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INDEPENDENCE OF THE INCREMENTS OF GAUSSIAN RANDOM FIELDS

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§1. Introduction

Let $X = \{X(A); A \in \boldsymbol{R}^n\}$ be a mean zero Gaussian random field ($n \geqslant 2$). We call X Euclidean if the probability law of the increments $X(A) - X(B)$ is invariant under the Euclidean motions. For such an X, the variance of $X(A) - X(B)$ can be expressed in the form $r(|A - B|)$ with a function $r(t)$ on $[0, \infty)$ and the Euclidean distance $|A - B|$.

We are interested in the dependence property of a Euclidean random field X and after P. Lévy [2] we introduce a set $\mathscr{F}_X(P_1|P_2)$ for a pair of points $P_1, P_2 \in \boldsymbol{R}^n$:

$$\mathscr{F}_X(P_1|P_2) = \{A \in \boldsymbol{R}^n; E[(X(A) - X(P_2))(X(P_1) - X(P_2))] = 0\} .$$

The set $\mathscr{F}_X(P_1|P_2)$, we expect, would characterize the Euclidean random field X. This is the case for a Lévy's Brownian motion X_1, where $r(t) = t$. Indeed, $\mathscr{F}_{X_1}(P_1|P_2)$ becomes the half-line emanating from P_2, i.e.,

$$\mathscr{F}_{X_1}(P_1|P_2) = \{A \in \boldsymbol{R}^n; |A - P_1| = |A - P_2| + |P_1 - P_2|\} ,$$

and the equality

$$\mathscr{F}_X(P_1|P_2) = \mathscr{F}_{X_1}(P_1|P_2), \qquad P_1, P_2 \in \boldsymbol{R}^n ,$$

implies that X has independent increments on any line in $\boldsymbol{R}^n$ and therefore that X is a Lévy's Brownian motion X_1 under the normalizing condition $r(1) = 1$. There are however some cases where the set $\mathscr{F}_X(P_1|P_2)$ is not rich enough to characterize X; for example we have $\mathscr{F}_X(P_1|P_2) = \{P_2\}$ when $r(t)$ is strictly concave on $(0, \infty)$. So we introduce in this paper a partition $\{\mathscr{C}_X(P_1, P_2; q); q \in \boldsymbol{R}\}$ satisfying the following property: The increments $X(A) - X(B)$ and $X(P_1) - X(P_2)$ are mutually independent if and only if A and B belong to the same set $\mathscr{C}_X(P_1, P_2; q)$ for some q. Our partition

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describes much finer structure of X than $\{\mathscr{F}_X(P_1|P_2)\}$ and has a relation $\mathscr{C}_X(P_1, P_2; 1) = \mathscr{F}_X(P_1|P_2)$. For a Lévy's Brownian motion X_1, the set $\mathscr{C}_{X_1}(P_1, P_2; q)$ with $0 < |q| < 1$ coincides with a sheet of the hyperboloid of two sheets of revolution with foci P_1 and P_2:

$$\mathscr{C}_{X_1}(P_1, P_2; q) = \{A \in \boldsymbol{R}^n; |A - P_1| = |A - P_2| + q\,|P_1 - P_2|\} .$$

We now raise the following question: From the equality

$$\mathscr{C}_X(P_1, P_2; q) = \mathscr{C}_{X_1}(P_1, P_2; q) \qquad \text{for any } P_1,\ P_2 \in \boldsymbol{R}^n ,$$

can one conclude that X with $r(1) = 1$ is a Lévy's Brownian motion X_1? Contrary to the above mentioned case $q = 1$, i.e., of $\mathscr{F}_{X_1}(P_1|P_2)$, this question is not easily answered. In addition, we shall be concerned with not only a Lévy's Brownian motion but also more general Euclidean random field X, and we consider the following

PROBLEM 1. For some fixed $q \in \boldsymbol{R}$, does a family of the sets $\{\mathscr{C}_X(P_1, P_2; q);$ $P_1, P_2 \in \boldsymbol{R}^n\}$ characterize the Euclidean random field X?

The second problem we consider is concerned with projective invariance, which characterizes X_α with $r(t) = t^\alpha$ $(0 < \alpha \leqslant 2)$ ([3]). It is easily seen that the projective invariance of X_α is inherited by $\mathscr{F}_{X_\alpha}(P_1|P_2)$ as follows: For any P_1, $P_2 \in \boldsymbol{R}^n$, the relation

$$\mathscr{F}_{X_\alpha}(TP_1\,|\,TP_2) = T\mathscr{F}_{X_\alpha}(P_1|P_2)$$

holds for each Euclidean motion, inversion with center P_2 and similar transformation T on $\boldsymbol{R}^n$. We are naturally led to the converse problem:

PROBLEM 2. Does the relation

$$\mathscr{F}_X(TP_1\,|\,TP_2) = T\mathscr{F}_X(P_1|P_2)$$

imply that the Euclidean random field X is an X_α?

The purpose of this paper is to give partial answers to these problems. In fact, we shall solve the Problem 1 for some class of Euclidean random fields X, in particular, for X_α with $0 < \alpha \leqslant 2$ (Theorems 2 and 3). We shall also show that the Problem 2 can be solved under some condition on X (Theorem 4).

We now give a summary of subsequent sections. Section 2 contains definitions and discussions of a general Gaussian random field X. We define the maximal conjugate set $\mathscr{F}_X(A\,|\,\mathscr{E})$ for any non-empty subset $\mathscr{E}$ of $\boldsymbol{R}^n$ (Definition 1) and then introduce the set $\mathscr{C}_X(P_1, P_2; q)$ (Definition 2) which plays an important role in our investigations.

In Section 3 we begin with a description of a Euclidean random field X in terms of $\mathscr{C}_X(P_1, P_2; 0)$; namely, a Gaussian random field X is Euclidean if and only if the relation

$$\mathscr{C}_X(P_1, P_2; 0) \supset \{A \in \boldsymbol{R}^n; |A - P_1| = |A - P_2|\}$$

holds for any P_1, $P_2 \in \boldsymbol{R}^n$ (Theorem 1).

We are mainly concerned with Euclidean random fields X_r on $\boldsymbol{R}^n$, which correspond to $r(t)$ expressed in the form

$$r(t) = ct^2 + \int_0^\infty (1 - e^{-t^2 u})u^{-1}d\gamma(u)$$

with $r(1) = 1$, where $c \geqslant 0$ and γ is a measure on $(0, \infty)$ such that $\int_0^\infty (1 + u)^{-1}d\gamma(u) < \infty$ ([4]). For such an X_r we find a parametrization of $\mathscr{C}_{X_r}(P_1, P_2; q)$ by a subset $T_r(|P_1 - P_2|; q)$ of $[0, \infty)$; for $a = |P_1 - P_2| > 0$,

$$T_r(a; q) = \{t \geqslant 0; r(|t - a|) \leqslant r(t) + qr(a) \leqslant r(t + a)\} .$$

The explicit form of $T_r(a; q)$ is given for some classes of $r(t)$ (Propositions 3A $\sim$ 3E). An important example of $r(t)$ is

$$r(t) = \int_0^2 t^\alpha d\lambda(\alpha)$$

with a probability measure λ on $(0, 2]$.

