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ON THE COHOMOLOGY OF CONGRUENCE SUBGROUPS OF SYMPLECTIC GROUPS

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§1. Introduction

This paper is concerned with the cohomology at "infinity" (in the sense of Harder [4], [5]) of a congruence subgroup of the symplectic group $G = Sp(2\ell, \boldsymbol{R})$. G is the subgroup of $GL(2\ell, \boldsymbol{R})$ consisting of matrices g satisfying ${}^t gJg = J$ where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and I is the $\ell \times \ell$ identity matrix. We consider G as the real points of the algebraic group $\underline{G} = Sp(2\ell)$ defined over $\boldsymbol{Q}$. Let p be a prime not equal to 2 and Γ be the kernel of the natural map

$$Sp(2\ell, \boldsymbol{Z}) \longrightarrow Sp(2\ell, \boldsymbol{Z}/p^r\boldsymbol{Z}) .$$

We assume that r is chosen large enough so that Γ is torsion free.

We are interested in the Eilenberg-Maclane cohomology groups $H^*(\Gamma, \boldsymbol{C})$ of Γ (cf: Borel [2]). It is well-known that $H^*(\Gamma, \boldsymbol{C}) \approx H^*(X/\Gamma, \boldsymbol{C})$ where X is the symmetric space of maximal compact subgroups of G and G acts in a natural way on X. In [3] Borel and Serre constructed a compactification $\overline{X}/\Gamma$ of X/Γ having the property that $H^*(\overline{X}/\Gamma, \boldsymbol{C}) \approx H^*(X/\Gamma, \boldsymbol{C})$. $\overline{X}/\Gamma$ is a manifold with corners and is a union of subsets $e'(P)$ (see [3] p. 476) where P runs over the Γ conjugacy classes of parabolic $\boldsymbol{Q}$-subgroups of $\underline{G}$. Let

$$r\colon H^*(\overline{X}/\Gamma, \boldsymbol{C}) \longrightarrow H^*(\partial(\overline{X}/\Gamma), \boldsymbol{C}) \tag{1.1}$$

be the homomorphism induced by the map $\partial(\overline{X}/\Gamma) \to \overline{X}/\Gamma$. The general programme is to investigate the existence of a subspace $H^*_{\text{inf}}(\Gamma, \boldsymbol{C})$ of $H^*(\Gamma, \boldsymbol{C}) \approx H^*(\overline{X}/\Gamma, \boldsymbol{C})$ which restricts isomorphically onto $\operatorname{Im} r$. The

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elements of $H^*_{\mathrm{inf}}(\Gamma, \boldsymbol{C})$ can be considered as describing the cohomology at "infinity".

Fix a standard Borel subgroup B of G (see § 1). Let T be the maximal torus in B and m be the dimension of the unipotent radical of B. Let D be a set of representatives for all the Γ-conjugacy classes of minimal parabolic subgroups of G which are defined over $\boldsymbol{Q}$. Choose D so that $B \in D$. Following G. Harder we use Langlands' theory of Eisenstein series to construct elements of $H^*_{\mathrm{inf}}(\Gamma, \boldsymbol{C})$ from cohomology classes in $\oplus_{P \in D} H^*(e'(P), \boldsymbol{C})$. For precise statement see § 10.

In the case of $\boldsymbol{Q}$-rank one algebraic groups, Harder has shown that cohomology classes constructed this way exhaust the entire space $H^*_{\mathrm{inf}}(\Gamma, \boldsymbol{C})$ (cf: [4], [5]). Schwermer has obtained similar results for $SL(3)$ and $SL(n)$ (cf: [11], [12]).

Notation. For an algebraic subgroup $\underline{G}$ of $GL(n)$ defined over a field k and r a subring of k, $\underline{G}(r)$ denotes the subgroup of $\underline{G}(k)$ (the group of k-rational points of $\underline{G}$) consisting of $n \times n$ matrices with entries in r and determinants units in r.

For a prime p of the rational numbers $\boldsymbol{Q}$ we write $\boldsymbol{Q}_p$ (resp. $\boldsymbol{Z}_p$) for the p-adic numbers (resp. integers). A (resp. A_0) denotes the adeles (resp. the finite adeles) of $\boldsymbol{Q}$.

