomposed as $X = N(L) + R(L)$ (see § 1), any $x \in X$ can be written as

$$x = Px + (I - P)x \equiv x_1 + x_2 ,$$

where $P$ is the projection from $X$ onto $N(L)$. Note that $(I - P)L = L$ and $Lx_1 = 0$. So (1.1) is equivalent to the following system:

$$\lambda PK(x_1 + x_2) - PM(x_1 + x_2) = 0 , \tag{2.1}$$

$$Lx_2 + \lambda(I - P)K(x_1 + x_2) - (I - P)M(x_1 + x_2) = 0 . \tag{2.2}$$

Now we put by the use of a parameter $\varepsilon \geqq 0$,

$$\lambda = \varepsilon^{m-k}a , \quad x_1 = \varepsilon y \quad \text{and} \quad x_2 = \varepsilon^m z , \quad \text{where} \quad \|y\| = 1 .$$

We substitute these expressions in (2.1) and (2.2) and devide them by $\varepsilon^m$. Then, by the homogeneity of $K$ and $M$ ((a.4) in § 1), we obtain

$$aPK(y + \varepsilon^{m-1}z) - PM(y + \varepsilon^{m-1}z) = 0 ,$$
$$Lz + a(I - P)K(y + \varepsilon^{m-1}z) - (I - P)M(y + \varepsilon^{m-1}z) = 0 .$$

By introducing a new parameter $e = \varepsilon^{m-1}$, it is easy to see that the above system is equivalent to

$$aPK(y + ez) - PM(y + ez) = 0 , \tag{2.3}$$

$$Lz + a(I - P)K(y + ez) - (I - P)M(y + ez) = 0 \tag{2.4}$$

with

$$\lambda = e^{(m-k)/(m-1)}a , \quad x_1 = e^{1/(m-1)}y \quad \text{and} \quad x_2 = e^{m/(m-1)}z , \tag{2.5}$$

where $e \geqq 0$ and $\|y\| = 1$.

Note that $L: D(L) \cap R(L) \to R(L)$ is an isomorphism. Then, for arbitrary $y, a$ and $e = 0$, (2.4) has a unique solution $z$. We denote this solution by $h_0(y, a) \in C^1(S \times R)$. The Fréchet derivative of the left-hand side of (2.4) with respect to $z$ at $(z, y, a, e) = (h_0(y, a), y, a, 0)$ is the isomorphism $L: D(L) \cap R(L) \to R(L)$. We can therefore apply the implicit function theorem for (2.4) and obtain the unique solution $z$ for $e$ small enough. We denote this solution by $h(y, a, e)$, where $h(y, a, 0) = h_0(y, a)$. Substituting this function in (2.3), we have

$$aPK(y + eh(y, a, e)) - PM(y + eh(y, a, e)) = 0 . \tag{2.6}$$

We call (2.6) the *bifurcation equation.* A solution of (2.6) is an element $(y, a, e) \in S \times R \times R$. By the preceding argument, we have:

PROPOSITION 2.1. *If* (2.6) *has a solution* $(y, a, e)$, *then* (1.1) *has a solution* $(x(y, a, e), \lambda(a, e)) \in X \times R$ *given by*

$$x(y, a, e) = e^{1/(m-1)}\{y + eh(y, a, e)\} \quad \textit{and} \quad \lambda(a, e) = e^{(m-k)/(m-1)}a ,$$

*where* $h \in C^1(D(h) \to D(L) \cap R(L))$ *with* $D(h) \subset S \times R \times R$ *is given above.*

*Particularly, if* (2.6) *has a family of solutions* $\{(y(e), a(e), e) \mid 0 \leqq e \leqq \rho\}$, *then* (1.1) *has a family of solutions* $\{(x(e), \lambda(e)) \mid 0 \leqq e \leqq \rho\}$, *where*

$$x(e) = x(y(e), a(e), e) \quad \textit{and} \quad \lambda(e) = \lambda(a(e), e) .$$

For any bounded interval of $a$, we have $y + eh_0(y, a) \in U$ (see (a.3) in § 1) if we choose $e$ small enough. Hence, $K(y + ez)$ and $M(y + ez)$ are differentiable with respect to $z$ at $(h_0(y, a), y, a, e)$ and $z$-derivative of them are sufficiently small with respect to the uniform topology of the space of bounded linear operators from $D(L) \cap R(L)$ to $R(L)$. Therefore, the $z$-derivative of the left-hand side of (2.4) has the inverse because it lies sufficiently near $L: D(L) \cap R(L) \to R(L)$. Then we obtain:

$$\begin{cases} \text{For any } r > 0, \text{ there exists a positive number } \rho = \rho(r) \\ \text{such that } h(y, a, e) \text{ can be defined in } S \times [-r, r] \times [-\rho, \rho] . \end{cases} \tag{2.7}$$

In other words, (2.4) has a unique solution $h(y, a, e)$ for every $(y, a, e) \in S \times [-r, r] \times [-\rho, \rho]$.

*Remark* 2.1. As $K$ and $M$ are not defined at the origin, we can not directly apply the implicit function theorem to (2.2) in order to solve for $x_2$.

*Remark* 2.2. If $K$ and $M$ are of $C^n$-class ($n = 1, 2, 3, \cdots$) or analytic in $U$, then $h(y, a, e)$ is of $C^n$-class or analytic.

**2.2.** A property of the bifurcation equation.

In the following, we shall obtain important properties derived from (c.1, 2) in § 1, which are used in order to define admissible domains to topological degree.

PROPOSITION 2.2. (1) *Suppose that* (c.1) *holds. Then there exists a domain* $D \equiv S \times (-r, r) \times (-\rho, \rho) \subset S \times R_a \times R_e$ *(where* $\rho = \rho(r)$ *depends on* $r$*) with the properties*:

(i) *If* $(y, a, e) \in D$, *then both* $K(y + eh(y, a, e))$ *and* $M(y + eh(y, a, e))$ *can be defined.*

(ii) $\|PM(y + eh(y, a, e))\|/\|PK(y + eh(y, a, e))\| < r$ *for all* $(y, a, e) \in D$.

(2) *Assume that* (c.2) *holds in addition to* (c.1). *Then* (ii) *is replaced by the following stronger inequality*

(iii) $r' < \|PM(y + eh(y, a, e))\|/\|PK(y + eh(y, a, e))\| < r$ *for all* $(y, a, e) \in D$, *where* $r > r' > 0$.