In Section 4 we consider the Problem 1 for X_r and $q \neq 0$ in a slightly general setting:

PROBLEM 1′. Suppose that, for some Euclidean random field X_{r_1} on $\boldsymbol{R}^n$ and some $q_1 \in \boldsymbol{R}$, the relation

$$\mathscr{C}_{X_r}(P_1, P_2; q) \subset \mathscr{C}_{X_{r_1}}(P_1, P_2; q_1)$$

holds for any P_1, $P_2 \in \boldsymbol{R}^n$. Then is it true that $r_1(t) = r(t)$?

This problem changes into the uniqueness problem of the solution $f(x) = x$ of the modified Cauchy's functional equation ([1]) with $f(1) = 1$ (Lemma 1):

$$f(qx + y) = q_1 f(x) + f(y)$$

for $x \in r((0, \infty))$ and $y \in r(T_r(r^{-1}(x); q))$. Here we put $r(F) = \{r(t); t \in F\}$ for a subset F of $[0, \infty)$ and $r^{-1}(t)$ is the inverse function of $r(t)$ strictly increasing. We can solve this equation for the above mentioned classes of X_r by using the properties of $T_r(a; q)$ (Theorems 2 and 3). In particular, we note that the Problem 1′ is completely answered for X_α $(0 < \alpha \leqslant 2)$.

The final section contains the solution of the Problem 2 for X_r under the condition that $T_r(a_0; 1) \supset [0, a_0]$ for some $a_0 > 0$ (Theorem 4).

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§ 2. The sets $\mathscr{F}_X(A \mid \mathscr{E})$ and $\mathscr{C}_X(P_1, P_2; q)$

Let $X = \{X(A); A \in \boldsymbol{R}^n\}$ $(n \geqslant 2)$ be a Gaussian random field such that $X(A) - X(B)$ has mean zero and variance $r(A, B)$. Then the covariance of the increments $X(A) - X(P)$ and $X(B) - X(P)$ is

$$E[(X(A) - X(P))(X(B) - X(P))] = \{r(A, P) + r(B, P) - r(A, B)\}/2 . \tag{1}$$

We see that $r(A, B)$ must satisfy the following conditions:

$$\begin{cases} r(A, B) = r(B, A), \quad r(A, A) = 0, \quad r(A, B) \geqslant 0 \quad \text{and} \\ \sum_{i,j=1}^{N} a_i a_j r(A_i, A_j) \leqslant 0 \quad \text{for any } A_i \in \boldsymbol{R}^n \quad \text{and for any } a_i \in \boldsymbol{R} \\ \text{such that } \sum_{i=1}^{N} a_i = 0 \qquad (1 \leqslant i \leqslant N < \infty) . \end{cases} \tag{2}$$

We assume that $r(A, B)$ is jointly continuous and not identically zero.

We now introduce a decomposition of $X(A)$ for any non-empty subset $\mathscr{E}$ of $\boldsymbol{R}^n$:

$$X(A) = \mu(A \mid \mathscr{E}) + \sigma(A \mid \mathscr{E})\xi(A \mid \mathscr{E}) , \tag{3}$$

where

$$\mu(A \mid \mathscr{E}) = E[X(A) \mid X(P); P \in \mathscr{E}] ,$$
$$\sigma^2(A \mid \mathscr{E}) = E[(X(A) - \mu(A \mid \mathscr{E}))^2]$$

and

$$\xi(A \mid \mathscr{E}) = \begin{cases} (X(A) - \mu(A \mid \mathscr{E}))/\sigma(A \mid \mathscr{E}) & \text{if } \sigma(A \mid \mathscr{E}) > 0 , \\ 0 & \text{if } \sigma(A \mid \mathscr{E}) = 0 . \end{cases}$$

Since X is Gaussian, we see that the random variable $\xi(A \mid \mathscr{E})$ is independent of $\{X(P); P \in \mathscr{E}\}$. The decomposition (3) is called the *canonical form* of $X(A)$ ([2]). Explicit forms of $\mu(A \mid \mathscr{E})$ and $\sigma(A \mid \mathscr{E})$ are easily given for the case $\mathscr{E} = \{P_1, P_2\}$. First suppose that $r(P_1, P_2) > 0$. Then

$$\mu(A \mid P_1, P_2) = (1-q)2^{-1}X(P_1) + (1+q)2^{-1}X(P_2), \tag{4}$$

and

$$\begin{aligned}\sigma^2(A \mid P_1, P_2) = (1-q)2^{-1}r(A, P_1) + (1+q)2^{-1}r(A, P_2) \\ - (1-q^2)4^{-1}r(P_1, P_2)\,,\end{aligned} \tag{5}$$

where the coefficient q is given by

$$q = (r(A, P_1) - r(A, P_2))/r(P_1, P_2)\,. \tag{6}$$

When $r(P_1, P_2) = 0$, we have $\mu(A \mid P_1, P_2) = X(P_1) = X(P_2)$ and the equality (4) holds for any $q \in \boldsymbol{R}$.

The correlation function of $\xi(A \mid \mathscr{E})$ is denoted by

$$\rho_X(A, B \mid \mathscr{E}) = E[\xi(A \mid \mathscr{E})\xi(B \mid \mathscr{E})]\,, \tag{7}$$

and is called the *conditional correlation function relative to* $\mathscr{E}$. After P. Lévy [2] we give the following

DEFINITION 1. For any $A \in \boldsymbol{R}^n$ and any non-empty subset $\mathscr{E}$ of $\boldsymbol{R}^n$,

$$\mathscr{F}_X(A \mid \mathscr{E}) = \{B \in \boldsymbol{R}^n;\, \rho_X(A, B \mid \mathscr{E}) = 0\}\,. \tag{8}$$

Two points A and B such that $\rho_X(A, B \mid \mathscr{E}) = 0$ are said to be *conjugate relative to* $\mathscr{E}$, and $\mathscr{F}_X(A \mid \mathscr{E})$ is called the *maximal conjugate set of A relative to* $\mathscr{E}$ ([2]). The set $\mathscr{F}_X(A \mid \mathscr{E})$ contains a point $B \in \boldsymbol{R}^n$ such that $\sigma(B \mid \mathscr{E}) = 0$, so that $\mathscr{F}_X(A \mid \mathscr{E}) \supset \bar{\mathscr{E}}$, $\bar{\mathscr{E}}$ being the closure of $\mathscr{E}$. If, in particular, $\sigma(A \mid \mathscr{E}) = 0$, we have $\mathscr{F}_X(A \mid \mathscr{E}) = \boldsymbol{R}^n$.