§ 2.

Let $\mathfrak{g}$ be the Lie algebra of G; $\mathfrak{g}$ consists of $2\ell \times 2\ell$ real matrices X satisfying the condition: ${}^tXJ + JX = 0$. Let $\mathfrak{a}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of real matrices

$$H = \begin{pmatrix} h_1 & & & & & & & \\ & h_2 & & & & & & \\ & & \ddots & & & & & \\ & & & h_\ell & & & & \\ & & & & -h_1 & & & \\ & & & & & -h_2 & & \\ & & & & & & \ddots & \\ & & & & & & & -h_\ell \end{pmatrix}.$$

For $1 \leq j \leq \ell$, define elements ε_j in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ by $\varepsilon_j(H) = h_j$. Then the root system of the pair $(\mathfrak{g}, \mathfrak{a})$ is given by

$$\Psi = \{\pm \varepsilon_i \pm \varepsilon_j \colon i \neq j,\ 1 \leq i,\ j \leq \ell\} \cup \{\pm 2\varepsilon_j \colon 1 \leq j \leq \ell\}$$

and the set of simple roots is

$$\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3 \cdots, \varepsilon_{\ell-1} - \varepsilon_\ell, 2\varepsilon_\ell\}$$

we also write $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$ for $1 \leq j \leq \ell - 1$ and $\alpha_\ell = 2\varepsilon_\ell$.

For $\mu \in \mathfrak{a}^*$, $H \in \mathfrak{a}$ we denote with $\langle H, \mu \rangle$ the value $\mu(H)$. The restriction of the Killing form $(\cdot, \cdot)$ of $\mathfrak{g}$ to $\mathfrak{a}$ is nonsingular and hence one can define a map $\mu \to H(\mu)$ of $\mathfrak{a}^*$ onto $\mathfrak{a}$ by the relation

$$(H, H(\mu)) = \langle H, \mu \rangle , \tag{2.1}$$

for all $H \in \mathfrak{a}$. The mapping defines a nonsingular bilinear form on $\mathfrak{a}^*$ given by

$$(\mu, \lambda) = \langle H(\mu), \lambda \rangle . \tag{2.2}$$

Denote by $\mathfrak{g}_\alpha$ the root subspace of $\mathfrak{g}$ corresponding to the root α. Let Ψ^+ (resp. Ψ^-) be the set of positive (resp. negative) roots with respect to the order determined by Δ. Let

$$\mathfrak{n} = \sum_{\alpha \in \Psi^+} \mathfrak{g}_\alpha$$

and $\mathfrak{b} = \mathfrak{a} + \mathfrak{n}$, the Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Let B be the Borel subgroup of G with Lie algebra $\mathfrak{b}$, N be the unipotent radical of B and T the maximal torus of G in B. Then $B = T.N$. Let K be the intersection of G and the special orthogonal group $SO(2\ell, \boldsymbol{R})$. K is a maximal compact subgroup of G consisting of matrices

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$$

satisfying the relations

$${}^tXX + {}^tYY = 1, \qquad {}^tXY = {}^tYX .$$

Moreover $G = K.B = KAN$ (Iawasawa decomposition), where A is the identity component of $T = \underline{T}(\boldsymbol{R})$.

X can be identified with $K\backslash G$ and we have a principal fibration

$$\pi\colon G/\Gamma \longrightarrow X/\Gamma$$

with structure group K. Let x_0 be the point in X fixed by K. Then we can identify the tangent space at x_0 with $\mathfrak{b}$ and the tangent bundle of X/Γ is the bundle induced (via π) by adjoint representation Ad of K on $\mathfrak{b}$ ([5] p. 131).

Let $\Omega^m(X/\Gamma)$ be the vector space of smooth $\boldsymbol{C}$-valued differential m-forms on X/Γ. Then the cohomology groups $H^*(X/\Gamma, \boldsymbol{C})$ are canonically isomorphic to the cohomology groups of the de Rham complex $(\Omega^*(X/\Gamma), d)$ (cf: [5], [9]). Moreover there is a natural identification between the space $\Omega^m(X/\Gamma)$ and the space of smooth functions

$$\phi\colon\ G/\Gamma \longrightarrow \mathrm{Hom}\,(\Lambda^m \mathfrak{b}, \boldsymbol{C})$$

which satisfy

$$\phi(kg) = \Lambda^m\ \mathrm{Ad}^*\,(k)\phi(g), \quad g \in G, \quad k \in K \tag{2.3}$$

where Ad* is the dual representation to the adjoint representation Ad of K on B (cf: [5] § 1).

