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ON SOME DIMENSION FORMULA FOR AUTOMORPHIC FORMS OF WEIGHT ONE I

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§ 0. Introduction

Let Γ be a fuchsian group of the first kind not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. We shall denote by d_0 the number of linearly independent automorphic forms of weight 1 for Γ. It would be interesting to have a certain formula for d_0. But, Hejhal said in his Lecture Notes 548, it is impossible to calculate d_0 using only the basic algebraic properties of Γ. On the other hand, Serre has given such a formula of d_0 recently in a paper delivered at the Durham symposium ([7]). His formula is closely connected with 2-dimensional Galois representations.

The purpose of this note is to give some formula of the number d_0 for the case of compact type, by making use of the Selberg trace formula ([6]). Our result is expressed by Theorem C (§ 2). It seems likely that the similar result holds for discontinuous groups of finite type ([2]).

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§ 1. The Selberg eigenspace $\mathfrak{M}(k, \lambda)$

Let

$$S = \{z = x + iy/x,\ y \text{ real and } y > 0\}$$

denote the complex upper half-plane and let $G = SL(2, \boldsymbol{R})$ be the real special linear group of the second degree. Consider direct products

$$\tilde{S} = S \times T\,,$$
$$\tilde{G} = G \times T\,,$$

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where T denotes the real torus, and let an element (g, α) of $\tilde{G}$ operate on $\tilde{S}$ as follows:

$$\tilde{S} \ni (z, \phi) \xrightarrow{(g,\alpha)} (z, \phi)(g, \alpha) = \left(\frac{az+b}{cz+d}, \phi + \arg(cz+d) - \alpha\right) \in \tilde{S} ,$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. The operation of $\tilde{G}$ on $\tilde{S}$ is transitive. $\tilde{S}$ is a weakly symmetric Riemannian space with the $\tilde{G}$-invariant metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + \left(d\phi - \frac{dx}{2y}\right)^2 ,$$

and with the isometry μ defined by

$$\mu(z, \phi) = (-\bar{z}, -\phi) .$$

The $\tilde{G}$-invariant measure $d(z, \phi)$ associated to the $\tilde{G}$-invariant metric is given by

$$d(z, \phi) \equiv d(x, y, \phi) = \frac{dx \wedge dy \wedge d\phi}{y^2} .$$

The ring $\mathfrak{R}(\tilde{S})$ of $\tilde{G}$-invariant differential operators on $\tilde{S}$ is generated by

$$\frac{\partial}{\partial\phi}$$

and

$$\Delta^{(\tilde{S})} \equiv \tilde{\Delta} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{5}{4}\frac{\partial^2}{\partial\phi^2} + y\frac{\partial}{\partial\phi}\frac{\partial}{\partial x} ,$$

where $\tilde{\Delta}$ is the Laplace operator of $\tilde{S}$.

Let Γ be a discrete subgroup of G not containing the element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and *suppose that $\Gamma\backslash G$ is compact.*

By the correspondence

$$G \ni g \leftrightarrow (g, 0) \in \tilde{G} = G \times T ,$$

we identify the group G with a subgroup $G\times\{0\}$ of $\tilde{G}$, and so the subgroup Γ identify with a subgroup $\Gamma \times \{0\}$ of $\tilde{G}$[1].

For an element $(g, \alpha) \in G$, we define a mapping $T_{(g,\alpha)}$ of $C^\infty(\tilde{S})$ into itself by

1) Therefore if $\Gamma\backslash G$ is compact, so is $\Gamma\backslash\tilde{G}$.

$$(T_{(g,\alpha)}f)(z, \phi) = f((z, \phi)(g, \alpha)) ,$$

where $f(z, \phi) \in C^\infty(\tilde{S})$. $(g, \alpha) \to T_{(g,\alpha)}$ is a representation of $\tilde{G}$. For an element $g \in G$ we put $T_{(g,0)} = T_g$. Then we have

$$(T_g f)(z, \phi) = f\left(\frac{az+b}{cz+d}, \phi + \arg(cz+d)\right),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Denote by $C^\infty(\Gamma \backslash \tilde{S})$ the set of all C^∞-class functions on $\tilde{S}$ invariant under Γ:

$$C^\infty(\Gamma \backslash \tilde{S}) = \{f(z, \phi) \in C^\infty(\tilde{S}) / T_g f = f \text{ for all } g \in \Gamma\} ;$$

and now consider the following simultaneous eigenvalue problem in $C^\infty(\Gamma \backslash \tilde{S})$:

$$\text{(A)} \qquad \begin{cases} f \in C^\infty(\Gamma \backslash \tilde{S}) , & (1) \\ \dfrac{\partial}{\partial \phi} f = -ikf , & (2) \\ \tilde{\Delta} f = \lambda f . & (3) \end{cases}$$