*Proof.* Let $(y, a, e) \in S \times [-r, r] \times [-\rho, \rho]$, where $r$ and $\rho$ satisfy (2.7). We shall determine $r$ and $\rho$ so that the statements of this proposition hold. We define a function $I(y, a, e)$ by

$$PK(y + h(y, a, e)) = PK(y) + eI(y, a, e) .$$

Then we have

$$I = \int_0^1 PK_x(y + teh(y, a, e))dt\, h(y, a, e) ,$$

where $K_x$ is the Fréchet differential of $K$. In fact,

$$\begin{aligned} PK(y + eh) - PK(y) &= \int_0^1 \frac{d}{dt} PK(t(y + eh) + (1 - t)y)dt \\ &= \int_0^1 PK_x(t(y + eh) + (1 - t)y)dt\, eh . \end{aligned}$$

Since $K \in C^1(U \to K)$, we get $I \in C^0$ if $y + teh(y, a, e) \in U$ for all $t \in [0, 1]$, which is possible by choosing $\rho$ small enough. Similarly we have

$$PM(y + eh(y, a, e)) = PM(y) + eJ(y, a, e) ,$$

where $J \in C^0$ is given by

$$J = \int_0^1 PM_x(y + teh(y, a, e)dt\, h(y, a, e)\ .$$

$I(y, a, e)$ and $J(y, a, e)$ are uniformly bounded for $y \in S$, $a \in (-r, r)$ (with any $r > 0$) and $e\,(\to 0)$. Therefore, if we choose $r$ and $r'$ such that

$$r > \max_S \|PM(y)\|/\min_S \|PK(y)\|\ , \qquad r' < \min_S \|PM(y)\|/\max_S \|PK(y)\|$$

and take $\rho$ small enough, we obtain

$$r' < \|PM(y + eh(y, a, e))\|/\|PK(y + eh(y, a, e))\| < r$$

for $S \times (-r, r) \times (\rho, \rho)$. If (c.2) also holds, we can choose $r' > 0$ since $\min_{y \in S} \|PM(y)\| > 0$. Thus the proof is completed.

By Proposition 2.2,

(2.8) The equation (2.6) has no solution on $S \times \{r, -r, r', -r'\} \times (-\rho, \rho)$ .

**2.3.** Preliminaries on degree theory.

Let $M$ and $N$ be two oriented manifolds of dimension $n$ with boundary $\partial M$ and $\partial N$ respectively. For a continuous map $f$ from $\overline{M}$ ($\equiv M \cup \partial M$) to $\overline{N}$ ($\equiv N \cup \partial N$), $M$ such that $\overline{M}$ is compact and a point $p \in N$ such that $f(\partial M) \not\ni p$, $\deg(f, M, p)$ is defined and it takes a value of integers (see Nirenberg [4] and Schwartz [5]). $\deg(f, M, p)$ is constant if $p$ runs over the same connected component of $N - f(\partial M)$. Therefore, if $N$ is connected and $f(\partial M) \subset \partial N$, in particular, if $\partial M = \phi$, then $\deg(f, M, p)$ is independent of $p \in N$. In this case, we define $\deg(f, M)$ by $\deg(f, M, p)$. If $f(q) = p$ and there exists a neighbourhood $\Omega$ of $q$ such that $f(\overline{\Omega} - \{q\}) \not\ni p$, we define $\mathrm{ind}(f, q)$ by $\mathrm{ind}(f, q) = \deg(f, \Omega, p)$.

Let $C(\overline{M} \to \overline{N})$ denote the set of all continuous functions from $\overline{M}$ to $\overline{N}$. For $f$ and $g \in C(\overline{M} \to \overline{N})$ satisfying $f(\partial M) \not\ni p$ and $g(\partial M) \not\ni p$, if there exists a continuous function $F \in C(\overline{M} \times I \to \overline{N})$, ($I = \{0 \leqq t \leqq 1\}$), such that $F|_{t=0} = f$, $F|_{t=1} = g$ and $F(\partial M \times I) \not\ni p$, then we say that $f$ is homotopic to $g$ with respect to $(M, p)$ and denote by $f \simeq g\,(M, p)$. $F$ is called a homotopy function. $\deg(\cdot, M, p)$ is constant on the same homotopy class. If $\deg(f, M, p) \neq 0$, then $f(x) = p$ for some $x \in M$.

For $f$ and $g \in C(\overline{M} \to \overline{N})$ satisfying $\{x \in \partial M \mid f(x) = g(x)\} = \phi$, the coincidence index $I(f, g; M, N)$ is defined, if $\overline{M}$ is compact, and takes a value of integers (see Nakaoka [9], chap. 3.) If $N$ is an open set of $R^n$, $I(f, g; M, N) = \deg(f - g, M, 0)$, where $(f - g)(x) = f(x) - g(x)$. If there is no confusion, we sometimes write $\deg f$ instead of $\deg(f, M)$ and $I(f, g)$ instead of $I(f, g; M, N)$.

For $f$ and $g \in C(S^n \to S^n)$, where $S^n$ denote the $n$-sphere, the formula

$$I(f, g) = \deg f + (-1)^n \deg g \tag{2.9}$$

holds. (2.9) is proved by using the Lefschetz coincidence formula [9]. If $I(f, g; M, N) \neq 0$, then $f(x) = g(x)$ for some $x \in M$.

Let $F \in C(\overline{M} \times I \to \overline{N})$ satisfy $F(\partial M \times I) \not\ni p$. A solution of $F(x, t) = p$ is a pair $(x, t) \in M \times I$. If $\deg (F|_{t=0}, M, p) \neq 0$, then there exists a connected set $C$ of solutions such that $P_I(C) = I$, where $P_I$ is the natural projection from $M \times I$ onto $I$.

**2.4.** Construction of the family of solutions.

Let $N(L)$ and $S$ be defined in § 1. Recall that $d = \dim N(L) < \infty$. For an orientation of $N(L)$, we define the orientation of $S \times R$, ($R = (-\infty, \infty)$) so that the natural injection: $(y, a) \to ay$ from $S \times (0, \infty)$ into $N(L)$ does not change the orientation. We define a continuous map $j: S \times R \to N(L)$ by $(y, a) \to ay$.

LEMMA 2.3. *For $q \in N(L)$ and $c > 0$ such that $0 < \|q\| < c$,*

(i) $\deg (j, S \times (0, c), q) = 1$,

(ii) $\deg (j, S \times (-c, 0), q) = (-1)^{d+1}$.