Matsushima-Murakami ([9]) defined an Laplacian operator Δ on the complex $\Omega^*(X/\Gamma)$. A form $\omega \in \Omega^*(X/\Gamma)$ is then said to be harmonic if $\Delta\omega = 0$. On the other hand, the universal enveloping algebra $U(g_C)$ of $\mathfrak{g}_C = \mathfrak{g} \otimes \boldsymbol{C}$ operates as an algebra of differential operators on the smooth functions on G/Γ with values in $\mathrm{Hom}\,(\Lambda^m\mathfrak{b}, \boldsymbol{C})$ ([6] chap I § 2). The Casimir operator C defined with respect to the Killing form lies in the centre $\mathscr{Z}$ of $U(\mathfrak{g})$ and it sends smooth functions satisfying (2.3) into smooth functions satisfying the same conditions. Moreover according to the lemma of Kuga ([9] § 6) we have for all smooth m-forms $\phi\colon G/\Gamma \to \mathrm{Hom}\,(\Lambda^m\mathfrak{b}, \boldsymbol{C})$ the formula

$$\Delta\phi = -\,C\phi\,. \tag{2.4}$$

§ 3.

Let $\underline{G}(A)$ be the adele group of $\underline{G}$ ([14]).

Let

$$\Gamma_p = \{(\gamma_q) \in \underline{G}(A) \,|\, \gamma_p \in G(\boldsymbol{Z}_p),\ \gamma_p \equiv 1 \bmod p^r,\ \text{and}\ \gamma_q = 1\ \text{if}\ q \neq p\}$$
$$K_0 = \textstyle\prod_p \underline{G}(\boldsymbol{Z}_p)$$
$$K_r = \Gamma_p \textstyle\prod_{q \neq p} \underline{G}(\boldsymbol{Z}_q)$$
$$\underset{\sim}{G} = K_r \backslash K_0\,.$$

Write $\underset{\sim}{B}$ (resp. $\underset{\sim}{T}, \underset{\sim}{N}$) for the image of $\underline{B}(A_0) \cap K_0$ (resp. $\underline{T}(A_0) \cap K_0$, $(\pm 1.\underline{N}(A_0)) \cap K_0$) in $\underset{\sim}{G}$. Then it is a well-known consequence of the strong approximation theorem that the Γ-conjugacy classes of Borel $\boldsymbol{Q}$-subgroups (Γ as defined in § 1) corresponds bijectively with the Cartesian product $X \times Y$ where $X = \underset{\sim}{B} \backslash \underset{\sim}{G}$ and

$$Y = K_0 \cap \underline{B}(A_0)\backslash\underline{B}(A_0)/\underline{B}(\boldsymbol{Q}) .$$

One can then deduce the following

LEMMA 3.1. *Let χ be a unitary character of $\underset{\sim}{T}$ satisfying*

$$\chi(t) = 1 \tag{3.1}$$

for all t in the image of $\underline{T}(\boldsymbol{Q}) \cap K_r$ in $\underset{\sim}{T}$.

Let V_χ be the complex vector space of $\boldsymbol{C}$-valued functions f on $\underset{\sim}{G}$ which are right invariant under $\underset{\sim}{N}$ and satisfy

$$f(kt) = \chi(t)f(k) \tag{3.2}$$

for $k \in \underset{\sim}{G}$ and $t \in \underset{\sim}{T}$. Then there exists an isomorphism

$$\oplus_{P \in D} H^*(e'(P), \boldsymbol{C}) \rightrightarrows H^*(e'(B), \boldsymbol{C}) \otimes (\oplus_\chi V_\chi) . \tag{3.3}$$

(The direct sum on the right is taken over all the unitary characters χ of $\underset{\sim}{T}$ satisfying (3.1).*)*

The explicit form of this isomorphism is not needed here and so we will not reproduce here the proof of this lemma (which follows the same lines as Satz 5.7 and p. 40 of Schwermer [12]). As a consequence of Lemma 3.1, an element of $\oplus_{P \in D} H^*(e'(P), \boldsymbol{C})$ can be identified with $\omega \otimes \sum f_\chi$ for some $\omega \in H^*(e'(B), \boldsymbol{C})$ and $f_\chi \in V_\chi$. Next we seek to represent ω by a function on $G/B \cap \Gamma$.

