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ON THE DOUADY SPACE OF A COMPACT COMPLEX SPACE IN THE CATEGORY $\mathscr{C}$

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Introduction

Let X be a complex space. Let D_X be the Douady space of compact complex subspaces of X [6] and $\rho_X: Z_X \to D_X$ the corresponding universal family of subspaces of X. Thus there is a natural embedding $Z_X \subseteqq D_X \times X$ such that ρ_X is induced by the projection $D_X \times X \to D_X$. Let $\pi_X: Z_X \to X$ be induced by the other projection $D_X \times X \to X$. For any irreducible component D_α of $D_{X,\mathrm{red}}$ we denote by $\rho_\alpha: Z_\alpha \to D_\alpha$ the universal family restricted to D_α, and set $\pi_\alpha = \pi_{X|Z_\alpha}: Z_\alpha \to X$, where $D_{X,\mathrm{red}}$ is the underlying reduced subspace of D_X. On the other hand, we have introduced in [9] a category $\mathscr{C}$ of compact complex spaces as follows (cf. also [10]). A compact complex space X is in $\mathscr{C}$ if and only if there exist a compact Kähler manifold Y and a generically surjective meromorphic map $h: Y \to X_{\mathrm{red}}$, X_{red} being as above. Then the main purpose of this paper is to prove the following theorem: *Let X be a compact complex space in $\mathscr{C}$. Then for every irreducible component D_α of $D_{X,\mathrm{red}}$ such that Z_α is reduced, D_α is compact and again belongs to $\mathscr{C}$.* The proof also shows that if X is Moishezon, then D_α also is Moishezon, which is a special case of a theorem of Artin [1]. Moreover since the Barlet space $B(X)$ of compact cycles of X [4] is a proper holomorphic image of the union of those irreducible components of $D_{X,\mathrm{red}}$ for which Z_α are reduced and of pure fiber dimension, the result also implies that every irreducible component of $B(X)$ is again in $\mathscr{C}$ if X is in $\mathscr{C}$. Here we note that the same result as above was also obtained by Campana [5] independently.

The arrangement of this article is as follows. In § 1 and § 2 we define respectively the notion of a Moishezon morphism and of a morphism in the category $\mathscr{C}/S$, which is a relative version of the category $\mathscr{C}$ above, and summarize some functorial properties of these morphisms. In § 3 we

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make some general study on the irreducibility of general fiber of a morphism, in part to be used in § 5. Then in § 4 we give the main ingredient of the proof of our theorem, Proposition 4, which states that if the general fiber of $\rho_\alpha: Z_\alpha \to D_\alpha$ is reduced and irreducible, then π_α defined above is Moishezon. In fact, combining this with the results in §§ 1 and 2 we obtain the theorem immediately in this special case. The reduction of the general case to this special case will then be given in § 5, thus completing the proof of the theorem. Actually our theorem is expected to be true for any irreducible component of $D_{X,\mathrm{red}}$. Presupposing the future investigation of this problem along the line of [9] and in view of an application [11] also, we have developed our exposition in the relative form as in [9] so that the above theorem is also true in this generalized form (see Theorem in § 5 for the precise statement). Finally in the Appendix we give a direct proof of Lemma 2.

Notation. Let $f: X \to S$ be a morphism of complex spaces. Then for any morphism $\alpha: T \to S$ we often write $X_T = X \times_S T$ and $f_T: X_T \to T$ for the natural projection. For instance if $U \subseteq S$ is open, f_U is the induced morphism $X_U = f^{-1}(U) \to U$. In particular if $T = \{s\}$ is a point of S we write X_s instead of $X_{\{s\}}$. For a complex space X, X_{red} denotes the underlying reduced analytic subspace.

§ 1. Moishezon morphisms

(1.1) We fix notation and terminology for meromorphic maps. Let $f: X \to S$ and $g: Y \to S$ be morphisms of reduced complex spaces. Then a *meromorphic S-map* $\alpha: X \to Y$ from X to Y is a reduced analytic subspace $\Gamma \subseteq X \times_S Y$ such that the natural projection $p: \Gamma \to X$ is a proper bimeromorphic morphism in the sense that p is proper and that there is a dense Zariski open subset U (resp. V) of Γ (resp. X) such that p induces an isomorphism of U and V. We call α a (proper) *bimeromorphic S-map*, or being *S-bimeromorphic*, if the natural projection $q: \Gamma \to Y$ also is a proper bimeromorphic morphism. We say that f and g are bimeromorphic if there is a bimeromorphic S-map of X to Y.

If f is proper in the above definition, $q(\Gamma)$ is an analytic subspace of Y and is called the image of X by α. On the other hand, α is called *generically surjective* (resp. *generically finite*) if $q(\Gamma)$ contains a dense Zariski open subset of Y (resp. q is generically finite). When f is proper, the generic surjectivity is equivalent to saying that $Y = q(\Gamma)$.

Given a meromorphic S-map $\alpha: X \to Y$ as above we often identify α with the induced S-morphism $\alpha' = qp^{-1}|_V: V \to Y$. Then the subspace Γ above is recovered from α' as the closure in $X \times_S Y$ of the graph $\Gamma_{\alpha'} \subseteqq V \times_S Y$ of α' and is called the graph of α'. Then an S-morphism is nothing but the meromorphic S-map α for which we can take $V = X$.

Let $f: X \to S$ and $g_i: Y_i \to S$, $1 \leqq i \leqq m$, be morphisms of complex spaces and $\alpha_i: X \to Y_i$ be meromorphic S-maps. Then we can define naturally a meromorphic S-map $\prod_i{}_S \alpha_i: X \to Y_1 \times_S \cdots \times_S Y_m$ called the product of α_i over S; one verifies readily that the closure of the graph of the S-morphism $\alpha'_1 \times_S \cdots \times_S \alpha'_m$ is analytic in $X \times_S Y_1 \times_S \cdots \times_S Y_m$ where α'_i for α_i has the same meaning as α' for α as above.

For later reference we recall here the analytic Chow lemma due to Hironaka [14], [15].

(1.1.1) Let $\alpha: X \to Y$ be a meromorphic S-map as above. Then there exist a complex manifold X^*, and a projective bimeromorphic morphism $h: X^* \to X$ such that the composition $\alpha h: X^* \to Y$ is a morphism.

(1.2) Let $f: X \to S$ be a proper morphism of complex spaces. We call f *locally projective* if for every relatively compact open subset Q of S there is an invertible sheaf $\mathscr{L} = \mathscr{L}(Q)$ defined on X such that $\mathscr{L}|_{X_Q}$ is f_Q-ample (cf. Notation). (In this case we simply say that $\mathscr{L}$ is f_Q-ample.) Thus if f is locally projective, then f_Q is projective for every relatively compact open subset $Q \subseteqq S$.

(1.2.1) A composition of two locally projective morphisms is again locally projective.

Proof. Let $f: X \to Y$, $g: Y \to Z$ be locally projective. Let $h = gf: X \to Z$. Let Q be any relatively compact open subset of Z. Take another relatively compact open subset Q' of Z with $Q \Subset Q'$. Then $\tilde{Q}' = g^{-1}(Q')$ is a relatively compact open subset of Y. Take an invertible sheaf $\mathscr{L}$ on X (resp. $\mathscr{F}$ on Y) which is $f_{\tilde{Q}'}$-ample (resp. $g_{Q'}$-ample). Then it is easy to see that $\mathscr{L} \otimes_{\mathscr{O}_X} f^*\mathscr{F}^m$ is h_Q-ample for all sufficiently large m (cf. EGA II, 4.6.13 (ii)). Thus h is locally projective.

(1.3) Let $f: X \to S$ be a locally projective morphism.

(1.3.1) If X has only a finite number of irreducible components, then f is S-bimeromorphic to a projective morphism.

Proof. Let Q be any relatively compact open subset of S such that

X_Q meets every irreducible component of X. Let $\mathscr{L}$ be an invertible sheaf on X which is f_Q-ample. Restricting Q and replacing $\mathscr{L}$ by its high power $\mathscr{L}^n$, $n \gg 0$, we may assume that $\mathscr{L}$ is even f_Q-very ample. Let $\alpha: X \to \boldsymbol{P}(f_*\mathscr{L})$ be the natural meromorphic S-map of X into the projective fiber space $\boldsymbol{P}(f_*\mathscr{L})$ over S associated to the coherent analytic sheaf $f_*\mathscr{L}$ on S (cf. [3], [13]). By our assumption α is an embedding on X_Q. Since X_Q meets every irreducible component of X, this implies that α is bimeromorphic onto its image. Hence f is bimeromorphic to a projective morphism. Q.E.D.

From the above proof follows also the following:

(1.3.2) Let $f: X \to S$ be as in (1.3.1). Then there exist an invertible sheaf $\mathscr{L}$ on X and a dense Zariski open subset W of S such that $\mathscr{L}$ is f_W-(very) ample.

(1.4) Definition. Let $f: X \to S$ be a proper morphism of reduced complex spaces. We call f *Moishezon* if f is bimeromorphic to a locally projective morphism $g: Y \to S$. By (1.3.1) when X has only a finite number of irreducible components, f is Moishezon if and only if f is bimeromorphic to a projective morphism.

Remark. In [17] Moishezon introduced the notion of an A-space over another complex space, and stated some of their fundamental properties. From his definition it follows readily that for a proper morphism $f: X \to S$ of reduced complex spaces X is an A-space over S if and only if f is locally Moishezon in the sense that for each point $s \in S$ there is a neighborhood $s \in U$ such that the induced morphism $f_U: X_U \to U$ is Moishezon in the sense defined above.

(1.5) Clearly the Moishezon property of a morphism is invariant under S-bimeromorphic equivalence. We now list some fundamental properties of Moishezon morphisms.

1) A composition of two Moishezon morphisms are again Moishezon.