*For $q \in N(L)$ and $c > 0$ such that $\|q\| < c$,*

(iii) $\deg (j, S \times (-c, c), q) = 2$ *if $d$ is odd.*

(iv) $\deg (j, S \times (-c, c), q) = 0$ *if $d$ is even.*

*Proof.* (i) is trivial by the definitions of the orientation of $S \times R$ and of the map $j$. We shall prove (ii). It is easy to see that $j^{-1}(q) = (\bar{q}, \|q\|), (-\bar{q}, -\|q\|)$, where $\bar{q} = q/\|q\|$. By (i), $\operatorname{ind} (j, (\bar{q}, \|q\|)) = 1$. Let $T$ and $T_1$ be the antipodal operators of $S$ and $R$ respectively, i.e., $Ty = -y$, $y \in S$ and $T_1 a = -a$, $a \in R$. It is well known that $\deg T = (-1)^d$ and $\deg (T_1, (-c, c)) = -1$. Since $(\bar{q}, \|q\|) = (T \times T_1)(-\bar{q}, -\|q\|)$,

$$\begin{aligned} \operatorname{ind} (j, (-\bar{q}, -\|q\|)) &= \operatorname{ind} (j, (\bar{q}, \|q\|)) \operatorname{ind} (T \times T_1, (-\bar{q}, -\|q\|)) \\ &= \operatorname{ind} (T, -\bar{q}) \operatorname{ind} (T_1, -\|q\|) = (\deg T) (\deg T_1) \\ &= (-1)^{d+1} , \end{aligned}$$

which proves (ii). If $q \neq 0$, by the additivity of the degree,

$$\begin{aligned} \deg (j, S \times (-c, c), q) &= \deg (j, S \times (-c, 0) \cup S \times (0, c), q) \\ &= \deg (j, S \times (-c, 0), q) + \deg (j, S \times (0, c), q) , \end{aligned}$$

which, together with (i) and (ii), implies (iii) and (iv). If $q = 0$, the continuity of the degree gives

$$\deg(j, S \times (-c, c), 0) = \lim_{q \to 0} \deg(j, S \times (-c, c), q) ,$$

from which the assertions (iii) and (iv) follow.

Let $f$ be a continuous function from $S$ into $N(L) - \{0\}$. We define a continuous function $\mathrm{id} \cdot f \colon S \times R \to N(L)$ by $(y, a) \to af(y)$. Let $\bar{f} = f(\cdot)/\|f(\cdot)\|$ for $f \in C(S \to N(L) - \{0\})$, so $\bar{f}$ is a function from $S$ to $S$.

LEMMA 2.4. *For $q \in N(L)$ and $c > 0$ such that*

$$0 < \|q\| < c \min_{y \in S} \|f(y)\| \equiv c' ,$$

(i) $\deg(\mathrm{id} \cdot f, S \times (0, c), q) = \deg \bar{f}$,

(ii) $\deg(\mathrm{id} \cdot f, S \times (-c, 0), q) = (-1)^{d+1} \deg \bar{f}$.

*For $q \in N(L)$ and $c > 0$ such that $\|q\| < c \min_{y \in S} \|f(y)\|$,*

(iii) $\deg(\mathrm{id} \cdot f, S \times (-c, c), q) = 2 \deg \bar{f}$ *if $d$ is odd,*

(iv) $\deg(\mathrm{id} \cdot f, S \times (-c, c), q) = 0$ *if $d$ is even.*

*Proof.* Define $g \colon S \times R \to S \times R$ by $(y, a) \to (\bar{f}(y), a\|f(y)\|)$. By using Lemma 2.3 and the properties of the degree for Cartesian products and compositions of maps, we have

$$\begin{aligned}
\deg(\mathrm{id} \cdot f, S \times (0, c), q) &= \deg(j \circ g, S \times (0, c), q) \\
&= \deg(g, S \times (0, c), (\bar{q}, \|q\|)) \\
&= \deg(\bar{f} \times \mathrm{id}, S \times (0, c'), (\bar{q}, \|q\|)) \\
&= \deg(\bar{f}, S, \bar{q}) \deg(\mathrm{id}, (0, c'), \|q\|) \\
&= \deg \bar{f} ,
\end{aligned}$$

which proves (i). We can prove similarly (ii), (iii) and (iv), so we omit the proof.

Now we shall prove the existence of solutions of the equation (2.6). Recall that we defined the maps $K_S$ and $M_S \colon S \to S$ in § 1.

THEOREM 2.5. *Suppose that* (c.1) *holds and that $d$ is odd and* $\deg K_S \neq 0$.

*Then the equation* (2.6) *has a family of solutions* $\{(y(e), a(e), e) \mid 0 \leqq e \leqq \rho\}$.

*Proof.* Let $(y, a, e) \in D = S \times (-r, r) \times (-\rho, \rho)$, which is defined in Proposition 2.2. We define $F_e(y, a) \colon S \times R \to N(L)$ by

$$F_e(y, a) = aPK(y + eh(y, a, e)) - PM(y + eh(y, a, e))$$

(see (2.7)). Then it follows from (2.8) that

(2.10) $$F_e \simeq F_0 \qquad (S \times (-r, r), 0)\ ,$$

where $F_0(y, a) = aPK(y) - PM(y)$ since $h(y, a, 0) = 0$. Similarly, the equation

$$f_t(y, a) \equiv aPK(y) - (1 - t)PM(y) = 0$$

has no solution on $S \times \{-r, r\}$ for all $0 \leqq t \leqq 1$. Hence, we have

(2.11) $$f_1 \simeq f_0 \qquad (S \times (-r, r), 0)\ .$$

By Lemma 2.4 (iii), for the map $\mathrm{id} \cdot PK: (y, a) \to aPK(y)$, we have

$$\deg (\mathrm{id} \cdot PK, S \times (-r, r), 0) = 2 \deg K_S \neq 0\ .^{*)}$$

Then the homotopy invariance, together with (2.10) and (2.11), implies

$$\begin{aligned} \deg (F_e, S \times (-r, r), 0) &= \deg (F_0, S \times (-r, r), 0) \\ &= \deg (f_1, S \times (-r, r), 0) \\ &= \deg (f_0, S \times (-r, r), 0) \\ &= 2 \deg K_S \neq 0\ , \end{aligned}$$

which asserts the existence of a family of solutions $\{(y(e), a(e), e) \mid 0 \leqq e \leqq \rho\}$ for (2.6).

THEOREM 2.6. *Suppose that* (c.1) *and* (c.2) *hold and that one of the following conditions holds*:

( i ) *$d$ is odd and* $\deg K_S \neq 0$ *or* $\deg M_S \neq 0$.

(ii) *$d$ is even and* $\deg K_S \neq \deg M_S$.

*Then the equation* (2.6) *has a family of solutions* $\{(y(e), a(e), e) \mid 0 \leqq e \leqq \rho\}$.