PROPOSITION 1. *The set $\mathscr{F}_X(A \mid \mathscr{E})$ is a maximal closed set $\mathscr{V}$ such that $\mu(A \mid \mathscr{V}) = \mu(A \mid \mathscr{E})$ and $\mathscr{V} \cap \mathscr{E} \neq \phi$. We also have*

$$\mathscr{F}_X(A \mid \mathscr{E}) = \{B \in \boldsymbol{R}^n;\, \mu(B \mid \mathscr{E} \cup \{A\}) = \mu(B \mid \mathscr{E})\}\,. \tag{9}$$

Proof. Set $V = \{\mathscr{V} \subset \boldsymbol{R}^n;\, \mu(A \mid \mathscr{V}) = \mu(A \mid \mathscr{E})$ and $\mathscr{V} \cap \mathscr{E} \neq \phi\}$. Then the first assertion is proved by the following facts:

(i) $\bar{\mathscr{V}} \in V$ when $\mathscr{V} \in V$; (ii) $\mathscr{F}_X(A \mid \mathscr{E}) \in V$; (iii) $\mathscr{V}_1 \cup \mathscr{V}_2 \in V$ when $\mathscr{V}_1, \mathscr{V}_2 \in V$; (iv) $\mathscr{V} \subset \mathscr{F}_X(A \mid \mathscr{E})$ when $\mathscr{V} \in V$.

The equality (9) is easily proved by taking the following formula into account:

$$\mu(B \mid \mathscr{E} \cup \{A\}) = \mu(B \mid \mathscr{E}) + \rho_X(A, B \mid \mathscr{E})\sigma(B \mid \mathscr{E})\xi(A \mid \mathscr{E})\,.$$

The proof is thus completed.

For the case $\mathscr{E} = \{P_2\}$, we see by (9) that

$$\mathscr{F}_X(P_1|P_2) = \{A \in \boldsymbol{R}^n;\ \mu(A|P_1, P_2) = X(P_2)\}\ ,$$

hence the equalities (4) and (6) give the following:

$$\mathscr{F}_X(P_1|P_2) = \{A \in \boldsymbol{R}^n;\ r(A, P_1) = r(A, P_2) + r(P_1, P_2)\}\ . \tag{10}$$

As will be shown in Theorem 2, there are some cases where $\mathscr{F}_X(P_1|P_2)$ is rich enough to characterize $\boldsymbol{X}$. But it may happen that $\mathscr{F}_X(P_1|P_2) = \{P_2\}$ (see Proposition 3C). Hence in order to characterize $\boldsymbol{X}$ even in such a case, it is necessary to introduce other kinds of subsets of the parameter space $\boldsymbol{R}^n$. Inspired by (4), we give the following

DEFINITION 2. For any P_1, $P_2 \in \boldsymbol{R}^n$ and any $q \in \boldsymbol{R}$,

$$\begin{aligned}\mathscr{C}_X(P_1, P_2; q) = \{A \in \boldsymbol{R}^n;\ \mu(A|P_1, P_2) = (1-q)2^{-1}X(P_1) \\ + (1+q)2^{-1}X(P_2)\}\ .\end{aligned} \tag{11}$$

This set can be expressed as follows:

$$\mathscr{C}_X(P_1, P_2; q) = \{A \in \boldsymbol{R}^n;\ r(A, P_1) = r(A, P_2) + qr(P_1, P_2)\}\ . \tag{12}$$

We note the following simple facts:

(i) $\bigcup_{q \in \boldsymbol{R}} \mathscr{C}_X(P_1, P_2; q) = \boldsymbol{R}^n$;

(ii) $\mathscr{C}_X(P_1, P_2; 1) = \mathscr{F}_X(P_1|P_2)$;

(iii) $\mathscr{C}_X(P_1, P_2; q) = \mathscr{C}_X(P_2, P_1; -q)$.

An interesting property of the set $\mathscr{C}_X(P_1, P_2; q)$ is illustrated by the following

PROPOSITION 2. *The increments* $X(A) - X(B)$ *and* $X(P_1) - X(P_2)$ *are mutually independent if and only if* A *and* B *belong to the same set* $\mathscr{C}_X(P_1, P_2; q)$ *for some* $q \in \boldsymbol{R}$.

Proof. Since $\boldsymbol{X}$ is Gaussian, the increments $X(A) - X(B)$ and $X(P_1) - X(P_2)$ are mutually independent if and only if

$$E[(X(A) - X(B))(X(P_1) - X(P_2))] = 0\ .$$

This is rephrased by the equation

$$r(A, P_1) - r(A, P_2) = r(B, P_1) - r(B, P_2)\ ,$$

which is equivalent, by (12), to the assertion that A and B belong to $\mathscr{C}_X(P_1, P_2; q)$ for some $q \in \boldsymbol{R}$. The proof is thus completed.

§ 3. The set $\mathscr{C}_{X_r}(P_1, P_2; q)$ for a Euclidean random field X_r

In this section we first give a description of a Euclidean random field

X in terms of $\mathscr{C}_X(P_1, P_2; 0)$, and then introduce a class $\boldsymbol{S}_\infty$ of functions $r(t)$ by using Schoenberg's theorem ([4]), and further investigate the set $\mathscr{C}_{X_r}(P_1, P_2; q)$ for such an $r(t) \in \boldsymbol{S}_\infty$.

Suppose that the probability law of a Gaussian random field X is invariant under each Euclidean motion T on $\boldsymbol{R}^n$, that is,

$$\rho_X(TA, TB \mid T\mathscr{E}) = \rho_X(A, B \mid \mathscr{E}) \tag{13}$$

for any A, $B \in \boldsymbol{R}^n$ and any $\mathscr{E} \subset \boldsymbol{R}^n$. Then the variance $r(A, B)$ of $X(A) - X(B)$ can be expressed in the form $r(A, B) = r(|A - B|)$ with a continuous function $r(t)$ on $[0, \infty)$. Such a Gaussian random field is called *Euclidean.* The Euclidean random field corresponding to $r(t)$ is denoted by X_r.

THEOREM 1. *A Gaussian random field X is Euclidean if and only if the relation*

$$\mathscr{C}_X(P_1, P_2; 0) \supset \{A \in \boldsymbol{R}^n; |A - P_1| = |A - P_2|\} \tag{14}$$

holds for any P_1, $P_2 \in \boldsymbol{R}^n$.

Proof. Since "only if" part is clear by (12), we shall prove "if" part. If $|A - P_1| = |A - P_2|$, then we have $r(A, P_1) = r(A, P_2)$. With this we must show that $r(A, B) = r(A', B')$ for any A, B, A', $B' \in \boldsymbol{R}^n$ such that $|A - B| = |A' - B'|$. Putting $|A - B| = d$, we can find a finite number of points $P_1, P_2, \cdots, P_N$ such that $|A - P_1| = |P_1 - P_2| = \cdots = |P_N - A'| = d$. Then we have

$$r(A, B) = r(A, P_1) = r(P_1, P_2) = \cdots = r(P_N, A') = r(A', B')\ ,$$

which completes the proof.

