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ON NON-ELLIPTIC BOUNDARY PROBLEMS

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Introduction

The purpose of this paper is to study the boundary value problems for the second order elliptic differential equation

$$(1)\qquad AU = -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j U) + \sum_{i=1}^{n} b_i\partial_i U + cU = F$$

in a bounded domain Ω in R^n ($n \geqq 3$) with the boundary condition

$$(2)\qquad BU = \sum_{i=1}^{n} \alpha_i\partial_i U + \beta U = f$$

on the boundary Γ of Ω, where we assume that

1) for every $x \in \Gamma$, the inequality

$$\sum_{i=1}^{n} \alpha_i(x)^2 > 0$$

holds,

2) let $(n_1(x), \cdots, n_n(x))$ be the exterior unit normal vector to Γ at x, then the subset of Γ,

$$\Gamma_0 = \left\{x \in \Gamma;\, \sum_{i=1}^{n} \alpha_i(x)n_i(x) = 0\right\}$$

is a C^∞-manifold of dimension $n-2$,

3) at every point $x \in \Gamma_0$, the n-vector $(\alpha_1(x), \cdots, \alpha_n(x))$ is not tangent to Γ_0.

Here ∂_i denotes $\partial/\partial x_i$, a_{ij} is symmetric on Ω, and Γ is assumed to be infinitely smooth and of dimension $n-1$. We further assume that the coefficients of the equations (1) and (2) are real-valued and infinitely differentiable on $\bar{\Omega} = \Omega \cup \Gamma$ and Γ, respectively, and that there exists a positive constant c_0 such that

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$$(3) \qquad \sum a_{ij}(x)\xi_i\xi_j \geqq c_0|\xi|^2$$

holds for all $x \in \bar{\Omega}$ and $\xi \in R^n$.

This problem was investigated by Maljutov [6] on probability considerations, by Egorov-Kondrat'ev [1] while they are developing some ideas of Hörmander [2], by Soga [7] and others. In the present paper we shall try to solve the problem (1)–(2) by using the similar argument as in [5].

If we set

$$\alpha'(x) = \sum_{i=1}^{n} \alpha_i(x)n_i(x)\,,$$

the boundary condition (2) can be written in the form

$$(2') \qquad BU = \alpha'\frac{\partial U}{\partial n} + \gamma' U + \beta U = f$$

with the suitable tangential vector field γ'. Using the conormal vector field $\nu=(\nu_1, \cdots, \nu_n)$ with

$$\nu_j(x) = \sum_{i=1}^{n} a_{ij}(x)n_i(x)\,,$$

and setting

$$(4) \qquad \frac{\partial U}{\partial n} = a_0(x)\frac{\partial U}{\partial \nu} + \gamma_0 U \qquad \left(\frac{\partial U}{\partial \nu} = \sum_{j=1}^{n} \nu_j \frac{\partial U}{\partial x_j}\right),$$

we can rewrite (2′) as

$$(2'') \qquad BU = \alpha\frac{\partial U}{\partial \nu} + \gamma U + \beta U = f$$

with $\alpha = \alpha' a_0$ and $\gamma = \alpha'\gamma_0 + \gamma'$, where $a_0(x)$ is a positive C^∞-function on Γ and γ_0 is also a tangential vector field. Assumptions 1), 2) and 3) yield that $\alpha(x)$ vanishes only on Γ_0, that γ is transversal to Γ_0, and that the boundary condition (2″) is elliptic (i.e. satisfies the Lopatinsky condition) on Γ except for Γ_0, that is, Γ_0 is a singular manifold for the boundary value problem (1)–(2). For the sake of simplicity, we assume that Γ_0 is connected. Following Egorov-Kondrat'ev we can then classify the singular manifold Γ_0, by denoting $\Gamma_+(\Gamma_-) = \{x \in \Gamma;\ \alpha(x) > 0\ (\alpha(x) < 0)\}$, as follows;

(I) $\Gamma_- = \phi$ (i.e. $\alpha \geqq 0$ throughout Γ) or
$\Gamma_+ = \phi$ (i.e. $\alpha \leqq 0$ throughout Γ),

(II) $\Gamma_+ \neq \phi$, $\Gamma_- \neq \phi$ and γ is transversal from Γ_- to Γ_+ on Γ_0,

(III) $\Gamma_+ \neq \phi$, $\Gamma_- \neq \phi$ and γ is transversal from Γ_+ to Γ_- on Γ_0.

It is clear that Γ_0 is a closed manifold in case (II) or (III).

In Chapter 1, we reduce the boundary problem (1)–(2″) to the pseudo-differential equation $(\alpha S + \gamma + \beta)u = f$ on Γ (Proposition 1.2) and introduce Hilbert spaces in which solutions of the equation are seeked by making use of the Lax-Milgram theorem. In Chapters 2, 3 and 4, we consider the boundary conditions, according to cases (I), (II) and (III), respectively. As in [5], we use the variational approach, and apply the elliptic regularization. A special feature of the proofs is to introduce the appropriate auxiliary functions h in the respective types (I), (II) and (III) so that $h\alpha$ is positive on $\Gamma \backslash \Gamma_0$ and vanishes on Γ_0, etc. (see Lemmas 2.1 and 3.1), and to consider the equations $Pu = f$ instead of the equation $(\alpha S + \gamma + \beta)u = f$, where $P = h(\alpha S + \gamma + \beta)$ in case (I), $P = (\alpha S + \gamma + \beta)h$ in case (II) and $P = h(\alpha S + \gamma + \beta)$ in case (III). Fortunately, we can choose in respective cases pseudo-differential operators H of order zero so that $(P + H)u = f$ are uniquely solvable for all f in some functional spaces. If we set $u = Kf$, it can be proved that the equation $Pu = f$ is altered to the equation $(1 - HK)g = f$ with $u = Kg$. In order to solve the latter equation, it is sufficient to show that the operator HK is compact (to apply the Riesz-Schauder theory).

In §§ 2.1, 3.1 and 4.1, we introduce h, H and treat $(P + H)u = f$. The equation $Pu = f$ is considered in §§ 2.2, 3.2 and 4.2. Sections 2.3, 3.3 and 4.3 are devoted to the uniqueness of solutions of $Pu = f$. Finally, in §§ 2.4, 3.4 and 4.4, we return to the original problem (1)–(2) and prove the uniqueness, the existence and the regularity.

For the more general case where the singular manifold Γ_0 consists of finite number of disjoint manifold of types (I), (II) and (III), we can also formulate the similar results by virtue of the results obtained in the respective types and their local character.

Recently, in [8] Winzell investigates the problem (1)–(2) ($\beta = 0$), allowing Γ_0 to be fairly complicated and to have a certain width.

Chapter 1. Preliminaries

1.0. Let Ω be a bounded domain of R^n with C^∞-boundary Γ of dimension $n-1$ and let A be the second order elliptic differential operator described in Introduction. Here we assume that Ω is of the form $\Gamma \times (0, 1)$ near Γ and that A is defined and is elliptic in a larger domain Ω_1 with C^∞-boundary such that $\bar{\Omega} \subset \Omega_1$. Near Γ we choose a coordinate system (x', x_n) such that $x' \in \Gamma$ and x_n is a normal coordinate on x'. Let $\{\omega_i\}_{i=1}^{\ell}$ be a finite open covering of Γ and κ_i be a C^∞-coordinate transformation $y = \kappa_i(x)$ such that ω_i is mapped onto an open ball B_i in R_y^{n-1} with the origin as center and $\omega_i \cap \Gamma_0$ onto $\{y_1 = 0\} \cap B_i$ if $\omega_i \cap \Gamma_0 \neq \phi$, and such that γ is transformed to $\partial/\partial y_1$ on ω_i such that $\omega_i \cap \Gamma_0 \neq \phi$. Let $\{\zeta_i\}_{i=1}^{\ell}$ be a partition of unity subordinate to the covering $\{\omega_i\}_{i=1}^{\ell}$ such that $\gamma(\zeta_j) = 0$ in a neighborhood of Γ_0 for all j.

Let E_s (s: real) be a pseudo-differential operator on R^{n-1} defined by

$$(E_s u)(y) = (2\pi)^{1-n} \int_{R^{n-1}} (1 + |\xi|^2)^{s/2} \hat{u}(\xi) e^{iy\xi} d\xi \tag{1.1}$$

where

$$\hat{u}(\xi) = \int_{R^{n-1}} u(y) e^{-iy\xi} dy\,.$$

Now $H_s(\Gamma)$ is the usual Sobolev space with the norm

$$\begin{cases} \|u\|_s = \left(\sum_{j=1}^{\ell} \int |E_s(\tilde{\zeta}_j \tilde{u})|^2 dy\right)^{1/2} & (s \neq 0) \\ \|u\|_0 = \int_\Gamma |u|^2 d\sigma\,, \end{cases}$$

where, as well as in the below, $\tilde{v}(y)$ denotes a function on B_j defined by

$$\tilde{v}(y) = v(\kappa_j^{-1}(y)), \qquad v \in \mathcal{D}'(\omega_j)$$

and $d\sigma$ is the Lebesgue measure on Γ.

1.1. Following [2], in a neighborhood of Γ we write the differential operator A in (1) in the form $A = \sum_{j=0}^{2} A_j D_n^j$, where $D_n = i^{-1}\partial/\partial x_n$ ($i = \sqrt{-1}$) and A_j is a differential operator of order $2-j$ acting along the parallel surface of Γ. Throughout this paper, we suppose the existence of the Green kernel G (pseudo-differential operator on Ω_1) of A for the Dirichlet problem on Ω_1. We denote by δ the surface measure on Γ. Then according to [2, Sections 2.1 and 2.2], we can prove the following

PROPOSITION 1.1. (a) *Let U be in $H_s(\Omega)$ with real s so that $AU = 0$ in Ω. Then the restriction of $D_n^j U$ on Γ, $u_j = D_n^j U|_\Gamma$, is well defined in*

$H_{s-j-1/2}(\Gamma)$ *for* $j = 0, 1$, *and* U *is written in the form*

$$U = i^{-1} \sum_{j=0}^{1} \sum_{k=0}^{1-j} GA_{j+k+1} D_n^j(u_k \delta) . \tag{1.2}$$

Furthermore, u_0 *and* u_1 *satisfy*

$$u_0 = Q_0 u_0 + Q_1 u_1 , \tag{1.3}$$

where $Q_k (k = 0, 1)$ *are the pseudo-differential operators on* Γ *of order* $-k$ *defined by*

$$Q_k v = i^{-1} \sum_{j=0}^{1-k} GA_{j+k+1} D_n^j(v\delta)|_\Gamma \qquad (k = 0, 1) . \tag{1.4}$$

(b) *Let* u_j $(j = 0, 1)$ *be in* $H_{s-j-1/2}(\Gamma)$ *(s real). If* u_0 *and* u_1 *satisfy* (1.3), *then the* U *given by* (1.2) *is a distribution in* $H_s(\Omega)$ *such that* $AU = 0$ *in* Ω, $u_0 = U|_\Gamma$ *and* $u_1 = D_n U|_\Gamma$. *Moreover there exists a constant* $c > 0$ *such that*

$$c^{-1} \| U \|_{s,\Omega} \leqq \| u_0 \|_{s-1/2} + \| u_1 \|_{s-3/2} \leqq c \| U \|_{s,\Omega} ,$$

where $\| \cdot \|_{s,\Omega}$ *represents the norm in the usual Sobolev space* $H_s(\Omega)$.

(c) *The* Q_1 *defined by* (1.4) *is actually elliptic and invertible, and the operator*

$$S_0 = i^{-1} Q_1^{-1}(1 - Q_0)$$

is a pseudo-differential operator on Γ *of order* 1. *Moreover there exist two constants* $c_1' > 0$ *and* c' *such that the inequality*

$$\mathrm{Re}(S_0 \phi, \phi) \geqq c_1' \| \phi \|_{1/2}^2 - c' \| \phi \|_0^2 \tag{1.5}$$

holds for every $\phi \in C^\infty(\Gamma)$, *where*

$$(u, v) = \int_\Gamma u \bar{v} \, d\sigma .$$

Proof. We refer to [2] for the proof of (a) and (b). Let π be a pseudo-differential operator given by

$$\pi(\phi) = G(\phi\delta)|_\Gamma , \qquad \phi \in C^\infty(\Gamma) .$$

Then we can write

$$\mathrm{Re}(\pi\phi, \phi) = \mathrm{Re} \int_{\Omega_1} G(\phi\delta) \cdot \overline{\phi\delta} \, dx , \tag{1.6}$$

noting that $\phi\delta \in H_{-1}(\Omega_1)$ and $G(\phi\delta) \in H_1(\Omega_1)$. On the other hand, for any $f \in C^\infty(\bar{\Omega}_1)$, we have setting $u = Gf$

$$\text{Re}\int_{\Omega_1} Gf\cdot\bar{f}dx = \text{Re}\int_{\Omega_1} u\cdot A\bar{u}\,dx \geqq c\|u\|_{1,\Omega_1}^2 \geqq c'\|f\|_{-1,\Omega_1}^2, \tag{1.7}$$

with suitable constants $c > 0$, $c' > 0$. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $C^\infty(\bar{\Omega}_1)$ such that f_n converges to $\phi\delta$ in $H_{-1}(\Omega_1)$ as $n\to\infty$. Applying (1.7) to $f = f_n$ and letting n to infinity, we obtain

$$\text{Re}\int_{\Omega_1} G(\phi\delta)\cdot\overline{\phi\delta}\,dx \geqq c'\|\phi\delta\|_{-1,\Omega_1}^2 \geqq c''\|\phi\|_{-1/2}^2,$$

with a constant $c'' > 0$ independent of ϕ. It then follows from (1.6) that the inequality

$$\text{Re}(\pi\phi, \phi) \geqq c''\|\phi\|_{-1/2}^2$$

holds for every $\phi \in C^\infty(\Gamma)$. This shows that π and hence $Q_1 = i^{-1}\pi A_2$ are elliptic of order -1 and invertible. Therefore the operator $S_0 = i^{-1}Q_1^{-1}(1 - Q_0)$ is a pseudo-differential operator of order 1.