*Proof.* We shall calculate

$$I_+ \equiv \deg (F_e, S \times (0, r), 0) \quad \text{and} \quad I_- \equiv \deg (F_e, S \times (-r, 0), 0)\ .$$

By (c.1), (c.2) and (2.8), we have

(2.12) $$F_e \simeq F_0 \quad (S \times (0, r), 0) \quad \text{and} \quad (S \times (-r, 0), 0)\ .$$

We define the map $\mathrm{id} \cdot K_S: S \times R \to N(L)$ by $(y, a) \to aK_S(y)$ and the map $c \cdot M_S: S \times R \to N(L)$ by $(y, a) \to cM_S(y)$, where $c$ is a constant of $R$. Then we have

(2.13) $$F_0 \simeq \mathrm{id} \cdot K_S - c \cdot M_S \quad (S \times (0, r), 0) \quad \text{and} \quad (S \times (-r, 0), 0)\ ,$$

*) If $\dim N(L) = 1$, $\deg (\mathrm{id} \cdot PK, \{y\} \times (-r, r), 0) = 1$ or $-1$ for each $y \in S$.

where $0 < c < r$ and $(\mathrm{id}\cdot K_S - c\cdot M_S)(y, a) = aK_S(y) - cM_S(y)$. By using (2.12), (2.13) and the homotopy invariance of the degree, we have

$$I_+ = \deg(\mathrm{id}\cdot K_S - c\cdot M_S, S \times (0, r), 0)\,, \tag{2.14, a}$$

$$I_- = \deg(\mathrm{id}\cdot K_S - c\cdot M_S, S \times (-r, 0), 0)\,. \tag{2.14, b}$$

First, we calculate $I_+$ by using the Lefschetz coincidence index. The fact summarized in § 2.3 yields

$$I_+ = I(\mathrm{id}\cdot K_S, c\cdot M_S; S \times (0, r), N(L))\,.$$

We consider $\mathrm{id}\cdot K_S$ and $c\cdot M_S$ as the following composition of maps:

$$\begin{cases} \mathrm{id}\cdot K_S\colon (y, a) \xrightarrow{K_S \times \mathrm{id}} (K_S(y), a) \xrightarrow{j} aK_S(y)\,, \\ c\cdot M_S\colon (y, a) \xrightarrow{M_S \times c} (M_S(y), c) \xrightarrow{j} cM_S(y)\,. \end{cases} \tag{2.15, a}$$

Since $\deg(j, S \times (0, r), q) = 1$ for $0 < \|q\| < r$, by Lemma 2.3 (i),

$$\begin{aligned} &I(\mathrm{id}\cdot K_S, c\cdot M_S; S \times (0, r), N(L)) \\ &\quad = I(K_S \times \mathrm{id}, M_S \times c; S \times (0, r), S \times (0, r))\,. \end{aligned}$$

By the product formula of the coincidence index, we have

$$\begin{aligned} &I(K_S \times \mathrm{id}, M_S \times c; S \times (0, r), S \times (0, r)) \\ &\quad = I(K_S, M_S; S, S) I(\mathrm{id}, c; (0, r), (0, r)) \\ &\quad = I(K_S, M_S; S, S) \deg(\mathrm{id} - c, (0, r), 0) \\ &\quad = I(K_S, M_S\colon S, S)\,. \end{aligned}$$

Since $\dim N(L) = d$, $S$ is regarded as $S^{d-1}$. Therefore, by (2.9), we have

$$I(K_S, M_S; S, S) = \deg K_S + (-1)^{d-1} \deg M_S\,.$$

Then,

$$I_+ = \deg K_S + (-1)^{d-1} \deg M_S\,. \tag{2.16}$$

Next, we calculate

$$I_- = I(\mathrm{id}\cdot K_S, c\cdot M_S; S \times (-r, 0), N(L))\,.$$

We regard $\mathrm{id}\cdot K_S$ and $c\cdot M_S$ as the following composition of maps:

$$\begin{cases} \mathrm{id}\cdot K_S\colon (y, a) \xrightarrow{K_S \times \mathrm{id}} (K_S(y), a) \xrightarrow{j} aK_S(y)\,, \\ c\cdot M_S\colon (y, a) \xrightarrow{TM_S \times (-c)} (TM_S(y), -c) \xrightarrow{j} cM_S(y)\,. \end{cases} \tag{2.15, b}$$

Similarly as above, we obtain

$$\begin{aligned}
&I(\mathrm{id}\cdot K_S, c\cdot M_S; S\times(-r,0), N(L))\\
&\quad= (-1)^{d+1}I(K_S\times \mathrm{id}, TM_S\times(-c); S\times(-r,0), S\times(-r,0))\\
&\quad= (-1)^{d+1}I(K_S, TM_S; S, S)I(\mathrm{id},(-c);(-r,0),(-r,0))\\
&\quad= (-1)^{d+1}I(K_S, TM_S; S, S)\\
&\quad= (-1)^{d+1}\deg K_S + (-1)^{d-1}\deg T\deg M_S\\
&\quad= (-1)^{d+1}(\deg K_S - \deg M_S)\,.
\end{aligned}$$

Here we used Lemma 2.3 (ii), (2.9), $\deg T=(-1)^d$ and product formula. Thus, we have

$$I_- = (-1)^{d+1}(\deg K_S - \deg M_S)\,. \tag{2.17}$$

Therefore, (2.16) and (2.17), with condition (i) or (ii) of this theorem, implies $I_+\neq 0$ or $I_-\neq 0$. We complete the proof.

*Remark* 2.3. If $d$ is odd, $I_+ + I_- = 2\deg K_S$, which is given in the proof of Theorem 2.5. If $d$ is even, $I_+ + I_- = 0$.

*Proof of Theorem* 1.1. By Theorems 2.5 and 2.6, we have a family of solutions $\{(y(e), a(e), e)\,|\,0\leqq e\leqq\rho\}$ of (2.6). So Proposition 2.1 implies the existence of a family $\{(x(e),\lambda(e))\,|\,0\leqq e\leqq\rho\}$ of solutions of (1.1), which is expressed in the following form,

$$\begin{aligned}
x(e) &= e^{1/(m-1)}\{y(e)+eh(y(e),a(e),e)\}\,,\\
\lambda(e) &= e^{(m-k)/(m-1)}a(e)\,,
\end{aligned}$$

where $y(e)\in S=\{y\in N(L)\,|\,\|y\|=1\}$ and $h\in C^1(S\times[-r,r]\times[-\rho,\rho]\to D(L)\cap R(L))$. Furthermore, it is easy to see from Proposition 2.2 and the construction of solutions that $|a(e)|$ is bounded from above in case of (i) of Theorem 1.1 and that it is bounded also from below in case of (ii) and (iii) of Theorem 1.1.*)

Corollary 1.2 is an immediate consequence of Theorem 1.1 by letting $e\to 0$.

## § 3. Non-homogeneous nonlinearity

We can now extend the result of Theorem 1.1 for the case of non-homogeneous nonlinearity. In this section we suppose that the operators

*) When dim $N(L)=1$, we can simplify the proof of Theorem 1. See Appendix.

$K$ and $M$ satisfy the following assumptions instead of (a.3) and (a.4) in §1:

(a′.3) $K$ and $M \in C^1(U \to X)$, where $U$ is a open set of $D(L)$ containing the origin.