2) $f: X \to S$ is Moishezon if and only if for each irreducible component X_i of X the restriction $f = f|_{X_i}: X_i \to S$ is Moishezon.

3) If f is Moishezon, there are a locally projective morphism $g: X^* \to S$ with X^* nonsingular and a bimeromorphic S-morphism $h: X^* \to X$.

4) Suppose that there exist a locally projective morphism $g: Y \to S$ and a generically finite meromorphic S-map $h: X \to Y$. Then f is Moishezon.

Proof. 1) and 3) follows from (1.1.1) and (1.2.1). Let $\mu: \tilde{X} \to X$ be the normalization of X. Since μ is bimeromorphic, f is Moishezon if and only if $f\mu$ is Moishezon. From this 2) follows readily. 4) Changing f under bimeromorphic equivalence we may assume that h is a morphism. Let $h = h_2 h_1$ with $h_1: X \to X^*$ and $h_2: X^* \to Y$ be the Stein factorization of h, where h_1 is a bimeromorphic, and h_2 is a finite, S-morphisms. Since a finite morphism is projective, $gh_2: X^* \to S$ is locally projective by (1.2.1), and hence 4).

(1.6) Less trivial to prove is the following:

PROPOSITION 1. *Let $f: X \to S$ be a Moishezon morphism, and $g: Y \to S$ a proper morphism, of reduced complex spaces. Suppose that there is a generically surjective meromorphic S-map $h: X \to Y$. Then g also is Moishezon.*

Proof. By (1.5) 2) we may assume that Y, and then X and S also, are irreducible. By (1.1.1) and (1.5) 3) we may further assume that f is locally projective, X is nonsingular and h is a morphism. Then there is a dense Zariski open subset V_0 of Y such that V_0 is nonsingular and $h_{V_0}: X_{V_0} \to V_0$ is smooth. Let $\mathscr{L}$ be an invertible sheaf on X which is f_W-ample for some dense Zariski open subset W of S (1.3.2). Restricting V_0 we may assume that $V_0 \subseteqq Y_W$. Then if n is sufficiently large, say, $n \geqq n_0$ for some $n_0 > 0$, there is a dense Zariski open subset V_n of Y such that $V_n \subseteqq V_0$ and $H^1(X_y, \mathscr{L}_y^n) = 0$ for all $y \in V_n$ where $\mathscr{L}_y^n = \mathscr{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{X_y}$. Let $\mathscr{E}_n = h_* \mathscr{L}^n$. Then $\mathscr{E}_n$ is a coherent analytic sheaf on Y which is locally free of rank, say r_n, on V_n (cf. [3, p. 122, Cor. 3.9]). Moreover taking n_0 larger if necessary we may assume that $r_n > 0$ for $n \geqq n_0$. On the other hand, by [20] we can find a proper surjective bimeromorphic morphism $\sigma_n: \tilde{Y}_n \to Y$ such that $\tilde{\mathscr{E}}_n = \sigma_n^* \mathscr{E}/\mathscr{T}_n$, $\mathscr{T}_n$ being the torsion part of $\sigma_n^* \mathscr{E}$, is locally free of rank r_n on $\tilde{Y}_n$. Moreover we can assume that σ_n gives an isomorphism of $\tilde{V}_n = \sigma_n^{-1}(V_n)$ onto V_n. Let $\tilde{g}_n = \sigma_n g: \tilde{Y}_n \to S$ and set $\mathscr{M}_n = \bigwedge^{r_n} \tilde{\mathscr{E}}_n$, where $\bigwedge^{r_n}$ denotes the r_n-th exterior product. Then $\mathscr{M}_n$ is an invertible sheaf on $\tilde{Y}_n$. Let $\alpha_n: \tilde{Y}_n \to \boldsymbol{P}(\tilde{g}_{n*}\mathscr{M}_n)$ be the natural meromorphic S-map from $\tilde{Y}_n$ to the projective fiber space $\boldsymbol{P}(\tilde{g}_{n*}\mathscr{M}_n)$ over S associated to the coherent analytic sheaf $\tilde{g}_{n*}\mathscr{M}_n$ on S (cf. [3, IV, § 1]). Then we show that for a sufficiently large n, α_n is generically finite; then by (1.5) d) the proposition would follow.

For this purpose it is enough to show that for some $n \geqq n_0$, for some $\tilde{y} \in \tilde{Y}_n$ and for some neighborhood U of $s = \tilde{g}_n(y)$ in S, the following holds true; there are sections $\varphi_1, \cdots, \varphi_d \in \Gamma(\tilde{Y}_{n,U}, \mathscr{M}_n)$ such that the meromorphic U-map $\Phi: \tilde{Y}_{n,U} \to U \times \boldsymbol{CP}^{d-1}$ associated to φ_α is holomorphic and locally biholomorphic at $\tilde{y}$, where $\tilde{Y}_n$ is over S by $\tilde{g}_n$. (Note that $\tilde{Y}_n$ is irreducible.) First we take $n, \tilde{y}$ and U in such a way that $\tilde{y} \in \tilde{V}_n$, U is a sufficiently small Stein open neighborhood of s, and that $H^1(X_U, m_y^2\mathscr{L}^n) = 0$, where $y = \sigma_n(\tilde{y})$ and m_y is the maximal ideal of $\mathcal{O}_Y$ at y. Clearly this is possible since $\mathscr{L}$ is f_W-ample and $y \in V_n \subseteqq Y_W$. Then in particular the restriction map $\beta_n: \Gamma(X_U, \mathscr{L}^n) \to \Gamma(X_U, \mathscr{L}^n_{(2)})$ is surjective, as follows from the long exact sequence associated to the short one

$$0 \longrightarrow m_y^2\mathscr{L}^n \longrightarrow \mathscr{L}^n \longrightarrow \mathscr{L}^n_{(2)} \longrightarrow 0$$

where we have put $\mathscr{L}^n_{(2)} = \mathscr{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_X/m_y^2\mathcal{O}_X$. Further since

$$H^1(X_U, m_y\mathscr{L}^n/m_y^2\mathscr{L}^n) \cong H^1(X_U, \mathscr{L}^n \otimes_{\mathcal{O}_Y} m_y/m_y^2) \cong H^1(X_y, \mathscr{L}^n_y) \otimes_{\boldsymbol{C}} m_y/m_y^2 = 0 ,$$

from the short exact sequence $0 \to m_y\mathscr{L}^n/m_y^2\mathscr{L}^n \to \mathscr{L}^n_{(2)} \to \mathscr{L}^n_y \to 0$ we have the exact sequence

$$0 \longrightarrow \Gamma(X_y, \mathscr{L}^u_y) \otimes_{\boldsymbol{C}} m_y/m_y^2 \xrightarrow{\gamma_n} \Gamma(X_U, \mathscr{L}^n_{(2)}) \xrightarrow{\delta_n} \Gamma(X_y, \mathscr{L}^n) \longrightarrow 0 .$$

Fix n and write $r = r_n$. Then take and fix a base $(\bar{\psi}^0_1, \cdots, \bar{\psi}^0_r)$ of $\Gamma(X_y, \mathscr{L}^n_y)$. Let $(y_1, \cdots, y_m)$, $m = \dim Y$, be a local coordinate system around y of Y and $\bar{y}_i$ the residue classes of y_i in $\mathcal{O}_Y/m_y^2\mathcal{O}_Y$. Then we take any base $(\bar{\psi}_1, \cdots, \bar{\psi}_d)$, $d = r(m+1)$, of $\Gamma(X_U, \mathscr{L}^n_{(2)})$ satisfying the following conditions; $\delta_n(\bar{\psi}_i) = \bar{\psi}^0_i$, $1 \leqq i \leqq r$, and $\bar{\psi}_{kr+j} = \gamma_n(\bar{y}_k\bar{\psi}_j)$, $1 \leqq k \leqq m$, $1 \leqq j \leqq r$, where $\bar{y}_k\bar{\psi}_j = \bar{\psi}_j \otimes \bar{y}_k \in \Gamma(X_y, \mathscr{L}_y) \otimes_{\boldsymbol{C}} m_y/m_y^2$. For each $1 \leqq k \leqq d$ take and fix $\psi_k \in \Gamma(X_U, \mathscr{L}^n)$ with $\beta_n(\psi_k) = \bar{\psi}_k$. With respect to the natural identification $\Gamma(X_U, \mathscr{L}^n) \cong \Gamma(Y_U, \mathscr{E}_n) \subseteqq \Gamma(\tilde{Y}_{n,U}, \tilde{\mathscr{E}}_n)$, we consider ψ_i naturally as sections of $\tilde{\mathscr{E}}_n$ on $\tilde{Y}_{n,U}$. Then for any $1 \leqq i_1 \cdots \leqq i_r \leqq d$ define $\varphi_{i_1\cdots i_r} \in \Gamma(\tilde{Y}_{n,U}, \mathscr{M}_n)$ by $\varphi_{i_1\cdots i_r} = \psi_{i_1} \wedge \cdots \wedge \psi_{i_r}$. We claim that these $\varphi_\alpha = \varphi_{i_1\cdots i_r}$ have the desired properties.

