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Nagoya Math. J.
Vol. 85 (1982), 223-230

AN ELIMINATION THEOREM OF UNIQUENESS CONDITIONS IN THE INTUITIONISTIC PREDICATE CALCULUS

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This paper is a sequel to Motohashi [4]. In [4], a series of theorems named "elimination theorems of uniqueness conditions" was shown to hold in the classical predicate calculus LK. But, these results have the following two defects: one is that they do not hold in the intuitionistic predicate calculus LJ, and the other is that they give no nice axiomatizations of some sets of sentences concerned. In order to explain these facts more explicitly, let us introduce some necessary notations and definitions. Let L be a first order classical predicate calculus LK or a first order intuitionistic predicate calculus LJ. n-ary formulas in L are formulas $F(\bar{a})$ in L with a sequence $\bar{a}$ of distinct free variables of length n such that every free variable in F occurs in $\bar{a}$. Sometimes, we shall omit the sequence $\bar{a}$ in an n-ary formula $F(\bar{a})$ if no confusions are likely to occur. Also, an n-ary predicate symbol R is frequently identified with the n-ary formula $R(\bar{a})$. (If necessary, we can assume that $\bar{a}$ is the sequence of first n free variables in a fixed enumeration of the free variables.) If $A(\bar{a}, a)$ and $E(\bar{a}, \bar{b})$ are $(n+1)$-ary formula and $2n$-ary formula, then the existence condition of A, denoted by $\mathrm{Ex}\, A(\bar{a}, b)$ or $\mathrm{Ex}\, A$, is the sentence; $\forall\bar{x}\exists y A(\bar{x}, y)$, the uniqueness condition of A with respect to E, denoted by $\mathrm{Un}\,(A(\bar{a}, b); E(\bar{a}, \bar{b}))$ or $\mathrm{Un}\,(A; E)$, is the sentence; $\forall\bar{x}\forall\bar{y}\forall x\forall y(E(\bar{x}, \bar{y}) \wedge A(\bar{x}, x) \wedge A(\bar{y}, y). \supset x = y)$, and the congruence condition of A with respect to E, denoted by $\mathrm{Co}\,(A(\bar{a}, b); E(\bar{a}, \bar{b}))$ or $\mathrm{Co}\,(A; E)$, is the sentence; $\forall\bar{x}\forall\bar{y}(E(\bar{x}, \bar{y}) \supset \forall x(A(\bar{x}, x) \equiv A(\bar{y}, x)))$. If $E(\bar{a}, \bar{b})$ is the formula $a_1 = b_1 \wedge \cdots \wedge a_n = b_n$, then $\mathrm{Un}\,(A; E)$ and $\mathrm{Co}\,(A; E)$ are written by $\mathrm{Un}\, A$ and $\mathrm{Co}\, A$, respectively. Note that $\mathrm{Co}\, A$ is provable in LJ. Let P be an m-ary predicate symbol. Then P-positive (P-negative) formulas are formulas which have no negative (positive) occurrences of P (cf. Takeuti [9]). P-positive formulas have the

Received April 30, 1980.

*) The author is partially supported in Grand-in-Aid for Co-operative Research, Project No. 434007.

following well-known property.

POSITIVE LEMMA. *Suppose that P and Q are m-ary predicate symbols. If $C(P)$ is a P-positive sentence, then the sentence* $\forall\bar{x}(P(\bar{x}) \supset Q(\bar{x})) \wedge C(P). \supset C(Q)$ *is provable in LJ, where $C(Q)$ is a sentence obtained from $C(P)$ by replacing some occurrences of P by Q.*

In [4], we proved the following elimination theorem of uniqueness condition (Theorem II in [4]).