Two Euclidean random fields X_{r_1} and X_{r_2} on $\boldsymbol{R}^n$ linked by $r_1(t) = (\text{const.}) r_2(t)$ have the same probabilistic structure:

$$\rho_{X_{r_1}}(A, B \mid \mathscr{E}) = \rho_{X_{r_2}}(A, B \mid \mathscr{E}),\ \mathscr{F}_{X_{r_1}}(A \mid \mathscr{E}) = \mathscr{F}_{X_{r_2}}(A \mid \mathscr{E}) \quad \text{and}$$
$$\mathscr{C}_{X_{r_1}}(P_1, P_2; q) = \mathscr{C}_{X_{r_2}}(P_1, P_2; q)$$

for any A, B, P_1, $P_2 \in \boldsymbol{R}^n$, any $\mathscr{E} \subset \boldsymbol{R}^n$ and any $q \in \boldsymbol{R}$.

As is easily seen, $r(t)$ never vanishes for $t > 0$, so we shall impose the normalizing condition $r(1) = 1$ in what follows.

We denote by $\boldsymbol{S}_n$ the class of functions $r(t)$ associated with Euclidean random fields X_r on $\boldsymbol{R}^n$. It is a well-known result (see, for example, [6]) that $r(t) \in \boldsymbol{S}_n$ has the following representation:

$$r(t) = c_n t^2 + \int_0^\infty \{1 - Y_n(tu)\} dL_n(u) , \tag{15}$$

where $c_n \geqq 0$, $Y_n(t) = \Gamma(n/2)(2/t)^{(n-2)/2} J_{(n-2)/2}(t)$ with the Bessel function $J_{(n-2)/2}(t)$ of order $(n-2)/2$ and where L_n is a measure on $(0, \infty)$ such that $\int_0^\infty u^2(1+u^2)^{-1} dL_n(u) < \infty$. Noting that $\boldsymbol{S}_n \supset \boldsymbol{S}_{n+1}$, I. J. Schoenberg [4] investigated the class $\boldsymbol{S}_\infty = \bigcap_{n \geqslant 2} \boldsymbol{S}_n$; namely, he proved that $r(t) \in \boldsymbol{S}_\infty$ is uniquely expressed in the following form:

$$r(t) = ct^2 + \int_0^\infty \{1 - e^{-t^2 u}\} u^{-1} d\gamma(u), \tag{16}$$

where $c \geqslant 0$ and γ is a measure on $(0, \infty)$ such that $\int_0^\infty (1+u)^{-1} d\gamma(u) < \infty$. The important subclass $\boldsymbol{L}_\infty$ of $\boldsymbol{S}_\infty$ is defined as the set of functions $r(t) = \int_0^2 t^\alpha d\lambda(\alpha)$ with probability measures λ on $(0, 2]$. We note that $r(t) \in \boldsymbol{S}_\infty$ is strictly increasing since

$$r'(t) = 2t\left\{c + \int_0^\infty e^{-t^2 u} d\gamma(u)\right\} > 0 \qquad \text{for } t > 0 ,$$

and hence the inclusion relation (14) becomes the equality

$$\mathscr{C}_{X_r}(P_1, P_2; 0) = \{A \in \boldsymbol{R}^n; |A - P_1| = |A - P_2|\} . \tag{17}$$

We also note that $r(t) \in \boldsymbol{S}_\infty$ can be extended analytically to the function $r(z)$ on the complex domain $\{z \in \boldsymbol{C}; |\arg z| < \pi/4\}$ ([5]). In the sequel we shall consider the set $\mathscr{C}_{X_r}(P_1, P_2; q)$ only for $q > 0$ and $r(t) \in \boldsymbol{S}_\infty$, because $\mathscr{C}_{X_r}(P_1, P_2; -q)$ is the mirror image of $\mathscr{C}_{X_r}(P_1, P_2; q)$ with respect to the hyperplane (17).

Now we shall illustrate the relation between the sets $\mathscr{C}_{X_r}(P_1, P_2; q)$ and $T_r(|P_1 - P_2|; q)$ which will be defined below by (18). Let $\boldsymbol{H}$ be an arbitrary two-dimensional half-plane in $\boldsymbol{R}^n$ such that P_1 and P_2 belong to the boundary-line of $\boldsymbol{H}$. We can give a natural parametrization to the set $\mathscr{C}_{X_r}(P_1, P_2; q) \cap \boldsymbol{H}$ in the following way. For any $A \in \mathscr{C}_{X_r}(P_1, P_2; q) \cap \boldsymbol{H}$, put $|P_1 - P_2| = a$ and $|A - P_2| = t$. Since $r(t)$ is strictly increasing, we have

$$r(|t - a|) \leqslant r(|A - P_1|) \leqslant r(t + a) .$$

Hence by (12),

$$r(|t - a|) \leqslant r(t) + qr(a) \leqslant r(t + a) .$$

Define the following subset of $[0, \infty)$ for each $a > 0$:

$$T_r(a; q) = \{t \geqslant 0; r(|t-a|) \leq r(t) + qr(a) \leqslant r(t+a)\} . \tag{18}$$

Then we see that for each $t \in T_r(|P_1 - P_2|; q)$ there exists uniquely a point $A(t) \in \mathscr{C}_{X_r}(P_1, P_2; q) \cap \boldsymbol{H}$ such that $|A(t) - P_2| = t$.

In the rest of this section we devote ourselves to the investigation of $T_r(a; q)$. First we see that

$$\{t \geqslant 0; r(|t-a|) \leqslant r(t) + qr(a)\} = \begin{cases} [D(a; q), \infty) & \text{if } 0 < q < 1 , \\ [0, \infty) & \text{if } q \geqslant 1 , \end{cases}$$

where $D(a; q)$ is the unique solution on $(0, a/2)$ of the equation $r(a-t) = r(t) + qr(a)$. Thus, putting

$$F_r(t; a, q) = r(t+a) - r(t) - qr(a) ,$$

we have

$$T_r(a; q) = \begin{cases} \{t \geqslant D(a; q); F_r(t; a, q) \geqslant 0\} & \text{if } 0 < q < 1 , \\ \{t \geqslant 0; F_r(t; a, q) \geqslant 0\} & \text{if } q \geqslant 1 . \end{cases} \tag{19}$$

We shall give further consideration on the following classes of $r(t) \in \boldsymbol{S}_\infty$:

A. $r(t) = t$, which corresponds to a Lévy's Brownian motion $\boldsymbol{X}_1$;
B. $r(t)$ is strictly convex on $(0, \infty)$;
C. $r(t)$ is strictly concave on $(0, \infty)$;
D. $r(t)$ is strictly convex on $(0, t_0)$ and strictly concave on (t_0, ∞) for some t_0 $(0 < t_0 < \infty)$.
E. $r(t)$ is strictly concave on $(0, t_0)$ and strictly convex on (t_0, ∞) for some t_0 $(0 < t_0 < \infty)$.

