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DIRICHLET INTEGRAL AND PICARD PRINCIPLE

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A *density* P on the punctured unit disk $\Omega: 0 < |z| < 1$ is a 2-form $P(z)dxdy$ whose coefficient $P(z)$ is a real valued nonnegative locally Hölder continuous function on the closed punctured unit disk $\bar{\Omega}: 0 < |z| \leqq 1$. Here we consider Ω as an end of the punctured sphere $0 < |z| \leqq +\infty$ so that the point $z = 0$ is viewed as the ideal boundary $\delta\Omega$ of Ω and the unit circle $|z| = 1$ as the relative boundary $\partial\Omega$ of Ω. We denote by $\mathscr{D} = \mathscr{D}(\Omega)$ the family of densities on Ω. A density P on Ω gives rise to an elliptic operator $L = L_P$ on $\bar{\Omega}$ defined by

$$Lu = L_P u = \Delta u - Pu\,, \qquad \Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2\,.$$

Since $\delta\Omega$ is of parabolic character, there exists a unique bounded solution $e = e_P$, referred to as the *P-unit* on Ω, of $Lu = 0$ on Ω with continuous boundary values 1 on $\partial\Omega$. With the operator $L = L_P$ we associate an elliptic operator $\hat{L} = \hat{L}_P$, referred to as the *associate operator* to L, given by

$$\hat{L}v = \hat{L}_P v = \Delta v + 2\nabla \log e_P \cdot \nabla v\,, \qquad \nabla = (\partial/\partial x, \partial/\partial y)\,.$$

We denote by $\mathscr{P} = \mathscr{P}_P$ the family of nonnegative solutions u of $Lu = 0$ on Ω with vanishing boundary values on $\partial\Omega$, by $\mathscr{B} = \mathscr{B}_P$ the family of bounded solutions u of $Lu = 0$ on Ω and similarly, by $\hat{\mathscr{B}} = \hat{\mathscr{B}}_P$ the family of bounded solutions v of $\hat{L}v = 0$ on Ω.

We are particularly interested in those densities P for which $\mathscr{P} = \mathscr{P}_P$ is generated by a single element $u_0: \mathscr{P} = \{\lambda u_0;\ \lambda \in \boldsymbol{R}^+\}$, where $\boldsymbol{R}$ is the real number field and $\boldsymbol{R}^+$ is the set of nonnegative real numbers. Since $P \equiv 0$ is the typical one of this character found by Picard, we say, after Bouligand (cf. Brelot [2]), that the *Picard principle* is valid for P at $\delta\Omega$ if $\mathscr{P}_P$ is generated by a single element, and we denote by $\mathscr{D}_{\mathfrak{P}} = \mathscr{D}_{\mathfrak{P}}(\Omega)$ the family of densities on Ω for which the Picard principle is valid. It is a fasci-

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nating problem to characterize the family $\mathscr{D}_{\mathfrak{P}}$. We compile some of papers answering to this question partially at the end of this paper. If the limit $\lim_{z\to\partial\Omega} u(z)$ exists for every u in $\hat{\mathscr{B}}_P$, then we say that the (weak) *Riemann theorem* is valid for the operator $\hat{L}_P$. We denote by $\mathscr{D}_{\mathfrak{R}}$ the family of densities P such that the Riemann theorem is valid for $\hat{L}_P$. We have the following *duality theorem* (cf. Heins [9], Hayashi [8], [26]):

$$\mathscr{D}_{\mathfrak{P}} = \mathscr{D}_{\mathfrak{R}} .$$

Therefore characterizing $\mathscr{D}_{\mathfrak{P}}$ is identical with characterizing $\mathscr{D}_{\mathfrak{R}}$. There are quite a few instances that the Dirichlet integral plays very important role to single out densities in $\mathscr{D}_{\mathfrak{R}}$ among $\mathscr{D}$.

The *purpose* of this paper is to clarify the efficiency of Dirichlet integrals and at the same time its limitation in the study of the Picard principle. For the purpose we further classify $\mathscr{D}$. A density P is said to be *finite* if

$$\int_\Omega P(z)dxdy < +\infty$$

and we denote by $\mathscr{D}_1$ the family of finite densities on Ω. The importance of the class $\mathscr{D}_1$ lies in the fact that $\mathscr{D}_1 \subset \mathscr{D}_{\mathfrak{P}}$ (cf. [27], Kawamura [13]). In connection with the class $\mathscr{D}_1$, we consider the class $\mathscr{D}_{\mathfrak{S}\mathfrak{D}}$ of, what we call, densities P of *strongly D-type* characterized by

$$\int_\Omega |\nabla \log e_P(z)|^2\, dxdy < +\infty .$$

It is known that $\mathscr{D}_1 \subset \mathscr{D}_{\mathfrak{S}\mathfrak{D}}$ (cf. [27]). It is easy to see that the *Dirichlet integral* $D_{\{0<|z|<r\}}(u)$ of any u in $\mathscr{B}_P$ is finite:

$$D_{\{0<|z|<r\}}(u) = \int_{0<|z|<r} |\nabla u(z)|^2\, dxdy < +\infty$$

for every r in $(0, 1)$. The same may or may not be true for the class $\hat{\mathscr{B}}_P$. If

$$D_{\{0<|z|<r\}}(v) = \int_{0<|z|<r} |\nabla v(z)|^2\, dxdy < +\infty$$

for any v in $\hat{\mathscr{B}}_P$ and any r in $(0, 1)$, then we say that P is of *D-type*, and we denote by $\mathscr{D}_{\mathfrak{D}}$ the family of densities of D-type on Ω. We know (cf. [27]) that $\mathscr{D}_{\mathfrak{S}\mathfrak{D}} \subset \mathscr{D}_{\mathfrak{D}} \subset \mathscr{D}_{\mathfrak{R}}$ from which we deduced the relation $\mathscr{D}_1 \subset \mathscr{D}_{\mathfrak{P}}$. Therefore it has been known that $\mathscr{D}_1 \subset \mathscr{D}_{\mathfrak{S}\mathfrak{D}} \subset \mathscr{D}_{\mathfrak{D}} \subset \mathscr{D}_{\mathfrak{R}} = \mathscr{D}_{\mathfrak{P}}$. We will study whether these inclusions are proper or not. The *conclusion* will be the following:

$$\mathscr{D}_1 = \mathscr{D}_{\mathfrak{S}\mathfrak{D}} < \mathscr{D}_{\mathfrak{D}} < \mathscr{D}_{\mathfrak{R}} = \mathscr{D}_{\mathfrak{P}} ,$$

where $<$ indicates the strict inclusion.

In § 1 we will prove $\mathscr{D}_1 = \mathscr{D}_{\mathfrak{S}\mathfrak{D}}$ by establishing an identity evaluating the Dirichlet integral of $\log e_P$ in terms of the integral involving P. In § 2 a necessary and sufficient condition is given for a rotation free density P to belong to $\mathscr{D}_{\mathfrak{D}}$. Here a density P is *rotation free*, by definition, if $P(z) = P(|z|)$ for every z in Ω. As an application of the result in § 2, we will see in § 3 that the simple density $P(z) = |z|^{-2}$ belongs to $\mathscr{D}_{\mathfrak{D}} - \mathscr{D}_{\mathfrak{S}\mathfrak{D}}$ and $P(z) = |z|^{-2}(\log|z|)^2$ belongs to $\mathscr{D}_{\mathfrak{R}} - \mathscr{D}_{\mathfrak{D}}$. Actually, as we will see in § 3, belonging to $\mathscr{D}_{\mathfrak{D}}$ is very delicate:

$$\begin{cases} c\,|z|^{-2}(\log|z|)^2 \in \mathscr{D}_{\mathfrak{D}} & \text{for } c \in [0, 1) , \\ c\,|z|^{-2}(\log|z|)^2 \notin \mathscr{D}_{\mathfrak{D}} & \text{for } c \in [1, +\infty) . \end{cases}$$

§ 1. An identity

1. Consider a subregion S of Ω with its relative boundary ∂S of a simple closed curve in Ω and with the ideal boundary $z = 0$. We do not exclude the case $S = \Omega$ so that $\partial S = \partial\Omega$. For every closed punctured disk $\overline{V}_\varepsilon : 0 < |z| \leqq \varepsilon$ contained in S, we denote by w_ε the harmonic measure of ∂S considered on $S - \overline{V}_\varepsilon$. Then the Stokes formula yields

$$\begin{aligned} \int_{\partial S} \frac{\partial e(z)}{\partial n} ds &= \int_{\partial(S - \overline{V}_\varepsilon)} w_\varepsilon(z) \frac{\partial e(z)}{\partial n} ds \\ &= \int_{S - \overline{V}_\varepsilon} \nabla w_\varepsilon(z) \cdot \nabla e(z) dxdy + \int_{S - \overline{V}_\varepsilon} w_\varepsilon(z) \Delta e(z) dxdy , \end{aligned}$$

where $\partial/\partial n$ is the outer normal derivative and ds the line element. By the maximum principle and the Harnack principle, we see that $w_\varepsilon \uparrow 1$ uniformly on each compact subset of $S \cup \partial S$. On setting $w_\varepsilon = 0$ on $\overline{V}_\varepsilon$, a simple application of the Stokes formula yields

$$\int_S |\nabla(w_\varepsilon - w_{\varepsilon'})(z)|^2\, dxdy = \int_S |\nabla w_\varepsilon(z)|^2\, dxdy - \int_S |\nabla w_{\varepsilon'}(z)|^2\, dxdy$$

for $\varepsilon < \varepsilon'$. Hence in particular we see that

$$\int_S |\nabla w_\varepsilon(z)|^2\, dxdy \downarrow 0 \qquad (\varepsilon \downarrow 0) .$$

By the Schwarz inequality

$$\left(\int_{S-\bar{V}_\varepsilon} \nabla w_\varepsilon(z)\cdot\nabla e(z)dxdy\right)^2 \leqq \int_{S-\bar{V}_\varepsilon} |\nabla w_\varepsilon(z)|^2\, dxdy\cdot\int_{S-\bar{V}_\varepsilon} |\nabla e(z)|^2\, dxdy\ .$$