§ 4.

According to Proposition 9.4 of Borel-Serre [3] we have

$$e'(B) = N/N \cap \Gamma .$$

Thus

$$H^*(e'(B), \boldsymbol{C}) = H^*(N/N \cap \Gamma, \boldsymbol{C}) . \tag{4.1}$$

The right hand side of (4.1) can be computed by using the de Rham complex $\Omega^*(N/N \cap \Gamma, \boldsymbol{C})$ and we can identify $\Omega^m(N/N \cap \Gamma, \boldsymbol{C})$ with the space of smooth functions $N \to \mathrm{Hom}(\Lambda^m \mathfrak{n}, \boldsymbol{C})$ which are right invariant under $N \cap \Gamma$. (Note that the tangent bundle of $N/N \cap \Gamma$ is trivial.) Thus a cohomology class of $H^m(e'(B), \boldsymbol{C})$ can be represented by a smooth function ϕ on N with values in $\mathrm{Hom}(\Lambda^m \mathfrak{n}, \boldsymbol{C})$. By composing with the embedding

$$\mathrm{Hom}(\Lambda^m \mathfrak{n}, \boldsymbol{C}) \longrightarrow \mathrm{Hom}(\Lambda^m \mathfrak{b}, \boldsymbol{C}) ,$$

we can think of ϕ as taking values in Hom $(\Lambda^m \mathfrak{b}, \boldsymbol{C})$. For $\lambda \in \mathfrak{a}_C^*$, we extend ϕ to a function

$$\phi_\lambda\colon G/B \cap \Gamma \longrightarrow \mathrm{Hom}\,(\Lambda^m \mathfrak{b}, \boldsymbol{C})$$

by

$$\phi_\lambda(g) = \Lambda^m \,\mathrm{Ad}^*\,(k)(\phi(1))\xi_{-\lambda-\rho}(a) \tag{4.2}$$

if $g = kan \in G = KAN$, $\rho = 1/2 \sum_{\alpha \in \Psi^+} \alpha$ and $\xi_{-\lambda-\rho}$ is the character on $\underline{T}$ associated to $-\lambda-\rho$.

§ 5.

Next we want to extend ϕ_λ to a function defined on $\underline{G}(A)$.

First we note that $\underline{G}(\boldsymbol{Q})$ is embedded as a discrete subgroup of $\underline{G}(A)$ and

$$\Gamma = \underline{G}(\boldsymbol{Q}) \cap K_r\,.$$

It is well-known that

$$\underline{G}(A) = K \cdot K_0 \cdot \underline{B}(A) \tag{5.1}$$

and

$$G/\Gamma = K_r \backslash \underline{G}(A)/\underline{G}(\boldsymbol{Q})\,. \tag{5.2}$$

(See for example [1]). In particular this means that any function on $K_r \backslash \underline{G}(A)/\underline{G}(\boldsymbol{Q})$ is determined by its restriction to G/Γ.

A function f in V_χ (see § 3) can be thought of as a function on $\underset{\sim}{G}$ right invariant under $\underset{\sim}{N}$. K_0 acts on the vector space $\boldsymbol{C}(\underset{\sim}{G})$ of complex valued functions on $\underset{\sim}{G}$ by left translation, $\underset{\sim}{L}$:

$$\underset{\sim}{L}(k)f(g) = f(k^{-1}g), \quad k \in K_0, \quad g \in \underset{\sim}{G}\,.$$

Let $\mathscr{P}_\chi$ be the projection from $\boldsymbol{C}(\underset{\sim}{G})$ to V_χ. Then $L(k)f = \mathscr{P}_\chi \underset{\sim}{L}(k)f$ defines an action of K_0 on V_χ. We can give $\boldsymbol{C}(\underset{\sim}{G})$ an inner product such that $L(k)$ is an unitary operator for all k.