We see that $r(t) = \int_0^2 t^\alpha d\lambda(\alpha) \in \boldsymbol{L}_\infty$ lies in **A**, **B** and **C** when the probability measure λ is concentrated on $\{1\}$, $[1, 2]$ and $(0, 1]$ respectively; otherwise $r(t) \in \boldsymbol{L}_\infty$ is always in **E**. Examples of $r(t)$ in **D**:

(i) $r(t) = (1 - e^{-ut^2})/(1 - e^{-u})$ $(u > 0)$;
(ii) $r(t) = \{2t/(t+1)\}^\alpha$ $(1 < \alpha \leqslant 2)$;
(iii) $r(t) = \log(1 + t^2)/\log 2$.

Note that $r(t) = \{2t/(t+1)\}^\alpha$ with $0 < \alpha \leqslant 1$ belongs to the class $\boldsymbol{C}$.

PROPOSITION 3A. *For $r(t) = t$, we have*

$$(20)\qquad T_r(a;q)=\begin{cases}[(1-q)a/2,\infty) & \text{if } 0<q\leqslant 1,\\ \phi & \text{if } q>1\,.\end{cases}$$

Proof is elementary, so is omitted.

For $r(t)$ in $\mathbf{B}\sim\mathbf{E}$, we shall introduce some notations. The limits $\lim_{t\to 0} r'(t)$ and $\lim_{t\to\infty} r'(t)$ exist in $[0,\infty]$, and are denoted by $r'(0+)$ and $r'(\infty)$, respectively. We denote by $C(a;q)$ the unique solution on $(0,\infty)$ of the equation $F_r(t;a,q)=0$ when a solution exists. We set

$$\begin{aligned}h(a;q)&\equiv\lim_{t\to\infty}F_r(t;a,q)=\lim_{t\to\infty}\int_0^a\{r'(t+s)-qr'(s)\}ds\\&=r'(\infty)a-qr(a)\,.\end{aligned}$$

Of course $h(a;q)\equiv\infty$ when $r'(\infty)=\infty$.

PROPOSITION 3B. *Suppose that $r(t)\in\boldsymbol{S}_\infty$ is strictly convex on $(0,\infty)$. Then we have*

$$(21)\qquad T_r(a;q)=\begin{cases}[D(a;q),\infty) & \text{if } 0<q<1\,,\\ [0,\infty) & \text{if } q=1\,,\\ [C(a;q),\infty) & \text{if } q>1 \text{ and } 0<a<a^*(q)\,,\\ \phi & \text{if } q>1 \text{ and } a\geqslant a^*(q)\,,\end{cases}$$

where $a^*(q)=\sup\{a\geqslant 0;h(a;q)\geqslant 0\}$. Moreover, for $q>1$, we have $a^*(q)=\infty$ if and only if $r'(\infty)=\infty$. In this case there exists an increasing continuous function $\phi_q(a)$ on $(0,\infty)$ such that $C(a;q)<\phi_q(a)$ for any $a>0$.

PROPOSITION 3C. *Suppose that $r(t)\in\boldsymbol{S}_\infty$ is strictly concave on $(0,\infty)$. Then we have*

$$(22)\qquad T_r(a;q)=\begin{cases}[D(a;q),C(a;q)] & \text{if } 0<q<1 \text{ and } 0<a<a_*(q)\,,\\ [D(a;q),\infty) & \text{if } 0<q<1 \text{ and } a\geqslant a_*(q)\,,\\ \{0\} & \text{if } q=1\,,\\ \phi & \text{if } q>1\,,\end{cases}$$

where $a_*(q)=\sup\{a\geqslant 0;h(a;q)\leqslant 0\}$. Moreover, for $0<q<1$, there exists an increasing continuous function $\psi_q(a)$ on $(0,\infty)$ such that $D(a;q)<\psi_q(a)<C(a;q)$ for $0<a<a_*(q)$ and $D(a;q)<\psi_q(a)$ for $a\geqslant a_*(q)$.

These two propositions can be proved in a similar manner, so we give only the proof of Proposition 3B.

The proof of Proposition 3B. Since $r'(t)$ is strictly increasing, we have $(d/dt)F_r(t;a,q)>0$. Noting that $F_r(0;a,q)=(1-q)r(a)$, we easily obtain (21) for $0<q\leqslant 1$.

Now consider the case $q > 1$. We devide the proof into two parts: (i) $r'(\infty) < \infty$ and (ii) $r'(\infty) = \infty$. First consider (i). We see that $(d/da)h(a; q)$ is positive on $(0, b)$ while negative on (b, ∞), where $b = \inf\{a > 0; qr'(a) > r'(\infty)\}$. Noting that the limit

$$\lim_{a\to\infty} h(a; q)/a = r'(\infty) - \lim_{a\to\infty} \frac{q}{a}\int_0^a r'(s)ds = (1-q)r'(\infty)$$

is negative, we see that $a^*(q)$ is finite and have

$$h(a; q)\begin{cases} > 0 & \text{if } 0 < a < a^*(q)\ , \\ \leqslant 0 & \text{if } a \geqslant a^*(q)\ . \end{cases}$$

If $h(a; q) > 0$, the solution $C(a; q)$ of the equation $F_r(t; a, q) = 0$ exists and $T_r(a; q) = [C(a; q), \infty)$ holds. While, if $h(a; q) \leqslant 0$, then $T_r(a; q) = \phi$. Thus (21) has been proved in the case (i).

Next consider (ii). It follows from $h(a; q) = \infty$ that $a^*(q) = \infty$ and $T_r(a; q) = [C(a; q), \infty)$ for any $a > 0$. The function $\phi_q(a) = r'^{-1}(qr'(a))$ satisfies the inequality $C(a; q) < \phi_q(a)$ for any $a > 0$, because

$$F_r(\phi_q(a); a, q) > a\{r'(\phi_q(a)) - qr'(a)\} = 0\ .$$

We note that $\phi_q(a)$ is increasing and continuous, and that $\phi_q(0+) = 0$ if and only if $r'(0+) = 0$. Thus all the assertions have been proved.

As for $r(t)$ in **D** or **E**, we are interested only in the case $q = 1$.

PROPOSITION 3D. *Suppose that $r(t) \in \boldsymbol{S}_\infty$ is strictly convex on $(0, t_0)$ and strictly concave on (t_0, ∞) for some t_0 $(0 < t_0 < \infty)$. Then we have*

$$T_r(a; 1) = \begin{cases} [0, \infty) & \text{if } 0 < a \leqslant a_*\ , \\ [0, C(a; 1)] & \text{if } a_* < a < a_1\ , \\ \{0\} & \text{if } a \geqslant a_1\ . \end{cases} \tag{23}$$

where $a_* = \inf\{a > 0; h(a; 1) \leqslant 0\}$ and $a_1 = \sup\{a > t_0; r'(a) > r'(0+)\}$. Moreover, if $r'(0+) \leqslant r'(\infty)$, then there exists a decreasing continuous function $\tau(a)$ on $(0, \infty)$ such that $0 < \tau(a) < C(a; 1)$ for $a > a_*$.