Now it easily follows that the principal symbol of S_0 is given by

$$\sigma_0(S_0) = i^{-1}\tau_+(x', \xi')$$

in a local coordinate system such that $x_n = 0$ on Γ, if we denote by $\tau_+(x', \xi')$ one root with positive imaginary part of the equation in t,

$$A_2(x')t^2 + A_1^0(x', \xi')t + A_0^0(x', \xi') = 0$$

where A_1^0 and A_0^0 are the principal part of A_1 and A_0 with respect to ξ', respectively, of order 1 and 2. The inequality (1.5) immediately follows from the fact that $\text{Re}\,\sigma_0(S_0) > 0$.

1.2. Let $a_0(x)$ and γ_0 be as given in (4) and set

$$S = a_0(x)^{-1}(S_0 - \gamma_0).$$

Then it easily follows from (1.5) that there exist two constants $c_1 > 0$ and M such that the inequality

$$\text{Re}(S\phi, \phi) \geqq c_1\|\phi\|_{1/2}^2 - M\|\phi\|_0^2 \tag{1.8}$$

holds for every $\phi \in c^\infty(\Gamma)$. Moreover it follows that if U is in $H_s(\Omega)$ and satisfies $AU = 0$ in Ω, then $u = U|_\Gamma$ belongs to $H_{s-1/2}(\Gamma)$ and $\partial U/\partial\nu|_\Gamma = Su$, and conversely if u is in $H_{s-1/2}(\Gamma)$, then there exists only one U in $H_s(\Omega)$

such that $AU = 0$ in Ω, $U|_\Gamma = u$ and $\partial U/\partial \nu|_\Gamma = Su$. Thus we can prove

PROPOSITION 1.2. *Solving in $H_s(\Omega)$ the boundary problem* (1)–(2) *with $F = 0$ is equivalent to finding solutions of the equation*

$$(\alpha S + \gamma + \beta)u = f \qquad \text{on } \Gamma$$

in $H_{s-1/2}(\Gamma)$.

It can be easily seen from (1.8) that the operator $E = \operatorname{Re} S + M$ ($\operatorname{Re} S = (S + S^*)/2$) is formally self-adjoint and positive, where S^* is the formally adjoint of S. Hence there exists the square root θ of the closure of E in $L^2(\Gamma)$ and so we have

$$\operatorname{Re} S = \theta^2 - M \,. \tag{1.9}$$

The θ is also regarded as a pseudo-differential operator of order 1/2 and invertible. The norms $\|\phi\|_{1/2}$ and $\|\theta\phi\|_0$ are equivalent.

1.3. Let ρ be in $C^\infty(\Gamma)$ so that $\rho(x) \geqq 0$ on Γ. By $\mathscr{U}^\rho$ and $\mathscr{F}^\rho$ we denote two Hilbert spaces obtained by the completion of $C^\infty(\Gamma)$ with respect to the norms

$$|||u||| = (\|\sqrt{\rho}\,\theta u\|_0^2 + \|u\|_0^2)^{1/2} \tag{1.10}$$

and

$$|||f|||' = \sup_{u \in \mathscr{U}, u \neq 0} \frac{|(f, u)|}{|||u|||} \,, \tag{1.11}$$

respectively. It should be noted that $\mathscr{F}^\rho$ is isometric to the dual space of $\mathscr{U}^\rho$ and that the multiplication mapping $u \mapsto \phi u$ with $\phi \in C^\infty(\Gamma)$ is continuous on $\mathscr{U}^\rho$ as well as on $\mathscr{F}^\rho$.

Now let s be a real number. By $\mathscr{U}_s^\rho$ and $\mathscr{F}_s^\rho$ we denote two Hilbert spaces obtained by the completion of $C^\infty(\Gamma)$ with respect to the respective norms

$$|||u|||_s = \left(\sum_{j=1}^{\ell} |||T_s^{(j)}u|||^2 + \|u\|_{s-1/2}^2\right)^{1/2}$$

and

$$|||f|||_s' = \left(\sum_{j=1}^{\ell} |||T_s^{(j)}u|||'^2 + \|f\|_{s-1}^2\right)^{1/2},$$

where $T_s^{(j)}$ $(j = 1, \cdots, \ell)$ are pseudo-differential operators on Γ of order s

which are defined as follows. We can write the operator E_s defined by (1.1) in the form $E_s = \tilde{T}_s - F_s$ where $\tilde{T}_s$ is properly supported, F_s has a C^∞-kernel and $\tilde{T}_0$ is the identity, and further assume that for each $j = 1, \cdots, \ell$ there exists a compact subset K_j of B_j such that $\operatorname{supp}[\tilde{T}_s\phi] \subset K_j$ for every $\phi \in C_0^\infty(R^{n-1})$ whose support is contained in the compact set $\operatorname{supp}[\tilde{\zeta}_j(y)]$ of B_j. A pseudo-differential operator $T_s^{(j)}$ on Γ of order s is defined by

$$(1.12) \qquad (T_s^{(j)}u)(x) = \begin{cases} \tilde{T}_s(\tilde{\zeta}_j\tilde{u})(\kappa_j(x)), & x \in \omega_j \\ 0, & x \notin \omega_j \end{cases}$$

for each $j = 1, \cdots, \ell$.

It is easily seen that $\mathscr{U}^\rho = \mathscr{U}_0^\rho$ and $\mathscr{F}^\rho = \mathscr{F}_0^\rho$, since $T_0^{(j)}u = \zeta_j u$. We can further prove

PROPOSITION 1.3. *For all real s,*

$$H_{s-1/2}(\Gamma) \supset \mathscr{F}_s^\rho \supset H_s(\Gamma) \supset \mathscr{U}_s^\rho \supset H_{s+1/2}(\Gamma)$$

is valid with the continuous injections.

Proof. First we note that there exist positive constants c_1, c_2 such that for all $u \in C^\infty(\Gamma)$

$$c_1^{-1}\|u\|_0 \leqq |||u||| \leqq c_1\|u\|_{1/2}, \qquad c_2^{-1}\|u\|_{-1/2} \leqq |||u|||' \leqq c_2\|u\|_0 .$$

Then it easily follows that $\mathscr{U}_s^\rho \supset H_{s+1/2}(\Gamma)$ and $\mathscr{F}_s^\rho \supset H_s(\Gamma)$ with continuous injections. Now for $u \in C^\infty(\Gamma)$, we have

$$(1.13) \qquad \|u\|_s^2 \leqq \text{const.}\left(\sum_{j=1}^{\ell}\|T_s^{(j)}u\|_0^2 + \|u\|_{s-1/2}^2\right) \leqq \text{const.}\,|||u|||_s^2$$

and

$$\|u\|_{s-1/2}^2 \leqq \text{const.}\left(\sum\|T_s^{(j)}u\|_{-1/2}^2 + \|u\|_{s-1}^2\right) \leqq \text{const.}\,|||u|||_s'^2 .$$

These imply $H_s(\Gamma) \supset \mathscr{U}_s^\rho$ and $H_{s-1/2} \supset \mathscr{F}_s^\rho$, respectively. Q.E.D.

Finally we state two propositions.

PROPOSITION 1.4. *Let L be a first order differential operator on Γ with C^∞-coefficients. Then for every s, there exists a constant $C_s > 0$ such that*

$$\|L(\rho)u\|_{s+1/2} \leqq C_s|||u|||_s$$

holds for every $u \in C^\infty(\Gamma)$, where ρ is the function introduced at the beginning of this section.

Proof. Let $u \in C^\infty(\Gamma)$. Then by (1.13) and Lemma A.2, we have

$$\begin{aligned}\|L(\rho)u\|_{s+1/2} &\leqq C\|\theta L(\rho)u\|_s \leqq C\Big(\sum_{j=1}^{\ell}\|T_s^{(j)}\theta L(\rho)u\|_0 + \|\theta L(\rho)u\|_{s-1/2}\Big)\\ &\leqq C\Big(\sum_{j=1}^{\ell}\|L(\rho)\theta T_s^{(j)}u\|_0 + \|u\|_{s-1/2} + \|L(\rho)u\|_s\Big)\\ &\leqq C\Big(\sum_{j=1}^{\ell}|||T_s^{(j)}u||| + \|u\|_{s-1/2} + \|L(\rho)u\|_s\Big)\\ &\leqq C(|||u|||_s + \|L(\rho)u\|_s)\,.\end{aligned}$$

where C denotes the various positive constants, from which we can conclude the proposition. Q.E.D.

PROPOSITION 1.5. *Let L be the same as in the preceding proposition. Then for every s, there exists a constant $C_s > 0$ such that*

$$|||L(\rho)u|||'_{s+1/2} \leqq C_s\|u\|_s$$

holds for every $u \in C^\infty(\Gamma)$.

Proof. It is enough to prove the inequality

$$|||T_{s+1/2}^{(j)}(L(\rho)u)|||' \leqq \text{const.}\,\|u\|_s$$

for $j = 1, \cdots, \ell$. For any $v \in C^\infty(\Gamma)$, we have

$$\begin{aligned}(T_{s+1/2}^{(j)}L(\rho)u, v) &= (\theta^{-1}T_{s+1/2}^{(j)}L(\rho)u, \theta v)\\ &= (\theta^{-1}T_{s+1/2}^{(j)}u, L(\rho)\theta v) + ([\theta^{-1}T_{s+1/2}^{(j)}, L(\rho)]u, \theta v)\,.\end{aligned}$$

Hence by Lemma A.2

$$\begin{aligned}|(T_{s+1/2}^{(j)}L(\rho)u, v)| &\leqq \text{const.}\,(\|u\|_s\|\sqrt{\rho}\,\theta v\|_0 + \|u\|_{s-1/2}\|v\|_0)\\ &\leqq \text{const.}\,\|u\|_s|||v|||\,,\end{aligned}$$

which completes the proof.

Chapter 2. The case of type (I)

2.0. In this chapter, we suppose the manifold Γ_0 to be of type (I). For simplicity, we assume $\alpha \geqq 0$ throughout Γ. This case was treated also in [4], but the formulation has a little difference.

2.1. The following lemma is nothing but Lemma 4 in [4].

LEMMA 2.1. *There exists a function h in $C^\infty(\Gamma)$ such that $h > 0$ on Γ and*

$$\tfrac{1}{2}\gamma^*(h) + h\beta = 1 \qquad on\ \Gamma_0,$$

where γ^* *is the adjoint of* γ *defined by the identity*

$$\int_\Gamma \gamma u\cdot\bar{v}d\sigma = \int_\Gamma u\cdot\overline{\gamma^* v}\,d\sigma\,, \qquad u,\ v\in C^\infty(\Gamma)\,.$$

Proof. First note that there exists C^∞-function $b(x)$ on Γ such that $\gamma^* = -\gamma + b(x)$. Then, we have only to find h such that $-\gamma(h) + (b + 2\beta)h = 2$ on Γ_0, which is written by the transformation $y = \kappa_j(x)$ as $-\partial\tilde{h}/\partial y_1 + (\tilde{b} + 2\tilde{\beta})\tilde{h} = 2$ in B_j when $\omega_j \cap \Gamma_0 \neq \phi$. Let $\tilde{h} = \tilde{h}_j$ be a positive solution of this equation. We then define as $h_j(x) = \tilde{h}_j(\kappa_j(x))$. On the other hand, on ω_j such that $\omega_j \cap \Gamma_0 = \phi$, we define $h_j(x) = 1$. The function $h = \sum_{j=1}^{\ell} \zeta_j h_j$ on Γ is the desired one. Q.E.D.

We are going to consider the equation

$$h(\alpha S + \gamma + \beta)u = f$$

instead of treating the equation $(\alpha S + \gamma + \beta)u = f$. Introducing a bilinear form

$$Q[u, v] = (h(\alpha S + \gamma + \beta)u, v)\,,$$

we have by (1.9), after simple calculation,

$$\begin{aligned}\operatorname{Re} Q[u, u] &= (h\alpha\theta u, \theta u)\\ &\quad + \Big(\Big(\frac{[[h\alpha, \theta], \theta]}{2} + \frac{[h\alpha, S - S^*]}{4} - h\alpha M\Big)u, u\Big)\\ &\quad + ((\tfrac{1}{2}\gamma^*(h) + h\beta)u, u)\end{aligned} \tag{2.1}$$

for $u \in C^\infty(\Gamma)$, where $[A, B] = AB - BA$. Since $h\alpha > 0$ on $\Gamma\backslash\Gamma_0$, it follows from Lemma 2.1 that there exist a constant $R > 0$ such that

$$Rh\alpha + \tfrac{1}{2}\gamma^*(h) + h\beta > 0 \qquad \text{on } \Gamma\,. \tag{2.2}$$

Let H be a pseudo-differential operator on Γ defined by

$$H = Rh\alpha + h\alpha M - \frac{[[h\alpha, \theta], \theta]}{2} - \frac{[h\alpha, S - S^*]}{4}$$

and set, for ε such that $0 < \varepsilon \leqq 1$,

$$Q_\varepsilon[u, v] = Q[u, v] + (Hu, v) + \varepsilon((S + M)u, v)\,.$$

Then it easily follows from (1.9), (2.1) and (2.2) that there exists two positive constants c_2, C independent of ε such that

$$\begin{cases} \operatorname{Re} Q_\varepsilon[u, u] \geqq c_2 |||u|||^2 + \varepsilon \|\theta u\|_0^2 \\ |Q_\varepsilon[u, v]| \geqq C \|u\|_{1/2} \|v\|_{1/2}\,, \end{cases} \tag{2.3}$$

for all $u, v \in H_{1/2}(\Gamma)$. Here and throughout this section, $|||u|||$ and $|||f|||'$ mean norms (1.10) and (1.11), respectively, with $\rho = h\alpha$. The Lax-Milgram theorem guarantees the existence of $u_\varepsilon \in H_{1/2}(\Gamma)$ for every $f \in C^\infty(\Gamma)$ such that

$$Q_\varepsilon[u_\varepsilon, v] = (f, v)\,, \qquad v \in H_{1/2}(\Gamma)\,.$$

Substitution $v = u_\varepsilon$ gives us the inequality

$$|||u_\varepsilon||| \leqq C_0 |||f|||' \tag{2.4}$$

with a constant $C_0 > 0$ not depending on ε. Since u_ε is a weak solution of the elliptic equation, we can assert $u_\varepsilon \in C^\infty(\Gamma)$, which satisfies the equation

$$\{h(\alpha S + \gamma + \beta) + H + \varepsilon(S + M)\}u_\varepsilon = f \qquad \text{on } \Gamma\,. \tag{2.5}$$

THEOREM 2.1. *Let $s \geqq \frac{1}{2}$. For every $f \in \mathscr{F}_s^{h\alpha}$ we can find one and only one $u \in \mathscr{U}_s^{h\alpha}$ satisfying the equation*

$$\{h(\alpha S + \gamma + \beta) + H\}u = f \qquad \text{on } \Gamma\,.$$

Moreover the inequality

$$|||u|||_s \leqq C_s |||f|||_s'$$

holds with a constant $C_s > 0$ independent of f.