Since the problem is local around y and σ_n gives a natural isomorphism of $\tilde{V}_n$ and V_n, in what follows we identify $\tilde{V}_n$ and V_n by σ_n and therefore $\tilde{y}$ with y and $\tilde{\mathscr{E}}_n|_{\tilde{V}_n}$ with $\mathscr{E}_n|_{V_n}$. Further we consider $\mathscr{M}_n|_{\tilde{V}_n}$ as an invertible sheaf on V_n and ψ_i as sections of $\mathscr{E}_n$ on $Y_U \cap V_n$. Now $\psi_1, \cdots, \psi_r$ define a trivialization $\mathscr{E}_n \cong \mathcal{O}_Y^r$ of $\mathscr{E}_n$, and hence also $\mathscr{M}_n \cong \mathcal{O}_Y$ of $\mathscr{M}_n$, in some neighborhood N of y. In particular we may consider each ψ_i (resp. $\varphi_{i_1\cdots i_r}$) as an r-tuple of holomorphic functions (resp. a holomorphic function) on N. Then we have by construction $\psi_i = (0, \cdots, 0, 1, 0, \cdots, 0)$

for $1 \leqq i \leqq r$ where 1 is on the i-th place and $\psi_{kr+j} \equiv (0, \cdots, 0, y_k, 0, \cdots, 0)$ modulo m_y^2, where y_k is on the j-th place. Hence we have $\varphi_{1\cdots r}(y) = \psi_1 \wedge \cdots \wedge \psi_r(0) \neq 0$ and $\varphi_{1\cdots\hat{k}\cdots r(kr+k)} = \psi_1 \wedge \cdots \wedge \hat{\psi}_k \wedge \cdots \wedge \psi_r \wedge \psi_{kr+k} \equiv y_k$ modulo m_y^2 where $\hat{u}$ implies the absense of u. The former implies that Φ is holomorphic at y and the latter implies that Φ is locally biholomorphic at y. Hence our claim is verified. Q.E.D.

(1.7) Let $f: X \to S$ be a Moishezon morphism. Then:

5) For every reduced analytic subspace $X' \subseteqq X$ the induced morphism $f' = f|_{X'}: X' \to S$ is Moishezon.

6) Let $\mu: \tilde{S} \to S$ be a morphism of reduced complex spaces. Then the induced map $f_{\tilde{S}, \mathrm{red}}: X_{\tilde{S}, \mathrm{red}} \to \tilde{S}$ is Moishezon.

7) Let $g: Y \to S$ be another Moishezon morphism. Then $f \times_S g: X \times_S Y \to S$ also is Moishezon.

Proof. Let $g: X^* \to S$ and $h: X^* \to X$ be as in (1.5) 3). Let $Z = h^{-1}(X')$ with reduced structure. Then $g|_Z: Z \to S$ is locally projective and $h|_Z: Z \to X'$ is surjective. Hence by the above proposition f is Moishezon. This proves 5). We show 6). Let g and h be as above. Then h induces a surjective morphism $h_{\tilde{S}, \mathrm{red}}: X^*_{\tilde{S}, \mathrm{red}} \to X_{\tilde{S}, \mathrm{red}}$ over S. Since $g_{\tilde{S}, \mathrm{red}}: X^*_{\tilde{S}, \mathrm{red}} \to \tilde{S}$ is locally projective, 6) also follows from the above proposition. Since $f \times_S g$ is the composition of the natural projection $X \times_S Y \to Y$ and g, 7) follows from (1.5) 1) and 6) above.

§ 2. Morphisms in $\mathscr{C}/S$

(2.1) Definition. Let $g: Y \to S$ be a proper morphism of complex spaces. Then: 1) ([9, Def. 4.1]) g is called *Kähler* if there exist an open covering $\{U_\alpha\}$ of Y and a C^∞ function p_α defined on each U_α such that for each α, p_α is strictly plurisubharmonic when restricted to each fiber of $g|_{U_\alpha}: U_\alpha \to S$ and that $p_\alpha - p_\beta$ is pluriharmonic on each $U_\alpha \cap U_\beta$. 2) g is called *locally Kähler* if for every relatively compact open subset Q of S there exist $\{U_\alpha\}$ and $\{p_\alpha\}$ satisfying the condition as above except that p_α is assumed to be strictly plurisubharmonic only when restricted to each fiber of $g|_{U_\alpha \cap g^{-1}(Q)}: U_\alpha \cap g^{-1}(Q) \to Q$.

In the above definition the real closed (1, 1)-form $\omega_\alpha = \sqrt{-1}\partial\bar{\partial}p_\alpha$, each defined on U_α, patch together to give a global real closed (1, 1)-form ω on Y, which we call a relative Kähler form for g (resp. for g over Q).

(2.2) In the following all the morphisms considered are proper.

1) Every (locally) projective morphism is (locally) Kähler.

2) Let $g: Y \to S$ be a (locally) Kähler morphism and $\alpha: \tilde{S} \to S$ a morphism of complex spaces. Then the induced morphism $g_{\tilde{S}}: Y_{\tilde{S}} \to \tilde{S}$ is (locally) Kähler.

3) Let $f: X \to Y$ and $g: Y \to S$ be locally Kähler morphism of complex spaces. Then the composition $gf: X \to S$ is again locally Kähler. Conversely if gf is (locally) Kähler, then f also is (locally) Kähler.

4) Let $f: X \to S$ and $g: Y \to S$ be locally Kähler morphisms. Then $f \times_S g: X \times_S Y \to S$ is locally Kähler.

Proof. See [9. Lemma 4.4] for 1). We show the former half of 3). Let $Q \Subset Q' \subset S$ and $\tilde{Q} = g^{-1}(Q')$ be as in the proof of (1.2.1) (with Z replaced by S). Let $\omega_{Q'}$ (resp. $\omega_{\tilde{Q}}$) be a relative Kähler form for g over Q' (resp. f over $\tilde{Q}$). Then for all sufficiently large $n > 0$, $\omega_{\tilde{Q}} + nf^*\omega_{Q'}|_{(gf)^{-1}(Q)}$ gives a relative Kähler form for the morphism gf over Q (cf. the proof of [9, Lemma 4.4]). Hence gf is locally Kähler. Since $f \times_S g$ is a composite of the natural projection $X \times_S Y \to Y$ and g, 4) follows from this and 2). The other assertions follow immediately from the definition.

(2.3) Definition. Let S be a reduced complex space. Then we define the category $\mathscr{C}/S$ as follows: An object of $\mathscr{C}/S$ is a proper morphism $f: X \to S$ of reduced complex spaces for which there exist a proper and locally Kähler morphism $g: Y \to S$ and a generically surjective meromorphic S-map $h: Y \to X$ (Notation: $f \in \mathscr{C}/S$); and a morphism in $\mathscr{C}/S$ is a morphism $u: X_1 \to X_2$ of complex spaces with $f_2 u = f_1$ where $f_i: X_i \to S \in \mathscr{C}/S$, $i = 1, 2$.

Remark. 1) Note the deviation from the notation adopted in [9, p. 51]; there we used the notation $\mathscr{C}/S$ for the category loc-$\mathscr{C}/S$ which is defined as follows: An object of loc-$\mathscr{C}/S$ is a proper morphism $f: X \to S$ of complex spaces for which there exists an open covering $\{U_\alpha\}$ of S such that $f_{U_\alpha}: X_{U_\alpha} \to U_\alpha \in \mathscr{C}/U_\alpha$ for each α, with morphisms defined as above. 2) When S is a point, we write $\mathscr{C}$ instead of $\mathscr{C}/S$. In this case the definition coincides with that given in [9, 4.3] except that we consider only reduced spaces here.

(2.4) We shall give some functorial properties of morphisms in $\mathscr{C}/S$ analogous to Moishezon morphisms.

1) Every Moishezon morphism belongs to $\mathscr{C}/S$.

Let $f: X \to S$ and $g: Y \to S$ be proper morphisms of reduced complex spaces. Suppose that $g \in \mathscr{C}/S$. Then:

2) $f \in \mathscr{C}/S$ if and only if there exist a proper and locally Kähler morphism $g^*: Y^* \to S$ with Y^* nonsingular, and a surjective S-morphism $h^*: Y^* \to S$.

3) For every analytic subspace Y' of Y the induced morphism $g|_{Y'}: Y' \to S$ is again in $\mathscr{C}/S$.

4) Suppose that there is a generically surjective meromorphic S-map $h: Y \to X$. Then $f \in \mathscr{C}/S$.

5) Suppose that there is an S-morphism $h: X \to Y$ with $h \in \mathscr{C}/Y$. Then $f \in \mathscr{C}/S$.

6) For any reduced complex space $\tilde{S}$ over S the induced morphism $g: Y_{\tilde{S}, \mathrm{red}} \to \tilde{S}$ is in $\mathscr{C}/\tilde{S}$.

7) Suppose that $f \in \mathscr{C}/S$. Then $f \times_S g: X \times_S Y \to S$ is again in $\mathscr{C}/S$.

Proof. 1) follows from (2.2) 1) and the definition of a Moishezon morphism. The proofs of 2), 3) and 4) are the same as those of 1), 2) and 3) of [9, Lemma 4.6] respectively, using (2.2) instead of [9, Lemma 4.4], and will be omitted.

5) By assumption and by 2) there exist a locally Kähler morphism $\tilde{g}: \tilde{Y} \to S$ (resp. $\tilde{h}: \tilde{X} \to Y$) and a surjective S-(resp. X-)morphism $\alpha: \tilde{Y} \to Y$ (resp. $\beta: \tilde{X} \to X$). Then the natural map $\gamma: \tilde{X} \times_Y \tilde{Y} \to S$ is locally Kähler by (2.2) 4). Moreover there is a natural surjective S-morphism $\tilde{X} \times_Y \tilde{Y} \to X$, which proves 5). Let $\tilde{g}: \tilde{Y} \to S$ and $\alpha: \tilde{Y} \to Y$ be as above. Then $\tilde{Y}_{\tilde{S}} \to \tilde{S}$ is locally Kähler by (2.2) 2) and there is a natural surjective $\tilde{S}$-morphism $\tilde{Y}_{\tilde{S}} \to Y_{\tilde{S}}$. This proves 6). 7) then follows from 5) and 6) as in the proof of (2.2) 4).

§ 3. Irreducibility of the general fiber of a morphism

(3.1) Let $f: X \to Y$ be a finite surjective morphism of reduced complex spaces. Then we call f a *finite* (*ramified*) *covering* if each irreducible component of X is mapped surjectively onto some irreducible component of Y.