(a′.4) $\|K(x)\| = O(|||x|||)$ and $\|M(x)\| = o(|||x|||)$ as $|||x||| \to 0$.

Furthermore, we put the following conditions instead of (c.1) and (c.2) in §1. Let $V$ be a cone containing a neighbourhood of $N(L)$.

(c′.1) $PK(x) \neq 0$ for $x \in N(L)$, $0 < \|x\| \leqq \rho$ with some $\rho > 0$.

(c′.2) $PM(x) \neq 0$ for $x \in N(L)$, $0 < \|x\| \leqq \rho$, where $\rho$ is given in (c′.1).

(c′.3) For $x \in V$, $\|PM(x)\|/\|PK(x)\| \to 0$ as $\|x\| \to 0$.

We define $K_S$ and $M_S\colon S \to S$ as follows:

$$K_S(y) = PK(\rho y)/\|PK(\rho y)\| \quad \text{and} \quad M_S(y) = PM(\rho y)/\|PM(\rho y)\|, \quad y \in S.$$

We give an analogue of Theorem 1.1.

THEOREM 3.1. *Suppose that one of the following assumptions is satisfied:*
(i) (c′.1) *and* (c′.3) *hold and* $d = \dim N(L)$ *is odd and* $\deg K_S \neq 0$.
(ii) (c′.1, 2, 3) *hold and* $d$ *is odd and* $\deg K_S \neq 0$ *or* $\deg M_S \neq 0$.
(iii) (c′.1, 2, 3) *hold and* $d$ *is even and* $\deg K_S \neq \deg M_S$.
*Then* $(0, 0) \in X \times R$ *is a bifurcation point of* (1.1). *In particular, in case of* (ii) *and* (iii), (0, 0) *is an isolated solution in* $X \times \{0\}$.

*Proof.* We have already shown that (1.1) is equivalent to (2.1) and (2.2) given in §1. By the implicit function theorem, (2.2) can be solved for $x_2 = u(x_1, \lambda)$ in a neibourhood of $(x_1, \lambda) = (0, 0)$ in $N(L) \times R$ (use (a′.3) and (a′.4)). Note that

$$\begin{aligned} Lx_2 &= -\lambda(1-P)K(x_1+x_2) + (1-P)M(x_1+x_2) \\ &= -\lambda(1-P)\left\{K(x_1) + \int_0^1 K_x(x_1+sx_2)ds\right\}x_2 \\ &\quad + (1-P)\left\{M(x_1) + \int_0^1 M_x(x_1+tx_2)dt\right\}x_2 . \end{aligned}$$

By (a′.4), we have

$$\left\{L + \lambda(1-P)\int_0^1 K_x(x_1+sx_2)ds - (1-P)\int_0^1 M_x(x_1+tx_2)dt\right\}x_2$$

$$= -\lambda(1-P)K(x_1) + (1-P)M(x_1)$$
$$= O(|\lambda| \, \|x_1\|) + o(\|x_1\|) .$$

Hence for small $|\lambda|$ and $x_1$, it is easy to see that

$$u(x_1, \lambda) = O(|\lambda| \, \|x_1\|) + o(\|x_1\|) . \tag{3.1}$$

Substituting $x_2 = u(x_1, \lambda)$ in (2.1), we get the bifurcation equation

$$\lambda PK(x_1 + u(x_1, \lambda)) - PM(x_1 + u(x_1, \lambda)) = 0 . \tag{3.2}$$

We put $x_1 = ry$ with $y \in S$ in (3.1) and (3.2). Then

$$\lambda PK(ry + u(ry, \lambda)) - PM(ry + u(ry, \lambda)) = 0 , \tag{3.3}$$

where $u(\cdot, \cdot)$ satisfies

$$u(ry, \lambda) = O(|\lambda| \, r) + o(r) . \tag{3.4}$$

Now suppose that (c$'$.1) and (c$'$.2) hold. We put $\lambda = ag(r)$, where

$$g(r) = \max \{\|PM(x)\|/\|PK(x)\| \mid \|x\| \leqq r,\ x \in V - \{0\}\} . \tag{3.5}$$

Note that $g(r) \to 0$ as $r \to 0$ (by (c$'$.3)). In the case (i), it may happen that $g(r) \equiv 0$. If so, we take as $g(r)$ any increasing continuous function with $g(0) = 0$. Then (3.3) is reduced to

$$ag(r)PK(ry + u(ry, ag(r)) - PM(ry + u(ry, ag(r)) = 0 . \tag{3.6}$$

From (3.4),

$$u(ry, ag(r)) = O(g(r)r) + o(r) = o(r) \qquad \text{as } r \to 0 . \tag{3.7}$$

(3.6) implies that

$$|a| \, g(r) = \|PM(ry + u(ry, ag(r))\|/\|PK(ry + u(ry, ag(r))\| .$$

From this equation, by the aid of (3.5) and (3.7), we obtain the uniform boundedness of $|a|$ as $r \to 0$. So we can choose some $\rho > 0$ such that there is no solution $(y, a)$ of (3.6) on $S \times \{2, -2\}$ for all $r \in (0, \rho)$. (In addition, if (c$'$.2) holds, there is no solution $(y, a)$ of (3.6) on $S \times \{2, 0, -2.\}$)

We define $F_r(y, a)$ by the left-hand side of (3.6). By the argument given above, $\deg(F_r, E, 0)$ is well defined for $E = S \times (-2, 2)$ ($S \times (0, 2)$ and $S \times (-2, 0)$ in case that (c$'$.2) holds). For continuous functions $h\colon R \to R$ and $f\colon S \to N(L)$, we define the map $h \cdot f\colon S \times R \to N(L)$ by $(y, a) \to h(a)f(y)$. By $PK_r$, we denote the map $y \to PK(ry)$ with $y \in S$. Then we obtain similarly as in the proof of Theorem 2.5 that

$$\begin{aligned} F_r &\simeq g(r)\,\mathrm{id}\cdot PK_r - PM_r && (S\times(-2,2),0)\\ &\simeq g(r)\,\mathrm{id}\cdot PK_r && (S\times(-2,2),0)\\ &\simeq \mathrm{id}\cdot K_S && (S\times(-2,2),0)\,. \end{aligned}$$

By Lemma 2.4 (iii) and the homotopy invariance of the degree, we have

$$\deg(F_r, S\times(-2,2),0) = 2\deg K_S \neq 0\,.$$

This proves Theorem 3.1 in the case (i).