The unitary character χ on $\underset{\sim}{T}$ can be trivially extended to an unitary character on

$$\underline{T}(\boldsymbol{R}) \cdot \underline{T}(A_0) \cap K_r \backslash \underline{T}(A)/\underline{T}(\boldsymbol{Q})\,.$$

We use the same symbol χ for the extended character. Note that if $t = (t_p) \in \underline{T}(A)$, $t_q \in \underline{T}(\boldsymbol{Z}_q)$ for $q \neq p$ and $t_p = 1$, then $\chi(t) = 1$.

By § 3 every element of $\oplus_{P\in D} H^m(e'(P), \boldsymbol{C})$ corresponds to a $[\phi]\otimes\sum_\chi f_\chi$ for some cohomology class $[\phi]$ of $H^m(e'(B), \boldsymbol{C})$ represented by ϕ and some $f_\chi \in V_\chi$. For $\lambda\in\mathfrak{a}_C^*$, we associate to $[\phi]\otimes f_\chi$ a function

$$\Phi_\chi\colon\ \underline{G}(A)/\underline{B}(\boldsymbol{Q}) \longrightarrow \mathrm{Hom}\,(\Lambda^m\mathfrak{b}, \boldsymbol{C})\otimes V_\chi$$

given by

$$\Phi_\chi(g) = (\Lambda^m\ \mathrm{Ad}^*\,(k)\phi(1))\otimes(L(k_0)f_\chi)\chi(t)\xi_{-\lambda-\rho}(t) \tag{5.3}$$

if $g=kk_0tn\in\underline{G}(A)=KK_0\underline{T}(A)\underline{N}(A)$. Here, ξ_λ is the character of $\underline{T}$ associated to λ and ξ_λ defines a character on $\underline{T}(A)$. We shall write Φ for Φ_χ unless specified otherwise. The following lemma is a straightforward consequence of the definition.

LEMMA 5.1. *For* $g=(g_q)\in\underline{G}(A)$ *and* $t\in\underline{T}(A)$ *we have*

(i) $\Phi(gt)=\Phi(g)\chi(t)\xi_{-\lambda-\rho}(t)$.

(ii) *If* $g_q=1$ *for all finite prime* q, *then*

$$\Phi(g)=\phi_\lambda(g_\infty)\otimes f_\chi\ .$$

(iii) *If* $g_{q'}=1$ *for all primes (including* ∞*)* q' *except one* $q\neq p$ *and* $g_q=k_qt_qn_q\in\underline{G}(\boldsymbol{Q}_q)=\underline{G}(\boldsymbol{Z}_q)\underline{T}(A_q)\underline{N}(A_q)$ *then*

$$\Phi(g)=\chi(t_q)\xi_{-\lambda-\rho}(t_q)(\phi(1)\otimes f_\chi)\ .$$

(iv) *If* $g_q=1$ *for all primes (including* ∞*) except the prime* p *and* $g_p=k_pt_pn_p$, *then*

$$\Phi(g)=(\phi(1)\otimes L(k_0)f_\chi)\chi(t_p)\xi_{-\lambda-\rho}(t_p)$$

where k *is the adele* $(1,\cdots,1,k_p,1,\cdots,1)$. *Moreover, we have*

$$\|\Phi(g)\|\le|\xi_{-\lambda-\rho}(t_p)|\,\|f_\chi\|\|\phi(1)\|\ .$$

§ 6.

We recall some results on differential operators.

Let W be the Weyl group of $(\mathfrak{g},\mathfrak{a})$. For $w\in W$, let $\sharp w$ denote the number of elements in the set $w\Psi^-\cap\Psi^+$. For any nonnegative integer m put

$$W(m)=\{w\in W\colon\ \sharp w=m\}\ .$$

The adjoint representation induces a representation of $\mathfrak{a}$ on $\mathrm{Hom}(\Lambda^*\mathfrak{n},\boldsymbol{C})$ which commutes with the coboundary operator (cf: [7] § 5.7) and so gives rise to a representation of $\mathfrak{a}$ on $H^*(\mathfrak{n}, \boldsymbol{C})$. For any weight μ of this representation, let $H^*(\mathfrak{n})^\mu$ be the space of all classes in $H^*(\mathfrak{n}, \boldsymbol{C})$ whose weight is μ. Kostant [7] proved that

$$H^*(\mathfrak{n}, \boldsymbol{C}) = \oplus H^*(\mathfrak{n})^\mu \tag{6.1}$$

where μ satisfies

$$(\rho, \rho) = (\rho + \mu, \rho + \mu) \tag{6.2}$$

and each $H^*(\mathfrak{n})^\mu$ is irreducible. Moreover

$$H^m(\mathfrak{n}, \boldsymbol{C}) = \oplus_{w \in W(m)} H(\mathfrak{n})^{w\rho - \rho} . \tag{6.3}$$