We denote by $\mathfrak{M}_\Gamma(k, \lambda) = \mathfrak{M}(k, \lambda)$ the set of all functions satisfying the above condition (A). It is well known that every eigenspace $\mathfrak{M}(k, \lambda)$ is finite dimensional and orthogonal to each other, and also the eigenspaces span together the Hilbert space $L^2(\Gamma \backslash \tilde{S})$ with norm

$$\|f\|^2 = \frac{1}{2\pi} \int_{\Gamma \backslash \tilde{S}} |f|^2 d(z, \phi) .$$

We put $\boldsymbol{\lambda} = (k, \lambda)$. For every invariant integral operator with a kernel function $k(z, \phi; z', \phi')$ on (k, λ), we have

$$(4) \qquad \int_{\tilde{S}} k(z, \phi; z', \phi') f(z', \phi') d(z', \phi') = h(\boldsymbol{\lambda}) f(z, \phi) ,$$

for $f \in \mathfrak{M}(k, \lambda)$.

It is to be noted that $h(\boldsymbol{\lambda})$ does not depend on f so long as f is in $\mathfrak{M}(k, \lambda)$. We also know that there is a basis $\{f^{(n)}\}_{n=1}^\infty$ of the space $L^2(\tilde{S}/\Gamma)$ under the condition that each $f^{(n)}$ satisfies (2) and (3) in (A). Then we put $\boldsymbol{\lambda}^{(n)} = (k, \lambda)$ for such a spectrum (k, λ).

We now obtain the following Selberg trace formula for $L^2(\Gamma \backslash \tilde{S})$:

$$(5) \qquad \sum_{n=1}^\infty h(\boldsymbol{\lambda}^{(n)}) = \sum_{M \in \Gamma} \int_{\tilde{D}} k(z, \phi; M(z, \phi)) d(z, \phi) ,$$

where $\tilde{D}$ denotes a compact fundamental domain of Γ in $\tilde{S}$ and $k(z, \phi; z', \phi')$ is a point-pair invariant kernel of (a)–(b) type in the sense of Selberg such that the series on the left-hand side of (5) is absolutely convergent ([4], [6]). Denote by $\Gamma(M)$ the centralizer of M in Γ, and put $\tilde{D}_M = \Gamma(M)\backslash\tilde{S}$. Then it is easy to see that

$$(6) \qquad \sum_{M\in\Gamma} \int_{\tilde{D}} k(z, \phi; M(z, \phi))d(z, \phi) = \sum_{\ell} \int_{\tilde{D}_{M_\ell}} k(z, \phi; M_\ell(z, \phi))d(z, \phi)\ ,$$

where the sum over $\{M_\ell\}$ is taken over the distinct conjugacy classes of Γ.

We shall denote by $\mathfrak{S}_1(\Gamma)$ the linear space of all holomorphic automorphic forms of weight 1 for the above fuchsian group Γ and put

$$d_0 = \dim \mathfrak{S}_1(\Gamma)\ .$$

Then the following equality comes from [1]:

$$d_0 = \dim \mathfrak{M}(1, -\tfrac{3}{2})\ .$$

§ 2. A formula for d_0

We consider an invariant integral operator on the Selberg eigenspace $\mathfrak{M}(k, \lambda)$ defined by a point-pair invariant kernel

$$\omega_s(z, \phi; z', \phi') = \left|\frac{(yy')^{1/2}}{(z-\bar{z}')/2i}\right|^s \frac{(yy')^{1/2}}{(z-\bar{z}')/2i} e^{-i(\phi-\phi')},\ (s > 1)\ .$$

By the relation (4), the integral operator ω_s vanishes on $\mathfrak{M}(k, \lambda)$ for all $k \neq 1$. The distribution of spectrum (k, λ) is given by Kuga in the compact case ([5]). It is discrete and

$$\begin{aligned} &(1, \mu_\beta), && (\mu_\beta < 0, \mu_\beta \neq -\tfrac{3}{2}, -\tfrac{1}{2}) \\ &(1, -\tfrac{3}{2}), \\ &(1, -\tfrac{1}{2}) \end{aligned}$$

give the complete set of spectra of the type $(1, *)$. But the spectra of types $-\frac{3}{2} < \mu_\beta < 0$ in $(1, \mu_\beta)$ and $(1, -\frac{1}{2})$ do not appear actually in the complete set (Bargmann)[2]. We put

$$\begin{aligned} &\mu_0 = -\tfrac{3}{2}, \mu_1, \mu_2, \cdots, \\ &d_\beta = \dim \mathfrak{M}(1, \mu_\beta), \qquad (\beta = 0, 1, 2, \cdots)\ . \end{aligned}$$

2) This remark was informed by Satake's letter to the author.