Since the Dirichlet integral of any function in $\mathscr{B}_P$ is finite,

$$\int_{S-\bar{V}_\varepsilon} |\nabla e(z)|^2\, dxdy$$

is dominated by

$$\int_\Omega |\nabla e(z)|^2\, dxdy < +\infty\ .$$

Thus we may conclude that

$$\lim_{\varepsilon\to 0}\int_{S-\bar{V}_\varepsilon} \nabla w_\varepsilon(z)\cdot\nabla e(z)dxdy = 0\ .$$

Observe that

$$\int_{S-\bar{V}_\varepsilon} w_\varepsilon(z)\Delta e(z)dxdy = \int_S w_\varepsilon(z)e(z)P(z)dxdy\ .$$

The Lebesgue-Fatou theorem implies that

$$\lim_{\varepsilon\to 0}\int_{S-\bar{V}_\varepsilon} w_\varepsilon(z)\Delta e(z)dxdy = \int_S e(z)P(z)dxdy\ .$$

We finally conclude that

$$(1)\qquad \int_{\partial S}\frac{\partial e(z)}{\partial n}ds = \int_S e(z)P(z)dxdy\ .$$

This means that $e(z)P(z)dxdy$ is a finite measure on Ω.

2. Consider a continuous function f on ∂S. We denote by H_f^S the uniquely determined bounded harmonic function on S with continuous boundary values $f(z)$ on ∂S and by h_ε the harmonic function on $S-\bar{V}_\varepsilon$ with continuous boundary values $f(z)$ on ∂S and 0 on ∂V_ε: $|z|=\varepsilon$. Then the Stokes formula yields

$$\int_{\partial S} f(z)\frac{\partial e(z)}{\partial n}ds = \int_{\partial(S-\bar{V}_\varepsilon)} h_\varepsilon(z)\frac{\partial e(z)}{\partial n}ds$$
$$= \int_{S-\bar{V}_\varepsilon} \nabla h_\varepsilon(z)\cdot\nabla e(z)dxdy + \int_{S-\bar{V}_\varepsilon} h_\varepsilon(z)\Delta e(z)dxdy\ .$$

Since the family of h_ε is uniformly bounded on S, converges to H_f^S uniformly on each compact subset of $S \cup \partial S$ as $\varepsilon \to 0$,

$$\int_S |\nabla(h_\varepsilon - h_{\varepsilon'})(z)|^2\, dxdy = \int_S |\nabla h_\varepsilon(z)|^2\, dxdy - \int_S |\nabla h_{\varepsilon'}(z)|^2\, dxdy$$

for $\varepsilon > \varepsilon' > 0$ by setting $h_\varepsilon = 0$ on V_ε, and $e(z)P(z)dxdy$ is a finite measure on Ω, the most right hand side of the above identity converges to

$$\int_S \nabla H_f^S(z) \cdot \nabla e(z) dxdy + \int_S H_f^S(z) e(z) P(z) dxdy$$

as $\varepsilon \to 0$ by the similar reasoning as in no. 1. Therefore we have a generalization of (1):

$$(2) \qquad \int_{\partial S} f(z) \frac{\partial e(z)}{\partial n} ds = \int_S \nabla H_f^S(z) \cdot \nabla e(z) dxdy + \int_S H_f^S(z) e(z) P(z) dxdy\,.$$

3. We will give an upper estimate of the Dirichlet integral of the harmonic function $H_{1/e}^S$ on S. Observe*) that $e \leqq H_e^S$. Since H_e^S attains its minimum value on ∂S we have

$$(H_e^S(z))^{-4} \leqq \left(\min_{\partial S} H_e^S\right)^{-4} = \left(\min_{\partial S} e\right)^{-4} = \max_{\partial S} e^{-4}$$

on S. Applying the Dirichlet principle to functions $H_{1/e}^S$, $1/H_e^S$, H_e^S, and e on S, we have

$$\begin{aligned}\int_S |\nabla H_{1/e}^S(z)|^2\, dxdy &\leqq \int_S |\nabla(1/H_e^S(z))|^2\, dxdy \\ &= \int_S (H_e^S(z))^{-4}\, |\nabla H_e^S(z)|^2\, dxdy\end{aligned}$$

and similarly

$$\int_S |\nabla H_e^S(z)|^2 dxdy \leqq \int_S |\nabla e(z)|^2\, dxdy\,.$$

Therefore we have the following estimate:

$$(3) \qquad \int_S |\nabla H_{1/e}^S(z)|^2\, dxdy \leqq \left(\max_{\partial S} e^{-4}\right) \int_S |\nabla e(z)|^2\, dxdy\,.$$

4. We next give an evaluation of the Dirichlet integral of $\log e$ on $\Omega - \bar{S}$. By the Stokes theorem we have

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*) Here and also in no. 6 we use the fact that e is subharmonic in $|z| < 1$ by defining $e(0) = \limsup_{z \to 0} e(z)$.