The vector space $\mathrm{Hom}\,(\Lambda^m \mathfrak{n}, \boldsymbol{C})$ can be considered as the space of smooth functions $N \to \mathrm{Hom}\,(\Lambda^m \mathfrak{n}, \boldsymbol{C})$ which are invariant under N. Therefore we have an embedding

$$\mathrm{Hom}\,(\Lambda^* \mathfrak{n}, \boldsymbol{C}) \longrightarrow \Omega^*(N/N \cap \Gamma, \boldsymbol{C})$$

(cf, § 4). It is an easy consequence of the theorems of van Est [13] that the above embedding induces isomorphism on cohomology:

$$H^*(\mathfrak{n}, \boldsymbol{C}) \approx H^*(N/N \cap \Gamma, \boldsymbol{C}) \tag{6.4}$$

(cf: [5] Theorem 2.2). Corresponding to (6.1) we have the decomposition

$$H^*(e'(B), \boldsymbol{C}) = \oplus H(e'(B))^\mu . \tag{6.5}$$

The elements of $H(e'(B))^\mu$ are called cohomology classes of weight μ. In particular if w_0 is the element of the Weyl group satisfying $w_0 \Psi^- = \Psi^+$ then $\sharp(w_0) = \dim \mathfrak{n}$ and (6.3) yields

$$H^m(e'(B), \boldsymbol{C}) = H(e'(B), \boldsymbol{C})^{-2\rho}, \qquad m = \dim \mathfrak{n}. \tag{6.6}$$

LEMMA 6.1.

(i) *Every cohomology class in* $H^m(e'(B), \boldsymbol{C})$ *can be represented by a harmonic differential form* ϕ.

(ii) *If* ϕ *is chosen as in* (i) *and* Φ_χ *is the function defined by* (5.3) *then the Eisenstein series*

$$E(g, \Phi_\chi) = \textstyle\sum_{\underline{G}(\boldsymbol{Q})/\underline{B}(\boldsymbol{Q})} \Phi_\chi(g\gamma)$$

is a "smooth" function on $G(A)$ *and is holomorphic* (*as a function of* λ) *in the domain defined by the condition*

$$\mathrm{Re}\,(\lambda, \alpha) > (\rho, \alpha) \qquad \textit{for all } \alpha \in \Psi^+ .$$

Moreover, $E(g, \Phi_\chi)$ has meromorphic continuation into the entire $\mathfrak{a}_C^*$.

(iii) *If ϕ is chosen as in* (i) *and $E(g, \Phi_\chi)$ is holomorphic at λ then $E(g, \Phi_\chi)$ defines a differential form on X/Γ. Moreover if ϕ is of weight μ, then we have*

$$\Delta E(g, \Phi_\chi) = ((\mu + \rho, \mu + \rho) - (\lambda, \lambda))E(g, \Phi_\chi) . \tag{6.7}$$

Proof. (i) follows trivially from the fact that $e'(B) = N/N \cap \Gamma$ is compact. If ϕ is harmonic, then by Kuga's lemma (§ 2) ϕ is an eigenfunction of the Casimir operator. Moreover ϕ is trivially a cusp form, so we can apply the theorem of Borel-Garland ([2] Theorem 6.2) to conclude that ϕ is an automorphic form in the sense of Harish-Chandra, Langlands ([6], [8]).