Then the left-hand side of the trace formula (5) implies

$$\sum_{n=1}^{\infty} h(\lambda^{(n)}) = \sum_{\beta=0}^{\infty} d_\beta \Lambda_\beta ,$$

where Λ_β denotes the eigenvalue of ω_s in $\mathfrak{M}(1, \mu_\beta)$. For the eigenvalue Λ_β, using the special eigenfunction

$$f(z, \phi) = e^{-i\phi} y^{r_\beta}, \ \mu_\beta = r_\beta(r_\beta - 1) - \tfrac{5}{4} ,$$

for a spectrum $(1, \mu_\beta)$ in $C^\infty(\tilde{S})$, we obtain

$$\Lambda_\beta = 2^{2+s}\pi \frac{\Gamma(1/2)\Gamma((1+s)/2)}{\Gamma(s)\Gamma(1+(s/2))} \Gamma\left(\frac{s-1}{2} + r_\beta\right)\Gamma\left(\frac{s-1}{2} - r_\beta\right).$$

If we put $r_\beta = \frac{1}{2} + iv_\beta$, then

$$\mu_\beta = -\frac{3}{2} - v_\beta^2, \ v_\beta = \frac{\sqrt{-(6+4\mu_\beta)}}{2} \geqq 0 ,$$

and

$$\Lambda_\beta = 2^{2+s}\pi \frac{\Gamma(1/2)\Gamma((1+s)/2)}{\Gamma(s)\Gamma(1+(s/2))} \Gamma\left(\frac{s}{2} + iv_\beta\right)\Gamma\left(\frac{s}{2} - iv_\beta\right).$$

Therefore there is a one-to-one correspondence between the functions Λ_β of μ_β and even function $h(v_\beta)$, the correspondence being given by $\Lambda_\beta = h(v_\beta)$. For the case of weight 2, Selberg introduced in [5] the point-pair invariant kernel $\omega_2 \cdot (((yy')^{1/2})/|(z - \bar{z}')/2i|)^s$ of (a)–(b) type under the condition $s > 0$. The above kernel ω_s is obtained by $s \to s-1$ in $\omega_2 \cdot (((yy')^{1/2})/|(z - \bar{z}')/2i|)^s$. Therefore our kernel ω_s is a point-pair invariant kernel of (a)–(b) type under the condition $s > 1$. In general, it is known that the series $\sum_{\beta=0}^{\infty} d_\beta \Lambda_\beta$ is absolutely convergent for $s > 1$. By the Stirling formula, we see that the above series is also absolutely and uniformly convergent for all bounded s except $s = 0$.

Now we shall calculate the components $J(I)$, $J(P)$, and $J(R)$ of traces appearing in the right-hand side of (6).

i) unit class: $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

It is clear that

$$\omega_s(z, \phi; M(z, \phi)) = 1 ,$$

and therefore

$$J(I) = \int_{\tilde{D}_M} d(z, \phi) = \int_{\tilde{D}} d(z, \phi) < \infty .$$

ii) Hyperbolic conjugacy classes.

We shall call a hyperbolic element P primitive, if it is not a power with exponent > 1 of any other element in Γ, and correspondingly we say the conjugacy class $\{P\}$ is primitive. When we write the primitive hyperbolic conjugacy classes as $\{P_\alpha\}$ $(\alpha = 1, 2, \cdots)$, the hyperbolic conjugacy classes in Γ can be expressed as $\{P_\alpha^k\}$ $(\alpha = 1, 2, \cdots; k = 1, 2, \cdots)$. It is noted that the Jordan canonical form of P is $\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix}$ with $\lambda_0 > 1$, and we can conclude that $\Gamma(P^k) = \Gamma(P)$ is an infinite cyclic group generated by the primitive element P. Put

$$g^{-1}Pg = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix} \quad (g \in G) \quad \text{and } \Gamma' = g^{-1}\Gamma g .$$

Then we have

$$\Gamma'\left(\begin{pmatrix} \lambda^0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix}\right) = g^{-1}\Gamma(P)g .$$