THEOREM A. *Suppose that R is an $(n+1)$-ary predicate symbol and $E(\bar{a}, \bar{b})$ is a $2n$-ary formula which has no occurrences of R. If C is an R-positive sentence, then the sentence* $\mathrm{Ex}\, R \wedge \mathrm{Un}(R; E). \supset C$ *is provable in LK if and only if the sentence* $\mathrm{Ex}\, R \wedge \mathrm{Co}(R; E). \supset C$ *is provable in LK.*

Firstly, we should remark that Theorem A does not hold in LJ: Counterexample, let $n = 1$, and C the R-positive sentence $\forall x_1 \exists y_1 \forall x_2 \exists y_2 (R(x_1, y_1) \wedge R(x_2, y_2) \wedge (x_1 = x_2 \supset y_1 = y_2))$, then $\mathrm{Ex}\, R \wedge \mathrm{Un}\, R. \supset C$ is provable in LJ but $\mathrm{Ex}\, R \wedge \mathrm{Co}\, R. \supset C$ is not. Secondly, we should remark that Theorem A shows us that the set S of R-positive sentences which are provable from the existence condition $\mathrm{Ex}\, R$ and the uniqueness condition $\mathrm{Un}(R; E)$, can be axiomatized by two axioms $\mathrm{Ex}\, R$ and $\mathrm{Co}(R; E)$. In this sense, Theorem A gives an axiomatization of the set S. But, this axiomatization is very unsatisfactory because $\mathrm{Co}(R; E)$ is not an R-positive sentence. So, it is desirable to give an R-positive axiomatization of the set S, i.e. an axiomatization of S whose axioms are all R-positive. As a special case of our Main Theorem in this paper, we have the following theorem which holds both in LJ and in LK, and gives an R-positive axiomatization of S. For each natural number k, the k-th existence condition of A with respect to E, denoted by $\mathrm{Ex}^k(A; E)$, is the sentence:

$$\forall \bar{x}_1 \exists y_1 \forall \bar{x}_2 \exists y_2 \cdots \forall \bar{x}_k \exists y_k \left[\bigwedge_{i=1}^{k} A(\bar{x}_i, y_i) \wedge \bigwedge_{i,j=1}^{k} (E(\bar{x}_i, \bar{x}_j) \supset y_i = y_j) \right].$$

As in the case $\mathrm{Ex}(A; E)$, $\mathrm{Ex}^k A$ denotes the sentence $\mathrm{Ex}^k(A; \bar{a} = \bar{b})$.

THEOREM B. *Suppose that R is an $(n+1)$-ary predicate symbol and $E(\bar{a}, \bar{b})$ $2n$-ary formula which has no occurrences of R. If C is an R-positive sentence, then the sentence* $\mathrm{Ex}\, R \wedge \mathrm{Un}(R; E). \supset C$ *is provable in L if and only if the sentence* $\mathrm{Ex}^k(R; E) \supset C$ *is provable in L for some k.*

Theorem B shows us that the set S above can be axiomatized by

$\mathrm{Ex}^k(R;E)$, $k = 1, 2, 3, \cdots$, which are all R-positive sentences. Since sentences $\mathrm{Ex}\,R \wedge \mathrm{Co}(R;E). \supset \mathrm{Ex}^k(R;E)$, $k = 1, 2, \cdots$, are all provable in LK, but not in LJ, Theorem A holds in LK, but not in LJ. As an application of Theorem B, we obtain a new proof of Minc's Theorem on Skolem functions in LJ (cf. [7] and [8]). Suppose that $A(\bar{a}, b)$ is an $(n+1)$-ary formula, C a sentence, f an n-ary function symbol which occurs neither in A nor in C, and R is an $(n+1)$-ary formula which occurs neither in A nor in C. Then,

$\forall \bar{x} A(\bar{x}, f(\bar{x})) \supset C$ is provable in LJ

$\Longleftrightarrow$

$\mathrm{Ex}\,R \wedge \mathrm{Un}\,R. \supset [\forall \bar{x} \forall y (R(\bar{x}, y) \supset A(\bar{x}, y)) \supset C]$ is provable in LJ

$\Longleftrightarrow$

$\mathrm{Ex}^k\,R \supset [\forall \bar{x} \forall y (R(\bar{x}, y) \supset A(\bar{x}, y)) \supset C]$ is provable in LJ for some k (by Theorem B)

$\Longleftrightarrow$

$\forall \bar{x} \forall y (R(\bar{x}, y) \supset A(\bar{x}, y)) \supset [\mathrm{Ex}^k\,R \supset C]$ is provable in LJ for some k

$\Longleftrightarrow$

$\mathrm{Ex}^k\,A \supset C$ is provable in LJ for some k (by Positive Lemma).