LEMMA 1. *Let $\beta: X \to Y$ be a finite covering of reduced complex spaces with Y irreducible. Then there are a normal complex space $\tilde{X}$ and a finite covering $\gamma: \tilde{X} \to Y$ such that the induced morphism $\beta_{\tilde{X}}^*: (X \times_Y \tilde{X})^* \to \tilde{X}$ is biholomorphic to the natural projection $E \times \tilde{X} \to \tilde{X}$, where $(X \times_Y \tilde{X})^*$ is the normalization of $X \times_Y \tilde{X}$, and E is a finite set considered as a 0-dimensional reduced complex space.*

Proof. Replacing X and Y by their normalizations X' and Y' respectively, and then considering separately the finite coverings $\beta_i: X_i \to Y'$ induced by β between the irreducible components X'_i of X' and Y', we infer readily that we may assume that both X and Y are normal and irreducible. Then by the argument in [21, p. 62] we can find a normal complex space $\tilde{X}$, a finite group G of biholomorphic automorphisms of $\tilde{X}$ and a subgroup H of G such that we have the natural isomorphisms $h: X \cong \tilde{X}/H$ and $g: Y \cong \tilde{X}/G$ with $\beta = g^{-1}\pi h$, where $\tilde{X}/H$ and $\tilde{X}/G$ are the quotients of $\tilde{X}$ by H and G respectively endowed with their natural structures of normal complex spaces, and $\pi: X/H \to X/G$ is the natural projection. Then identifying π with β by the above isomorphisms, this implies the lemma as follows. Let Δ be the diagonal of $\tilde{X} \times_{\tilde{X}/G} \tilde{X}$ and let G act on $\tilde{X} \times_{\tilde{X}/G} \tilde{X}$ by $(x_1, x_2) \to (gx_1, x_2)$ for each $g \in G$. Then $\tilde{X} \times_{\tilde{X}/G} \tilde{X} = \bigcup_{g \in G} g\Delta$ so that $(\tilde{X} \times_{\tilde{X}/G} \tilde{X})^* \cong \coprod_{g \in G} g\Delta$ and each $g\Delta$ is mapped isomorphically onto $\tilde{X}$ by the second projection. Accordingly, we have $(\tilde{X}/H \times_{\tilde{X}/G} \tilde{X})^* \cong \coprod_{g \in E} (\hat{\pi} \times \mathrm{id}_{\tilde{X}})(g\Delta) \cong \coprod_{g \in E} g\Delta \cong E \times \tilde{X}$ where $\hat{\pi}: \tilde{X} \to \tilde{X}/H$ is the natural projection and E is any complete set of representatives of G/H in G. Q.E.D.

(3.2) Let $f: X \to S$ be a proper surjective morphism of reduced complex spaces. In what follows the 'general' fiber of f is always considered with respect to the Zariski topology of S. For example 'the general fiber of f is reduced and irreducible' means that X_s is reduced and irreducible for every $s \in U$ for some dense Zariski open subset U of S.

PROPOSITION 2. *Let $f: X \to S$ be as above. Then there exist a finite surjective morphism $\beta: \tilde{S} \to S$ with $\tilde{S}$ reduced, and a reduced analytic subspace $\tilde{X}$ of $X \times_S \tilde{S}$ such that if $f: \tilde{X} \to \tilde{S}$ and $\alpha: \tilde{X} \to X$ are the naturally induced morphisms, then* 1) *the irreducible components of $\tilde{X}$ are mutually disjoint,* 2) *α is bimeromorphic, and in particular every irreducible component of $\tilde{X}$ is mapped bimeromorphically onto an irreducible component of X and* 3) *the general fiber of $\tilde{f}$ is reduced and irreducible. Moreover if f is flat, then we can take β to be a finite covering.*

Proof. Let $\nu: X' \to X$ be the normalization of X and let $f\nu = \beta g$ with $g: X' \to \tilde{S}$ and $\beta: \tilde{S} \to S$ be the Stein factorization of $f\nu: X' \to S$. Then we set $\tilde{X} = (\nu \times g)(X') \subseteq X \times_S \tilde{S}$, and define α and $\tilde{f}$ as above. Then clearly β is finite surjective and 2) is satisfied. We shall show 1). Suppose that $\tilde{X}_i \cap \tilde{X}_j \neq \emptyset$ for some distinct irreducible components $\tilde{X}_i$ and $\tilde{X}_j$ of $\tilde{X}$.

Let $\tilde{x} \in \tilde{X}_i \cap \tilde{X}_j$ be any point and $\tilde{s} = g(\tilde{x})$. Then if X'_i and X'_j are the irreducible components of X' with $(\nu \times g)(X'_i) = \tilde{X}_i$ and $(\nu \times g)(X'_j) = \tilde{X}_j$ respectively, then we have $X'_{i,\tilde{s}} \neq \emptyset$ and $X'_{j,\tilde{s}} \neq \emptyset$. Since $X'_{\tilde{s}}$ is connected by the definition of Stein factorization, this implies that there is some irreducible component $X'_k \neq X'_i$ of X' such that $X'_i \cap X'_k \neq \emptyset$. This is a contradiction since X' is normal. Hence 1) is proved. Then the reducedness of the general fiber of $\tilde{f}$ follows from [9, Lemma 1.5]. So it remains to show that the general fiber of $\tilde{f}$ is irreducible. This in turn follows from that of g, and the latter can be seen as follows. Let $r: X^* \to X'$ be a resolution of X' and $g^* = gr: X^* \to S$. Then there is a dense Zariski open subset V of S such that $g^*_V: X^*_V \to V$ is smooth, and hence irreducible since each fiber of g^* is connected as well as that of g, X' being normal. Hence $X'_{\tilde{s}} = r(X^*_{\tilde{s}})$ are also irreducible for all $\tilde{s} \in V$. Q.E.D.

Remark. In the above proof, to show the irreducibility of the general fiber of g, instead of resolution we can also use the fact that if $h: X \to S$ is a proper morphism with X normal, then the set $\{s \in S; X_s$ is normal and f is flat at each point of $X_s\}$ is dense and Zariski open in S, which can be shown as in [9, Lemmas 1.4, 1.5] starting from a result of [2].

(3.3) We shall show that a general fiber of a proper flat morphism is irreducible if at least one fiber is reduced and irreducible. Though the result is not absolutely necessary for the proof of Theorem, it provides us with a useful criterion for the applicability of Proposition 4 in § 4. First we need some lemmas.

LEMMA 2. *Let $f: X \to Y$ be a proper morphism of complex spaces and $y \in Y$. Then f is flat at each point of X if and only if for any morphism $h: D \to Y$ with $h(0) = y$, the induced morphism $f_D: X_D \to D$ is flat at each point of $X_{D,0}$ where $D = \{t \in \boldsymbol{C}; |t| < 1\}$ is the unit disc and $0 \in D$ is the origin.*

Proof. This is an immediate consequence of the existence of 'platificateur' in [16, Th. 1′] (cf. also [14, Th. 2.4]). We shall also give a direct proof of the lemma in the Appendix.

COROLLARY. *Let $f: X \to Y$ be a proper surjective morphism of reduced and irreducible complex spaces. Let $y \in Y$. Suppose that X_y is reduced and irreducible, and that* $\dim X_y = \dim X - \dim Y$. *Then f is flat at every point of X_y.*

Proof. It suffices to show that for any $h: D \to Y$ with $h(0) = y$, $f_D: X_D \to D$ is flat along $X_{D,0}$. Since $X_{D,0} \cong X_y$ is reduced and irreducible, by Nakayama we may assume that X_D is reduced. Let X_{Di}, $1 \leqq i \leqq m$, be the irreducible components of X_D. Restricting D smaller we may further assume that $f_D(X_{Di}) = D$ or $\{0\}$ for each i. Since $X_{D,0}$ is reduced and irreducible, if $f_D(X_{Di}) = \{0\}$ for some i, we must have $X_{Di} = X_{D,0}$, and i is unique, say $i = m$. Note that since f_D is surjective, $m > 1$. Then for $1 \leqq i < m$ we have $\dim X_{Di,t} \leqq \dim X_{Di,0} < \dim X_{D,0}$, and hence $\dim X_{D,t} < \dim X_{D,0}$, or $\dim X_{h(t)} \leqq \dim X_y$, for $t \neq 0$. On the other hand, our dimensional assumption implies that $\dim X_y = \dim X_{y'}$ for all y' sufficiently near to y since X is irreducible. This is a contradiction. Hence $f(X_{Di}) = D$ for all i so that f_D is flat along $X_{D,0}$. Q.E.D.

LEMMA 3. *Let $f: X \to S$ be a proper flat morphism of complex spaces. Suppose that S is reduced and irreducible. Suppose further that for some $o \in S$, X_o is reduced and pure dimensional. Then X also is reduced and pure dimensional.*

Proof. Since X_o is reduced, by Nakayama and the flatness of f we infer readily that X is reduced (cf. the proof of [9, Lemma 1.4]). To show the pure dimensionality it suffices to show that there is no irreducible component, say X_1, of X such that if q_1 is the dimension of the general fiber of the induced map $X_1 \to S$, then $q_1 < q_o = \dim X_o$. Suppose that such an X_o exists. Let $S_k(f) = \{x \in X; \operatorname{codh}_x X_{f(x)} \leqq k\}$. Then $S_k(f)$ is an analytic subset of X by [2]. Hence $S_{q_1}(f) \supseteq X_1$, X_1 being reduced, and so $\dim S_{q_1}(f)_o \geqq q_1$. Since $S_{q_1}(f)_o = \{x \in X_o; \operatorname{codh}_x X_o \leqq q_1\}$, this implies that on X_o there is a nonzero holomorphic function φ with support of dimension $\leqq q_1$ (cf. [3, p. 76, Cor. 5.2 d) $\to$ b)] applied to $\mathscr{F} = \mathcal{O}_X$ and $d = q_1$). This is a contradiction to the reducedness and pure dimensionality together of X_o. Hence X is pure dimensional. Q.E.D.