Suppose that (c′.1, 2, 3) hold. The analoguous calculations as in the proof of Theorem 2.6 yield

$$\begin{aligned} I_+ &\equiv \deg(F_r, S\times(0,2),0)\\ &= I(g(r)\,\mathrm{id}\cdot PK_r, PM_r; S\times(0,r), N(L))\\ &= \deg K_S + (-1)^{d-1}\deg M_S \end{aligned}$$

and

$$\begin{aligned} I_- &= \deg(F_r, S\times(-2,0),0)\\ &= I(g(r)\,\mathrm{id}\cdot PK_r, PM_r; S\times(-2,0), N(L))\\ &= (-1)^{d+1}(\deg K_S - \deg M_S)\,. \end{aligned}$$

In the cases (ii) and (iii), we have $I_+ \neq 0$ or $I_- \neq 0$. Hence we can obtain the conclusion of Theorem 3.1.

## § 4. Stability for small perturbation of nonlinearity

In this section, we consider the equations of the form

$$Lx + \lambda K(x) - M(x) + R(x,\lambda) = 0\,, \tag{4.1}$$

where $R(x,\lambda)$ is a nonlinear operator which is small in the sense of the assumptions given below (see (r.2) and (r.4)). The assertions in Theorem 1.1 and Theorem 3.1 are also true for the equation (4.1) with a small perturbed nonlinear operator $R(x,\lambda)$.

First we shall extend the result of Theorem 1.1 for (4.1) by putting the following assumptions on $R(x,\lambda)$:

(r.1) $R(x,\lambda)\in C^1(V\times I\to X)$ with $V=\{\alpha x\in D(L)\mid x\in U,\ \alpha>0\}$ where $U$ is the neighbourhood of $S$ defined in (a.3) of § 1 and $I=(-\rho,\rho)$ if $(m-k)(m-1)>0$, $I=(-\infty,-\rho)\cup(\rho,\infty)$ if $(m-k)(m-1)<0$ with some $\rho>0$.

(r.2) $R(e^{1/(m-1)}x,\ e^{(m-k)/(m-1)}\lambda) = o(e^{m/(m-1)})$ as $e \to 0$, uniformly on any bounded set of $V \times I$.

THEOREM 4.1. *Let* $R(x, \lambda)$ *satisfy* (r.1) *and* (r.2). *Then the statements of Theorem* 1.1 *hold true with* (1.1) *replaced by* (4.1).

*Proof.* We can obtain the bifurcation equation for (4.1) by the same reduction as in § 2.1 once we note that the implicit function theorem is applicable by (r.1) and (r.2). Moreover, we can neglect the term generated from $R(x, \lambda)$ in the bifurcation equation by using the homotopy invariance of the degree and (r.2). So Theorem 5.1 follows immediately from Theorems 2.5 and 2.6.

We can generalize Theorem 4.1 by the similar arguments as above. We make the following assumptions on $R(x, \lambda)$ instead of (r.1) and (r.2):

(r.3) $R(x, \lambda) \in C^1(W \to X)$, where $W$ is a neighbourhood of the origin of $D(L) \times R$.

(r.4) $R(rx, \lambda g(r)) = o(\|M(rx)\|)$ for any fixed $x \in V$ and $\lambda \in [-2, 2]$ as $r \to 0$, where $g(r)$ is the function defined by (3.5).

THEOREM 4.2. *Suppose that the assumptions of Theorem* 4.1 *hold. Let* $R(x, \lambda)$ *satisfy* (r.3) *and* (r.4). *Then the statements of Theorem* 3.1 *hold true with* (1.1) *replaced by* (4.1).

*Proof.* The term $R(x, \lambda)$ can be neglected by the similar arguments as in the proof of Theorem 4.1 by using (r.3) and (r.4). The proof is completed in the same way as in the proof of Theorem 3.1.

## § 5. Applications

The purpose of this section is to show how our theorems of previous sections are applied to problems of nonlinear elliptic differential equations. In this section, let $\Omega$ be a bounded domain in $R^n$ with smooth boundary $\partial\Omega$. We introduce the usual Hölder space $C^{m+\alpha}(\Omega)$ with norm

$$\|u\|_{m+\alpha} = \sup_{\substack{|\beta| \leq m \\ x \in \Omega}} |D^\beta u(x)| + \sup_{\substack{|\beta| = m \\ x, y \in \Omega}} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha} \qquad (0 < \alpha < 1),$$

where $\beta$ denotes multi-indices $\beta = (\beta_1, \cdots, \beta_n)$ and $|\beta| = \beta_1 + \cdots + \beta_n$.

**5.1.** We consider the following nonlinear elliptic equation

$$\text{(5.1)}\qquad \begin{cases} (\varDelta - \mu_1)u + \lambda f(x)\,|u|^k - g(x)\,|\varDelta u|^m = 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $k$ and $m$ are real numbers with $m \neq 0, 1, k$, $\mu_1$ is the first eigenvalue of $\varDelta$ with zero-Dirichlet condition and $\lambda$ is a real parameter. It is well known that $\mu_1$ is simple and the corresponding eigenfunction $\phi_1$ is positive in $\Omega$.

We want to obtain a family $\{(u, \lambda)\}$ of classical solutions and parameter of (5.1). Put $X = \{u \in C^\alpha(\Omega)\,|\, u = 0 \text{ on } \partial\Omega\}$, $D = \{u \in C^{2+\alpha}(\Omega)\,|\, u = \varDelta u = 0 \text{ on } \partial\Omega\}$, $L = \varDelta - \mu_1$, $K(u) = f\,|u|^k$ and $M(u) = g\,|\varDelta u|^m$, where $f, g \in C^\alpha(\Omega)$, $f$ or $g$ is of compact support when $k < 0$ or $m < 1$, respectively. Then (5.1) is formally transformed to the equation

$$\text{(5.2)}\qquad Lu + \lambda K(u) - M(u) = 0 \qquad \text{in } X\,.$$

Application of Theorem 1.1 yields:

THEOREM 5.1. *If* $\int_\Omega f(x)\phi_1^{k+1}dx \neq 0$, *then the equation* (5.1) *has a family of solutions* $\{(u(e), \lambda(e)) \in X \times R\,|\, 0 \leqq e \leqq \rho\}$ *with some* $\rho > 0$ *such that*

$$u(e) = e^{1/(m-1)}\{\phi_1 + ez(e)\}\,, \qquad \int_\Omega z(e)\phi_1 dx = 0\,,$$
$$\lambda(e) = e^{(m-k)/(m-1)}a(e)\,,$$

*where* $z(e)$ *and* $a(e)$ *are bounded. In particular, if*

$$\int_\Omega g(x)\phi_1^{k+1}dx \neq 0\,,$$

*then* $r_2 \leqq |a(e)| \leqq r_1$ *with some* $r_1$ *and* $r_2$, $r_1 > r_2 > 0$.