PROPOSITION 3E. *Suppose that $r(t) \in \boldsymbol{S}_\infty$ is strictly concave on $(0, t_0)$ and strictly convex on (t_0, ∞) for some t_0 $(0 < t_0 < \infty)$. Then we have*

$$T_r(a; 1) = \begin{cases} \{0\} & \text{if } 0 < a \leqslant a^*\ , \\ \{0\} \cup [C(a; 1), \infty) & \text{if } a^* < a < a_2\ , \\ [0, \infty) & \text{if } a \geqslant a_2\ , \end{cases} \tag{24}$$

where $a^* = \inf\{a > 0; h(a; 1) \geqslant 0\}$ *and* $a_2 = \sup\{a > t_0; r'(a) < r'(0+)\}$. *Moreover,* $a^* = 0$ *if and only if* $r'(0+) \leqslant r'(\infty)$. *In case* $r'(0+) = r'(\infty)$, *there exists* $a_0 \in (t_0, \infty)$ *such that* $C(a; 1) \leqslant a_0$ *for* $a \geqslant a_0$.

The above two propositions can be proved in a similar manner, so we give only the proof of Proposition 3E.

The Proof of Proposition 3E. When $a \geqslant a_2$ ($a_2 < \infty$), we easily see that $(d/dt)F_r(t; a, 1) > 0$ for any $t > 0$. From this we have $T_r(a; 1) = [0, \infty)$, which implies that $a^* < a_2$. On the other hand, when $a < a_2$, $(d/dt)F_r(t; a, 1)$ is negative for $0 < t < t_a$ while positive for $t > t_a$, where $t_a \in (0, t_0)$ is the unique solution of the equation $r'(t + a) = r'(t)$. Therefore, if $h(a; 1) > 0$, the solution $C(a; 1)$ of the equation $F_r(t; a, 1) = 0$ exists and $T_r(a; 1) = \{0\} \cup [C(a; 1), \infty)$ holds. While, if $h(a; 1) \leqslant 0$, then $T_r(a; 1) = \{0\}$. We are now in a position to see that

$$h(a; 1)\begin{cases} \leqslant 0 & \text{if } 0 < a \leqslant a^* , \\ > 0 & \text{if } a > a^* . \end{cases}$$

For $(d/da)h(a; 1)$ is negative on $(0, b)$ while positive on (b, ∞), where $b = \inf\{a \in (0, t_0); r'(a) < r'(\infty)\} < a^*$. Thus we have proved (24).

We now proceed to the proof of the second part. We first note that $a^* = 0$ if and only if $b = 0$, which is equivalent to the condition $r'(0+) \leqslant r'(\infty)$. In case $r'(0+) = r'(\infty)$ (i.e., $a^* = 0$ and $a_2 = \infty$), we can choose $a_0 \in (t_0, \infty)$ such that $r(2a_0) \geqslant 2r(a_0)$, because $g(a) = r(2a) - 2r(a)$ is strictly increasing on (t_0, ∞) and the limit

$$\lim_{a\to\infty} g(a) = \lim_{a\to\infty} \int_0^a \{r'(s + a) - r'(s)\}ds = \int_0^\infty \{r'(\infty) - r'(s)\}ds$$

is positive. It is easily verified that $F_r(a_0; a, 1) \geqslant 0$ for $a \geqslant a_0$, which implies that $C(a; 1) \leqslant a_0$ for $a \geqslant a_0$. Thus the proof is completed.

§4. Characterization of X_r by means of $\mathscr{C}_{X_r}(P_1, P_2; q)$

In this section we consider the Problem 1 concerning the characterization of a Euclidean random field X_r on $\boldsymbol{R}^n$ by means of $\mathscr{C}_{X_r}(P_1, P_2; q)$. First we note that the family $\{\mathscr{C}_{X_r}(P_1, P_2; q); P_1, P_2 \in \boldsymbol{R}^n, q \in \boldsymbol{R}\}$ uniquely determines the probability law of X_r. That is, if functions $r(t)$, $r_1(t) \in \boldsymbol{S}_n$ satisfy the equality

$$\mathscr{C}_{X_r}(P_1, P_2; q) = \mathscr{C}_{X_{r_1}}(P_1, P_2; q) \tag{25}$$

for any P_1, $P_2 \in \boldsymbol{R}^n$ and any $q \in \boldsymbol{R}$, then we have $r(t) = r_1(t)$. This is easily verified by noting that (25) is equivalent to the following:

$$\begin{aligned}(26)\qquad &\{r(|A-P_1|)-r(|A-P_2|)\}/r(|P_1-P_2|)\\ &\quad=\{r_1(|A-P_1|)-r_1(|A-P_2|)\}/r_1(|P_1-P_2|)\end{aligned}$$

for any $A,\ P_1,\ P_2 \in \boldsymbol{R}^n$.

Our conjecture is that the family $\{\mathscr{C}_{X_r}(P_1, P_2; q); P_1, P_2 \in \boldsymbol{R}^n\}$ with some *fixed* $q > 0$ would suffice for the characterization of $\boldsymbol{X}_r$.

PROBLEM 1′. Let $r(t) \in \boldsymbol{S}_\infty$, $q > 0$ and $n \geqslant 2$ be fixed. Suppose that $r_1(t) \in \boldsymbol{S}_n$ and $q_1 \in \boldsymbol{R}$ satisfy the relation

$$(27)\qquad \mathscr{C}_{X_r}(P_1, P_2; q) \subset \mathscr{C}_{X_{r_1}}(P_1, P_2; q_1)$$

for any $P_1,\ P_2 \in \boldsymbol{R}^n$. Then is it true that $r_1(t) = r(t)$ and $q_1 = q$?

Proposition 2 tells us the following: For any $A,\ B \in \mathscr{C}_{X_r}(P_1, P_2; q)$ the increments $X(A) - X(B)$ and $X(P_1) - X(P_2)$, viewed as the differences of members of $\boldsymbol{X}_r$, are mutually independent. By the relation (27), this property is still true even if those increments are viewed as the differences of members of $\boldsymbol{X}_{r_1}$. Therefore, if the Problem 1′ is affirmative, the parameter set of the form $\mathscr{C}_{X_r}(P_1, P_2; q)$ is thought of as a characteristic of a Gaussian random field, so far as the independence property of the increments is concerned. We shall solve this problem for the classes $\mathbf{A} \sim \mathbf{E}$ of $r(t) \in \boldsymbol{S}_\infty$ by using the properties of $T_r(a; q)$.

We deduce a functional equation for $f(x) = r_1(r^{-1}(x))$ from the relation (27). For each $t \in T_r(|P_1 - P_2|; q)$, there exists a point $A(t) \in \mathscr{C}_{X_r}(P_1, P_2; q)$ such that $|A(t) - P_2| = t$. By (12), we see that

$$r(|A(t) - P_1|) = r(t) + qr(|P_1 - P_2|)\ .$$

Since the point $A(t)$ belongs also to $\mathscr{C}_{X_{r_1}}(P_1, P_2; q_1)$, the equality

$$r_1(|A(t) - P_1|) = r_1(t) + q_1 r_1(|P_1 - P_2|)$$

holds. From these equations, putting $x = r(|P_1 - P_2|)$ and $y = r(t)$, we obtain

$$(28)\qquad f(qx + y) = q_1 f(x) + f(y)\ ,$$

where

$$(29)\qquad x \in r((0, \infty)), \qquad y \in r(T_r(r^{-1}(x); q))\ .$$

What has been discussed can be summarized as

LEMMA 1. *Suppose that the relation* (27) *holds for any* P_1, $P_2 \in \boldsymbol{R}^n$. *Then the continuous function* $f(x) = r_1(r^{-1}(x))$ *satisfies the functional equation* (28).