PROPOSITION 3. *Let $f: X \to S$ be a proper flat and surjective morphism of complex spaces. Suppose that S is reduced and irreducible. Suppose further that for some $o \in S$ the fiber X_o is reduced and irreducible. Then the general fiber of f is irreducible.*

Proof. By Lemma 3 X is reduced and pure dimensional. Apply Proposition 2 to f and obtain a proper surjective morphism $\tilde{f}: \tilde{X} \to \tilde{S}$ and finite coverings $\alpha: \tilde{X} \to X$ and $\beta: \tilde{S} \to S$ with $\beta \tilde{f} = f\alpha$ satisfying the properties stated in the proposition. Let $\beta^{-1}(o) = \{\tilde{o}_1, \cdots, \tilde{o}_m\}$. Then it suffices to show that β is locally biholomorphic at each $\tilde{o}_k$ and that $m = 1$. In fact, then β must be bimeromorphic and hence the irreducibility of the general fiber of f follows from that of $\tilde{f}$ together with the surjectivity of α. Now to prove the above assertion first we note that since f is flat, X is pure dimensional and S is irreducible, every fiber of f is pure dimensional of dimension $q = \dim X_o$, and, further, since X_o is reduced and irreducible, every irreducible component of X contains X_o. Combining this with 2) of Proposition 2 and the fact that $\alpha|_{\tilde{X}_{\tilde{s}}}: \tilde{X}_{\tilde{s}} \to X_{\beta(\tilde{s})}$ is an embedding for each $\tilde{s} \in \tilde{S}$, we get that α induces the isomorphisms $\tilde{X}_{\tilde{o}_k} \cong X_o$ for all k. This then implies that β is locally biholomorphic at each $\tilde{o}_k$, for otherwise $\tilde{X}_{\tilde{o}_k}$ is nonreduced at each of its points since so is $\tilde{S}$ at $\tilde{o}_k$ already. By Corollary above it also follows from $\tilde{X}_{\tilde{o}_k} = X_o$, that $\tilde{f}$ is flat in a neighborhood of $\tilde{X}_{\tilde{o}_k}$ for each k. Now we need the following result from [9, Cor. 3.3]; let $g: Y \to Z$ be a proper flat morphism of reduced complex spaces. Suppose that every fiber of g has pure dimension q which is independent of z. For any $z \in Z$ let $Y_{z,i}$, $i = 1, \cdots, n = n(z)$, be the irreducible components of $Y_{z,\mathrm{red}}$ and $m_{z,i}$ the multiplicities of Y_z along $Y_{z,\mathrm{red}}$ (cf. [9, 3.1]). Then for any continuous (q, q)-form χ on X the function

$$\lambda_\chi(z) = \sum_{i=1}^{n} m_{z,i} \int_{Y_{z,i}} \chi$$

is a continuous function on Z. Using this we shall now show that $m = 1$. Let ω be any Hermitian $(1, 1)$-form on X (cf. [9, Def. 1.2]) and set $\chi = \omega \wedge \cdots \wedge \omega$ (q-times) and $\tilde{\chi} = \alpha^*\chi$. Then $\lambda_\chi(s)$ (resp. $\lambda_{\tilde{\chi}}(\tilde{s})$) are functions which are defined on S (resp. $\tilde{S}$) and continuous in a neighborhood of o (resp. $\beta^{-1}(o)$) by the result quoted above. Let U (resp. $\tilde{U}_k$) be a neighborhood of o (resp. $\tilde{o}_k$) such that β induces isomorphisms $\beta_k: \tilde{U}_k \cong U$ for each k. For any $s \in U$ we write $\tilde{s}_k = \beta_k^{-1}(s)$. On the other hand, since α is bimeromorphic, there is a dense Zariski open subset V of U such that for each $s \in V$, $\tilde{X}_{\tilde{s}_i}$, $1 \leqq i \leqq m$, are reduced and irreducible and $\alpha(\tilde{X}_{\tilde{s}_k}) \neq \alpha(\tilde{X}_{\tilde{s}_\ell})$ if $k \neq \ell$. Hence noting that $X_s = \bigcup_k \tilde{X}_{\tilde{s}_k}$ we have $\lambda_\chi(s) = \sum_{k=1}^{m} \lambda_{\tilde{\chi}}(\tilde{s}_k)$ for every $s \in V$. Now take a sequence $\{s^{(i)}\}$ of points of V converging to o in U. Then since λ_χ (resp. $\lambda_{\tilde{\chi}}$) is continuous at o (resp. $\tilde{o}_k$), we get that $\lambda_\chi(o) = \lim_i \lambda_\chi(s^{(i)}) = \lim_i \sum_{k=1}^{m} \lambda_{\tilde{\chi}}(\tilde{s}_k^{(i)}) = \sum_{k=1}^{m} \lambda_{\tilde{\chi}}(\tilde{o}_k)$. Since

$$\lambda_\chi(o) = \int_{X_o} \chi = \int_{\tilde{X}_{\tilde{o}_k}} \tilde{\chi} = \lambda_{\tilde{\chi}}(\tilde{o}_k) ,$$

this implies that $m = 1$, for $\int_{X_o} \chi > 0$. Q.E.D.

§ 4. Moishezonness of π_A in a special case

(4.1) Let $f: X \to S$ be a proper morphism of complex spaces. Let $\beta_{X/S}: D_{X/S} \to S$ be the relative Douady space of X over S parametrizing analytic subspaces of X contained in the fibers of f (cf. [17], [9]). Let $\rho_{X/S}: Z_{X/S} \to D_{X/S}$ be the corresponding universal family, so that there is a natural embedding $Z_{X/S} \subseteqq D_{X/S} \times_S X$ with $\rho_{X/S}$ induced by the natural projection $p_1: D_{X/S} \times_S X \to D_{X/S}$. We denote by $\pi_{X/S}$ the natural morphism $Z_{X/S} \to X$ induced by the projection $p_2: D_{X/S} \times_S X \to X$. Then $\pi_{X/S}$, restricted to each fiber of $\rho_{X/S}$, is an embedding. Let $\alpha: \tilde{S} \to S$ be a morphism of complex spaces with $\tilde{S}$ reduced and $Z \subseteqq \tilde{S} \times_S X$ a subspace. Let $\rho: Z \to \tilde{S}$ be the natural projection. If ρ is flat, then we call ρ a flat family of subspaces of X over S parametrized by $\tilde{S}$. In the general case, by Frisch [8] there is a dense Zariski open subset W of $\tilde{S}$ such that $\rho_W: Z_W \to W$ is flat. (In what follows we use this result of Frisch without further reference.) Then there is a unique S-morphism $\tau: W \to D_{X/S}$ such that ρ_W is isomorphic to the map induced from $\rho_{X/S}$ via τ, where W is over S by $\alpha|_W$. We call such a map τ simply the *universal S-map associated to ρ_W.*

Now we recall the following consequence of Hironaka's flattening theorem [14] which is of frequent use in the sequel.

LEMMA 4. *The universal S-map τ extends to a meromorphic S-map $\tau^*: \tilde{S} \to D_{X/S,\,\mathrm{red}}$. In particular if α is proper, then the closure of $\tau(W)$ in $D_{X/S,\mathrm{red}}$ is an analytic subspace of $D_{X/S,\,\mathrm{red}}$ which is proper over S.*

Proof. See [9, Lemma 5.1].

(4.2) In the case of a projective morphism a special way of constructing $D_{X/S}$ is available by Grothendieck [12], [13]; what we need here from his construction is the following:

LEMMA 5. *Let $f: X \to S$ be a projective morphism and $\beta_{X/S}: D_{X/S} \to S$ be the relative Douady space of X over S. Let Q be any relatively compact open subset of S and A any connected component of $\beta_{X/S}^{-1}(Q)_{\mathrm{red}}$. Then the induced morphism $h: A \to Q$ is projective.*

Proof. Let Q' be any relatively compact open subset of S with $Q \Subset Q'$. Let $\mathscr{L}$ be an $f_{Q'}$-very ample invertible sheaf on X such that $f_*\mathscr{L}$ is locally free on Q'. So we have an Q'-embedding $j: X_{Q'} \to \boldsymbol{P}(f_*\mathscr{L})_{Q'}$ with $\mathscr{L} \cong j^*\mathcal{O}_{\boldsymbol{P}}(1)$, $\boldsymbol{P} = \boldsymbol{P}(f_*\mathscr{L})$. Then replacing S by Q' we may assume that $X = \boldsymbol{P}(\mathscr{E})$ for some locally free coherent analytic sheaf $\mathscr{E}$ on S. Now for

$d \in D_{X/S}$ write $Z_d = Z_{X/S,d}$ and consider $Z_d \subsetneqq X_{\beta(d)} \cong \boldsymbol{CP}^{r-1}$ by $\pi_{X/S}$, where $\beta = \beta_{X/S}$ and $r = \operatorname{rank} \mathscr{E}$. For every $d \in D_{X/S}$ define a polynomial $P_d = P_d(n)$ in n by $P_d = \sum_{i \geqq 0}(-1)^i H^i(Z_d, \mathscr{O}_{Z_d}(n))$. Then P_a is independent of $a \in A$ (cf. [3]) and we set $P_A = P_a$ for any $a \in A$. Set $\tilde{A} = \{d \in D_{X/S,\,\mathrm{red}};\ P_d = P_A\}$. Then A is a connected component of $\tilde{A}_Q$. Hence it suffices to show that $\tilde{A}_Q$ is projective over Q. In fact, the proof of [13, IX, Théorème 1.1] (and [12, 221, § 3]) shows that for each point $s \in S$ there exists a neighborhood $s \in U$ in S and an integer $\nu_0 = \nu_0(s)$ such that for all $\nu \geqq \nu_0$ the natural map $\beta^* f_* \mathscr{O}_X(\nu) \cong p_{1*} p_2^* \mathscr{O}_X(\nu) \to \rho_{X/S*} \mathscr{O}_{Z_{X/S}}(\nu)$, $\mathscr{O}_{Z_{X/S}}(\nu) = \pi_{X/S}^* \mathscr{O}_X(\nu)$, is surjective on $\tilde{A}_U$ and the corresponding morphism $\tilde{A}_U \to \operatorname{Grass}_m (f_* \mathscr{O}_X(\nu))_U$ is a closed embedding over U, where $\operatorname{Grass}_m (f_* \mathscr{O}_X(\nu))$ is the Grassmann variety of locally free quotients of $f_* \mathscr{O}_X(\nu)$ of rank m, where $m = m(\nu) = \operatorname{rank}(\rho_{X/S*} \mathscr{O}_{Z_{X/S}}(\nu))$ [13]. Hence for all sufficiently large ν, $\tilde{A}_Q$ can be embedded in $\operatorname{Grass}_m (f_* \mathscr{O}_X(\nu))_Q$ over Q and hence is projective over Q. Q.E.D.