Proof. First suppose $f \in C^\infty(\Gamma)$ and substitute $u = T_s^{(j)} u_\varepsilon$ (see (1.12) for $T_s^{(j)}$) in (2.3). Then we have, for $j = 1, \cdots, \ell$, by (2.5)

$$\begin{aligned} c_2 |||T_s^{(j)} u_\varepsilon|||^2 + \varepsilon \|\theta T_s^{(j)} u_\varepsilon\|_0 &\leqq \operatorname{Re} Q_\varepsilon[T_s^{(j)} u_\varepsilon, T_s^{(j)} u_\varepsilon] \\ &= \operatorname{Re}(\{\alpha S + \gamma + \beta + h^{-1}H + \varepsilon h^{-1}(S + M)\}T_s^{(j)} u_\varepsilon, hT_s^{(j)} u_\varepsilon) \\ &= \operatorname{Re}(T_s^{(j)} h^{-1} f, hT_s^{(j)} u_\varepsilon) + \operatorname{Re}([\{\cdots\}, T_s^{(j)}]u_\varepsilon, hT_s^{(j)} u_\varepsilon) \\ &= \operatorname{Re}(T_s^{(j)} h^{-1} f, hT_s^{(j)} u_\varepsilon) + \operatorname{Re}([\beta + h^{-1}H + \varepsilon h^{-1}M, T_s^{(j)}]u_\varepsilon, hT_s^{(j)} u_\varepsilon) \\ &\quad + \operatorname{Re}(X_j + Y_j + Z_j) \\ &\leqq |||T_s^{(j)} h^{-1} f|||' \, |||T_s^{(j)} u_\varepsilon||| + \operatorname{Re}(X_j + Y_j + Z_j) + O(\|u\|_{s-1/2}) \end{aligned}$$

with

$$\begin{aligned} X_j &= ([\alpha S, T_s^{(j)}]u_\varepsilon, hT_s^{(j)} u_\varepsilon)\,, \\ Y_j &= ([\varepsilon h^{-1} S, T_s^{(j)}]u_\varepsilon, hT_s^{(j)} u_\varepsilon)\,, \\ Z_j &= ([\gamma, T_s^{(j)}]u_\varepsilon, hT_s^{(j)} u_\varepsilon)\,. \end{aligned}$$

In this section, we denote generally by C various constants independent of ε, $0 < \varepsilon \leqq 1$, and write, for brevity, as $u = u_\varepsilon$ and $T = T_s^{(j)}$ in the below. Since

$$X_j = (\theta^{-1}[S, T]u, \theta h\alpha Tu) + ([\alpha, T]Su, hTu)$$

it easily follows from Lemmas A.1 and A.2 that

$$|X_j| \leqq C\|u\|_{s-1/2}(\|h\alpha\theta Tu\|_0 + \|\sqrt{\alpha}\,\theta Tu\|_0 + \|u\|_{s-1/2}).$$

Accordingly, for every $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$|X_j| \leqq \delta\|\sqrt{h\alpha}\,\theta Tu\|_0^2 + C_\delta\|u\|_{s-1/2}^2.$$

Similarly, since

$$Y_j = (\theta^{-1}[\varepsilon h^{-1}S, T]u, \theta hTu),$$

we obtain

$$|Y_j| \leqq \varepsilon C\|u\|_{s-1/2}\|\theta Tu\|_0 \leqq \varepsilon(\delta\|\theta Tu\|_0^2 + C_\delta\|u\|_{s-1/2}^2).$$

Thus we have

$$\begin{aligned} c_2|||Tu|||^2 + \varepsilon\|\theta Tu\|_0^2 &\leqq |||Tf|||'\,|||Tu||| + \delta\|\sqrt{h\alpha}\,\theta Tu\|_0^2 \\ &\quad + \delta\varepsilon\|\theta Tu\|_0^2 + C_\delta\|u\|_{s-1/2}^2 + \operatorname{Re} Z_j \end{aligned}$$

which implies

$$|||Tu|||^2 \leqq C(|||Tf|||'^2 + \|u\|_{s-1/2}^2 + \operatorname{Re} Z_j).$$

Consequently,

$$|||u|||_s^2 \leqq C\Big(|||f|||_s'^2 + \|u\|_{s-1/2}^2 + \sum_{j=1}^{\ell} \operatorname{Re} Z_j\Big). \tag{2.6}$$

Now we shall show the existence of a constant C_δ for any $\delta > 0$ such that

$$\sum_{j=1}^{\ell} \operatorname{Re} Z_j \leqq \delta|||u|||_s^2 + C_\delta\|u\|_{s-1/2}^2. \tag{2.7}$$

On ω_j such that $\omega_j \cap \Gamma_0 \neq \phi$, the operator $[\gamma, T]$ is transformed by κ_j to

$$\begin{aligned} \Big[\frac{\partial}{\partial y_1}, \tilde{T}_s\tilde{\zeta}_j\Big] &= \Big[\frac{\partial}{\partial y_1}, \tilde{T}_s\Big]\tilde{\zeta}_j + \tilde{T}_s\Big[\frac{\partial}{\partial y_1}, \tilde{\zeta}_j\Big] \\ &= \Big[\frac{\partial}{\partial y_1}, E_s + F_s\Big]\tilde{\zeta}_j + \tilde{T}_s\frac{\partial\tilde{\zeta}_j}{\partial y_1} = \Big[\frac{\partial}{\partial y_1}, F_s\Big]\tilde{\zeta}_j + \tilde{T}_s\frac{\partial\tilde{\zeta}_j}{\partial y_1}, \end{aligned}$$

since $T = T_s^{(j)}$ and $[\partial/\partial y_1, E_s] = 0$. Hence we have

$$|Z_j| \leqq C(\|\gamma(\zeta_j)u\|_s + \|u\|_{s-1/2})\|u\|_s .$$

From the fact that $\gamma(\zeta_j) = 0$ in some neighborhood of Γ_0, it follows by (1.13) that

$$\begin{aligned}\|\gamma(\zeta_j)u\|_s \leqq C\|h\alpha u\|_s &\leqq C\Big(\sum_{i=1}^{\ell} \|Th\alpha u\|_0 + \|u\|_{s-1/2}\Big) \\ &\leqq C\Big(\sum_{i=1}^{\ell} \|h\alpha Tu\|_0 + \|u\|_{s-1/2}\Big) .\end{aligned} \tag{2.8}$$

Using the interpolation inequality, we can assert the existence of C_δ, for any $\delta > 0$, such that

$$\|\gamma(\zeta_j)u\|_s \leqq \delta \sum_{i=1}^{\ell} \|\theta h\alpha Tu\|_0 + C_\delta\|u\|_{s-1/2} .$$

On the other hand, on ω_j such that $\omega_j \cap \Gamma_0 = \phi$ we can immediately obtain

$$|Z_j| \leqq C(\|\gamma(\zeta_j)u\|_s + \|\zeta_j u\|_s)\|u\|_s .$$

Since ζ_j as well as $\gamma(\zeta_j)$ vanishes near Γ_0, we can estimate $\|\gamma(\zeta_j)u\|_s$ and $\|\zeta_j u\|_s$ as in (2.8). Thus we can establish (2.7), and hence (2.6) becomes

$$|||u|||_s \leqq C(|||f|||'_s + \|u\|_{s-1/2}) . \tag{2.9}$$

If $s \geqq 1/2$, it then follows from (2.4) and the interpolation inequality that for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\|u\|_{s-1/2} \leqq \delta|||u|||_s + C_\delta|||f|||'_s ,$$

which together with (2.9) implies the inequality

$$|||u_\varepsilon|||_s \leqq C_s|||f|||'_s \tag{2.10}$$

for all $s \geqq 1/2$ with a constant $C_s > 0$ independent of ε and f, where we put again $u = u_\varepsilon$.

By (2.10), we can choose a sequence $\varepsilon_1 > \varepsilon_2 > \cdots \to 0$ such that u_{ε_j} converges in $C^\infty(\Gamma)$. Let u be the limit function. Then we have from (2.5) and (2.10)

$$\begin{cases} h(\alpha S + \gamma + \beta) + H)u = f & \text{on } \Gamma \\ |||u|||_s \leqq C_s|||f|||'_s . \end{cases} \tag{2.11}$$

Now let $f \in \mathscr{F}_s^{h\alpha}(s \geqq 1/2)$ and choose f_j in $C^\infty(\Gamma)$ so that $f_j \to f$ in $\mathscr{F}_s^{h\alpha}$ as

$j \to \infty$. We have just proved that, for each f_j, there exists $u_j \in C^\infty(\Gamma)$ satisfying (2.11) with $u = u_j$ and $f = f_j$. It is not hard to prove that the sequence u_j has a limit u belonging to $\mathscr{U}_s^{h\alpha}$ and satisfying (2.11).

To complete the proof, we must show the uniqueness. Let u be a solution in $\mathscr{U}_s^{h\alpha}(s \geqq 1/2)$ of the equation $\{h(\alpha S + \gamma + \beta) + H\}u = 0$ on Γ. It follows from Proposition 1.3 that $u \in H_{1/2}(\Gamma)$. Hence, by (2.3) with $\varepsilon = 0$, we have

$$0 = (h(\alpha S + \gamma + \beta)u + Hu, u) = Q_0[u, u] = c_2 |||u|||^2 .$$

This implies $u = 0$.

2.2. If we write the solution u in Theorem 2.1 as $u = Kf$, then K is a continuous mapping of $\mathscr{F}_s^{h\alpha}$ into $\mathscr{U}_s^{h\alpha}$ $(s \geqq 1/2)$ and satisfies

$$(h(\alpha S + \gamma + \beta) + H)K = 1 \qquad \text{on } \mathscr{F}_s^{h\alpha} .$$

Proposition 1.3 guarantees that K is also a continuous mapping of $H_s(\Gamma)$ $(s \geqq 1/2)$ into itself. Let f be in $H_s(\Gamma)$. If $g \in H_s(\Gamma)$ and satisfies

$$(1 - HK)g = f , \tag{2.12}$$

then $u = Kg$ satisfies

$$h(\alpha S + \gamma + \beta)u = f . \tag{2.13}$$

Conversely, if $u \in H_s(\Gamma)$ and satisfies (2.13), we have $h(\alpha S + \gamma + \beta)u + Hu = f + Hu$. Therefore $u = K(f + Hu)$. Put $g = f + Hu$. Then $h(\alpha S + \gamma + \beta)Kg = f$. So we have (2.12). Thus it is enough to treat the equation (2.12) in order to solve the equation (2.13). If $g \in H_s(\Gamma)$, then $Kg \in \mathscr{U}_s^{h\alpha}$. Using Proposition 1.4, Lemmas A.1 and A.2, we have $HKg \in H_{s+1/2}(\Gamma)$. Moreover it easily follows that HK is a continuous mapping of $H_s(\Gamma)$ into $H_{s+1/2}(\Gamma)$. Accordingly, HK is a compact operator on $H_s(\Gamma)$. Applying the Riesz-Schauder theory, we can establish the main theorem of this section.

THEOREM 2.2. (i) *Let $s \geqq 1/2$ and $f \in H_s(\Gamma)$. Then the equation* (2.13) *admits a solution $u \in H_s(\Gamma)$ if and only if f is orthogonal to a finite-dimensional subspaces $\tilde{N}_s$ of $H_s(\Gamma)$, which has the same dimension as $N = \{u \in H_s(\Gamma); (\alpha S + \gamma + \beta)u = 0\}$.* (ii) *Every solution $u \in H_s(\Gamma)(s \geqq 1/2)$ of* (2.13) *belongs to $H_t(\Gamma)$ if $f \in H_t(\Gamma)$ with $t > s$.*

Proof. For the proof of (i), see pp. 284–5 of [9]. So we shall prove only (ii). Let f be in $H_t(\Gamma)$ with $t > s$. If $u \in H_s(\Gamma)$ and satisfies (2.13), then $h(\alpha S + \gamma + \beta)u + Hu = f + Hu$. By Proposition 1.5 ($\rho = h\alpha$), we have $Hu \in \mathscr{F}^{h\alpha}_{s=1/2}$. Therefore $u = Kf + KHu \in H_t(\Gamma)$ if $t \leqq s + 1/2$, and $u \in H_{s+1/2}(\Gamma)$ if $t > s + 1/2$. If the later takes place, we have only to repeat the above process. Q.E.D.

Remark 2.1. The dimension of N is independent of s. In fact, by virtue of Theorem 2.2 (ii), we have $N \subset C^\infty(\Gamma)$.

2.3. We shall study the possibility of $\dim N = 0$ in the preceding theorem. For this purpose we first state a lemma which is similar to Lemma 2 in [4]. Using Lemma 2.1, we can easily prove it.

LEMMA 2.2. *We can find a function* $q(x) \in C^\infty(\bar{\Omega})$ *satisfying*

(i) $q(x) > 0$ *in* Ω *and* $q(x) = h(x)\alpha(x)$ *on* Γ,

(ii) *there exist two positive constants* k *and* d *such that*

$$k \operatorname{dis}(x, \Gamma) \leqq q(x) \qquad \text{in } \Omega_d = \{x \in \bar{\Omega}; \operatorname{dis}(x, \Gamma) < d\},$$

(iii) *the inequality*

$$\frac{1}{2}\frac{\partial q}{\partial \nu} + \frac{1}{2}\gamma^*(h) + h\beta \geqq c_3$$

holds on Γ *with a constant* $c_3 > 0$.