*Proof.* We have only to examine that all the assumptions of Theorem 1.1 are satisfied. We define $D(L) = D$. It is well known that $L$ is a Fredholm operator of index zero and $\dim N(L) = 1$ by the assumption, so (a.1) is satisfied. Since $\varDelta - \mu_1$ is formally self-adjoint, we have easily $N(L) = N(L^n)$, $n = 1, 2, \cdots$. Furthermore, any eigenvalue of $\varDelta$ is isolated. Thus, by Ize [2, p. 36, Theorem 5.1], we have (a.2). We define

$$U = \{u \in D\,|\, b\phi_1 < u < c\phi_1^\alpha,\ b'\phi_1 < \varDelta u < c'\phi_1^\alpha \text{ for some } bc > 0 \text{ and } b'c' > 0\}\,.$$

$U$ is an open set of $D$. It is easy to see that $K(\cdot)$ and $M(\cdot) \in C^1(U \to X)$ for all $k, m$. Let $\alpha < k$ if $0 < k < 1$. We note that $f$ or $g$ has compact support in $\Omega$ for $k < 0$ or $m < 1$ respectively. We shall give the proof in case of $M(u)$. By the mean value theorem, we have

$$\begin{aligned}&\|g\{(\Delta(u+v))^m-(\Delta u)^m-m(\Delta u)^{m-1}\Delta v\}\|_X\\&=m\left\|g\left\{\int_0^1(\Delta(u+tv))^{m-1}dt-(\Delta u)^{m-1}\right\}\Delta v\right\|_X\\&=o(\|v\|_D)\qquad\text{as } \|v\|_D\to 0\,.\end{aligned}$$

This means that $M(u)$ is Fréchet differentiable. Therefore $M(u)\in C^1(U\to X)$ for all $m$. Similarly we can prove $K(u)\in C^1(U\to X)$ for all $k$. Thus (a.3) holds. (a.4) is trivially satisfied. Since

$$PK(u)=\phi\int_\Omega f(x)|u|^k\phi dx\quad\text{and}\quad PM(u)=\phi\int_\Omega g(x)|\Delta u|^m\phi dx\,,$$

we see that

$$\int_\Omega f(x)\phi^{k+1}dx\neq 0\quad\text{and}\quad\int_\Omega g(x)\phi^{m+1}dx\neq 0$$

are equivalent to (c.1) and (c.2) respectively. Finally, since $d=1$, all the assumptions of Theorem 1.1 (i), (ii) are satisfied.

As a corollary of Theorem 5.1, we can obtain a solution curve of the nonlinear equations of the form

$$\text{(5.3)}\qquad\begin{cases}|\Delta u|^a(\Delta-\mu_1)u+\lambda|u|^b=g(x) & \text{in } \Omega\,,\\ u=0 & \text{on } \partial\Omega\,,\end{cases}$$

where $a\neq 0,-1$ and $b>a$, or

$$\text{(5.4)}\qquad\begin{cases}|\Delta u|^a(\Delta-\mu_1)u+|u|^b=\lambda f(x) & \text{in } \Omega\,,\\ u=0 & \text{on } \partial\Omega\,,\end{cases}$$

where $b\neq 0$, $a+1$ and $b>a$. In fact, we can reduce (5.3) to (5.2) by putting

$$L=\Delta-\mu_1\,,\quad K(u)=|u|^b|\Delta u|^{-a}\,,\quad M(u)=g(x)|\Delta u|^{-a}\,,$$

where we assume that $g(x)$ is of compact support if $a>-1$. Similarly we can reduce (5.4) to (5.2) by putting

$$L=\Delta-\mu_1\,,\quad K(u)=-f(x)|\Delta u|^{-a}\,,\quad M(u)=-|u|^b|\Delta u|^{-a}\,,$$

where we assume that $f(x)$ is of compact support if $a>-1$. It is easy to see that $K, M\in C^1(U\to X)$ by the similar argument as in the proof of Theorem 5.1. Thus we have the following:

COROLLARY 5.2. (5.3) *has a family of solutions* $\{(u(e), \lambda(e) \mid 0 \leqq e \leqq \rho\}$ *with*

$$u(e) = e^{-1/(a+1)}\{\phi_1 + ez(e)\}\ , \qquad \lambda(e) = e^{b/(a+1)}a(e)\ ,$$

*where* $z(e)$ *and* $a(e)$ *have all the properties expressed in Theorem* 5.1.

COROLLARY 5.3. *If* $\int f(x)\phi(x)^{-a+1}dx \neq 0$, *then* (5.4) *has a family of solutions* $\{(u(e), \lambda(e) \mid 0 \leqq e \leqq \rho\}$ *with*

$$u(e) = e^{1/(b-a-1)}\{\phi_1 + ez(e)\}\ , \qquad \lambda(e) = e^{b/(b-a-1)}a(e)\ ,$$

*where* $z(e)$ *and* $a(e)$ *have all the properties expressed in Theorem* 5.1.

**5.2.** We consider the nonlinear elliptic equation

$$\begin{cases} (\Delta - \mu_0)u + \lambda u^3 = f(x)|\nabla u|^4 & \text{in } \Omega\ , \\ u = 0 & \text{on } \partial\Omega\ , \end{cases} \tag{5.5}$$

where $\mu_0$ is an eigenvalue of $\Delta$ with multiplicity $d \geqq 1$ and $f \in C^\alpha(\Omega)$.

THEOREM 5.2. *Let* $\phi_j$ $(j = 1, \cdots, d)$ *be the basis of* $N(\Delta - \mu_0)$. *If*
( i ) *d is odd, or*
(ii) *d is even and for any* $u \in N(\Delta - \mu_0) - \{0\}$, *there exists some* $\phi_j$ *such that* $\int f|\nabla u|^4 \phi_j dx \neq 0$, *then* $(u, \lambda) = (0, 0)$ *is a bifurcation point of* (5.5).

*Proof.* We define $L = \Delta - \mu_0$, $K(u) = u^3$, $M(u) = f|\nabla u|^4$, $X = C^\alpha(\Omega)$ and $D(L) = \{u \in C^{2+\alpha}(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$. $L$ satisfies (a.1) and (a.2) (see § 5.1). The conditions (a.3) and (a.4) are easily verified by the fact that $C^\alpha(\Omega)$ is a Banach algebra, i.e., if $f, g \in C^\alpha(\Omega)$, then $\|fg\| \leqq \|f\|\,\|g\|$. Let $u \in S$, the unit sphere of $N(L)$. We define the projection $P_u$ by $P_u v = (v, u)u$, where $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$. Then

$$P_u K(u) = P_u(u^3) = (u^3, u)u = u\int_\Omega u^4 dx \neq 0\ .$$

Hence we have $PK(u) \neq 0$, which means (c.1). Moreover we have

$$tu + (1-t)PK(u) \neq 0 \qquad \text{for } u \in S,\ 0 \leqq t \leqq 1\ .$$

Then $K_S\colon S \to S$ is homotopic to the identity $I\colon S \to S$, where $K_S$ is defined by $K_S(u) = PK(u)/\|PK(u)\|$. Thus $\deg K_S = 1$. Since (c.2) holds by the assumption (ii), $M_S\colon S \to S$ is also well defined. It is well known that if $M_S$ is an even map (i.e. $M_S(u) = M_S(-u)$) then $\deg M_S$ is even. Then we have $\deg K_S \neq \deg M_S$, which means (iii) of Theorem 1.1. Therefore the assumption (i) or (iii) of Theorem 1.1 is satisfied, which completes the proof.