$$\int_{\partial\Omega}\frac{\partial e(z)}{\partial n}ds-\int_{\partial S}(1/e(z))\frac{\partial e(z)}{\partial n}ds=\int_{\partial(\Omega-\bar{S})}(1/e(z))\frac{\partial e(z)}{\partial n}ds$$
$$=\int_{\Omega-\bar{S}}\nabla(1/e(z))\cdot\nabla e(z)dxdy+\int_{\Omega-\bar{S}}(1/e(z))\Delta e(z)dxdy\,.$$

If we set $S=\Omega$ in (1), then we have

$$\int_{\partial\Omega}\frac{\partial e(z)}{\partial n}ds=\int_{\Omega}e(z)P(z)dxdy\,.$$

In view of the identities $\nabla(1/e(z))\cdot\nabla e(z)=-|\nabla\log e(z)|^2$ and $(1/e(z))\Delta e(z)=P(z)$, we deduce

$$\int_{\Omega-\bar{S}}|\nabla\log e(z)|^2\,dxdy=\int_{\Omega-\bar{S}}P(z)dxdy-\int_{\Omega}e(z)P(z)dxdy+\int_{\partial S}(1/e(z))\frac{\partial e(z)}{\partial n}ds\,. \tag{4}$$

5. The identity (4) shows that the Dirichlet integral of $\log e$ over Ω is essentially controlled by the integral of $(1/e)(\partial e/\partial n)$ over ∂S. Therefore we have to study the behavior of the integral of $(1/e)(\partial e/\partial n)$ over ∂S as $\Omega-S$ exhausts Ω, or, what amounts to the same, $\bar{S}\downarrow\emptyset$. For the purpose we consider two cases separately: $\limsup_{z\to0}e(z)=0$ and >0. First we consider the case $\limsup_{z\to0}e(z)=0$, i.e. $\lim_{z\to0}e(z)=0$. For every t in $(0,1)$ consider the subregion $S_t: e(z)<t$ of Ω, then $\bar{S}_t\downarrow\emptyset$ as $t\to0$. Moreover from (1) it follows that

$$0\leqq\int_{S_t}e(z)P(z)dxdy=\int_{\partial S_t}\frac{\partial e(z)}{\partial n}ds\leqq\int_{\partial S_t}(1/e(z))\frac{\partial e(z)}{\partial n}ds$$
$$=\frac{1}{t}\int_{\partial S_t}\frac{\partial e(z)}{\partial n}ds=\frac{1}{t}\int_{S_t}e(z)P(z)dxdy\leqq\int_{S_t}P(z)dxdy\,.$$

Therefore the integral of $(1/e)(\partial e/\partial n)$ over ∂S_t, which is nonnegative, converges to 0 as $t\to0$ if $P(z)dxdy$ is a finite measure on Ω.

6. Assume next that $\limsup_{z\to0}e(z)\equiv a>0$. There exists a closed set E thin at $z=0$ in Ω such that $e(z)\to a$ as $z\to0$ with $z\notin E$ (cf. Brelot [4]). Then we may take a decreasing sequence $\{t_m\}$ in $(0,1)$ with $E\cap\{z;|z|=t_m\}=\emptyset$ for every m and $\lim_{m\to\infty}t_m=0$. Applying (2) to the function $1/e$ and the subregion $S_m: 0<|z|<t_m$ of Ω we have

$$\int_{\partial S_m}(1/e(z))\frac{\partial e(z)}{\partial n}ds=\int_{S_m}\nabla H_{1/e}^{S_m}(z)\cdot\nabla e(z)dxdy+\int_{S_m}H_{1/e}^{S_m}(z)e(z)P(z)dxdy\,.$$

The second term on the right hand side of the above equality is dominated by

$$\left(\max_{\partial S_m} e^{-1}\right)\int_{S_m} e(z)P(z)dxdy\ ,$$

and moreover by (3) we have

$$\left(\int_{S_m} \nabla H_{1/e}^{S_m}(z)\cdot \nabla e(z)dxdy\right)^2 \leqq \int_{S_m} |\nabla H_{1/e}^{S_m}(z)|^2\, dxdy\cdot \int_{S_m} |\nabla e(z)|^2\, dxdy$$
$$\leqq \left(\max_{\partial S_m} e^{-4}\right)\left(\int_{S_m} |\nabla e(z)|^2\, dxdy\right)^2 .$$

Therefore we have

$$\lim_{m\to\infty}\int_{\partial S_m} (1/e(z))\frac{\partial e(z)}{\partial n}ds = 0\ .$$

7. Apply (4) to $S = S_t$ in the case of no. 5 or S_m in the case of no. 6 and make $t\to 0$ or $m\to\infty$ accordingly. Then we obtain the following evaluation of the Dirichlet integral of $\log e$ on Ω:

THEOREM. *For every density $P(z)dxdy$ on Ω*

$$\int_{\Omega} |\nabla \log e(z)|^2\, dxdy = \int_{\Omega} (1-e(z))P(z)dxdy\ .$$