Now we consider a bilinear form

$$B[U, V] = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}\partial_j U\cdot\overline{\partial_i(qV)} + \sum_{i=1}^n b_i\partial_i U\cdot\overline{qV} + cU\cdot\overline{qV}\Big)dx + \int_\Gamma h(\gamma u + \beta u)\cdot\bar{v}\, d\sigma,$$

with $u = U|_\Gamma$ and $v = V|_\Gamma$. Integrating by part, we obtain

$$B[U, V] = \int_\Omega qAU\cdot\bar{V}dx + \int_\Gamma h\Big(\alpha\frac{\partial U}{\partial\nu} + \gamma u + \beta u\Big)\cdot\bar{v}\, d\sigma. \tag{2.14}$$

On the other hand, by (3) we have

$$\operatorname{Re} B[U, U] = \operatorname{Re}\int_\Omega q(\Sigma a_{ij}\partial_j U\cdot\overline{\partial_i U} + \Sigma b_i\partial_i U\cdot\bar{U} + cU\bar{U})dx + \frac{1}{2}\int_\Omega \Sigma a_{ij}\partial_i q\partial_j(U\bar{U})dx + \int_\Gamma\Big(\frac{1}{2}\gamma^*(h) + h\beta\Big)u\cdot\bar{u}\, d\sigma$$

$$\geqq \frac{c_0}{2}\|p\partial U\|_{0,\Omega}^2 + \int_\Omega (c-\mu)|pU|^2 dx + \frac{1}{2}\int_\Omega A_0 q\cdot|U|^2 dx$$
$$+ \int_\Gamma \left(\frac{1}{2}\frac{\partial q}{\partial\nu} + \frac{1}{2}\gamma^*(h) + h\beta\right)|u|^2 d\sigma\,,$$

where μ is a constant, $p(x)=\sqrt{q(x)}$, $A_0 q = -\sum_{i,j=i}^n \partial_i(a_{ij}\partial_j q)$ and

$$\|p\partial U\|_{0,\Omega}^2 = \sum_{j=1}^n \int q|\partial_j U|^2 dx\,.$$

It then follows from Lemma 2.2 and Lemma A.3 that there exist two constants $c_4>0$ and λ_0 such that

$$\text{(2.15)} \qquad \operatorname{Re} B[U,U] \geqq c_4\|p\partial U\|_{0,\Omega}^2 - \lambda_0\|pU\|_{0,\Omega}^2 + c_3\|u\|_0^2$$

for all $U\in C^\infty(\bar{\Omega})$.

Denote by S_λ the operator S corresponding to the operator $A_\lambda = A + \lambda$. Then we have

THEOREM 2.3. *Let* $N(\lambda)=\{u\in H_s(\Gamma)(s\geqq 1/2);\,(\alpha S_\lambda+\gamma+\beta)u=0\}$. *Then* $\dim N(\lambda)=0$ *for all* $\lambda\geqq\lambda_0$.

Proof. In view of Theorem 2.2 (ii), we can immediately prove that $N_\lambda\subset C^\infty(\Gamma)$. Let U be a C^∞-solution of the Dirichlet problem; $A_\lambda U=0$ in Ω and $U=u$ on Γ with $u\in N(\lambda)$. From Proposition 1.2, it follows $\alpha\partial U/\partial\nu+\gamma U+\beta U=0$ on Γ. This implies $U=0$, if we apply (2.14) and (2.15) with this U. So we have $u=0$.

2.4. Finally we return to the original problem (1)–(2). Corresponding to Theorem 2.2, we can state

THEOREM 2.4. (i) *Let* $s\geqq 0$ *and* (F,f) *belong to* $H_s(\Omega)\times H_{s+1/2}(\Gamma)$. *Then the problem*

$$\text{(2.16)} \qquad \begin{cases} AU=F & \text{in } \Omega \\ \alpha\dfrac{\partial U}{\partial\nu}+\gamma U+\beta U=f & \text{on } \Gamma \end{cases}$$

admits a solution $U\in H_{s+1}(\Omega)$ *if and only if* (F,f) *is orthogonal to a finite-dimensional subspace of* $H_s(\Omega)\times H_{s+1/2}(\Gamma)$, *and the space of solutions of* (2.16) *with* $F=f=0$ *has the finite dimension.* (ii) *If* $(F,f)\in H_t(\Omega)\times H_{t+1/2}(\Gamma)$ $(t>s\geqq 0)$, *every solution* $U\in H_{s+1}(\Omega)$ *of* (2.15) *belongs to* $H_{t+1}(\Omega)$.

Proof. (i) Let $V = V_F$ be the unique solution in $H_{s+2}(\Omega)$ of the Dirichlet problem; $AV = F$ in Ω and $V = 0$ on Γ. Then $v' = \partial V/\partial\nu|_\Gamma$ is in $H_{s+1/2}(\Gamma)$ and the inequality

$$(2.17) \qquad \|v'\|_{s+1/2} \leqq C\|V\|_{s+2,\Omega} \leqq C\|F\|_{s,\Omega}$$

holds for every $F \in H_s(\Omega)$. Applying Theorem 2.2 (i), we can find a solution $u \in H_{s+1/2}(\Gamma)$ of the equation

$$(2.18) \qquad (\alpha S + \gamma + \beta)u = f - \alpha v'$$

if and only if $h(f - \alpha v')$ is orthogonal to $\tilde{N}_{s+1/2}$.

Let u be a solution in $H_{s+1/2}(\Gamma)$ of (2.18). It then follows from Propositions 1.1 and 1.2 that a solution W of the Dirichlet problem, $AW = 0$ in Ω and $W = u$ on Γ, satisfies

$$(2.19) \qquad \alpha\frac{\partial W}{\partial\nu} + \gamma W + \beta W = f - \alpha v'$$

on Γ. We can easily see that $U = V + W$ satisfies (2.16). Conversely, let $U \in H_{s+1}(\Omega)$ be a solution of the problem (2.16) and $V = V_F$ be the same function as above. Then it follows that $W = U - V$ satisfies $AW = 0$ in Ω and (2.19). Hence $u = W|_\Gamma$ is in $H_{s+1/2}(\Gamma)$ and satisfies (2.18). Thus we showed the problem (2.16) admits a solution in $H_{s+1}(\Omega)$ if and only if $h(f - \alpha v')$ is orthogonal to $\tilde{N}_{s+1/2}$. By (2.17), the linear mapping $(F, f) \mapsto f - \alpha v'$ is a continuous operator of $H_s(\Omega) \times H_{s+1/2}(\Gamma)$ into $H_{s+1/2}(\Gamma)$. Hence there exists a finite number of linear functionals Φ_i on $H_s(\Omega) \times H_{s+1/2}(\Gamma)$ such that (2.16) admits a solution in $H_{s+1}(\Omega)$ if and only if $\Phi_i(F, f) = 0$ for all i. Now if U satisfies (2.16) with $F = 0$ and $f = 0$, then we have $(\alpha S + \gamma + \beta)u = 0$ $(u = U|_\Gamma)$. These complete the proof of (i).

(ii) Let $(F, f) \in H_t(\Omega) \times H_{t+1/2}(\Gamma)$ and U be a solution in $H_{s+1}(\Omega)$ of (2.16) Set $W = U - V$. Here $V = V_F$ and note that $V \in H_{t+2}(\Omega)$. Then $w = W|_\Gamma$ satisfies (2.18), where $f - \alpha v'$ belongs to $H_{t+1/2}(\Gamma)$. According to Theorem 2.2 (ii), we have $w \in H_{t+1/2}(\Gamma)$, which proves $W \in H_{t+1}(\Omega)$. Hence $U \in H_{t+1}(\Omega)$. Q.E.D.

As a corollary of Theorems 2.3 and 2.4, we can prove

THEOREM 2.5. *Let λ_0 be the number introduced in* (2.15) *and λ be any real number such that $\lambda \geqq \lambda_0$. Then for every $(F, f) \in H_s(\Omega) \times H_{s+1/2}(\Gamma)$ $(s \geqq 0)$, we can find one and only one $U \in H_{s+1}(\Omega)$ satisfying*

$$
(2.20) \qquad \begin{cases} (A+\lambda)U = F & \text{in } \Omega \\ \alpha \dfrac{\partial U}{\partial \nu} + \gamma U + \beta U = f & \text{on } \Gamma\,. \end{cases}
$$

Moreover the inequality

$$\|U\|_{s+1,\Omega} \leqq C_s(\|F\|_{s,\Omega} + \|f\|_{s+1/2})$$

holds with a suitable constant $C_s > 0$.

Proof. Let $\lambda \geqq \lambda_0$. Theorem 2.2 with $S = S_\lambda$ and Theorem 2.3 guarantee that for every $(F, f) \in H_s(\Omega) \times H_{s+1/2}(\Gamma)$ $(s \geqq 0)$, the equation (2.20) has one and only one solution U in $H_{s+1}(\Omega)$. Now we set $W = U - V$. Then it follows that $w = W|_\Gamma$ satisfies (2.18) and is estimated by

$$\|w\|_{s+1/2} \leqq \text{const.}\|f - \alpha v'\|_{s+1/2} \leqq \text{const.}(\|F\|_{s,\Omega} + \|f\|_{s+1/2})\,.$$

Therefore by (2.17), we have

$$\|U\|_{s+1,\Omega} \leqq \|V\|_{s+1,\Omega} + \|W\|_{s+1,\Omega} \leqq \text{const.}(\|F\|_{s,\Omega} + \|W\|_{s+1/2})\,,$$

which completes the proof.

Chapter 3. The case of type (II)

3.0. This chapter is devoted to the manifold Γ_0 of type (II). Suppose that Γ_0 is a closed manifold which devides Γ into two open sets Γ_-, Γ_+ so that $\alpha < 0$ on Γ_-, $\alpha > 0$ on Γ_+ and $\alpha = 0$ on Γ_0. We then consider the boundary condition (2″) with γ transversal from Γ_- to Γ_+ on Γ_0.

3.1. After Lemma 2.1, we first introduce an auxiliary function h. Note that this h is different from h in Lemma 2.1.

LEMMA 3.1. *There exists a function h in $C^\infty(\Gamma)$ such that*

(i) *$h < 0$ on Γ_-, $h > 0$ on Γ_+ and $h = 0$ on Γ_0,*

(ii) *$\gamma(h) = 1$ near Γ_0,*

(iii) *on ω_j such that $\omega_j \cap \Gamma_0 \neq \phi$, $h(x)$ is transformed to y_1 by κ_j, i.e. $h(\kappa_j^{-1}(y)) = y_1$ on the ball B_j in R^{n-1} with the origin as center.*

Proof. Setting $y = \kappa_j(x)$, we define

$$h(x) = \sum_{j=1}^{\ell} \zeta_j(x) h_j(x)\,,$$

where $h_j(x) = y_1$ on ω_j such that $\omega_j \cap \Gamma_0 \neq \phi$, $= 1$ on ω_j such that $\omega_j \subset$

Γ_+ and $= -1$ on ω_j such that $\omega_j \subset \Gamma_-$. Then (i) and (iii) are obvious. Since $\gamma(\zeta_j) = 0$ near Γ_0 and $\gamma(h_j) = 1$ on ω_j for all j such that $\omega_j \cap \Gamma_0 \neq \phi$, we have $\gamma(h) = \sum_{j=1}^{\ell} \zeta_j \gamma(h_j) = 1$ near Γ_0, which prove (ii). Q.E.D.

Now we consider the equation

$$(\alpha S + \gamma + \beta)hu = f$$

instead of $(\alpha S + \gamma + \beta)u = f$. We can then obtain similar results as in Theorems 2.1, 2.2 and 2.3. To do so, we introduce a bilinear form

$$Q[u, v] = ((h\alpha S + h\gamma + \beta_0)u, v)\,,$$

where β_0 is defined by

$$(\alpha S + \gamma + \beta)h = h\alpha S + h\gamma + \beta_0\,,$$

i.e., $\beta_0 = \gamma(h) + h\beta + \alpha[S, h]$. By the same way as in (2.1), we have

$$\begin{aligned} \operatorname{Re} Q[u, u] &= (h\alpha\theta u, \theta u) \\ &\quad + \left(\left(\frac{[[h\alpha, \theta], \theta]}{2} + \frac{[h\alpha, S - S^*]}{4} - h\alpha M\right)u, u\right) \\ &\quad + \left(\left(\frac{1}{2}\gamma^*(h) + \frac{\beta_0 + \beta_0^*}{2}\right)u, u\right). \end{aligned} \tag{3.1}$$

LEMMA 3.2. *There exist two positive constants R and c such that*

$$((Rh\alpha + \tfrac{1}{2}\gamma(h) + \tfrac{1}{2}hb + h\beta)u, u) - |(\alpha[S, h]u, u)| \geqq c\|u\|_0^2, \quad u \in C^\infty(\Gamma)\,,$$

where b is a C^∞-function on Γ defined by $\gamma^* = -\gamma + b$.

Proof. Since $h\alpha > 0$ in $\Gamma\backslash\Gamma_0$, $\gamma(h) = 1$ on Γ_0 and $h = 0$ on Γ_0, we can find two positive constants R_0 and c such that

$$R_0 h\alpha + \tfrac{1}{2}\gamma(h) + \tfrac{1}{2}hb + h\beta \geqq 2c \qquad \text{on } \Gamma\,.$$

Accordingly, we have

$$\begin{aligned} |(((R_0 + R_1)h\alpha + \tfrac{1}{2}\gamma(h) + \tfrac{1}{2}hb + h\beta)u, u)| &\geqq 2c\|u\|_0^2 + R_1 \int_\Gamma h\alpha|u|^2 d\sigma \\ &\geqq 2c\|u\|_0^2 + R_1 \min_{\Gamma - V}(h\alpha)\|u\|_{0,\Gamma-V}^2\,, \end{aligned}$$

where V is a neighborhood of Γ_0 and R_1 is any positive constant. Taking V as $V_\delta = \{x \in \Gamma; |\alpha(x)| < \delta\}$ for $\delta > 0$, we can establish, for all $u \in C^\infty(\Gamma)$,

$$\left|\int_\Gamma \alpha[S, h]u \cdot \bar{u}\, d\sigma\right| \leqq K_0\|u\|_0(\|\alpha u\|_{0,V_\delta} + \|\alpha u\|_{0,\Gamma - V_\delta}) \tag{3.2}$$

$$\leqq K_0 \| u \|_0 (\delta \| u \|_0 + \max_{\Gamma} |\alpha| \cdot \| u \|_{0, \Gamma - V_\delta})$$
$$\leqq 2K_0 \delta \| u \|_0^2 + \frac{K_0}{4\delta} \max_{\Gamma} |\alpha| \cdot \| u \|_{0, \Gamma - V_\delta}^2 ,$$

where K_0 is positive constant such that

$$\| [S, h] u \|_0 \leqq K_0 \| u \|_0 , \qquad u \in C^\infty(\Gamma) .$$

Choosing $\delta = c/2K_0$ and

$$R_1 = \frac{K_0 \max_\Gamma |\alpha|}{4\delta \min_{\Gamma - V_\delta} (\alpha h)} ,$$

we can conclude the lemma, with $R = R_0 + R_1$. Q.E.D.