Hence, we have.

THEOREM C (Minc). *The sentence* $\forall \bar{x} A(\bar{x}, f(\bar{x})) \supset C$ *is provable in* LJ *if and only if* $\mathrm{Ex}^k\,A \supset C$ *is provable in* LJ *for some* k.

Finally, the author wants to comment on a connection which exists among the existence condition $\mathrm{Ex}\,R$, the uniqueness condition $\mathrm{Un}(A;E)$, and k-th existence conditions $\mathrm{Ex}^k(A;E)$. By considering this connection, the author obtained a new theory, an approximation theory of uniqueness conditions by existence conditions. According to this theory, $\mathrm{Ex}^k(A;E)$ can be considered as examples of approximations of the uniqueness condition $\mathrm{Un}(A;E)$ by the existence condition $\mathrm{Ex}\,R$. Recently, the author realized that this theory implies Barwise's results on Henkin quantifier in [1], Harnik-Makkai's results on Vaught sentences in [2], and their analogues in LJ (cf. [7] and [8]). The details of them will appear in [6]. In § 1 of this paper, we shall state our Main Theorem, which gives a new expression of Theorem V in [4]. An outline of a proof of our Main Theorem will be given in § 2.

§ 1. Main Theorem

In this section, we shall state a new elimination theorem of uniqueness conditions, which holds both in LJ and in LK. Suppose that $A(\bar{a}, a)$, $B(\bar{b}, b)$, and $E(\bar{a}, \bar{b})$ are $(n+1)$-ary formula, $(m+1)$-ary formula, and $(n+m)$-ary formula, respectively. Then, the uniqueness condition of A and B with respect to E, denoted by $\mathrm{Un}(A, B; E)$, is the sentence:

$$\forall\bar{x}\forall x\forall\bar{y}\forall y(E(\bar{x}, \bar{y}) \wedge A(\bar{x}, x) \wedge B(\bar{y}, y). \supset x = y).$$

Suppose that $A_i(\bar{a}_i, b)$ are (n_j+1)-ary formulas and $E_i(\bar{a}_0, \bar{b}_i)$ are (n_0+n_i)-ary formulas $(i = 0, 1, \cdots, N)$. Then, k-th existence condition of A_0 and E_0 with respect to $A_1, \cdots, A_N$ and $E_1, \cdots, E_N$, denoted by $\mathrm{Ex}^k(A_0; E_0; A_1, \cdots, A_N; E_1, \cdots, E_N)$, is the sentence;

$$\forall\bar{x}_1\exists y_1\forall\bar{x}_2\exists y_2 \cdots \forall\bar{x}_k\exists y_k \left[\bigwedge_{j=1}^{k} A_0(\bar{x}_j, y_j) \wedge \bigwedge_{j,s=1}^{k} (E_0(\bar{x}_j, \bar{x}_s) \supset y_j = y_s)\right.$$
$$\left.\wedge \bigwedge_{j=1}^{k}\bigwedge_{i=1}^{N} \forall\bar{z}_i\forall z(E_i(\bar{x}_j, \bar{z}_i) \wedge A_i(\bar{z}_i, z). \supset z = y_j)\right].$$

Note that $\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=0}^{N} \mathrm{Un}(A_0, A_i; E_i). \supset \mathrm{Ex}^k(A_0; E_0; A_1, \cdots, A_N; E_1, \cdots, E_N)$ $(k = 1, 2, \cdots)$ are all provable in LJ.