Here in the above equality it may happen $+\infty = +\infty$, which is exactly the case P is not finite. As a direct consequence of this we obtain the following:

$$\mathscr{D}_1 = \mathscr{D}_{\mathfrak{S}\mathfrak{D}}\ .$$

§ 2. Rotation free densities

8. Consider a *rotation free* density $P(z)dxdy$ on Ω, i.e. the density with $P(z) = P(|z|)$ on $\bar{\Omega}$. For every nonnegative integer n we set $P_n(z) = P(z) + n^2/|z|^2$, which is also a rotation free density on Ω. Since the P_n-unit e_n, i.e. the unique bounded solution of $\Delta u = P_n u$ on Ω with the boundary values 1 on $\partial\Omega$, is also rotation free, it may be viewed as a function of r in $(0, 1]$. In other words, $e_n(r)$ may be considered as the unique bounded solution of

$$\ell_n\psi(r) \equiv \ell_{n,P}\psi(r) \equiv \frac{d^2}{dr^2}\psi(r) + \frac{1}{r}\frac{d}{dr}\psi(r) - P_n(r)\psi(r) = 0$$

on $(0, 1)$ with $e_n(1) = 1$, where we follow the convention $P_0 = P$ and $e_0 = e$.

We recall some of fundamental properties of e_n (cf. [21], Imai [10]): For any $\rho \in (0, 1]$,

$$\frac{e_{n+1}(r)}{e_{n+1}(\rho)} \leqq \frac{e_n(r)}{e_n(\rho)} \qquad (n = 0, 1, \cdots) \tag{5}$$

for every r in $(0, \rho]$; If we denote by ψ' the derivative $d\psi/dr$, then

$$0 \leqq \frac{e'_{n+1}(r)}{e_{n+1}(r)} - \frac{e'_n(r)}{e_n(r)} \leqq \frac{1}{r} \qquad (n = 0, 1, \cdots) \tag{6}$$

on $(0, 1]$; If $P \leqq Q$ on $\Omega_\rho : 0 < |z| < \rho$ $(0 < \rho \leqq 1)$ for another rotation free density $Q(z)dxdy$ on Ω, then

$$\frac{e_n(\rho)e_{n+1}(r)}{e_{n+1}(\rho)e_n(r)} \leqq \frac{f_n(\rho)f_{n+1}(r)}{f_{n+1}(\rho)f_n(r)} \qquad (n = 0, 1, \cdots) \tag{7}$$

on $(0, \rho]$, where $Q_n(z) = Q(z) + n^2/|z|^2$, f_n the Q_n-unit with the convention $f_0 = f$ being Q-unit; The Picard principle is valid for P if and only if

$$\lim_{r \to 0} \frac{e_1(r)}{e_0(r)} = 0 . \tag{8}$$

In particular (7) was first shown by Imai [10; p. 182].

9. Consider a bounded solution u of $Lu = 0$ on Ω, i.e. $u \in \mathscr{B}_P$. In this and following nos. we will study the Dirichlet integral of u/e in a neighborhood of $z = 0$. For a continuous function w on Ω the Fourier coefficients

$$\begin{cases} c_0(r) = c_0(r;\ w) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} w(re^{i\theta})d\theta , \\ a_n(r) = a_n(r;\ w) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} w(re^{i\theta}) \cos n\theta d\theta , \\ b_n(r) = b_n(r;\ w) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} w(re^{i\theta}) \sin n\theta d\theta \end{cases}$$

of w are functions of r alone in $(0, 1)$. Since u is a bounded solution of $Lu = 0$, the Fourier coefficients of u satisfy that

$$\begin{aligned} \frac{d^2}{dr^2} c_0(r;\ u) + \frac{1}{r}\frac{d}{dr} c_0(r;\ u) &= c_0\left(r;\ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \\ &= c_0\left(r;\ \Delta u - \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = P(r)c_0(r;\ u) , \end{aligned}$$

$$\frac{d^2}{dr^2}a_n(r;u)+\frac{1}{r}\frac{d}{dr}a_n(r;u)=a_n\Big(r;\Delta u-\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\Big)$$
$$=P(r)a_n(r;u)-\frac{n}{r^2}b_n\Big(r;\frac{\partial u}{\partial\theta}\Big)=\Big(P(r)+\frac{n^2}{r^2}\Big)a_n(r;u)\,,$$

and similarly

$$\frac{d^2}{dr^2}b_n(r;u)+\frac{1}{r}\frac{d}{dr}b_n(r;u)=\Big(P(r)+\frac{n^2}{r^2}\Big)b_n(r;u)\,.$$

Therefore they are bounded solutions of $\ell_0\psi=0$ or $\ell_n\psi=0$. For any fixed ρ in $(0,1)$ we have

$$\begin{cases} c_0(r;u)=\dfrac{c_0(\rho;u)}{e(\rho)}e(r)\,,\\ a_n(r;u)=\dfrac{a_n(\rho;u)}{e_n(\rho)}e_n(r)\,,\\ b_n(r;u)=\dfrac{b_n(\rho;u)}{e_n(\rho)}e_n(r)\end{cases}$$

on $(0,\rho]$. Therefore the Fourier coefficients of $\partial u/\partial\theta$ may be represented in terms of e_n in the following way:

$$c_0\Big(r;\frac{\partial u}{\partial\theta}\Big)=0\,,$$
$$a_n\Big(r;\frac{\partial u}{\partial\theta}\Big)=nb_n(r;u)=nb_n(\rho;u)\frac{e_n(r)}{e_n(\rho)}\,,$$

and similarly

$$b_n\Big(r;\frac{\partial u}{\partial\theta}\Big)=-na_n(\rho;u)\frac{e_n(r)}{e_n(\rho)}\,.$$