Let H be a pseudo-differential operator on Γ defined by

$$H = Rh\alpha + h\alpha M - \frac{[[h\alpha, \theta], \theta]}{2} - \frac{[h\alpha, S - S^*]}{4}$$

and set, for ε such that $0 < \varepsilon \leqq 1$,

$$Q_\varepsilon[u, v] = Q[u, v] + (Hu, v) + \varepsilon((S + M)u, v) .$$

It then follows from (3.1) that

$$\operatorname{Re} Q_\varepsilon[u, u] = (h\alpha\theta u, \theta u) + \left(\left(\frac{1}{2}\gamma^*(h) + \frac{\beta_0 + \beta_0^*}{2} + Rh\alpha\right)u, u\right) + \varepsilon\| \theta u \|_0$$
$$\geqq (h\alpha\theta u, \theta u) + \left(\left(Rh\alpha + \frac{1}{2}\gamma(h) + \frac{1}{2}hb + h\beta\right)u, u\right)$$
$$- |(\alpha[S, h]u, u)| + \varepsilon \| \theta u \|_0^2 .$$

Therefore, by Lemma 3.2 we have

$$\operatorname{Re} Q_\varepsilon[u, u] \geqq (h\alpha\theta u, \theta u) + c\| u \|_0^2 + \varepsilon \| \theta u \|_0^2 \geqq c_2 |||\omega|||^2 + \varepsilon \| \theta u \|_0^2 \tag{3.3}$$

with a suitable constant $c_2 > 0$. Using the same argument as in § 2, we can obtain $u_\varepsilon \in C^\infty(\Gamma)$ for every $f \in C^\infty(\Gamma)$ such that $|||u_\varepsilon||| \leqq C_0 |||f|||'$ and

$$\{(\alpha S + \gamma + \beta)h + H + \varepsilon(S + M)\}u_\varepsilon = f \qquad \text{on } \Gamma . \tag{3.4}$$

THEOREM 3.1. *Let $s \geqq 1/2$. For every $f \in \mathscr{F}_s^{h\alpha}$ we can find one and only one $u \in \mathscr{U}_s^{h\alpha}$ satisfying the equation*

$$\{(\alpha S + \gamma + \beta)h + H\}u = f \qquad \text{on } \Gamma .$$

Moreover the inequality

$$|||u|||_s \leqq C_s|||f|||'_s$$

holds with a constant $C_s > 0$ independent of f.

Proof. Suppose $f \in C^\infty(\Gamma)$ and substitute $u = T_s^{(j)}u_\varepsilon (s \geqq 1/2)$ in (3.3), which we write, for simplicity, as Tu in the below. If we go through the same procedure as in the proof of Theorem 2.1, it follows from (3.4) that for each $j = 1, \cdots, \ell$

$$\begin{aligned} c_2|||Tu|||^2 + \varepsilon\|\theta Tu\|_0^2 &\leqq \mathrm{Re}\,(Tf, Tu) + \mathrm{Re}\,([\beta_0 + H + \varepsilon M, T]u, Tu) \\ &\quad + \mathrm{Re}\,([h\alpha S + h\gamma + \varepsilon S, T]u, Tu) \\ &\leqq |||Tf|||'\,|||Tu||| + \mathrm{Re}\,(X_j + Y_j + Z_j) + C\|u\|_{s-1/2} \end{aligned}$$

with

$$\begin{cases} X_j = ([h\alpha S, T]u, Tu) \\ Y_j = ([\varepsilon S, T]u, Tu) \\ Z_j = ([h\gamma, T]u, Tu) \end{cases},$$

and that for every $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\text{(3.5)} \qquad \begin{cases} |X_j| \leqq \delta\|\sqrt{h\alpha}\,\theta Tu\|_0^2 + C_\delta\|u\|_{s-1/2}^2 \\ |Y_j| \leqq \varepsilon(\delta\|\theta Tu\|_0^2 + C_\delta\|u\|_{s-1/2}^2) \\ |(h[\gamma, T]u, Tu)| \leqq \delta|||u|||_s^2 + C_\delta\|u\|_{s-1/2}^2 \end{cases},$$

from which we can easily deduce

$$\text{(3.6)} \qquad |||u|||_s^2 \leqq C(|||f|||_s'^2 + \|u\|_{s-1/2}^2 + \mathrm{Re}\sum_{j=1}^{l}([h, T]\gamma u, Tu))\,.$$

Here and in the following, the letters $C, C_0, C_1 \cdots$ stand for positive constants.

Now we shall estimate the last term of (3.6). On ω_j such that $\omega_j \cap \Gamma_0 \neq \phi$, the operator $[h, T_s^{(j)}]\gamma$ is transformed by κ_j to

$$\text{(3.7)} \qquad \begin{aligned} [y_1, \tilde{T}_s\tilde{\zeta}_j]\frac{\partial}{\partial y_1} &= [y_1, E_s\tilde{\zeta}_j]\frac{\partial}{\partial y_1} + [y_1, F_s\tilde{\zeta}_j]\frac{\partial}{\partial y_1} \\ &= [y_1, E_s]\frac{\partial}{\partial y_1}\cdot\tilde{\zeta}_j - [y_1, E_s]\frac{\partial\tilde{\zeta}_j}{\partial y_1} + [y_1, F_s\tilde{\zeta}_j]\frac{\partial}{\partial y_1} \\ &= s\Big(\frac{\partial}{\partial y_1}\Big)^2 E_{s-2}\tilde{\zeta}_j + p.d.0.\text{ of order } s-1\,. \end{aligned}$$

Since $E_s = (1 - \Delta_y)E_{s-2}(\Delta_y) = \sum_{i=1}^n (\partial/\partial y_i)^2)$, we have

$$([h, T]\gamma u, Tu) = s\int_{B_j}\left(\frac{\partial}{\partial y_1}\right)^2 E_{s-2}\tilde{\zeta}_j\tilde{u}\cdot\overline{(1-\Delta_y)E_{s-2}\tilde{\zeta}_j\tilde{u}}\,|J_j|\,dy + O(\|u\|_s\|u\|_{s-1})$$
$$= -s\int\left(\frac{\partial}{\partial y_1}\right)^2 \tilde{T}_{s-2}\tilde{\zeta}_j\tilde{u}\cdot\overline{\Delta_y\tilde{T}_{s-2}\tilde{\zeta}_j\tilde{u}}\,|J_j|\,dy + O(\|u\|_s\|u\|_{s-1})$$
$$= -s\sum_{i=1}^{n}\int\left|\frac{\partial^2}{\partial y_1\partial y_i}\tilde{T}_{s-2}\tilde{\zeta}_j\tilde{u}\right|^2|J_j|\,dy + O(\|u\|_s\|u\|_{s-1})$$

where J_j is the Jacobian of the mapping κ_j and $d\sigma = |J_j|dy$ on ω_j. Consequently

$$\operatorname{Re}([h, T]\gamma u, Tu) \leqq C\|u\|_s\|u\|_{s-1}\,.$$

Now for j such that $\omega_j \cap \Gamma_0 = \phi$, we can immediately obtain

$$|([h, T]\gamma u, Tu)| \leqq C(\|\zeta_j u\|_s + \|u\|_{s-1/2})\|u\|_s\,.$$

Thus we can obtain the inequality similar to (2.7) by the same argument as in (2.8). Combining this inequality with (3.6), we obtain

$$|||u_\varepsilon|||_s \leqq C(|||f|||'_s + \|u_\varepsilon\|_{s-1/2})\,,$$

where we wrote again $u = u_\varepsilon$.

To complete the proof, we have only to proceed likewise in Theorem 2.1.

3.2. Let K be a continuous mapping of $\mathscr{F}_s^{h\alpha}$ into $\mathscr{U}_s^{h\alpha}$ $(s \geqq 1/2)$ such that

$$\{(\alpha S + \gamma + \beta)h + H\}Kf = f\,, \qquad f \in \mathscr{F}_s^{h\alpha}\,.$$

This K is well defined by Theorem 3.1. By the same way as in Theorem 2.2, we can apply the Riesz-Schauder theory to the equation

$$(1 - HK)u = f$$

and can deduce

THEOREM 3.2. (i) *Let $s \geqq 1/2$ and $f \in H_s(\Gamma)$. Then the equation*

$$(\alpha S + \gamma + \beta)hu = f \qquad \text{on } \Gamma \tag{3.8}$$

admits a solution $u \in H_s(\Gamma)$ if and only if f is orthogonal to a finite-dimensional subspace $\tilde{N}_s$ of $H_s(\Gamma)$, which has the same dimension as $N = \{u \in H_s(\Gamma); (\alpha S + \gamma + \beta)hu = 0\}$. (ii) *Every solution $u \in H_s(\Gamma)$ $(s \geqq 1/2)$ of (3.8) belongs to $H_t(\Gamma)$ if $f \in H_t(\Gamma)$ with $t > s$.*

Remark 3.1. The null space N is independent of s and is contained in $C^\infty(\Gamma)$.

3.3. We shall investigate the possibility of $\dim N = 0$ in Theorem 3.2. For this purpose, we introduce auxiliary functions q_0, q_1 in Ω, associated with the function h defined in Lemma 3.1. Corresponding to Lemma 2.2, we can state

LEMMA 3.3. *Let V be a neighborhood of Γ_0 and K_1 be a positive constant. Then we can find two functions $q_0(x)$ and $q_1(x)$ in $C^\infty(\bar\Omega)$, satisfying for $j = 0, 1$,*

(i) *$q_j(x) > 0$ in Ω and $q_j(x) = \frac{1}{2}h(x)\alpha(x)$ on Γ,*

(ii) *there exist two positive constants k and d such that*

$$k \operatorname{dis}(x, \Gamma) \leqq q_0(x) \qquad \text{in } \Omega_d = \{x \in \bar\Omega;\ \operatorname{dis}(x, \Gamma) < d\}$$

(iii)$_0$ *there exists a constant $c_3 > 0$ such that the inequality*

$$\frac{1}{2}\frac{\partial q_0}{\partial \nu} + \frac{1}{2}\gamma(h) + \frac{1}{2}hb + h\beta > c_3$$

holds on Γ.

(iii)$_1$ *$\partial q_1/\partial\nu \geqq 0$ on Γ and for every $u \in C^\infty(\Gamma)$*

$$\int_\Gamma \frac{1}{2}\frac{\partial q_1}{\partial \nu}|u|^2 d\sigma \geqq K_1 \|u\|_{0,\Gamma - V}\,.$$

For the proof, we have only to note

$$\frac{1}{2}\gamma(h) + \frac{1}{2}hb + h\beta = \frac{1}{2} \qquad \text{on } \Gamma_0\,,$$

which follows from Lemma 3.1.

Setting $q(x) = q_0(x) + q_1(x)$, we consider bilinear form

$$B[U, V] = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_j U \cdot \overline{\partial_i(qV)} + \sum_{i=1}^n b_i \partial_i U \cdot \overline{qV} + cU \cdot \overline{qV}\right)dx$$
$$+ \int_\Gamma (\gamma(hu) + h\beta u + \alpha[S, h]u)\bar v \, d\sigma$$

with $u = U|_\Gamma$ and $v = V|_\Gamma$. Integrating by part, we have

$$B[U, V] = \int_\Omega qAU \cdot \bar V dx + \int_\Gamma \left(h\alpha\frac{\partial U}{\partial \nu} + \alpha[S, h]u + \gamma(hu) + h\beta u\right)\bar v d\sigma\,. \tag{3.9}$$

On the other hand, it follows from (3), Lemma A.3, Lemma 3.3 and (3.2) that there exist constants $c_4 > 0$ and λ_0 such that

$$
\begin{aligned}
\operatorname{Re} B[U, U] &= \operatorname{Re} \int_\Omega q\Big(\sum_{i,j=1}^n a_{ij}\partial_j U\cdot\overline{\partial_i U} + \sum_{i=1}^n b_i\partial_i U\cdot\overline{U} + cU\overline{U}\Big)dx \\
&+ \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i q\cdot\partial_j(U\overline{U})dx \\
&+ \operatorname{Re}\int_\Gamma (\gamma(hu) + h\beta u + \alpha[S, h]u)\bar{u}d\sigma \geqq \frac{c_0}{2}\|p\partial U\|_{0,\Omega}^2 \\
&+ \int_\Omega (c-\mu)|pU|^2dx + \frac{1}{2}\int_\Omega A_0 q\cdot|U|^2dx \\
&+ \int_\Gamma\Big(\frac{1}{2}\frac{\partial q_0}{\partial\nu} + \frac{1}{2}\gamma^*(h) + \gamma(h) + h\beta\Big)|u|^2d\sigma \\
&+ \int_\Gamma \frac{1}{2}\frac{\partial q_1}{\partial\nu}|u|^2d\sigma - \Big|\int_\Gamma \alpha[S, h]u\cdot\bar{u}\,d\sigma\Big| \\
&\geqq c_4\|p\partial U\|_{0,\Omega}^2 - \lambda_0\|pU\|_{0,\Omega}^2 + c_3\|u\|_0^2 \\
&+ \int_\Gamma \frac{1}{2}\frac{\partial q_1}{\partial\nu}|u|^2d\sigma - 2K_0\delta\|u\|_0^2 - \frac{K_0}{4\delta}\max_\Gamma|\alpha|\cdot\|u\|_{0,\Gamma-V_\delta}^2.
\end{aligned} \tag{3.10}
$$

Taking, in Lemma 3.3, $V = V_\delta$ with $\delta = c_3/4K_0$ and $K_1 = K_0\max_\Gamma|\alpha|/4\delta$, we obtain, for all $U\in C^\infty(\overline{\Omega})$,

$$
\operatorname{Re} B[U, U] \geqq c_4\|p\partial U\|_{0,\Omega}^2 - \lambda_0\|pU\|_{0,\Omega}^2 + \frac{c_3}{2}\|u\|_0^2\,. \tag{3.11}
$$

THEOREM 3.3. *Let* $N(\lambda) = \{u\in H_s(\Gamma)\ (s\geqq 1/2);\ (\alpha S_\lambda + \gamma + \beta)hu = 0\}$. *Then* $\dim N(\lambda) = 0$ *for all* $\lambda \geqq \lambda_0$.