MAIN THEOREM. *Suppose that A_0 is an (n_0+1)-ary predicate symbol which occurs in none of $A_1, \cdots, A_N$, $E_0, E_1, \cdots, E_N$. If C is an A_0-positive sentence, then the sentence $\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=0}^{N} \mathrm{Un}(A_0, A_i; E_i). \supset C$ is provable in L if and only if a sentence $\mathrm{Ex}^k(A_0; E_0; A_1, \cdots, A_N; E_1, \cdots, E_N) \supset C$ is provable in L for some k.*

In case that $N = 0$, this theorem is Theorem B in the introduction of this paper. As is explained in the introduction, this Main Theorem gives an A_0-positive axiomatization of the set of A_0-sentences which are provable from $\mathrm{Ex}\, A_0$ and $\bigwedge_{i=0}^{N} \mathrm{Un}(A_0, A_i; E_i)$ in L. For each $i, j = 1, \cdots, N$, let $E_i^{n_0}E_j(\bar{a}_i, \bar{b}_j)$ be the (n_i+n_j)-ary formula $\exists\bar{v}(E_i(\bar{v}, \bar{a}_i) \wedge E_j(\bar{v}, \bar{b}_j))$, and $\mathrm{Co}(A_0, A_i; E_i)$ the sentence $\forall\bar{x}_0\forall\bar{y}_i\forall x(E_i(\bar{x}_0, \bar{y}_i) \wedge A_i(\bar{y}_i, x). \supset A_0(\bar{x}_0, x))$. If sentences $\forall\bar{x}\forall\bar{y}(E_0(\bar{x}, \bar{y}) \supset \forall\bar{z}_i(E_i(\bar{x}, \bar{z}_i) \equiv E_i(\bar{y}, \bar{z}_i))$, $i = 1, 2, \cdots, N$, are all provable in LK, then sentences

$$\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=0}^{N} \mathrm{Un}(A_0, A_i; E_i). \supset \mathrm{Co}(A_0; E_0),$$

$$\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=1}^{N} \mathrm{Un}(A_0, A_i; E_i). \supset \bigwedge_{i=1}^{N} \mathrm{Co}(A_0, A_i; E_i),$$

$$\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=1}^{N} \mathrm{Un}\,(A_0, A_i; E_i). \supset \bigwedge_{i,j=1}^{N} \mathrm{Un}\,(A_i, A_j; E_i^{n_0} E_j)$$

and

$$\left.\begin{array}{l}\mathrm{Ex}\, A_0 \wedge \mathrm{Co}(A_0; E_0) \wedge \bigwedge_{i=1}^{N} \mathrm{Co}\,(A_0, A_i; E_i) \wedge \bigwedge_{i,j=1}^{N} \mathrm{Un}\,(A_i, A_j; E_i^{n_0} E_j). \\ \quad \supset \mathrm{Ex}^k\,(A_0; E_0; A_1, \cdots, A_N; E_1, \cdots, E_N) \qquad (k = 1, 2, \cdots)\end{array}\right\} \quad (*)$$

are all provable in LK. Hence, we have.

COROLLARY (Theorem V in [4]). *Suppose that A is an (n_0+1)-ary predicate symbol which occurs in none of $A_1, \cdots, A_N, E_0, E_1, \cdots, E_N$, and sentences*

$$\forall \bar{x} \forall \bar{y}(E_0(\bar{x}, \bar{y}) \supset \forall \bar{z}_i(E_i(\bar{x}, \bar{z}_i) \equiv E_i(\bar{y}, \bar{z}_i))), \qquad i = 1, 2, \cdots, N \textit{ are all}$$

provable in LK. If C is an A_0-positive sentence, then the sentence

$$\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=0}^{N} \mathrm{Un}\,(A_0, A_i; E_i). \supset C$$

is provable in LK if and only if the sentence

$$\mathrm{Ex}\, A_0 \wedge \mathrm{Co}\,(A_0; E_0) \wedge \bigwedge_{i=0}^{N} \mathrm{Co}\,(A_0, A_i; E_i) \wedge \bigwedge_{i,j=1}^{N} \mathrm{Un}\,(A_i, A_j; E_i^{n_0} E_j). \supset C$$

is provable in LK.

Remark 1. By considering the proof of Maehara's ε-theorem ([3]) from Theorem V in [4], we can easily see that the following three properties (i), (ii), and (iii) of LK are sufficient to prove Maehara's ε-theorem.

(i) Main Theorem.