If we set $r=\rho$ then the Parseval identity yields that

$$\sum_{n=1}^{\infty}n^2(a_n(\rho;u)^2+b_n(\rho;u)^2)=\frac{1}{\pi}\int_0^{2\pi}\Big(\frac{\partial}{\partial\theta}u(\rho e^{i\theta})\Big)^2d\theta<+\infty\,.$$

Moreover from (5) it follows that

$$\begin{cases} a_n\Big(r;\dfrac{\partial u}{\partial\theta}\Big)^2\leqq n^2b_n(\rho;u)^2\dfrac{e_1(r)^2}{e_1(\rho)^2}\,,\\ b_n\Big(r;\dfrac{\partial u}{\partial\theta}\Big)^2\leqq n^2a_n(\rho;u)^2\dfrac{e_1(r)^2}{e_1(\rho)^2}\end{cases}$$

for every positive integer n. Thus applying the Parseval identity to $\partial u/\partial\theta$ we have

$$\begin{aligned}(9)\qquad &\int_0^{2\pi}\int_0^{\rho}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\frac{u(re^{i\theta})}{e(r)}\right)^2 rdrd\theta \\ &\leqq \frac{\pi}{e_1(\rho)^2}\sum_{n=1}^{\infty} n^2(a_n(\rho;u)^2+b_n(\rho;u)^2)\int_0^{\rho}\frac{1}{r}\left(\frac{e_1(r)}{e(r)}\right)^2 dr\end{aligned}$$

for every ρ in $(0,1)$.

10. The Fourier coefficients of $\partial(u/e)/\partial r$ are represented in terms of e_n:

$$\begin{aligned}c_0\left(r;\frac{\partial}{\partial r}\frac{u}{e}\right) &= c_0\left(r;\frac{1}{e}\frac{\partial u}{\partial r}-\frac{e'}{e^2}u\right) = \frac{1}{e(r)}\frac{d}{dr}c_0(r;u)-\frac{e'(r)}{e(r)^2}c_0(r;u) \\ &= \frac{c_0(\rho;u)e'(r)}{e(r)e(\rho)}-\frac{e'(r)c_0(\rho;u)e(r)}{e(r)^2e(\rho)}=0\,, \\ a_n\left(r;\frac{\partial}{\partial r}\frac{u}{e}\right) &= \frac{1}{e(r)}\frac{d}{dr}a_n(r;u)-\frac{e'(r)}{e(r)^2}a_n(r;u) \\ &= \frac{a_n(\rho;u)e_n(r)}{e(r)e_n(\rho)}\left(\frac{e_n'(r)}{e_n(r)}-\frac{e'(r)}{e(r)}\right),\end{aligned}$$

and similarly

$$b_n\left(r;\frac{\partial}{\partial r}\frac{u}{e}\right) = \frac{b_n(\rho;u)e_n(r)}{e(r)e_n(\rho)}\left(\frac{e_n'(r)}{e_n(r)}-\frac{e'(r)}{e(r)}\right).$$

Then by (6) we have

$$\begin{aligned}a_n\left(r;\frac{\partial}{\partial r}\frac{u}{e}\right)^2 &\leqq \left(\frac{na_n(\rho;u)e_n(r)}{re(r)e_n(\rho)}\right)^2 \\ &\leqq \frac{n^2a_n(\rho;u)^2}{r^2e_1(\rho)^2}\left(\frac{e_1(r)}{e(r)}\right)^2\end{aligned}$$

and similarly

$$b_n\left(r;\frac{\partial}{\partial r}\frac{u}{e}\right)^2 \leqq \frac{n^2b_n(\rho;u)^2}{r^2e_1(\rho)^2}\left(\frac{e_1(r)}{e(r)}\right)^2$$

for every positive integer n, where $e_0=e$. Therefore applying the Parseval identity to $\partial(u/e)/\partial r$ we have

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\rho}\left(\frac{\partial}{\partial r}\frac{u(re^{i\theta})}{e(r)}\right)^2 rdrd\theta \\ &\leqq \frac{\pi}{e_1(\rho)^2}\sum_{n=1}^{\infty} n^2(a_n(\rho;u)^2+b_n(\rho;u)^2)\int_0^{\rho}\frac{1}{r}\left(\frac{e_1(r)}{e(r)}\right)^2 dr\end{aligned}$$

for every ρ in $(0, 1)$. Thus in view of (9) and the above inequality the Dirichlet integral of u/e on Ω_ρ satisfies the following:

$$\int_{\Omega_\rho}\left|\nabla\frac{u(z)}{e(z)}\right|^2 dxdy \tag{10}$$
$$\leqq \frac{2\pi}{e_1(\rho)^2}\sum_{n=1}^{\infty} n^2(a_n(\rho; u)^2 + b_n(\rho; u)^2)\int_0^\rho \frac{1}{r}\left(\frac{e_1(r)}{e(r)}\right)^2 dr$$

for every ρ in $(0, 1)$.