**5.3.** We consider the system of the nonlinear elliptic equations

$$(5.6)\qquad \begin{cases} (\Delta - \mu_0)u + \lambda(au + bv) + u^2 - v^2 = 0 & \text{in } \Omega\,, \\ (\Delta - \mu_0)v + \lambda(cu + dv) + uv = 0 & \text{in } \Omega\,, \\ u = v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\mu_0$ is a simple eigenvalue of $\Delta$ and $ad - bc \neq 0$.

THEOREM 5.3. *Let $\phi$ be the eigenfunction corresponding to $\mu_0$. If $\int_\Omega \phi^3 dx \neq 0$, then $(u, v, \lambda) = (0, 0, 0)$ is a bifurcation point of* (5.6).

*Proof.* Put $X = \{C^\alpha(\Omega)\}^2$, $D = \{u \in C^{2+\alpha}(\Omega) \,|\, u = 0 \text{ on } \partial\Omega\}^2$,

$$L = \begin{pmatrix} \Delta - \mu_0 & \\ & \Delta - \mu_0 \end{pmatrix} \text{ with } D(L) = D\,, \quad K = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$M(U) = \begin{pmatrix} -(u^2 - v^2) \\ -uv \end{pmatrix},$$

where $U = (u, v)^t$. Then (5.6) can be expressed in the form (5.2). $L$ satisfies (a.1) and (a.2) with $\dim N(L) = 2$. Clearly (a.3) and (a.4) hold true. It is easy to see that (c.1) holds and that $\deg K_S = \pm 1$ if $ad - bc \gtrless 0$ respectively. We shall prove that if $\int_\Omega \phi^3 dx \neq 0$, then (c.2) holds and $\deg M_S = 2$. Since $PU = (u, \phi)\phi_1 + (v, \phi)\phi_2$, where $\phi_1 = (\phi, 0)^t$, $\phi_2 = (0, \phi)^t$. If we identify $U = s\phi_1 + t\phi_2$ with $(s, t)^t$, then

$$PM\colon (s, t)^t \longrightarrow (g(s^2 - t^2), gst)\,, \qquad g = \int_\Omega \phi^3 dx\,.$$

Therefore we can see that the condition (iii) of Theorem 1.1 is satisfied. This proves the theorem.

**5.4.** We consider the system of elliptic equations with nonhomogeneous nonlinear terms

$$(5.7)\qquad \begin{cases} (\Delta - \mu_0)u + \lambda v + u^7 = 0 & \text{in } \Omega\,, \\ (\Delta - \mu_0)v + \lambda u^3 + v^5 = 0 & \text{in } \Omega\,, \\ u = v = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\lambda$ is a real parameter and $\mu_0$ is an eigenvalue of $\Delta$ with multiplicity $d$.

THEOREM 5.4. *If* $d = \dim N(\Delta - \mu_0)$ *is odd, then* $(u, v, \lambda) = (0, 0, 0)$ *is a bifurcation point of* (5.7).

*Proof.* We define $X, D, L$ and $U$ as in § 5.3. Further we define $K(U) = (v, u^3)^t$ and $M(U) = (-u^7, -v^5)^t$. Then (5.7) can be written in the form (5.2). We shall prove $\deg K_S = -1$ and $\deg M_S = 1$, where the maps $K_S$ and $M_S\colon S \to S$ are defined as in § 5.3. Since

$$PK(U) = P\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u^3 \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P\begin{pmatrix} u^3 \\ v \end{pmatrix},$$

we have $\deg K_S = \deg A_S \deg K'_S$, where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K'(U) = \begin{pmatrix} u^3 \\ v \end{pmatrix}$$

(1 is the identity on $N(\Delta - \mu_0)$, i.e. the identity matrix of size $d$). Clearly $\deg A_S = \det A = -1$. We shall prove $\deg K'_S = 1$. Define the map $P_U$ by $P_U U' = (U', U)U$, where

$$(U', U) = \int_\Omega (u'u + v'v)dx$$

with $U = (u, v)^t$ and $U' = (u', v')^t$. Since

$$P_U K'(U) = U \int_\Omega (u^4 + v^2)dx\,, \qquad P_U K'(U) \neq 0$$

for all $U \in S$. Hence $PK'(U) \neq 0$ for all $U \in S$. Furthermore we have

$$tU + (1 - t)PK'(U) \neq 0 \qquad \text{for } U \in S,\ 0 \leqq t \leqq 1\,.$$

Then $K'_S$ is homotopic to the identity $I\colon S \to S$, which implies $\deg K'_S = 1$. Hence $\deg K_S = \deg A_S \deg K'_S = (-1) \times 1 = -1$. Similarly we have $\deg M_S = 1$. It remains to verify the assumption (c.3). We have

$$\max_{U \in S} \frac{\|PM(rU)\|_X}{\|PK(rU)\|_X} \leqq C \frac{r^7 \|u\|_\alpha^7 + r^5 \|v\|_\alpha^5}{\|P_U K'(rU)\|_X}$$
$$\leqq C \frac{r^7 \|u\|_\alpha^7 + r^5 \|v\|_\alpha^5}{r^3 \|u\|_\alpha \int u^4 dx + r \|v\|_\alpha \int v^2 dx} \to 0 \qquad \text{as } r \to 0$$

for any $U = (u, v)^t \in S$, where $\|\cdot\|_\alpha$ denotes the norm of $C^\alpha(\Omega)$. This implies (c′.3). Thus all the assumptions (a.1), (a.2), (a′.3), (a′.4), (c′.1), (c′.2), (c′.3) and (iii) of Theorem 3.1 are satisfied.

**Appendix.** We can simplify the proof of Theorem 1.1 if $\dim N(L) = 1$. In this case, $S$ is composed of two points, say, $\pm y_0$. Thus Equation (2.6) (put $F_e(y, a) = 0$) is directly solved by the implicit function theorem with respect to $a$ in terms of $e$ for each $\pm y_0$, because $(\partial/\partial a)F_0(\pm y_0, a) = PK(\pm y_0) \neq 0$. Hence, Theorem 1 immediately follows from Proposition 2.1.

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