(ii) (*) are provable for each $A_0, \cdots, A_N, E_0, \cdots, E_N$ such that $\forall \bar{x} \forall \bar{y}(E_0(\bar{x}, \bar{y}) \supset \forall \bar{z}_i(E_i(\bar{x}, \bar{z}_i) \equiv E_i(\bar{y}, \bar{z}_i))$, $i = 1, \cdots, N$ are all provable,

(iii) formulas of the form $\forall \bar{x} \exists y(\exists u A(\bar{x}, u) \supset A(\bar{x}, y))$ are all provable.

Remark 2. Elimination theorems in [4] do not hold in LJ as is shown in this paper. But, if we add some conditions on C, we can prove some of those elimination theorems in their original forms. For example, by using "elimination of positive occurrences of the equality symbol" in [5], we have; If R is an $(n+1)$-ary predicate symbol in and C is an R-positive sentence which has no negative occurrences of the equality symbol, then $\mathrm{Ex}\, R \wedge \mathrm{Un}\, R. \supset C$ is provable in LJ if and only if $\mathrm{Ex}\, R \supset C$ is provable in LJ.

§ 2. A proof

We prove our Main Theorem only in the case that $L = LJ$. We assume that LJ is formulated in the Gentzen style (cf. [9]) with a slight modification that every sequent in LJ is a pair (Γ, Θ), denoted by $\Gamma \to \Theta$, of finite sets of formulas such that Θ has at most one formulas. Suppose that $A_i(\bar{a}_i, \bar{b})$, $i = 0, 1, \cdots, N$, are $(n_i + 1)$-ary formulas, and $E_i(\bar{a}_0, \bar{b}_i)$, $i = 0, 1, \cdots, N$, are $(n_0 + n_i)$-ary formulas such that $A_0(\bar{a}_0, b)$ is an $(n_0 + 1)$-ary predicate symbol which occurs in none of $A_1, \cdots, A_N, E_0, E_1, \cdots, E_N$. Let C be an arbitrary A_0-positive sentence in LJ. It is sufficient to prove that one of the sentences

$$\mathrm{Ex}^k (A_0; E_0; A_1, \cdots, A_N; E_1, \cdots, E_N) \supset C, \qquad k = 1, 2, \cdots$$

is provable in LJ by assuming that the sentence

$$\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=0}^{N} \mathrm{Un}\,(A_0, A_i; E_i). \supset C$$

is provable in LJ. Assume that the sentence

$$\mathrm{Ex}\, A_0 \wedge \bigwedge_{i=0}^{N} \mathrm{Un}\,(A_0, A_i; E_i). \supset C$$

is provable. Then the sequent $\mathrm{Ex}\, A_0, \{\mathrm{Un}\,(A_0, A_i; E_i)\}_{i=0}^{N} \to C$ is provable in LJ. By using the technique in § 3 of [4], we have a proof-figure D of the sequent $\to C$ such that:

(i) every sequent in D is an A_0-sequent, i.e. a sequent of the form $\Gamma_0, \Gamma \to \Theta$, where Γ_0 is a set of A_0-atomic formulas (formulas of form $A_0(\bar{t}, t)$) and $\Gamma \to \Theta$ is A_0-positive (every formula in Γ is A_0-negative and every formula in Θ is A-positive):

(ii) D has no occurrences of A_0-equality axiom sequents,

$$t_0 = s_0, t_1 = s_1, \cdots, t_{n_0} = s_{n_0}, A_0(t_0, \cdots, t_{n_0}) \longrightarrow A_0(s_0, \cdots, s_{n_0}),$$

(iii) every inference rule in D is one of the followings;