The hyperbolic component $J(P)$ is calculated as follows:

$$\begin{aligned} J(P^k) &= \int_{\tilde{D}_P} \omega_s(z, \phi; P^k(z, \phi))d(z, \phi) \\ &= \int_{g^{-1}\tilde{D}_P} \omega_s(g(z, \phi), P^k g(z, \phi))d(z, \phi) \\ &= \int_{g^{-1}\tilde{D}_P} \omega_s(z, \phi; g^{-1}P^k g(z, \phi))d(z, \phi) \end{aligned}$$

$\left(g^{-1}\tilde{D}_P \text{ is a fundamental domain of } \Gamma'\left(\begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0^{-1} \end{pmatrix}\right) \text{ in } \tilde{S}\right)$

$$= (2\pi)(2^{s+1}i)\,|\lambda_0^k|^{(s+1)} \int_{g^{-1}D_P} \frac{y^{s-1}}{(z - \lambda_0^{2k}\bar{z})\,|z - \lambda_0^{2k}\bar{z}|^s} dxdy$$

$(z = \rho e^{i\theta}, \rho > 0, 0 < \theta < \pi)$

$$= (2^{s+2}\pi i)|\lambda_0^k|^{(s+1)} \int_1^{\lambda_0^2} \frac{d\rho}{\rho} \int_0^\pi \frac{(\sin\theta)^{s-1}d\theta}{(e^{i\theta} - \lambda_0^{2k}e^{-i\theta})\,|e^{i\theta} - \lambda_0^{2k}e^{-i\theta}|^s}$$

$\left(\alpha = \dfrac{1 + \lambda_0^{2k}}{1 - \lambda_0^{2k}}\right)$

$$= (2^{s+2}\pi i)\,|\lambda_0^k|^{(s+1)} \log \lambda_0^2 \frac{1}{(1 - \lambda_0^{2k})\,|1 - \lambda_0^{2k}|^s} \int_0^\pi \frac{(\sin\theta)^{s-1}(\cos\theta - i\alpha\sin\theta)}{(\cos^2\theta + \alpha^2\sin^2\theta)^{(s/2)+1}} d\theta$$

$(t = \cot\theta)$

$$= (2^{s+2}\pi i)|\lambda_0^k|^{(s+1)} \log\lambda_0^2 \frac{-2\alpha i}{(1-\lambda_0^{2k})|1-\lambda_0^{2k}|^s} \frac{1}{2|\alpha|^{s+1}} \frac{\Gamma((s+1)/2)\Gamma(1/2)}{\Gamma((s+2)/2)} .$$

Thus,

$$J(P^k) = (2^{s+3}\pi)\frac{\Gamma(1/2)\Gamma((s+1)/2)}{\Gamma((s+2)/2)} \frac{\log|\lambda_0|}{|\lambda_0^{-k}-\lambda_0^k|(\lambda_0^{-k}+\lambda_0^k)^s} .$$

Consequently, we have

$$\begin{aligned} J(P) &= \sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty} J(P_\alpha^k) \\ &= \frac{8\pi^{3/2}2^s\Gamma((s+1)/2)}{\Gamma((s+2)/2)} \sum_{\alpha=1}^{\infty}\sum_{k=1}^{\infty} \frac{\log|\lambda_{0,\alpha}|}{|\lambda_{0,\alpha}^k - \lambda_{0,\alpha}^{-k}|}|\lambda_{0,\alpha}^k + \lambda_{0,\alpha}^{-k}|^{-s} . \end{aligned}$$

iii) Elliptic conjugacy classes.

Let ρ, $\bar\rho$ be the fixed points of an elliptic element $M(\rho \in S)$ and ζ, $\bar\zeta$ be the eigenvalues of M. Let φ be a linear transformation such that maps S into a unit circle:

$$w = \varphi(z) = \frac{z-\rho}{z-\bar\rho} .$$

Then we have

$$\varphi M\varphi^{-1} = \begin{pmatrix}\zeta & 0\\ 0 & \bar\zeta\end{pmatrix} .$$

and

$$\frac{Mz-\rho}{Mz-\bar\rho} = \frac{\zeta}{\bar\zeta}\frac{z-\rho}{z-\bar\rho} .$$

The elliptic component $J(R)$ is calculated as follows:

$$\begin{aligned} J(M) &= \int_{\tilde D_M} \omega_s(z,\phi; M(z,\phi))d(z,\phi) \\ &= \frac{1}{[\Gamma(M):1]}\int_{\tilde S} \omega_s(z,\phi; M(z,\phi))d(z,\phi) \\ &= \frac{2^{s+1}i}{[\Gamma(M):1]}\int_{\tilde S} \frac{(yy')^{(s+1)/2}}{(z-\bar z')|z-\bar z'|^s} e^{-i(\phi-\varphi')}d(z,\phi) \qquad ((z',\phi') = M(z,\phi)) \\ &= \frac{8\pi\bar\zeta}{[\Gamma(M):1]}\int_{|w|<1} \frac{(1-w\bar w)^{s-1}}{(1-\bar\zeta^2 w\bar w)|1-\bar\zeta^2 w\bar w|^s} dudv \qquad (w = u+iv) \end{aligned}$$