Since the equality $q_1 = q$ easily follows from $r_1(t) = r(t)$, our goal is to prove that $f(x) = x$ is the unique solution of (28) with $f(1) = 1$.

(a) *The case* $q = 1$. In this case the Problem 1′ becomes somewhat simple; the relation (27) implies that $q_1 = 1$. We thus have Cauchy's functional equation:

$$(28)_1 \qquad f(x + y) = f(x) + f(y) \ ,$$

$$(29)_1 \qquad x \in r((0, \infty)), \qquad y \in r(T_r(r^{-1}(x); 1)) \ .$$

When $r(t)$ is strictly concave (i.e., in the class **C**), $\mathscr{F}_{X_r}(P_1 | P_2) = \{P_2\}$ holds, so that we cannot obtain $r_1(t) = r(t)$. On the other hand, when $r(t)$ is strictly convex (in **B**) or $r(t) = t$ (in **A**), Cauchy's functional equation $(28)_1$ holds for *any* $x, y \geqslant 0$. Then it is a classical result that $f(x) = x$ is the unique solution with $f(1) = 1$ ([1]). Furthermore we shall show that this is true also for $r(t)$ in **D** or **E** under the condition $r'(0+) \leqslant r'(\infty)$, by using the theorem of J. Aczél (p. 46 in [1]).

First, let $r(t) \in \boldsymbol{S}_\infty$ be in **D** with the condition $r'(0+) \leqslant r'(\infty)$. Then we see by Proposition 3D that the domain $(29)_1$ includes the following set:

$$(30) \qquad D_\Phi = \{(x, y); 0 < x < \beta, 0 < y \leqslant \Phi(x), x + y < \beta\}$$

with the decreasing continuous function $\Phi(x) = r(\tau(r^{-1}(x)))$ on $(0, \beta)$, where $\beta = r(\infty) \in (1, \infty]$ and where $\tau(a)$ is the function on $(0, \infty)$ in Proposition 3D. When $\beta < \infty$, we may assume that $\Phi(x) < \beta - x$ without loss of generality.

LEMMA 2. *Suppose that a continuous function* $f(x)$ *with* $f(1) = 1$ *satisfies Cauchy's functional equation* $(28)_1$ *for any* $(x, y) \in D_\Phi$ *with a decreasing continuous function* $\Phi(x)$ *on* $(0, \beta)$ *such that* $0 < \Phi(x) < \beta - x$ $(1 < \beta \leqslant \infty)$. *Then we have* $f(x) = x$.

Proof. Take x_0 such that $\Phi(x_0) = x_0$. Then, $(0, x_0] \times (0, x_0] \subset D_\Phi$, which means that $(28)_1$ holds for any $x, y \in [0, x_0]$. Hence by Aczél's theorem, we have $f(x) = cx$ on $[0, x_0]$ with some constant c. When $x > x_0$, it follows from $(28)_1$ that

$$\{f(x + y) - f(x)\}/y = f(y)/y = c \qquad \text{for } 0 < y < \Phi(x) \ ,$$

so that the right derivative of f at $x \in (x_0, \beta)$ exists and is equal to the constant c. From this we obtain $f(x) = cx$ on $[0, \beta)$, and $c = 1$ since $f(1) = 1$. The proof is thus completed.

Next, let $r(t) \in \boldsymbol{S}_\infty$ be in $\mathbf{E}$ with the condition $r'(0+) \leqslant r'(\infty)$. Then we see by Proposition 3E that the domain $(29)_1$ includes the following set:

$$D^{\Psi} = \{(x, y); 0 < x < \infty, \Psi(x) \leqslant y < \infty\} , \tag{31}$$

where $\Psi(x)$ is the nonnegative continuous function defined by

$$\Psi(x) = \begin{cases} r(C(r^{-1}(x); 1)) & \text{for } 0 < x < r(a_2) , \\ 0 & \text{for } x \geqslant r(a_2) , \end{cases}$$

and satisfies the property that there exists $x_0 \in (0, \infty)$ such that $\Psi(x) \leqslant x_0$ for $x \geqslant x_0$.

LEMMA 3. *Suppose that a continuous function $f(x)$ with $f(1) = 1$ satisfies Cauchy's functional equation $(28)_1$ for any $(x, y) \in D^{\Psi}$ with a nonnegative continuous function $\Psi(x)$ on $(0, \infty)$ satisfying the property that there exists $x_0 \in (0, \infty)$ such that $[x_0, \infty) \times [x_0, \infty) \subset D^{\Psi}$. Then we have $f(x) = x$.*

This is a simple consequence of Aczél's theorem, so we omit the proof.

Thus we have proved the following

THEOREM 2. *Suppose that $r(t) \in \boldsymbol{S}_\infty$ satisfies one of the following four conditions:*

(i) *$r(t) = t$;*

(ii) *$r(t)$ is strictly convex on $(0, \infty)$;*

(iii) *$r(t)$ is strictly convex on $(0, t_0)$, strictly concave on (t_0, ∞) for some t_0 $(0 < t_0 < \infty)$ and $r'(0+) \leqslant r'(\infty)$;*

(iv) *$r(t)$ is strictly concave on $(0, t_0)$, strictly convex on (t_0, ∞) for some t_0 $(0 < t_0 < \infty)$ and $r'(0+) \leqslant r'(\infty)$.*

Then, $r_1(t) \in \boldsymbol{S}_n$ satisfies the relation

$$\mathscr{F}_{X_r}(P_1 | P_2) \subset \mathscr{F}_{X_{r_1}}(P_1 | P_2) \qquad \text{for any } P_1, P_2 \in \boldsymbol{R}^n$$

if and only if $r_1(t) = r(t)$.

In the above cases (iii) and (iv), we have assumed, for convenience, that $r'(0+) \leqslant r'(\infty)$. Without this assumption, difficulties arise, for one thing the equality $\mathscr{F}_{X_r}(P_1 | P_2) = \{P_2\}$ holds for $|P_1 - P_2| \geqslant a_1$ in the case (iii) (see Proposition 3D) and for $|P_1 - P_2| \leqslant a^*$ in the case (iv) (Proposition 3E).