(a) logical rules in LJ (cf. [9]),

(b) weakening rules and cut-rules whose cut formulas have no occurrences of A_0,

(c) (Ex)-rule and (A_i), $i = 0, 1, \cdots, N$ rules below,

$$(\mathrm{Ex})\ \frac{A_0(\bar{t}, a), \Gamma \to \Theta}{\Gamma \to \Theta}, \text{ where } a \text{ occurs neither in } \bar{t} \text{ nor } \Gamma \cup \Theta,$$

$$(A_0)\ \frac{\Gamma \to E_0(\bar{t}, \bar{s}) \quad t = s, \Gamma \to \Theta}{\Gamma \to \Theta}, \text{ where } A_0(\bar{t}, t), A_0(\bar{s}, s) \in \Gamma.$$

$$(A_i)\ \frac{\Gamma \to E_i(\bar{t}, \bar{s}) \quad \Gamma \to A_i(\bar{s}, s) \quad t = s, \Gamma \to \Theta}{\Gamma \to \Theta}, \text{ where}$$

$$A_0(\bar{t}, t) \in \Gamma \text{ and } i = 1, 2, \cdots, N.$$

For each finite set Γ of A_0-atomic formulas, let $I(\Gamma)$ be the set $\{\langle \bar{t}, t\rangle \mid A_0(\bar{t}, t) \in \Gamma\}$, $\mathrm{Ex}^0(\Gamma)$ the formula;

$$\bigwedge_{\langle \bar{t}, t\rangle \in I(\Gamma)} A_0(\bar{t}, t) \wedge \bigwedge_{\langle \bar{t}, t\rangle, \langle \bar{s}, s\rangle \in I(\Gamma)} (E_0(\bar{t}, \bar{s}) \supset t = s)$$

$$\bigwedge_{\langle \bar{t}, t\rangle \in I(\Gamma)} \bigwedge_{i=1}^{N} \forall \bar{z}_i \forall z (E_i(\bar{t}, \bar{z}_i) \wedge A_i(\bar{z}_i, z). \supset z = t),$$

and $\mathrm{Ex}^k(\Gamma)$ the formula

$$\forall \bar{x}_1 \exists y_1 \cdots \forall \bar{x}_k \exists y_k \,\mathrm{Ex}^0(\Gamma \cup \{A_0(\bar{x}_1, y_1), \cdots, A_0(\bar{x}_k, y_k)\}).$$

Then, clearly sentences $\mathrm{Ex}^k(\Gamma) \to \mathrm{Ex}^s(\Theta)$ are provable in LJ if $s \leqq k$ and $\Theta \subseteqq \Gamma$.

Also, $\mathrm{Ex}^k(\phi)$ is the sentence $\mathrm{Ex}^k(A_0; E_0; A_1, \cdots, E_N)$, where ϕ is the empty set.

Let D_0 be an arbitrary subproof-figure of D, whose end sequent is $\Gamma_0, \Gamma \to \Theta$, where Γ_0 is a set of A_0-atomic formulas and $\Gamma \to \Theta$ is an A_0-positive sequent.

By induction on D_0, we can easily see that the sequent $\mathrm{Ex}^k(\Gamma_0), \Gamma \to \Theta$ is provable in LJ for some k. By applying this fact to the proof-figure D we have that the sequent $\mathrm{Ex}^k(\phi) \to C$ is provable in LJ for some k. This completes our proof of our Main Theorem.

References

[1] J. Barwise, Some applications of Henkin quantifier, Israel J. of Math., **25** (1976), 47–63.

[2] V. Harnik and M. Makkai, Applications of Vaught sentences and the covering theorem, J.S.L., **41** (1976), 171–187.

[3] S. Maehara, Equality axioms on Hilbert ε-theorem, J. of the Faculty of Science, Univ. of Tokyo, Sect. 1, **7** (1957), 419–435.

[4] N. Motohashi, Elimination theorems of uniqueness conditions, to appear in Zeitschrift für Math. Logik und Grundlagen der Math.

[5] ——, Some proof-theoretic results on equivalence conditions, congruence conditions, and uniqueness conditions, to appear.

[6] ——, Approximations of uniqueness conditions by existence conditions, to appear in Fund. Math.

[7] C. Smorynski, On axiomatizing fragments, J.S.L., **43** (1977), 530–544.
[8] ——, The axiomatization problem for fragments, Ann. of Math. Logic, **14** (1978), 193–221.
[9] G. Takeuti, Proof Theory, North-Holland, Amsterdam, 1975.

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