Proof. Let $u\in N(\lambda)$. By the same argument as in Theorem 2.3, we can first establish $u\in C^\infty(\Gamma)$. The solution U of the Dirichlet problem, $A_\lambda U = 0$ in Ω and $U = u$ on Γ, is in $C^\infty(\overline{\Omega})$ and satisfies $h\alpha\partial U/\partial\nu + \gamma(hu) + h\beta u + \alpha[S_\lambda, h]u = 0$, because $h\alpha S_\lambda u + \gamma(hu) + h\beta u + \alpha[S_\lambda, h]u = 0$ on Γ. Therefore, by (3.9) we have

$$
B[U, V] + \lambda\int_\Omega qU\overline{V}dx = 0
$$

for all $V\in C^\infty(\Omega)$. Applying (3.11) to this U, we have $u = 0$. Q.E.D.

3.4. We return again to the problem (1)–(2). We shall say a function U contained in $H_{s+1}(\Omega)$ to vanish on Γ_0, if there exists $u_1\in H_{s+1/2}(\Gamma)$ such that $U = hu_1$ on Γ. In that case we write briefly $U = 0$ on Γ_0.

THEOREM 3.4. (i) *Let* $s\geqq 0$ *and* (F, f) *belong to* $H_s(\Omega)\times H_{s+1/2}(\Gamma)$.

Then the problem

$$(3.12)\qquad \begin{cases} AU = F & \text{in } \Omega \\ \alpha\dfrac{\partial U}{\partial\nu} + \gamma U + \beta U = f & \text{on } \Gamma \\ U = 0 & \text{on } \Gamma_0 \end{cases}$$

admits a solution $U \in H_{s+1}(\Omega)$ *if and only if* (F, f) *is orthogonal to a finite dimensional subspace of* $H_s(\Omega) \times H_{s+1/2}(\Gamma)$, *and the space of solutions of* (3.12) *with* $F = f = 0$ *has the finite dimension.* (ii) *If* $(F, f) \in H_t(\Omega) \times H_{t+1/2}(\Gamma)$ $(t > s \geqq 0)$, *every solution* $U \in H_{s+1}(\Omega)$ *of* (3.12) *belongs to* $H_{t+1}(\Omega)$.

The proof is established by similar argument as in the proof of Theorem 2.4.

Now we can state a uniqueness theorem as a corollary of Theorems 3.3 and 3.4.

THEOREM 3.5. *Let* λ_0 *be the number appearing in* (3.11) *and* λ *be any number such that* $\lambda \geqq \lambda_0$. *Then for every* $(F, f) \in H_s(\Omega) \times H_{s+1/2}(\Gamma)$ $(s \geqq 0)$, *we can find one and only one* $U \in H_{s+1}(\Omega)$ *satisfying*

$$\begin{cases} (A + \lambda)U = F & \text{in } \Omega \\ \alpha\dfrac{\partial U}{\partial\nu} + \gamma U + \beta U = f & \text{on } \Gamma \\ U = 0 & \text{on } \Gamma_0 . \end{cases}$$

Moreover the inequality

$$\|U\|_{s+1,\Omega} \leqq C_s(\|F\|_{s,\Omega} + \|f\|_{s+1/2})$$

holds with a suitable constant $C_s > 0$.

Chapter 4. The case of type (III)

4.0. In this final chapter, we consider the manifold Γ_0 of type (III). Suppose that Γ_0 is a closed manifold which devides Γ into two open sets Γ_-, Γ_+ so that $\alpha < 0$ on Γ_-, $\alpha > 0$ on Γ_+ and $\alpha = 0$ on Γ_0, and that the tangential vector field $-\gamma$ is transversal from Γ_+ to Γ_- on Γ_0. So that we can assume that α, γ and β are the same things as in § 3. Then we must treat the boundary condition

$$\alpha\frac{\partial U}{\partial\nu} - \gamma U + \beta U = f \qquad \text{on } \Gamma ,$$

that is, solve the equation

$$(\alpha S - \gamma + \beta)u = f \qquad \text{on } \Gamma .$$

To do so we consider the equation

$$h(\alpha S - \gamma + \beta)u = f \tag{4.1}$$

instead of it, with the same function h as the one defined in Lemma 3.1, and shall give a solution u in $H_0(\Gamma)$ which is smooth in $\Gamma \backslash \Gamma_0$.

4.1. We first study the equation (4.1) and set

$$Q[u, v] = (h(\alpha S - \gamma + \beta)u, v) .$$

a simple calculation gives

$$\begin{aligned} \operatorname{Re} Q[u, u] = (h\alpha\theta u, \theta u) + \Big(\Big(\frac{[[h\alpha, \theta], \theta]}{2} + \frac{[h\alpha, S - S^*]}{4} - h\alpha M\Big)u, u\Big) \\ + \Big(\Big(\frac{1}{2}\gamma(h) - \frac{1}{2}hb + h\beta\Big)u, u\Big) . \end{aligned} \tag{4.2}$$

Since $\gamma(h) = 1$ near Γ_0 (see Lemma 3.1), there exists a constant $R > 0$ such that

$$Rh\alpha + \frac{1}{2}\gamma(h) + h\beta - \frac{1}{2}bh > 0 \qquad \text{on } \Gamma . \tag{4.3}$$

Let H be a pseudo-differential operator on Γ defined by

$$H = Rh\alpha + h\alpha M - \frac{[[h\alpha, \theta], \theta]}{2} - \frac{[h\alpha, S - S^*]}{4}$$

and set, for ε such that $0 < \varepsilon \leqq 1$,

$$Q_\varepsilon[u, v] = Q[u, v] + (Hu, v) + \varepsilon((S + M)u, v) .$$

It then follows from (4.2) and (4.3) that there exists a constant $c_2 > 0$ such that

$$\operatorname{Re} Q_\varepsilon[u, u] \geqq c_2 |||u|||^2 + \varepsilon \|\theta u\|_0^2 . \tag{4.4}$$

Accordingly, for every $f \in C^\infty(\Gamma)$ we can find $u_\varepsilon \in C^\infty(\Gamma)$ satisfying

$$\{h(\alpha S - \gamma + \beta) + H + \varepsilon(S + M)\}u_\varepsilon = f \qquad \text{on } \Gamma$$

and

$$|||u_\varepsilon||| \leqq C_0 |||f|||' \tag{4.5}$$

with a constant $C_0 > 0$ not depending on ε.

We now introduce sequences $\hat{\mathscr{U}}_{m/2}$ and $\hat{\mathscr{F}}_{m/2}$ $(m = 0, 1, 2, \cdots)$ of subspaces of $\mathscr{U}^{h\alpha}$ and $\mathscr{F}^{h\alpha}$, respectively; $\hat{\mathscr{U}}_{m/2}$ and $\hat{\mathscr{F}}_{m/2}$ are the Hilbert spaces obtained by the completion of $C^\infty(\Gamma)$ with respect to the norms

$$\langle\!\langle u \rangle\!\rangle_{m/2} = \left(\sum_{k=0}^{m} |||h^k u|||_{k/2}^2 \right)^{1/2}$$

and

$$\langle\!\langle f \rangle\!\rangle'_{m/2} = \left(\sum_{k=0}^{m} |||h^k f|||_{k/2}'^2 \right)^{1/2},$$

respectively. It is easily seen that

$$\mathscr{U}^{h\alpha} = \hat{\mathscr{U}}_0 \supset \hat{\mathscr{U}}_{1/2} \supset \cdots, \qquad \mathscr{F}^{h\alpha} = \hat{\mathscr{F}}_0 \supset \hat{\mathscr{F}}_{1/2} \supset \cdots$$

with continuous injections. Then we can state

THEOREM 4.1. *Let m be a non-nagative integer. For every $f \in \hat{\mathscr{F}}_{m/2}$, we can find one and only one $u \in \hat{\mathscr{U}}_{m/2}$ satisfying the equation*

$$\{h(\alpha S - \gamma + \beta) + H\}u = f \qquad \text{on } \Gamma .$$

and the inequality

$$\langle\!\langle u \rangle\!\rangle_{m/2} \leqq C_{m/2} \langle\!\langle f \rangle\!\rangle'_{m/2}$$

with a constant $C_{m/2} > 0$ independent of f. Moreover the u is unique in $H_0(\Gamma)$.

Proof. Suppose first $f \in C^\infty(\Gamma)$ and substitute $u = T_s^{(j)}(h^m u_\varepsilon)$ $(s = \mathrm{m}/2)$ in (4.4). Writting, for simplicity, $T_s^{(j)}$ as T and u_ε as u, we have, for $j = 1, \cdots, \ell$,

$$\begin{aligned} c_2 |||Th^m u|||^2 + \varepsilon \|\theta Th^m u\|_0^2 &\leqq \operatorname{Re} Q_\varepsilon[Th^m u, Th^m u] \\ &= \operatorname{Re} (\{h(\alpha S - \gamma + \beta) + H + \varepsilon(S + M)\} Th^m u, Th^m u) \\ &= \operatorname{Re} (Th^m \{\quad\} u, Th^m u) + \operatorname{Re} ([\{\quad\}, Th^m] u, Th^m u) \\ &= \operatorname{Re} (Th^m f, Th^m u) + \operatorname{Re}([h\beta + H, Th^m] u, Th^m u) \\ &\quad + \operatorname{Re} ([h\alpha S - h\gamma + \varepsilon S, Th^m] u, Th^m u) = \mathrm{I} + \mathrm{II} + \mathrm{III} . \end{aligned}$$

First we shall estimate the second term II of the right hand side which is written as $\operatorname{Re} A_j$. Rewritting as

$$\begin{aligned} [h\beta + H, Th^m] &= [h\beta + H, T] h^m + T[H, h^m] \\ &= [h\beta + H, T] h^m + T \sum_{k=0}^{m-1} H_{k-m} h^k , \end{aligned}$$

we have

$$\mathrm{Re}\, A_j = \mathrm{Re}\,(\theta[h\beta + H, Th^m]u, \theta^{-1}Th^m u) \leqq C\langle\!\langle u\rangle\!\rangle_{s-1/2}^2\,, \tag{4.6}$$

where H_{k-m} is a pseudo-differential operator of order $k - m$ and, as well as in the following, $C, C_0, C_1, \cdots$ denote positive constants not depending on ε.

Now we shall estimate the third term III. To do so we represent this term as the sum of $\mathrm{Re}\, X_j$, $\mathrm{Re}\, Y_j$ and $\mathrm{Re}\, Z_j$, where

$$\begin{aligned} X_j &= ([h\alpha S, Th^m]u, Th^m u)\,, \\ Y_j &= ([\varepsilon S, Th^m]u, Th^m u)\,, \\ Z_j &= -([h\gamma, Th^m]u, Th^m u)\,. \end{aligned}$$

Since

$$X_j = (Th\alpha[S, h^m]u, Th^m u) + ([h\alpha S, T]h^m u, Th^m u)$$

and the second term has the same form as X_j in § 3, we have only calculate the first. Noting that

$$[S, h^m] = \sum_{k=0}^{m-1} S_{1+k-m} h^k\,,$$

with pseudo-differential operators S_{1+k-m} of order $1 + k - m$, we obtain

$$\begin{aligned} &\mathrm{Re}\,(Th\alpha[S, h^m]u, Th^m u) \\ &\quad \leqq \mathrm{Re} \sum_{k=0}^{m-1} \{(TS_{1+k-m}h^k u, h\alpha Th^m u) + (\theta[T, h\alpha]S_{1+k-m}h^k u, \theta^{-1}Th^m u)\} \\ &\quad = \mathrm{Re} \sum_{k=0}^{m-1} (\theta^{-1}TS_{1+k-m}h^k u, h\alpha\theta Th^m u) + O(\langle\!\langle u\rangle\!\rangle_{s-1/2}^2) \\ &\quad \leqq C\Big(\sum_{k=0}^{m-1} \|h^k u\|_{k/2} \|h\alpha\theta Th^m u\|_0 + \langle\!\langle u\rangle\!\rangle_{s-1/2}^2\Big)\,. \end{aligned}$$

This and (3.5) show that for every δ there exists a constant $C_\delta > 0$ such that

$$\mathrm{Re}\, X_j \leqq \delta\|\sqrt{h\alpha}\,\theta Th^m u\|_0^2 + C_\delta\langle\!\langle u\rangle\!\rangle_{s-1/2}^2\,. \tag{4.7}$$

By the same argument as in the above, we can obtain, for every $\delta > 0$,

$$\mathrm{Re}\, Y_j \leqq \varepsilon(\delta\|\theta Th^m u\|_0^2 + C_\delta\langle\!\langle u\rangle\!\rangle_{s-1/2}^2) \tag{4.8}$$

with a constant $C_\delta > 0$.