(4.3) Let $f: X \to S$ be a morphism of complex spaces. Let $\beta_{X/S}: D_{X/S} \to S$, $\rho_{X/S}: Z_{X/S} \to D_{X/S}$ and $\pi_{X/S}: Z_{X/S} \to X$ be as in (4.1). For any locally closed analytic subspace A of $D_{X/S,\,\mathrm{red}}$ we shall denote by $\rho_A: Z_A \to A$ the restriction of $\rho_{X/S}$ to $Z_A = \rho_{X/S}^{-1}(A)$ and $\pi_A: Z_A \to X$ the S-morphism induced by $\pi_{X/S}$, where Z_A is over S by $\beta_{X/S}\rho_A$.

Proposition 4. *Let $f: X \to S$ be a proper morphism of complex spaces and Q a relatively compact open subset of S. Let A be a reduced and irreducible analytic subspace of $\beta_{X/S}^{-1}(Q)$ which is proper over Q and for which the general fiber of ρ_A is reduced and irreducible. Then π_A is Moishezon.*

Proof. Changing the notation we set $S = Q$, $X = X_Q$ and $f = f_Q$ so that A is an analytic subspace of $D_{X/S}$. (The original X, S and f do not appear explicitly in the following, so no confusion may arise). We first note that from our assumption it follows immediately that Z_A is reduced and irreducible. Consider the Z_A-embedding $j: Z_A \times_A Z_A \subsetneqq Z_A \times_X (X \times_S X)$ defined by $j(z_1, z_2) = (z_1, \pi_A(z_1), \pi_A(z_2))$ where $Z_A \times_A Z_A$ (resp. $X \times_S X$) is over Z_A (resp. X) with respect to the projection to the first factor. Note that j is in fact obtained by the composition $Z_A \times_A Z_A \subsetneqq Z_A \times_A (A \times_S X) \cong Z_A \times_S X \cong Z_A \times_X (X \times_S X)$ where the isomorphisms are all natural ones. On the other hand, let $\Delta = \Delta_{X/S}$ be the diagonal of $X \times_S X$ and $\mathscr{I}$ the sheaf of ideals of Δ in $X \times_S X$. Let $\Delta_{(n)} = (\Delta, \mathscr{O}_{X \times_S X}/\mathscr{I}^{n+1})$ be the n-th infinitesimal neighborhood of Δ in $X \times_S X$, and $\beta_n: \Delta_{(n)} \to X$ be induced by the projection $X \times_S X \to X$ to the first factor. Then β_n are finite, and hence projective, morphisms. Let $\delta_n: D_{(n)} \to X$ with $D_{(n)} = D_{\Delta_{(n)}/X}$ be the relative Douady spaces associated to β_n. Then for any connected component $D_{(n),k}$ of $D_{(n)}$ the induced morphism $D_{(n),k} \to X$ is projective by Lemma 5 since f is proper.

Let $Y_{(n)} = (Z_A \times_A Z_A) \cap (Z_A \times_X \Delta_{(n)}) \subseteqq Z_A \times_X \Delta_{(n)}$ and $\gamma_n: Y_{(n)} \to Z_A$ be the natural S-morphisms induced by the projections $Z_A \times_X \Delta_{(n)} \to Z_A$, where the intersection is taken in $Z_A \times_X (X \times_S X)$ considering $Z_A \times_A Z_A$ as a subspace of $Z_A \times_X (X \times_S X)$ via j. Then γ_n are finite surjective morphisms, the fibers over $z \in Z_A$ being naturally identified with the subspace $B_{z,n} = \pi_A(Z_{A,a}) \cap x_{(n)}$ of $x_{(n)} = (x, \mathcal{O}_X/m_x^{n+1})$ where $a = \rho_A(z)$, $x = \pi_A(z)$ and m_x is the maximal ideal of $\mathcal{O}_X$ at x. Now for each n there is a dense Zariski open subset U_n of Z_A such that $\gamma_{n,U_n}: Y_{(n),U_n} \to U_n$ is flat, so that it may be considered as a flat family of subspaces of $\Delta_{(n)}$ over X parametrized by U_n. Let $\tau_n: U_n \to D_{(n)}$ be the universal X-map associated to γ_{n,U_n} (cf. (4.1)). Then by Lemma 4 τ_n extends to a meromorphic X-map $\tau_n^*: Z_A \to D_{(n)}$ and the closure E_n of $\tau_n(U_n)$ in $D_{(n)}$ is analytic in $D_{(n)}$ and is proper over S as well as Z_A.

Now we shall show that (*) *τ_n^* are bimeromorphic X-maps onto its image for all sufficiently large n.* Then since π_A is proper and the images of τ_n^* are contained in some $D_{(n),k}$, Z_A being irreducible, this would imply that $\pi_A: Z_A \to X$ is Moishezon by (1.7) 5), completing the proof of the proposition. To show (*) we first observe the following: ($''$) If $z, z' \in U_m \cap U_n$ and if $m \leqq n$, then $\tau_n(z) = \tau_n(z')$ implies that $\tau_m(z) = \tau_m(z')$. In fact, for $z \in U_n$, $\tau_n(z)$ is the point of $D_{(n)}$ corresponding to the subspace $B_{z,n}$ of $x_{(n)}$ defined above, and that $B_{z,n} = B_{z',n}$ clearly implies that $B_{z,m} = B_{z',m}$. This shows ($''$). Now since Z_A is irreducible, for each n there exist an integer $d_n \geqq 0$ and a dense Zariski open subset V_n of U_n such that $\dim_z \tau_n^{-1}\tau_n(z) = d_n$ for all $z \in V_n$. Then we see that $d_n \leqq d_m$ for $m \leqq n$ by ($''$). Hence there are integers $n_0 > 0$ and $d \geqq 0$ such that $d_n = d$ for all $n \geqq n_0$.

Next we show that $d = 0$. Let W be a dense Zariski open subset of A such that $Z_{A,a}$ is reduced and irreducible for all $a \in W$. Let $V = \bigcap_n V_n$. Then V is everywhere dense in Z_A. Suppose now that $d > 0$. Then there exist points $z, z' \in V \cap \rho_A^{-1}(W)$, $z \neq z'$, such that z' belongs to an irreducible component C of $\tau_{n_0}^{-1}\tau_{n_0}(z)$ containing z. (In particular $\pi_A(z) = \pi_A(z')$.) Then since both $Z_{A,\rho_A(z)}$ and $Z_{A,\rho_A(z')}$ are reduced and irreducible, there is an integer $n_1 \geqq n_0$ such that $B_{z,n_1} \neq B_{z',n_1}$, or equivalently, $\tau_{n_1}(z) \neq \tau_{n_1}(z')$. Hence

$\tau_{n_1,C} = \tau_{n_1}|_{C\cap U_{n_1}}$ is nontrivial, i.e., the fibers of $\tau_{n_1,C}$ have dimension $< d = \dim C$. Note here that $C \cap U_{n_1} \supseteqq C \cap V \neq \emptyset$. On the other hand, by ($''$) $\tau_{n_1,C}^{-1}\tau_{n_1}(z)$ is one of the irreducible components of $\tau_{n_1}^{-1}\tau_{n_1}(z)$ at z and hence there is an irreducible component $C' \subsetneqq C$ of $\tau_{n_1}^{-1}\tau_{n_1}(z)$ containing z, so that $\dim_{z''} \tau_{n_1}^{-1}\tau_{n_1}(z'') < d$ for some $z'' \in C'$. This implies that $d_{n_1} < d$ by the upper semi-continuity of the function $\dim_z \tau_{n_1}^{-1}\tau_{n_1}(z)$, $z \in U_{n_1}$, which is a contradiction. Hence we get that τ_n is generically finite for $n \geqq n_0$.

Thus for each $n \geqq n_0$, there exist an integer $k_n > 0$ and a dense Zariski open subset Q_n of E_n contained in $\tau_n(U_n)$ such that $\tau_{n,Q_n}: \tau_n^{-1}(Q_n) \to Q_n$ is an unramified covering of degree k_n. Here one needs to recall that τ_n extends to a meromorphic X-map from Z_A to E_n which are both proper over X. Then again by ($''$), $k_n \leqq k_m$ if $n \geqq m$ so that $k_n = k$ for all $n \geqq n_2$ for some $k \geqq 1$ and $n_2 \geqq n_0$. We show that $k = 1$. Let $\tilde{Q} = \bigcap_n \tau_n^{-1}(Q)$ which is everywhere dense in Z_A. For $\tilde{q} \in \tilde{Q}$, $\tau_n^{-1}\tau_n(\tilde{q})$, as a set, is independent of $n \geqq n_2$. Suppose that $k > 1$. Then there are points z, $z' \in \tilde{Q} \cap \rho_A^{-1}(W)$, $z \neq z'$, such that $\tau_{n_2}(z) = \tau_{n_2}(z')$. Then by the same argument as above we can find $n > n_2$ such that $\tau_n(z) \neq \tau_n(z')$, implying that $k_n < k$, since $z, z' \in \tilde{Q}$. This contradicts our choice of n_2. Hence $k = 1$, i.e., τ_n^* is X-bimeromorphic onto its image for all $n \geqq n_2$, and (*) is proved. Q.E.D.