11. Consider the function $v_1(re^{i\theta}) = e_1(r)\cos\theta/e(r)$ on Ω and observe that $L(e_1(r)\cos\theta) = 0$. Then v_1 is a bounded solution of $\hat{L}v = 0$ on Ω. Moreover from the fact that

$$|\nabla v_1(z)|^2 \geqq \frac{1}{r^2}\left(\frac{e_1(r)}{e(r)}\right)^2 \sin^2\theta$$

it follows that

$$\int_{\Omega_\rho}|\nabla v_1(z)|^2\, dxdy \geqq \pi\int_0^\rho \frac{1}{r}\left(\frac{e_1(r)}{e(r)}\right)^2 dr \tag{11}$$

for any ρ in $(0, 1)$, where $z = re^{i\theta}$. Here note that

$$\int_\rho^1 r^{-1}(e_1(r)/e(r))^2 dr < +\infty$$

for every ρ in $(0, 1)$.

12. In view of (11) the divergence of the integral of $r^{-1}(e_1(r)/e(r))^2$ over $(0, 1)$ implies the existence of a bounded solution of $\hat{L}v = 0$ on Ω whose Dirichlet integral over a neighborhood of $z = 0$ is infinite. Conversely assume that the integral of $r^{-1}(e_1(r)/e(r))^2$ over $(0, 1)$ is finite. Take an arbitrary bounded solution v of $\hat{L}v = 0$ on Ω. Then the function ve is a bounded solution of $Lu = 0$ on Ω. In view of (10) the Dirichlet integral of $v = ve/e$ on a neighborhood of $z = 0$ is finite. Therefore we obtain the following

THEOREM. *Let $P(z)dxdy$ be a rotation free density on Ω. Then the Dirichlet integral of every bounded solution of $\hat{L}_P v = 0$ on a neighborhood of $z = 0$ is finite if and only if*

$$\int_0^1 \frac{1}{r}\left(\frac{e_1(r)}{e(r)}\right)^2 dr < +\infty\,. \tag{12}$$

We have thus characterized $\mathscr{D}_{\mathfrak{D}} \cap$ {rotation free densities} completely: It is exactly the set of rotation free densities with (12). We feel characterizing the general $\mathscr{D}_{\mathfrak{D}}$ is very difficult and we do not have even the foggiest idea at present.

§ 3. Examples

13. Consider the rotation free density $P(z)dxdy = |z|^{-2}\,dxdy$. The P-unit d and the $(P(z)+1/|z|^2)$-unit d_1 are given by $d(r) = r$ and $d_1(r) = r^{\sqrt{2}}$. Observe that

$$\int_{\Omega} |\nabla \log d(z)|^2\, dxdy = 2\pi \int_0^1 \frac{1}{r} dr = +\infty$$

and yet

$$\int_0^1 \frac{1}{r}\left(\frac{d_1(r)}{d(r)}\right)^2 dr = \int_0^1 r^{2\sqrt{2}-3} dr < +\infty\,.$$

Then from Theorem in no. 12 it follows that $P \in \mathscr{D}_{\mathfrak{D}} - \mathscr{D}_{\mathfrak{S}\mathfrak{D}}$ and therefore

$$\mathscr{D}_{\mathfrak{S}\mathfrak{D}} < \mathscr{D}_{\mathfrak{D}}\,.$$

14. We will give a rotation free density belonging to $\mathscr{D}_{\mathfrak{R}} - \mathscr{D}_{\mathfrak{D}}$. Let $0 \leqq \alpha < 1/2$,

$$\rho_\alpha = \max\left(\left(\frac{1}{\alpha}(5+2\alpha)(1+2\alpha)\right)^{1/2},\ \left(\frac{1}{8\alpha}(3+2\alpha)(1+2\alpha)^2(1-2\alpha)\right)^{1/4}\right)$$

for $\alpha > 0$, and $\rho_0 = 2$. Then the function

$$F_\alpha(x) = 1 - \tfrac{1}{2}(5+2\alpha)(1+2\alpha)x^{-2} - \tfrac{1}{16}(3+2\alpha)(1+2\alpha)^2(1-2\alpha)x^{-4}$$

of x in $[\rho_\alpha, +\infty)$ satisfies that $F_\alpha \geqq 1-\alpha$ for $\alpha > 0$ and $0 \leqq F_0 \leqq 1$. Consider rotation free densities $P_\alpha(z)dxdy$ and $P_{\alpha 1}(z)dxdy$ defined by

$$P_\alpha(z) = \begin{cases} \dfrac{1}{(1+2\alpha)^2} F_\alpha(-\log|z|)\dfrac{(\log|z|)^2}{|z|} & (0 < |z| \leqq \exp(-\rho_\alpha))\,, \\ P_\alpha(\exp(-\rho_\alpha)) & (\exp(-\rho_\alpha) < |z| \leqq 1)\,, \end{cases}$$