$(w = re^{i\theta})$

$$= \frac{16\pi^2\zeta}{[\Gamma(M):1]} \int_0^1 \frac{(1-r^2)^{s-1} r}{(1-\bar{\zeta}^2 r^2)\,|1-\bar{\zeta}^2 r^2|^s}\, dr \,.$$

By a simple calculation, we have

$$\lim_{s\to 0} sJ(M) = \frac{8\pi^2}{[\Gamma(M):1]} \frac{-\bar{\zeta}}{(\bar{\zeta}^2-1)^2} \,.$$

We put

$$\zeta^*(s) = \sum_{\alpha=1}^{\infty} \sum_{k=1}^{\infty} \frac{\log \lambda_{0,\alpha}}{\lambda_{0,\alpha}^k - \lambda_{0,\alpha}^{-k}} (\lambda_{0,\alpha}^k + \lambda_{0,\alpha}^{-k})^{-s} \,.$$

Then, by the trace formula, the Dirichlet series $\zeta^*(s)$ has a meromorphic continuation to all s, the only singularity being a simple pole at $s=0$ whose residue will appear in (7).

Finally, multiply the both sides of (5) by s and tend s to zero, then the limit is expressed, by the above i), ii), iii), as follows:

$$(7) \qquad d_0 = \frac{1}{2} \sum_{\{M\}} \frac{1}{[\Gamma(M):1]} \frac{-\bar{\zeta}}{(\bar{\zeta}^2-1)^2} + \frac{1}{2} \operatorname*{Res}_{s=0} \zeta^*(s) \,,$$

where the sum over $\{M\}$ is taken over the distinct elliptic conjugacy classes of Γ.

Remark on the Dirichlet series $\zeta^*(s)$. In his work ([3]), H. Huber introduced some zeta function defined by

$$H(s) = \sum_{\alpha=1}^{\infty} \sum_{k=1}^{\infty} \frac{\log |\lambda_{0,\alpha}|}{|\lambda_{0,\alpha}^k - \lambda_{0,\alpha}^{-k}|} (\lambda_{0,\alpha}^{2k} + \lambda_{0\alpha}^{-2k})^{-s+(1/2)} \,.$$

It is clear that

$$\operatorname*{Res}_{s=0} \zeta^*(s) = \operatorname*{Res}_{s=1/2} H(s) \,.$$

These functions are "Selberg type zeta-functions" connected with the distribution problems of hyperbolic conjugacy classes in a discrete group.

As more information of d_0, we consider an integral operator $\tilde{\omega}_s$ on $\mathfrak{M}(0, \lambda)$ defined by

$$\tilde{\omega}_s(z, \phi; z', \phi') = \frac{(yy')^{(s+1)/2}}{|(z-\bar{z}')/2i|^{s+1}} \,, \qquad (s > 1) \,.$$

Then, by a similar calculation as in the above we have

$$\operatorname*{Res}_{s=0} \zeta^*(s) = 2 \dim \mathfrak{M}(0, -\tfrac{1}{4}) \,.$$

Now our main result can be stated as follows.

THEOREM C. *Let Γ be a fuchsian group of the first kind not containing the element* $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *and suppose that Γ has a compact fundamental domain in the upper half plane S. Let d_0 be the dimension for the linear space consisting of all holomorphic automorphic forms of weight* 1 *with respect to the group Γ. Then the number d_0 is given by the formula*:

$$d_0 = \frac{1}{2} \sum_{\{M\}} \frac{1}{[\Gamma(M):1]} \frac{-\bar{\zeta}}{(\zeta^2-1)} + \dim \mathfrak{M}\left(0, -\frac{1}{4}\right),$$

where the sum over $\{M\}$ is taken over the distinct elliptic conjugacy classes of Γ, $\Gamma(M)$ denotes the centralizer of M in Γ, ζ is one of the eigenvalues of M, and $\mathfrak{M}(0, -1/4)$ denotes the eigenspace with the eigenvalue $-1/4$ for the Laplacian $y^2((\partial^2/\partial x^2) + ((\partial^2/\partial y^2))$ on the space $C^\infty(\Gamma \backslash S)$.

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