Remark. A meromorphic map $g: Y \to Y'$ of reduced complex spaces is called generically light if there is a dense Zariski open subset $U \subseteqq \Gamma$ such that $\dim_\gamma q^{-1}q(\Gamma) = 0$ for every $\gamma \in U$ where Γ is the graph of g and $q: \Gamma \to Y'$ is the natural projection (cf. (1.1)). Then the above proof shows that even in the general case where A may not be proper over S, there is a generically light meromorphic X-map $\lambda: Z_A \to B$ of complex spaces over X with B projective over X.

§ 5. Reduction of the general case and proof of Theorem

(5.1) We use the notation of (4.3).

PROPOSITION 5. *Let $f: X \to S$ be a morphism of complex spaces. Let A be an irreducible component of $D_{X/S,\,\mathrm{red}}$ which is proper over S and for which Z_A is reduced. Then there exist* 1) *reduced and irreducible analytic subspaces B_i, $i = 1, \cdots, n$, of $D_{X/S}$ such that B_i is proper over S and the general fiber of $\rho_{B_i}: Z_{B_i} \to B_i$ is reduced and irreducible,* 2) *a reduced and irreducible analytic subspace B of $\hat{B} = B_1 \times_S \cdots \times_S B_n$ and* 3) *a generically* *surjective meromorphic S-map* $h: B \to A$.

Proof. We write $Z = Z_A$ and $\rho = \rho_A$. Let $\tilde{\rho}: \tilde{Z} \to \tilde{A}$, $\alpha: \tilde{Z} \to Z$ and $\beta: \tilde{A} \to A$ be as in Proposition 2 applied to $f = \rho$. In particular $\tilde{Z} \subseteq \tilde{A} \times_A Z$ with $\tilde{\rho}$ induced by the natural projection $\tilde{A} \times_A Z \to \tilde{A}$. Further since ρ is flat, we may assume that β is a finite covering. Moreover for each $s \in A$, $Z_s = \bigcup_{\tilde{s} \in \beta^{-1}(s)} \alpha(\tilde{Z}_{\tilde{s}})$ by 2) of the proposition. Then we apply Lemma 1 to β and obtain a normal complex space A', a finite covering $\gamma: A' \to A$ and an A'-isomorphism $\lambda: (\tilde{A} \times_A A')^* \cong E \times A'$, where $(\tilde{A} \times_A A')^*$ is the normalization of $\tilde{A} \times_A A'$ and E is a finite set considered as a zero dimensional analytic space. Write $A^* = (\tilde{A} \times_A A')^*$. Let $\rho^*: Z^* \to A^*$ be the pull-back of $\tilde{\rho}$ to A^* with respect to the natural projection $A^* \to \tilde{A}$. Identifying E with $\{1, \cdots, n\}$, $n = \sharp E$, in a certain fixed way and A^* with $E \times A'$ via λ, we write for each i, $Z_i^* = \rho^{*-1}(\{i\} \times A')$, and $\rho_i^* = \rho^*|_{Z_i^*}: Z_i^* \to \{i\} \times A' = A'$ and define $\pi_i^*: Z_i^* \to X$ to be the natural map. We thus get the following commutative diagram

$$\begin{tikzcd}
\coprod_i Z_i^* = Z^* \arrow[r] \arrow[d, "\coprod \rho_i^*"', "\rho^*"] & \tilde{Z} \arrow[d, "\tilde{\rho}"] \arrow[dr, "\alpha"] & \\
 & & Z \arrow[r, "\pi_A"] \arrow[d, "\rho"] & X \\
\coprod_i (A' \times \{i\}) = A^* \arrow[r] \arrow[dr, "\delta"'] & \tilde{A} \arrow[dr, "\beta"] & \\
 & A' \arrow[r, "\gamma"'] & A
\end{tikzcd}$$

Let U be any dense Zariski open subset of A' such that $\rho_{i,U}^*: Z_{i,U}^* \to U$ are flat for all i. Then by the definition of $\tilde{Z}$ we may consider $\rho_{i,U}^*$ naturally as a flat family of subspaces of X over S parametrized by U. Let $\tau_i: U \to D_{X/S}$ be the universal S-map associated to $\rho_{i,U}^*$ and $\tau = \prod_i \tau_i: U \to D_{X/S} \times_S \cdots \times_S D_{X/S}$ (n-times). Let B (resp. B_i) be the closure of $\tau(U)$ (resp. $\tau_i(U)$) in $\hat{D} = D_{X/S} \times_S \cdots \times_S D_{X/S}$ (resp. $D_{X/S}$). Then by Lemma 4 (cf. also (1.1)) B and B_i are reduced analytic subspaces of $\hat{D}$ and $D_{X/S}$ respectively which are proper over S. They are irreducible since so is U, and we have $B \subseteq \hat{B} = B_1 \times_S \cdots \times_S B_n$ and $\dim B \leqq \dim A$. Moreover since by 3) of Proposition 2 together with the definition of τ_i, $Z_{X/S,d}$ is reduced and irreducible for each $d \in \tau_i(U)$ (after restricting U if necessary), the general fiber of $\rho_{B_i}: Z_{B_i} \to B_i$ is reduced and irreducible. (For instance, since $\tau_i(U)$ is everywhere dense in B_i it follows that Z_{B_i} is reduced and irreducible. Then we can apply Proposition 3.)

Now let $\rho_B^{(i)}: Z_B^{(i)} \to B$ be the pull-back of ρ_{B_i} with respect to the map

$B \to B_i$ induced by the natural projection $\hat{B} \to B_i$. Take the union $\check{Z}_B = \bigcup_i Z_B^{(i)}$ in $X \times_S B$ ($\supseteqq Z_B^{(i)} = Z_{B_i} \times_{B_i} B$). Let $\psi: \check{Z}_B \to B$ be the natural projection and take any dense Zariski open subset V of B such that $\psi_V: \psi^{-1}(V) \to V$ is flat. Let $\tau': V \to D_{X/S}$ be the universal S-map associated with ψ_V. Let $\bar{\pi}: \check{Z}_B \to X$ be the natural map induced by the projection $X \times_S B \to X$. Then from the construction above we have in X the equality $\bar{\pi}(\check{Z}_{B,b}) = \pi_A(Z_{A,\gamma(u)})$ for each $b \in \tau(U)$ where $u \in U$ is any point with $\tau(u) = b$. In fact by 2) of Proposition 2 we have $\pi_A(Z_{A,\gamma(u)}) = \bigcup_{\tilde{a} \in \beta^{-1}\gamma(u)} \tilde{\pi}(\tilde{Z}_a) = \bigcup_{a^* \in \delta^{-1}(u)} \pi^*(Z^*_{a^*}) = \bigcup_i \pi^*_i(Z^*_{i,u}) = \bar{\pi}(\check{Z}_{B,b})$ in X, where $\tilde{\pi} = \pi_A \alpha$ and π^* is the composite of $\tilde{\pi}$ and the natural map $Z^* \to \tilde{Z}$. This implies that $\tau'\tau|_{\tau^{-1}(V)} = j\gamma|_{\tau^{-1}(V)}$ where $j: A \to D_{X/S}$ is the natural inclusion. In particular $\tau'(V)$ contains $\gamma(\tau^{-1}(V))$ and hence a nonempty Zariski open subset of A since $\tau^{-1}(V) \neq \emptyset$. Thus the closure $\tau'(V)^-$ of $\tau'(V)$ in $D_{X/S}$, which is an analytic subset of $D_{X/S}$ by Lemma 4, contains A so that $\dim B \geqq \dim A$. Combining with the opposite inequality noted above we have $\dim B = \dim A$, and thus $\tau'(V)^- = A$. Hence $h = \tau'$ is a generically surjective meromorphic S-map from B onto A. Q.E.D.

Remark. In fact the above h is bimeromorphic as one shows readily.

(5.2) THEOREM. *Let $f: X \to S$ be a proper morphism of reduced complex spaces, and Q a relatively compact open subset of S. Let $\beta: D_{X/S} \to S$ be the relative Douady space of X over S. Suppose that $f \in \mathscr{C}/S$ (resp. is Moishezon). Then for any irreducible component A of $\beta^{-1}(Q)_{\mathrm{red}}$ such that Z_A is reduced, the induced morphism $\beta|_A: A \to Q$ is proper and again belongs to $\mathscr{C}/S$ (resp. is Moishezon).*

Proof. We shall write $f \in \mathscr{M}/S$ if f is Moishezon. First we show that A is proper over S. Since $f \in \mathscr{C}/S$ (resp. $\mathscr{M}/S$), there is a proper and locally Kähler (resp. locally projective) morphism $g: Y \to S$ of complex spaces and a surjective S-morphism $h: Y \to X$ (cf. (2.4) 2) and (1.5) 3)). Let $X' = f^{-1}(Q)$, $f' = f|_{X'}: X' \to Q$, $Y' = g^{-1}(Q)$ and $g' = g|_{Y'}: Y' \to Q$. Then g' is Kähler (cf. (2.2) 1)). Hence by [9, Theorem 4.3] g' has property BP, i.e., every irreducible component of the relative Barlet space $B(Y'/Q)$ (cf. [9]) is proper over Q. Then by [9, Prop. 4.8] f' also has property BP, which in turn implies that f' has property $\bar{D}P$, i.e., every irreducible component of $\bar{D}_{X'/Q}$ is proper over Q, by [9, Prop. 3.4], where $\bar{D}_{X'/Q}$ is the union of those irreducible components D_γ of $D_{X'/Q,\mathrm{red}}$ such that $Z_\gamma = Z_{D_\gamma}$ are reduced and pure dimensional. Then by [9, Lemma 3.5] and the remark following it (where Z_α and D_α should read D_α and S respectively), this further implies that the given A is proper over S since Z_A is reduced.