and $P_{\alpha 1}(z) = P_\alpha(z) + 1/|z|^2$. Observe that the function

$$G_\alpha(r) = \frac{1}{1+2\alpha}\,\frac{\log r^{-1}}{r}\left(1 - \frac{1}{4}(3+2\alpha)(1+2\alpha)\left(\log\frac{1}{r}\right)^{-2}\right)$$

of r in $(0, \exp(-\rho_\alpha))$ satisfies $G_\alpha \geqq 0$ and

$$\frac{d}{dr}G_\alpha(r) + G_\alpha(r)^2 + \frac{1}{r}G_\alpha(r) = P_\alpha(r) .$$

Then the function

$$E_\alpha(r) = \exp\left(-\int_r^{\exp(-\rho_\alpha)} G_\alpha(t)dt\right)$$

of r in $(0, \exp(-\rho_\alpha)]$ is a bounded solution of

$$\ell_{0,P_\alpha}\psi(r) = 0 \quad \text{with} \quad E_\alpha(\exp(-\rho_\alpha)) = 1 .$$

Moreover it is easy to show the fact that $E_\alpha(r)(\rho_\alpha/\log r^{-1})^{1/2+\alpha}$ is a bounded solution of $\ell_{1,P_\alpha}\psi(r) = 0$ on $(0, \exp(-\rho_\alpha))$ with the boundary values 1 at $r = \exp(-\rho_\alpha)$. Therefore the P_α-unit ($P_{\alpha 1}$-unit, resp.) $e_{\alpha 0}$ ($e_{\alpha 1}$, resp.) may be represented in terms of E_α on $(0, \exp(-\rho_\alpha))$ as follows:

$$e_{\alpha 0}(r) = E_\alpha(r)e_{\alpha 0}(\exp(-\rho_\alpha))$$
$$\left(e_{\alpha 1}(r) = E_\alpha(r)\left(\frac{\rho_\alpha}{\log r^{-1}}\right)^{1/2+\alpha} e_{\alpha 1}(\exp(-\rho_\alpha)), \text{ resp.}\right) .$$

By the above representation we have

$$\frac{e_{\alpha 1}(r)}{e_{\alpha 0}(r)} = \frac{e_{\alpha 1}(\exp(-\rho_\alpha))}{e_{\alpha 0}(\exp(-\rho_\alpha))}\rho_\alpha^{1/2+\alpha}\left(\frac{1}{\log r^{-1}}\right)^{1/2+\alpha}$$

and hence in view of (8) and Theorem in no. 12 we deduce $P_\alpha \in \mathscr{D}_{\mathfrak{D}}$ $(\alpha > 0)$ and $P_0 \in \mathscr{D}_{\mathfrak{R}} - \mathscr{D}_{\mathfrak{D}}$, where $P_0 = P_\alpha$ with $\alpha = 0$.

15. Since the function F_α satisfies that $F_\alpha \geqq 1 - \alpha$ for $\alpha > 0$ and $F_0 \leqq 1$ on $[\rho_\alpha, +\infty)$ P_α satisfies that

$$P_\alpha(z) \geqq \frac{1-\alpha}{(1+2\alpha)^2}\frac{(\log|z|)^2}{|z|^2}$$

for $\alpha > 0$ and

$$P_0(z) \leqq \frac{(\log|z|)^2}{|z|^2}$$

on $0 < |z| \leqq \exp(-\rho_\alpha)$, where $P_0(z) = P_\alpha(z)$ with $\alpha = 0$. Observe that $\lim_{\alpha\to 0}(1-\alpha)(1+2\alpha)^{-2} = 1$. Then in view of (7) and Theorem in no. 12 the rotation free density $c|z|^{-2}(\log|z|)^2dxdy$ satisfies

$$\begin{cases} c|z|^{-2}(\log|z|)^2 \in \mathscr{D}_{\mathfrak{D}} & \text{for } c \in [0, 1) , \\ c|z|^{-2}(\log|z|)^2 \notin \mathscr{D}_{\mathfrak{D}} & \text{for } c \in [1, +\infty) . \end{cases} \tag{13}$$

However $c|z|^{-2}(\log|z|)^2 \in \mathscr{D}_{\mathfrak{R}}$ for every $c \in [0, +\infty)$. The relation (13) suggests the delicacy of the class $\mathscr{D}_{\mathfrak{D}}$. It is not convex. It is known that $\mathscr{D}_{\mathfrak{R}} = \mathscr{D}_{\mathfrak{P}}$ is also not convex (cf. [23], Kawamura [15]). We have thus completed the classification as announced in the introduction:

$$\mathscr{D}_1 = \mathscr{D}_{\mathfrak{S}\mathfrak{D}} < \mathscr{D}_{\mathfrak{D}} < \mathscr{D}_{\mathfrak{R}} = \mathscr{D}_{\mathfrak{P}} < \mathscr{D} . \tag{14}$$

As for the last strict inclusion see e.g. [21].

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