Finally we consider Z_j which is written as

$$\begin{aligned} Z_j &= -([h\gamma, T]h^m u, Th^m u) - (Th[\gamma, h^m]u, Th^m u) \\ &= -([h, T]\gamma h^m u, Th^m u) - (h[\gamma, T]h^m u, Th^m u) \\ &\quad - m(Th^m u, Th^m u) - m(T(\gamma(h) - 1)h^m u, Th^m u)\,. \end{aligned}$$

Consequently

$$\begin{aligned} \operatorname{Re} Z_j \leqq & - \operatorname{Re}([h, T]\gamma h^m u, Th^m u) - m\| Th^m u\|_0^2 \\ & + |(h[\gamma, T]h^m u, Th^m u) + C\|(\gamma(h) - 1)h^m u\|_s \, \|h^m u\|_s\,. \end{aligned}$$

It then follows from (3.5) and the fact that $\gamma(h) = 1$ near Γ_0 that for every $\delta > 0$ there exists a constant $C_\delta > 0$ such that

$$\begin{aligned} \operatorname{Re} Z_j \leqq & - \operatorname{Re}([h, T]\gamma h^m u, Th^m u) - m\| Th^m u\|_0^2 \\ & + \delta |||h^m u|||_s^2 + C_\delta \|h^m u\|_{s-1/2}^2\,. \end{aligned} \tag{4.9}$$

By (3.7) we have, for j such that $\omega_j \cap \Gamma_0 \neq \phi$,

$$\begin{aligned} & - \operatorname{Re}([h, T_s^{(j)}]\gamma h^m u, T_s^{(j)} h^m u) \\ & = -s\int_{B_j} \left(\frac{\partial}{\partial y_1}\right)^2 E_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u}) \cdot \overline{(1 - \Delta_y)E_{s-2}(\zeta_j y_1^m \tilde{u})}|J_j|\,dy \\ & \quad + O(\|h^m u\|_s \|h^m u\|_{s-1}) \\ & = s\int_{B_j} \left(\frac{\partial}{\partial y_1}\right)^2 E_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u}) \cdot \overline{\Delta_y E_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u})}|J_j|\,dy \\ & \quad + O(\|h^m u\|_s \|h^m u\|_{s-1}) \\ & = s\int |\Delta_y \tilde{T}_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u})|^2 |J_j|\,dy - s\sum_{k=2}^{n} \int \left(\frac{\partial}{\partial y_k}\right)^2 \tilde{T}_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u}) \\ & \quad \cdot \overline{\Delta_y \tilde{T}_{s-2}(\tilde{\zeta}_j y_1 \tilde{u})}|J_j|\,dy + O(\|h^m u\|_s \, \|h^m u\|_{s-1}) \\ & = s\int |\Delta_y \tilde{T}_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u})|^2 |J_j|\,dy - s\sum_{k=2}^{n}\sum_{i=1}^{n} \int \left| \frac{\partial^2}{\partial y_k \partial y_1} \tilde{T}_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u})\right|^2 |J_j|\,dy \\ & \quad + O(\|h^m u\|_s \| h^m u\|_{s-1})\,. \end{aligned}$$

Since

$$\| T_s^{(j)} h^m u\|_0^2 = \int \Delta_y \tilde{T}_{s-2} \tilde{\zeta}_j y_1^m \tilde{u}|^2 |J_j|\,dy + O(\|h^m u\|_s \|h^m u\|_{s-1})$$

we have by (4.9)

$$\begin{aligned} \operatorname{Re} Z_j \leqq (s - m) \int \Delta_y \tilde{T}_{s-2}(\tilde{\zeta}_j y_1^m \tilde{u})|^2 |J_j|\,dy + C\|h^m u\|_s \|h^m u\|_{s-1} \\ + \delta ||| h^m u|||_s^2 + C_\delta \|h^m u\|_{s-1/2}^2\,. \end{aligned}$$

Accordingly, for any $\delta > 0$ there exists another constant $C_\delta > 0$ such that

(4.10) $$\operatorname{Re} Z_j \leqq \delta |||h^m u|||_s^2 + C_\delta \| h^m u \|_{s-1/2}^2$$

by taking account of $s - m = -m/2$. This remain valid also for j such that $\omega_j \cap \Gamma_0 = \phi$. Thus it follows from (4.6), (4.7), (4.8) and (4.10) that

$$\langle\!\langle u_\varepsilon \rangle\!\rangle_s^2 \leqq C_s(\langle\!\langle f \rangle\!\rangle_s'^2 + \langle\!\langle u_\varepsilon \rangle\!\rangle_{s-1/2}^2)$$

is valid for all $s = m/2$ ($m = 0, 1, 2, \cdots$). By (4.5) and induction on m, we obtain

$$\langle\!\langle u_\varepsilon \rangle\!\rangle_s \leqq C_s \langle\!\langle f \rangle\!\rangle_s',$$

for all $s = m/2$ ($m = 0, 1, 2, \cdots$), with a constant $C_s > 0$ not depending on ε.

Let m be fixed. Then it follows from the theorem of Banach-Sacks that there exists a decreasing sequence $\varepsilon_1, \varepsilon_2, \cdots$ converging to zero such that

$$v_n = \frac{u_{\varepsilon_1} + \cdots + u_{\varepsilon_n}}{n}$$

converges to some u in $\hat{\mathscr{U}}_s$ ($s = m/2$). Accordingly the u satisfies

(4.11) $$\begin{cases} \{h(\alpha S - \gamma + \beta) + H\} u = f & \text{on } \Gamma \\ \langle\!\langle u \rangle\!\rangle_s \leqq C_m \langle\!\langle f \rangle\!\rangle_s'. \end{cases}$$

Now let $f \in \hat{\mathscr{F}}_s$ and choose f_j in $C^\infty(\Gamma)$ so that $f_j \to f$ in $\hat{\mathscr{F}}_s$ as $j \to \infty$. For each f_j, we can find u_j in $\hat{\mathscr{U}}_s$ satisfying (4.11) with $f = f_j$. As is easily seen, u_j converges in $\hat{\mathscr{U}}_s$ as $j \to \infty$ and the limit u satisfies (4.11).

Finally we shall prove the uniqueness of u in $H_0(\Gamma)$. To do so, we consider the dual problem

(4.12) $$\{(\alpha S - \gamma + \beta)^* h + H\} v = g.$$

Let g be in $C^\infty(\Gamma)$. Then, by (4.4), we can find $v_\varepsilon \in C^\infty(\Gamma)$ such that

$$\{(\alpha S - \gamma + \beta)^* h + H + \varepsilon(S^* + M)\} v_\varepsilon = g \qquad \text{on } \Gamma$$

and $|||v_\varepsilon||| \leqq C_0 |||g|||'$ with a constant C_0 independent of ε. Substitute $u = T_s^{(j)} v_\varepsilon$ (s real $\geqq 1/2$) in (4.4). Then following the same argument as in the proof of Theorem 3.1, we can derive the inequality

$$|||v_\varepsilon|||_s \leqq C_s |||g|||_s'.$$

Thus we can prove that for every $g \in \mathscr{F}_s^{h\alpha}$ there exists one and only one

$v \in \mathscr{U}_s^{h\alpha}$ satisfying (4.12). Furthermore the result of Theorem 3.2 remains valid for the equation $(\alpha S - \gamma + \beta)^* hv = g$.

Now let u be in $H_0(\Gamma)$ and satisfy $\{h(\alpha S - \gamma + \beta) + H\}u = 0$ on Γ. This means that

$$(u, \{(\alpha S - \gamma + \beta)^* h + H\}v) = 0$$

for all $v \in C^\infty(\Gamma)$. Hence we have $(u, g) = 0$ for all $g \in C^\infty(\Gamma)$. Thus the proof of Theorem 4.1 is completed.

4.2. Let m be a non-negative integer. By $\mathscr{H}_s(\Gamma)$ $(s = m/2)$ we denote the Hilbert space obtained by the completion of $C^\infty(\Gamma)$ with respect to the norm

$$\langle u \rangle_s = \left(\sum_{k=0}^{m} \| h^k u \|_{k/2}^2 \right)^{1/2}.$$

It easily follows from Proposition 1.3 that for all $s = m/2$

$$\hat{\mathscr{F}}_s \supset \mathscr{H}_s(\Gamma) \supset \hat{\mathscr{U}}_s \tag{4.13}$$

is valid and the injections are continuous. Using Theorem 4.1, we can define a continuous mapping K of $\hat{\mathscr{F}}_s$ into $\hat{\mathscr{U}}_s$ such that

$$\{h(\alpha S - \gamma + \beta) + H\}K = 1 .$$

Hence (4.13) guarantees that K is also a continuous mapping of $\mathscr{H}_s(\Gamma)$ into itself. Thus we have only to consider the equation

$$(1 - HK)g = f$$

in order to solve the original equation

$$h(\alpha S - \gamma + \beta)u = f . \tag{4.14}$$

THEOREM 4.2. (i) *Let m be a non-negative integer and f be in $\mathscr{H}_s(\Gamma)$ $(s = m/2)$. Then the equation* (4.14) *admits a solution $u \in \mathscr{H}_s(\Gamma)$ if and only if f is orthogonal to a finite-dimensional subspace $\hat{N}_s$ of $\mathscr{H}_s(\Gamma)$ which has the same dimension as $N = \{u \in \mathscr{H}_s(\Gamma);\ h(\alpha S - \gamma + \beta)u = 0\}$.* (ii) *Every solution u in $\mathscr{H}_s(\Gamma)$ $(s = m/2)$ of* (4.14) *belongs to $\mathscr{H}_t(\Gamma)$ if $f \in \mathscr{H}_t(\Gamma)$ $(t = \ell/2)$ with integer $\ell > m$.*

Proof. In order to show (i), it is sufficient to prove the compactness of the operator HK on $\mathscr{H}_s(\Gamma)$, where $s = m/2$. If $g \in \mathscr{H}_s(\Gamma)$, then $Kg \in \hat{\mathscr{U}}_s$, that is, $h^k Kg \in \mathscr{U}_{k/2}^{h\alpha}$ for $k = 0, 1, \cdots, m$. Hence we have

$$h^k HKg = Hh^k Kg + [h^k, H]Kg = Hh^k Kg - \sum_{i=0}^{k-1} H_{i-k} h^i Kg\,,$$

where H_{i-k} is a pseudo-differential operator of order $i - k$. Proposition 1.4 implies $h^k HKg \in H_{(k+1)/2}(\Gamma)$ for $k = 0, 1, \cdots, m$, from which it follows that HK is a compact operator on $\mathscr{H}_s(\Gamma)$.

(ii) If u is in $\mathscr{H}_s(\Gamma)$ $(s = m/2)$ and satisfy (4.14), then $H(\alpha S - \gamma + \beta)u + Hu = f + Hu$. By Proposition 1.5, we have $Hu \in \hat{\mathscr{F}}_{s+1/2}$, since $h^k u \in H_{k/2}(\Gamma)$ for $k = 0, 1, \cdots, m$ and

$$h^k Hu = Hh^k u - \sum_{i=0}^{k-1} H_{i-k} h^i u\,.$$

Therefore $u = Kf + KHu \in \mathscr{H}_\ell(\Gamma)$ if $\ell \leq m + 1$. If $\ell > m + 1$, $u \in \mathscr{H}_{s+1/2}(\Gamma)$. After repeating this argument, we obtain $u \in \mathscr{H}_{\ell/2}(\Gamma)$. Q.E.D.

Remark 4.1. The null space N is independent of s and is contained in $\bigcap_{m-1}^{\infty} \mathscr{H}_{m/2}(\Gamma)$, which easily follows from Theorem 4.2 (ii).

4.3. We shall again study the possibility of $\dim N = 0$ in Theorem 4.2. To do so we first introduce a C^∞-function $q(x)$ in Ω in like manner as in Lemmas 2.2 and 3.2.

LEMMA 4.2. *We can find a function $q(x)$ in $C^\infty(\Omega)$ satisfying* (i) *and* (ii) *of Lemma* 2.2, *and*

(iii) *the inequality*

$$\frac{1}{2}\frac{\partial q}{\partial \nu} + \frac{1}{2}\gamma(h) - \frac{1}{2}hb + h\beta > c_3$$

holds on Γ with a constant $c_3 > 0$, where h is the function introduced in Lemma 3.1.

Now we consider a bilinear form

$$B[U, V] = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}\partial_j U\cdot\overline{\partial_i(qV)} + \sum_{i=1}^n b_i\partial_i U\cdot\overline{qV} + cU\cdot\overline{qV}\Big)dx$$
$$+ \int_\Gamma (-h\gamma(u) + h\beta u)\bar{v}\,d\sigma\,,$$

u and v being the restrictions of U and V on Γ, respectively. Similar calculation as in (2.14) and (2.15) leads to

$$B[U, V] = \int_\Omega qAU\cdot\overline{V}\,dx + \int_\Gamma \Big(h\alpha\frac{\partial U}{\partial \nu} - h\gamma(u) + h\beta u)\bar{v}\,d\sigma \tag{4.15}$$

and

$$\operatorname{Re} B[U, V] \geqq c_4 \|p\partial U\|_{0,\Omega}^2 - \lambda_0 \|pU\|_{0,\Omega}^2 + c_3 \|u\|_0^2 \tag{4.16}$$

for all U and V in $C^\infty(\bar{\Omega})$, where c_4 is a positive constant, $p = \sqrt{q}$ and λ_0 is a real number.

THEOREM 4.3. *Let* $N(\lambda) = \{u \in \mathscr{H}_{m/2}(\Gamma)$ *(m integer* $\geqq 0$*);* $h(\alpha S_\lambda - \gamma + \beta)u = 0\}$. *Then* $\dim N(\lambda) = 0$ *for all* $\lambda \geqq \lambda_0$.

Proof. First we prove that if v is in $H_s(\Gamma)$ $(s \geqq 1/2)$ and satisfies $(\alpha S_\lambda - \lambda + \beta)^* hv = 0$ with $\lambda \geqq \lambda_0$, then $v = 0$. It is obvious that $v \in C^\infty(\Gamma)$ (see the final part of Proof of Theorem 4.1). Let V be in $C^\infty(\bar{\Omega})$, and satisfy $A_\lambda V = 0$ in Ω and $V = v$ on Γ. It then follows from (4.15) that for every $U \in C^\infty(\bar{\Omega})$ such that $A_\lambda U = 0$ in Ω

$$\begin{aligned} B[U, V] + \lambda \int_\Omega qU \cdot \bar{V}\, dx &= (h(\alpha S_\lambda - \gamma + \beta)u, v) \\ &= (u, (\alpha S_\lambda - \gamma + \beta)^* hv) = 0\,, \end{aligned} \tag{4.17}$$

where $u = U|_\Gamma$. Taking V as U in (4.17) and applying (4.16) for $U = V$, we have $v = 0$. Accordingly, for every $g \in C^\infty(\Gamma)$, we can find $v \in C^\infty(\Gamma)$ so that $(\alpha S_\lambda - \gamma + \beta)^* hv = g$ (cf. Theorem 3.2). Now let $u \in N(\lambda)$. We then have

$$0 = (u, (\alpha S_\lambda - \gamma + \beta)^* hv) = (u, g)\,.$$

Hence $u = 0$.