Now apply Proposition 5 to our A and obtain $B \subseteqq B_1 \times_Q \cdots \times_Q B_n$ as in that proposition (with S replaced by Q). In particular, B_i are proper over Q, the general fiber of $\rho_{B_i}: Z_{B_i} \to B_i$ is reduced and irreducible, and there is a generically surjective meromorphic S-map $B \to A$. The first two facts, together with Proposition 4, shows that $\pi_{B_i}: Z_{B_i} \to X'$ is Moishezon, $1 \leqq i \leqq n$. Hence $f'\rho_{B_i}: Z_{B_i} \to Q \in \mathscr{C}/Q$ (resp. $\mathscr{M}/Q$) by (2.4) 5) (resp. (1.5) 1)). Then by (2.4) 3) 4) and 7) (resp. (1.6) and (1.7) 5) 7)) the natural map $B_i \to Q$, and hence $B \to Q$ also, belong to $\mathscr{C}/Q$ (resp. are Moishezon). Finally by (2.4) 4) (resp. (1.6)) $\beta|_A: A \to Q \in \mathscr{C}/Q$ (resp. $\mathscr{M}/Q$).

Q.E.D.

Remark. Taking S to be a point and then setting $S = Q$, we obtain the theorem stated in the introduction.

Appendix

We shall give a direct proof of Lemma 2, in § 3.

Let $D = \{t \in \boldsymbol{C}; |t| < 1\}$ be the unit disc. For any complex space Y and $y \in Y$ we denote by $S(Y, y)$ the set of morphisms $h: D \to Y$ with $h(0) = y$. Let $f: X \to Y$ be a morphism of complex spaces and $y \in Y$. Then for any $h \in S(Y, y)$ we write X_h for $X \times_Y D$ and f_h (resp. p_h) for the natural projection $X_h \to D$ (resp. $X_h \to X$). Further for any coherent analytic sheaf $\mathscr{F}$ on X we denote by $\mathscr{F}_h$ the $\mathscr{O}_{X_h}$-module $p_h^*\mathscr{F}$. Then Lemma 2 is a special case of the following:

Proposition. *Let $f: X \to Y$ be a morphism of complex spaces and $\mathscr{F}$ a coherent analytic sheaf on X. Let $x \in X$ and $y = f(x)$. Suppose that Y is reduced. Then the following conditions are equivalent*: 1) *$\mathscr{F}$ is f-flat at x.* 2) *For every $h \in S(Y, y)$, $\mathscr{F}_h$ is f_h-flat at $x_h = (x, 0)$.*

Proof. 2) is clearly a consequence of 1). So suppose that 2) is true. We use an analytic analogue of the technique due to Raynaud and Gruson [19, 2.1]. Let $S(\mathscr{F})$ be the support of $\mathscr{F}$ considered as the analytic subspace of X defined by the ideal sheaf of annihilators of $\mathscr{F}$. Then we proceed by induction on $n = \dim_x (X_y \cap S(\mathscr{F}))$. First replacing X by $S(\mathscr{F})$ if necessary we may assume that $X = S(\mathscr{F})$, so that $n = \dim_x X_y$. Then there is a neighborhood U of x in X and a commutative diagram of complex spaces

$$
\begin{tikzcd}
U \arrow[r, "\tau"] \arrow[dr, "f"'] & Y \times C^n \arrow[d, "p_1"] \\
& Y
\end{tikzcd}
$$

such that τ is finite at x (cf. [7, 3.3]). Then since $\mathscr{F}_x$ and $(\tau_*\mathscr{F})_{\tau(x)}$ are isomorphic as $\mathcal{O}_{Y,y}$-modules, we can replace f and $\mathscr{F}$ by p_1 and $\tau_*\mathscr{F}$ respectively. Thus we may assume that $X = Y \times V$ with Y Stein and V a polydisc in C^n containing the origin 0, $x = (y, 0) \in Y \times C^n$ and $f: X \to Y$ is the natural projection.

SUBLEMMA. *$\mathscr{F}$ is locally free at some point of X_y.*

Proof. For any $a \in X$ we set $d(a) = \dim_{C} \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X/m_a\mathcal{O}_X$ where m_a is the maximal ideal of $\mathcal{O}_X$ at a. Then $d(a)$ is upper semicontinuous with respect to the Zariski topology. In particular, if we set $d_0 = \min\{d(a); a \in X\}$, then the set $U_0 = \{a \in A; d(a) = d_0\}$ is Zariski open in X and $\mathscr{F}$ is locally free on U_0. We may assume that $x \in \overline{U}_0$, the closure of U_0. Similarly if we put $d_{y0} = \min\{d(a); a \in X_y\}$, then $U_{y0} = \{a \in X_y; d(a) = d_{y0}\}$ is dense and Zariski open in $X_y \cong V$. We show that $d_0 = d_{y0}$. Take $h \in S(Y, y)$ in such a way that $p_h^{-1}(U_0) \neq \emptyset$. By our assumption $\mathscr{F}_h$ is f_h-flat at x_h and hence f_h-flat in some neighborhood W of x_h. Since D is smooth of dimension 1, this is equivalent to saying that $\mathscr{H}^0_{X_{h,0}}(\mathscr{F}_h) = 0$ on W. On the other hand, the latter implies that $\dim S_n(\mathscr{F}_h) < n$ (a special case of a theorem of Trautmann [3, p. 66]) where $S_n(\mathscr{F}) = \{u \in X; \operatorname{codh}_u \mathscr{F} \leqq n\}$, codh denoting the cohomological dimension. Hence for the general point $w \in X_{h,0}$, $\operatorname{codh}_w \mathscr{F}_h = n + 1$, i.e., $\mathscr{F}_h$ is locally free at w. Thus if U_h is the maximal dense Zariski open subset of X_h on which $\mathscr{F}_h$ is locally free, then $U_h \cap X_y \neq \emptyset$. Hence if r is the rank of $\mathscr{F}_h$ on U_h, then taking any $a' \in U_h \cap p_h^{-1}(U_{y0}) \subseteqq X_{h,0}$ and $w' \in U_h \cap p_h^{-1}(U_0)$ we have $d_{y0} = d(p_h(a')) = d_h(a') = r = d_h(w') = d_0$, where d_h is defined for $\mathscr{F}_h$ in the same way as $d(a)$. Hence $U_{y0} \subseteqq U_0$ and $\mathscr{F}$ is locally free at each point of $U_{y0} \subseteqq X_y$.

Q.E.D.

By Sublemma there exists a $v \in V$ such that $\mathscr{F}$ is free of rank, say r, as an $\mathcal{O}_X$-module at $x' = (y, v)$. We take $e_1, \cdots, e_r \in \Gamma(X, \mathscr{F})$ which give free generators of $\mathscr{F}$ at x'. This is possible since X is Stein. Let $\alpha: \mathcal{O}_X^{\oplus r} \to \mathscr{F}$ be the map defined by e_i, and $\mathscr{K}$ (resp. $\mathscr{P}$) the kernel (resp. cokernel) of α. Since α is isomorphic in a neighborhood of x', $\mathscr{K} = \mathscr{P} = 0$ at x'. In particular they are torsion $\mathcal{O}_X$-modules. Hence as a subsheaf of a free sheaf on the reduced space X, $\mathscr{K}$ must vanish identically on X. Thus we get an exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}_X^{\oplus r} \xrightarrow{\alpha} \mathscr{F} \longrightarrow \mathscr{P} \longrightarrow 0$$

on X. Now we show that 2) is satisfied also for $\mathscr{P}$. For any $h \in S(Y, y)$, pulling back $(*)$ to X_h we obtain the following exact sequence on X

$$0 \longrightarrow \mathcal{O}_X^{\oplus r} \xrightarrow{\alpha_h} \mathscr{F}_h \longrightarrow \mathscr{P}_h \longrightarrow 0 .$$

In fact, by the same reasoning as above, firstly α_h is isomorphic at $x'_h = (x', 0) \in X_h$ and then injective on the whole X_h since $X_h \cong V \times D$ is reduced. Thus to show the flatness of $\mathscr{P}_h$ it is enough to show that for every integer $k \geqq 1$ the natural map $\alpha_h^{(k)}: \mathcal{O}_{X_h}/n^k\mathcal{O}_{X_h} \to \mathscr{F}_h/n^k\mathscr{F}_h$ induced by α_h is injective, where n is the maximal ideal of $\mathcal{O}_{D,0}$. In fact, by the flatness of $\mathscr{F}_h$ this implies that $\mathrm{Tor}_1^R(\mathscr{P}, R/a) = 0$ for all ideals a of $R = \mathcal{O}_{D,0}$. Since α_h is isomorphic at x'_h, so are $\alpha_h^{(k)}$ for all $k > 0$. Thus if $\mathscr{K}_k = \mathrm{Ker}\,\alpha_h^{(k)}$, $\mathscr{K}_k = 0$ at X'_h. Thus the support of $\mathscr{K}_k$ is a proper analytic subset of $X_{h,0}$. Since $\mathscr{K}_k \subseteqq \mathcal{O}_{X_h}/n^k\mathcal{O}_{X_h}$, it follows that $\mathscr{K}_k = 0$. Hence $\mathscr{P}_h$ is f_h-flat, and 2) is verified for $\mathscr{P}$.

Now we finish the proof as follows. Recall that $\mathscr{P}_{x'} = 0$ so that $\dim_x (X_y \cap S(\mathscr{P})) < n$. If $n = 0$, then $\mathscr{P} = 0$ at x so that $\mathscr{F}$ is free at x. So suppose that $n > 0$. Then by induction and 2) for $\mathscr{P}$, $\mathscr{P}$ is f-flat at x. Then the flatness of $\mathscr{F}$ follows from $(*)$. Q.E.D.

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