(b) *The cases* $0 < q < 1$ *or* $q > 1$. When $r(t)$ is strictly concave (in $\boldsymbol{C}$) or $r(t) = t$ (in A), we have $\mathscr{C}_{X_r}(P_1, P_2; q) = \phi$ for $q > 1$, so the answer to the Problem 1′ is obviously "No". But we have an affirmative answer in the following four cases:

$$(32) \quad \begin{cases} \text{(i)} \ \ 0 < q < 1 \text{ and } r(t) = t \ ; \\ \text{(ii)} \ \ 0 < q < 1 \text{ and } r(t) \text{ is strictly convex on } (0, \infty) \ ; \\ \text{(iii)} \ \ 0 < q < 1 \text{ and } r(t) \text{ is strictly concave on } (0, \infty) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{with } r(\infty) = \infty \ ; \\ \text{(iv)} \ \ q > 1 \text{ and } r(t) \text{ is strictly convex on } (0, \infty) \text{ with } r'(0+) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } r'(\infty) = \infty \ . \end{cases}$$

In these cases, we see by Propositions 3A, 3B and 3C that in the interior of the domain (29) there exists an increasing continuous curve Γ: $y = \phi(x)$, $0 < x < \infty$, with $\phi(0+) = 0$. Therefore, under the restriction that $r_1(t) \in \boldsymbol{S}_n$ is twice differentiable, we can easily verify that $f(x) = x$ is the unique solution of (28) with $f(1) = 1$. Thus we have obtained the following

THEOREM 3. *Suppose that* $r(t) \in \boldsymbol{S}_\infty$ *and* $q > 0$ *satisfy one of the four conditions in* (32). *Then, a twice differentiable function* $r_1(t) \in \boldsymbol{S}_n$ *and* $q_1 \in \boldsymbol{R}$ *satisfy the relation*

$$\mathscr{C}_{X_r}(P_1, P_2; q) \subset \mathscr{C}_{X_{r_1}}(P_1, P_2; q_1) \qquad \textit{for any } P_1,\ P_2 \in \boldsymbol{R}^n$$

if and only if $r_1(t) = r(t)$ *and* $q_1 = q$.

Remark 1. We see by Theorems 2 and 3 that the answer to the Problem 1′ for $r(t) = t^\alpha$ $(0 < \alpha \leqslant 2)$ is "Yes" in the following cases: (i) $0 < q < 1$ and $0 < \alpha \leqslant 2$; (ii) $q = 1$ and $1 \leqslant \alpha \leqslant 2$; (iii) $q > 1$ and $1 < \alpha \leqslant 2$. In the other cases, the answer is "No".

Remark 2. Theorem 3 holds even in the case where a parameter $q_1 \in \boldsymbol{R}$ depends on P_1, $P_2 \in \boldsymbol{R}^n$.

§5. The projective invariance of $\mathscr{F}_{X_\alpha}(P_1 | P_2)$

In this section we consider the Problem 2 mentioned in §1. The probability law of X_α is invariant under each Euclidean motion, similar transformation and inversion T on $\boldsymbol{R}^n$, that is, the equality

$$(33) \qquad \rho_{X_\alpha}(TA, TB \,|\, T\mathscr{E}) = \rho_{X_\alpha}(A, B \,|\, \mathscr{E})$$

holds for any A, $B \in \boldsymbol{R}^n$ and any $\mathscr{E} \subset \boldsymbol{R}^n$. Here we take an inversion T with center in $\mathscr{E}$, that is, for some $a > 0$ and some $P \in \mathscr{E}$,

$$\begin{cases} TA = a^2(A - P)|A - P|^{-2} + P & \text{if } A \neq P\,, \\ TP = P\,. \end{cases}$$

The property (33) is the characteristic property of X_α called projective invariance ([3]). It easily follows from (33) that

$$\mathscr{F}_{X_\alpha}(TA\,|\,T\mathscr{E}) = T\mathscr{F}_{X_\alpha}(A\,|\,\mathscr{E}) \qquad \text{for any } A \in \boldsymbol{R}^n \text{ and any } \mathscr{E} \subset \boldsymbol{R}^n. \tag{34}$$

Now we wish to show that there is no other X_r with the above property (34). Namely, we are ready to discuss

PROBLEM 2. Suppose that $r(t) \in \boldsymbol{S}_\infty$ satisfies the equality

$$\mathscr{F}_{X_r}(TP_1\,|\,TP_2) = T\mathscr{F}_{X_r}(P_1\,|\,P_2) \qquad \text{for any } P_1,\ P_2 \in \boldsymbol{R}^n, \tag{35}$$

where a transformation T on $\boldsymbol{R}^n$ runs over all similar transformations and inversions with center P_2. Then is it true that $r(t) = t^\alpha$?

We can solve this problem under the following condition:

$$\text{There exists } a_0 > 0 \text{ such that } r(t) + r(a_0) \leqslant r(t + a_0) \qquad \text{for } 0 \leqslant t \leqslant a_0\,, \tag{36}$$

which means that $T_r(a_0; 1) \supset [0, a_0]$. It follows from (35) that $T_r(a; 1) = \{at/a_0;\, t \in T_r(a_0; 1)\}$, $a > 0$, and that the set $T_r(a; 1)\backslash\{0\}$, $a > 0$, is invariant under the inversion $t^* = a^2/t$ on $(0, \infty)$. By using the condition (36), we have $T_r(a; 1) = [0, \infty)$ for any $a > 0$.

THEOREM 4. *Suppose that $r(t) \in \boldsymbol{S}_\infty$ satisfies the condition* (36). *Then the equality* (35) *holds for any similar transformation and inversion with center P_2 if and only if $r(t) = t^\alpha$ $(1 \leqslant \alpha \leqslant 2)$.*

Proof. It suffices to prove "only if" part. From the equality (35) for any similar transformation T on $\boldsymbol{R}^n$, we obtain the equation

$$r(kr^{-1}(r(t) + 1)) = r(kt) + r(k) \tag{37}$$

for any $k > 0$ and any $t \in T_r(1; 1) = [0, \infty)$. With this we show the following equation for any natural number m:

$$r(kr^{-1}(m)) = mr(k) \qquad \text{for any } k > 0\,. \tag{38}$$

This equation clearly holds for $m = 1$. Suppose the equation (38) holds for m. Then, putting $t = r^{-1}(m)$ in (37), we see that

$$r(kr^{-1}(m+1)) = r(kr^{-1}(m)) + r(k) = (m+1)r(k)\ .$$

By induction on m, the equation (38) holds for all m.

If we set $r(k) = a$ in (38), then we have $r^{-1}(ma) = r^{-1}(m)r^{-1}(a)$. It easily follows that $r^{-1}(pa) = r^{-1}(p)r^{-1}(a)$ for any rational number p and any $a > 0$. Since $r^{-1}(t)$ is continuous, we obtain

$$r^{-1}(ab) = r^{-1}(a)r^{-1}(b) \qquad \text{for any } a,\ b \geqslant 0\ ,$$

which implies that $r^{-1}(t) = t^{1/\alpha}$ for some $\alpha > 0$. Thus, excluding the case $0 < \alpha < 1$ because of (36), we have $r(t) = t^{\alpha}$ with $1 \leqslant \alpha \leqslant 2$. The proof is completed.

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