4.4. Let $\eta(x)$ be a C^∞-function in Ω_1 such that $\eta(x) = h(x)^2$ on Γ and $\eta(x) > 0$ in Ω. For every non-negative integer m and real number μ, we denote by $\mathscr{H}_{m,\mu}(\Omega)$ the Hilbert space obtained by the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|U\|_{m,\mu,\Omega} = \left(\sum_{k=0}^m \|\eta^k U\|_{k+\mu,\Omega}^2\right)^{1/2}.$$

Then it is easily seen that if $U \in \mathscr{H}_{m,\mu}(\Omega)$, then $\partial_j U \in \mathscr{H}_{m,\mu-1}(\Omega)$ for $j = 1, \cdots, n$. Moreover we can prove that if $\mu > 1/2$, then the restriction on Γ of $U \in \mathscr{H}_{m,\mu}(\Omega)$ is in $\mathscr{H}_m(\Gamma)$, by using the following

PROPOSITION 4.1. *Let k be an integer* $\geqq 1$. *Then there exists a constant* $C_k > 0$ *such that the inequality*

$$\|hu\|_{k-1/2} \leqq C_k(\|u\|_{k-1} + \|h^2 u\|_k) \tag{4.18}$$

holds for every $u \in C^\infty(\Gamma)$.

Proof. It is sufficient to prove (4.18) when u has its support in ω_j such that $\omega_j \cap \Gamma_0 \neq \phi$. Then, by the transformation $y = \kappa_j(x)$, the inequality (4.18) is altered to

$$\|y_1\phi\|_{k-1/2,R^{n-1}} \leqq C_k(\|\phi\|_{k-1,R^{n-1}} + \|y_1^2\phi\|_{k,R^{n-1}}), \tag{4.19}$$

where $\phi(y) = u(\kappa_j^{-1}(y))$. Integrating by part, we have

$$\begin{aligned}\|y_1\phi\|^2_{k-1/2,R^{n-1}} &= \int (1+|\xi|^2)^{k-1/2} \frac{\partial \hat{\phi}}{\partial \xi_1} \frac{\partial \hat{\phi}}{\partial \xi} d\xi \\ &= -\int \left\{(2k-1)\xi_1(1+|\xi|^2)^{k-3/2}\frac{\partial \hat{\phi}}{\partial \xi_1} + (1+|\xi|^2)^{k-1/2}\frac{\partial^2 \hat{\phi}}{\partial \xi_1^2}\right\}\bar{\hat{\phi}}\, d\xi \\ &\leqq (2k-1)\int (1+|\xi|^2)^{k-1}|\hat{\phi}|\left|\frac{\partial \hat{\phi}}{\partial \xi_1}\right| d\xi \\ &\quad + \int (1+|\xi|^2)^{k-1/2}|\hat{\phi}|\left|\frac{\partial^2 \hat{\phi}}{\partial \xi_1^2}\right| d\xi\,.\end{aligned}$$

Accordingly, by the Schwarz inequality,

$$\|y_1\phi\|^2_{k-1/2} \leqq (2k-1)\|\phi\|_{k-1}\|y_1\phi\|_{k-1} + \|\phi\|_{k-1}\|y_1^2\phi\|_k\,,$$

where we omit the suffix R^{n-1}. This immediately implies (4.19), and hence (4.18) is proved.

Now we can state

THEOREM 4.4. (i) *Let* m *be a non-negative integer and* (F, f) *belong to* $\mathscr{H}_{m,0}(\Omega) \times \mathscr{H}_m(\Gamma)$. *Then the problem*

$$\begin{cases} AU = F & \text{in } \Omega \\ h\left(\alpha\dfrac{\partial U}{\partial \nu} - \gamma U + \beta U\right) = hf & \text{on } \Gamma \end{cases} \tag{4.20}$$

admits a solution U *in* $\mathscr{H}_{m,1/2}(\Omega)$ *if and only if* (F, f) *is orthogonal to a finite-dimensional subspace of* $\mathscr{H}_{m,0}(\Omega) \times \mathscr{H}_m(\Gamma)$ *and the space of solutions in* $\mathscr{H}_{m,1/2}(\Omega)$ *of* (4.20) *with* $F = f = 0$ *has the finite dimension.* (ii) *If* (F, f) *belongs to* $\mathscr{H}_{\ell,0}(\Omega) \times \mathscr{H}_\ell(\Gamma)$ (ℓ *integer* $> m$), *then* $U \in \mathscr{H}_{\ell,1/2}(\Omega)$.

Proof (cf. Proof of Theorem 2.4). (i) Let (F, f) be in $\mathscr{H}_{m,0}(\Omega) \times \mathscr{H}_m(\Gamma)$. By V we mean a solution in $H_2(\Omega)$ of the Dirichlet problem; $AV = F$ in Ω and $V = 0$ an Γ. Since we have

$$A(\eta^k V) = \eta^k F + [A, \eta^k]V = \eta^k F + P_1\eta^{k-1}V + P_0\eta^{k-2}V$$

for $k \geqq 1$ ($P_0 = 0$ if $k = 1$), with some differential operators P_i $(i = 0, 1)$ of order i, it then follows from (2.17) that

$$\|\eta^k V\|_{k+2} \leqq C(\|\eta^k F\|_k + \|\eta^{k-1} V\|_{k-1+2} + \|\eta^{k-2} V\|_{k-2+2}).$$

Accordingly we have

$$\|V\|_{m,2,\Omega} \leqq C\|F\|_{m,0,\Omega}. \tag{4.21}$$

This implies $V \in \mathscr{H}_{m,2}(\Omega)$. Hence $\partial V/\partial\nu \in \mathscr{H}_{m,1}(\Omega)$ and so $v' = \partial V/\partial\nu|_\Gamma$ is in $\mathscr{H}_m(\Gamma)$. As a matter of fact, we can obtain the inequality

$$\langle v\rangle_m \leqq C\sum_{k=0}^{m}\|h^{2k}v\|_k, \qquad v \in C^\infty(\Gamma),$$

applying (4.18) with $u = h^{2k-2}v$. From this the inequality

$$\langle v'\rangle_m \leqq C\|V\|_{m,2,\Omega} \tag{4.22}$$

is easily derived. Accordingly, Theorem 4.2 (i) guarantees that the equation

$$h(\alpha S - \gamma + \beta)w = h(f - \alpha v') \tag{4.23}$$

admits a solution w in $\mathscr{H}_m(\Gamma)$ if and only if $h(f - \alpha v')$ is orthogonal to $\tilde{N}_m$.

Let w be a solution in $\mathscr{H}_m(\Gamma)$ of (4.23). It then follows from Propositions 1.1 and 1.2 that the distribution on Ω_1,

$$W = i^{-1}\sum_{j=0}^{1}\sum_{k=0}^{1-j} GA_{j+k+1}D_m^j(w_k\delta) \ (w_0 = w,\ w_1 = iS_0w) \tag{4.24}$$

belongs to $\mathscr{H}_{m,1/2}(\Omega)$, and satisfies $AW = 0$ in Ω, $W = w$ on Γ and

$$h\left(\alpha\frac{\partial W}{\partial\nu} + \gamma W + \beta W\right) = h(f - \alpha v') \tag{4.25}$$

on Γ. In fact, we have

$$\begin{aligned}\eta^k W &= i^{-1}\sum_{j=0}^{1}\sum_{\ell=0}^{1-j}\{GA_{j+\ell+1}D_n^j\eta^k(w_l\delta) + [\eta^k, GA_{j+\ell+1}D_n^j](w_\ell\delta)\}\\ &= i^{-1}\sum_{j=0}^{1}\sum_{\ell=0}^{1-j}\Big\{GA_{j+\ell+1}D_n^j(h^{2k}w_\ell\delta) + \sum_{i=0}^{k-1}P_{-(\ell+k+1-i)}\eta^i(w_\ell\delta)\Big\},\end{aligned}$$

where P_{-j} $(i = \ell + 2, \cdots, \ell + k + 1)$ are pseudo-differential operators on Ω_1 of order $-j$. Following [2, Section 2.1], we can obtain

$$\|\eta^k W\|_{k+1/2,\Omega} \leqq C_k\sum_{i=0}^{k}(\|h^{2i}w_0\|_i + \|h^{2i}w_1\|_{i-1})$$

with a suitable constant C_k. Accordingly

$$\|W\|_{m,1/2,\Omega} \leqq C\langle w\rangle_m , \tag{4.26}$$

which immediately implies $W \in \mathscr{H}_{m,1/2}(\Omega)$. Thus it follows that $U = V + W$ is in $\mathscr{H}_{m,1/2}(\Omega)$ and satisfies (4.20). Conversely, let $U \in \mathscr{H}_{m,1/2}(\Omega)$ be a solution of (4.20). Then $W = U - V$ satisfies $AW = 0$ in Ω and (4.25), and hence $w = W|_\Gamma$ satisfies (4.23). Since $W \in \mathscr{H}_{m,1/2}(\Omega) \subset H_{1/2}(\Omega)$, we have, by Proposition 1.1 (a), $w \in H_0(\Gamma) = \mathscr{H}_0(\Gamma)$. Therefore Theorem 4.2 (ii) guarantees $w \in \mathscr{H}_m(\Gamma)$, since right hand side of (4.23) is contained in $\mathscr{H}_m(\Gamma)$. Thus we could show that the problem (4.20) admits a solution in $\mathscr{H}_{m,1/2}(\Omega)$ if and only if $h(f - \alpha v')$ is orthogonal to $\tilde{N}_m$. Now if U is in $\mathscr{H}_{m,1/2}(\Omega)$ and satisfies (4.20) with $F = f = 0$, as we have seen above, $u = U|_\Gamma$ belongs to $\mathscr{H}_m(\Gamma)$ and satisfies $h(\alpha S - \gamma + \beta)u = 0$. These complete the proof of (i).

(ii) Let $(F, f) \in \mathscr{H}_{\ell,0}(\Omega) \times \mathscr{H}_\ell(\Gamma)$ and U be a solution in $\mathscr{H}_{m,1/2}(\Omega)$ of (4.20). Set $W = U - V$. Note that $V \in \mathscr{H}_{m,2}(\Omega)$. Then $w = W|_\Gamma$ satisfies (4.23) whose right hand side belongs to $\mathscr{H}_\ell(\Gamma)$. Therefore in virtue of Theorem 4.2 (ii), we have $w \in \mathscr{H}_\ell(\Gamma)$, which together with (4.26) proves $U \in \mathscr{H}_{m,1/2}(\Omega)$. Q.E.D.

As a corollary of Theorems 4.3 and 4.4, we can state

THEOREM 4.5. *Let λ_0 be the number introduced in* (4.16) *and λ be any real number such that $\lambda \geqq \lambda_0$. Then for every $(F, f) \in \mathscr{H}_{m,0}(\Omega) \times \mathscr{H}_m(\Gamma)$ (m integer $\geqq 0$), we can find one and only one $U \in \mathscr{H}_{m,1/2}(\Omega)$ satisfying*

$$\begin{cases} (A+\lambda)U = F & \text{in} \\ h\Big(\alpha\dfrac{\partial U}{\partial \nu} - \gamma U + \beta U\Big)U = hf & \text{on } \Gamma . \end{cases}$$

Moreover the inequality

$$\|U\|_{m,1/2,\Omega} \leqq C_m(\|F\|_{m,0,\Omega} + \langle f\rangle_m) . \tag{4.27}$$

holds with a suitable constant $C_m > 0$.

Proof. The first half of the theorem is obvious. We only prove (4.27). Let V and v be the same as in the proof of the preceding theorem and w be a solution in $\mathscr{H}_m(\Gamma)$ of (4.23). Then we can write as $U = V + W$, where W is defined by (4.24). Consequently, we have by (4.21), (4.22) and (4.26)

$$\|U\|_{m,1/2,\Omega} \leqq \|V\|_{m,1/2,\Omega} + \|W\|_{m,1/2,\Omega} \leqq \|V\|_{m,1/2,\Omega} + C_1\langle w\rangle_m$$
$$\leqq C_2(\|V\|_{m,2,\Omega} + \langle f\rangle_m) \leqq C_3(\|F\|_{m,0,\Omega} + \langle f\rangle_m)\,.$$

Appendix

LEMMA A.1 (see Lemma 1 in [5]). *Let $f(x)$ be in $C^\infty(R^m)$ and P be a pseudo-differential operator on R^m of order t. Then there exist pseudo-differential operators P_j $(j = 1, \cdots, m)$ and Q on R^m of order $t - 1$ and $t - 2$, respectively, such that*

$$[f, P] = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} P_j + Q\,.$$

LEMMA A.2 (see Lemma A.1 in [3]). *Let $f(x)$ be in $C_0^\infty(R^m)$ such that $f(x) \geqq 0$ in R^m. Then*

$$\left|\frac{\partial f}{\partial x_j}(x)\right|^2 \leqq 2K_j f(x)\,, \qquad x \in R^m \ (j = 1, \cdots, m)\,,$$

where

$$K_j = \sup_{x \in R^m}\left|\frac{\partial^2 f}{\partial x_j^2}(x)\right|.$$

LEMMA A.3 (see Lemma 3 in [4]). *Let Ω be a bounded domain in R^m with C^∞ boundary of dimension $m - 1$ and let $q(x)$ be in $C^\infty(\bar\Omega)$ such that $q(x) > 0$ in Ω and $C \operatorname{dis}(x, \Gamma) \leqq q(x)$ in $\Omega_d = \{x \in \bar\Omega;\, \operatorname{dis}(x, \Gamma) < d\}$ with suitable constant $C > 0$ and $d > 0$. Then for any $\delta > 0$ there exists a constant $C_\delta > 0$ such that*

$$\|U\|_{0,\Omega}^2 \leqq \delta\|p\partial U\|_{0,\Omega}^2 + C_\delta\|pU\|_0^2, \qquad U \in C^\infty(\bar\Omega),$$

where $p = \sqrt{q}$.

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