(ii) now follows from standard results on Eisenstein series (cf: [6] Chap. 11 § 2, [8] Chap. 4 and Appendix II). It is clear from the definition of f_χ and (5.3) that $E(g, \Phi_\chi)$ is a function on $K_r\backslash \underline{G}(A)/\underline{G}(\boldsymbol{Q})$. Thus it defines, via (5.2) a function on G/Γ with values in $\mathrm{Hom}(\Lambda^m\mathfrak{b}, \boldsymbol{C}) \otimes V_\chi$ and by § 2 is a differential form on X/Γ. The formula (6.7) for the Laplacian operator is a trivial consequence of Kuga's lemma.

Suppose that under the map (3.3) the element ω of $\oplus_{P\in D} H^*(e'(P), \boldsymbol{C})$ corresponds to $[\phi] \otimes \sum_\chi f_\chi$ where ϕ is chosen to be harmonic. Let $E(g, \omega)$ be $\sum_\chi E(g, \Phi_\chi)$. Define the constant term of $E(g, \omega)$ by

$$E^B(g, \omega) = \textstyle\sum_\chi E^B(g, \Phi_\chi)$$

where

$$E^B(g, \Phi_\chi) = \int_{\underline{N}(A)/\underline{N}(\boldsymbol{Q})} E(gn, \Phi_\chi)dn . \tag{6.8}$$

Then Harder ([5]) has proved the following lemma.

LEMMA 6.2. *If at $\lambda = \lambda_0$ $E(g, \Phi_\chi)$ is holomorphic and $dE^B(g, \Phi_\chi) = 0$, then the value of $E(g, \omega)$ at $\lambda = \lambda_0$ is a closed form. Moreover $E(g, \omega)$ and $E^B(g, \omega)$ represents the same cohomology class on the boundary $\partial(\overline{X}/\Gamma)$.*

Next we calculate $E^B(g, \Phi_\chi)$.

§ 7.

For $\alpha \in \Psi$, let $\underline{G}_\alpha$ be the derived group of the centralizer in $\underline{G}$ of the connected component of the kernel of ξ_α. Let $\underline{G}_\alpha(\boldsymbol{R}) = K_\alpha A_\alpha N_\alpha$ be the Iwasawa decomposition (compatible with that of $\underline{G}(\boldsymbol{R})$); $N_\alpha = \underline{N}_\alpha(\boldsymbol{R})$ where

$\underline{N}_\alpha$ is a one parameter subgroup in $\underline{G}$. For $w \in W$, let

$$\underline{N}^w = \prod_{\substack{\alpha \in \Psi^+ \\ w^{-1}\alpha \notin \Psi^+}} \underline{N}_\alpha .$$

Then we have the Bruhat decomposition

$$\underline{G}(\boldsymbol{Q}) = \bigcup_{w \in W} \underline{N}^w(\boldsymbol{Q}) r_w \underline{B}(\boldsymbol{Q})$$

where r_w belongs to the group of $\boldsymbol{Q}$-rational points of the normalizer of $\underline{T}$. Moreover r_w can be chosen in K. We shall simply write w for r_w. The following lemma is an easy consequence of the Bruhat decomposition (see for example [8] p. 85 and 277)

LEMMA 7.1. *For $g \in \underline{G}(A)$, let*

$$c(w, \lambda)\Phi(g) = \int_{\underline{N}^w(A)} \Phi(gnw)dn . \tag{7.1}$$

Then

$$E^B(g, \Phi) = \sum_{w \in W} c(w, \lambda)\Phi(g) . \tag{7.2}$$

The integral in (7.1) actually gives the effect of the linear transformation $c(w, \lambda)$ on the function Φ. For our purposes the exact space of functions (see [5] p. 149) on which $c(w, \lambda)$ acts is not important. However it is known that $c(w, \lambda)$ satisfies the following functional equation ([8] p. 120)

$$c(w_1 w_2, \lambda) = c(w_1, w_2\lambda)c(w_2, \lambda) \tag{7.3}$$

for $w_1, w_2 \in W$. Since W is generated by the reflections w_α for $\alpha \in \Delta$, the functional equation (7.3) allows us to restrict our attention to $c(w_\alpha, \lambda)\Phi(g)$ for $\alpha \in \Delta$. In fact it is sufficient to calculate $c(w_\alpha, \lambda)\Phi(t)$ for $t \in \underline{T}(A)$. In this case we have

$$\begin{aligned} c(w, \lambda)\Phi(t) &= \int_{\underline{N}^w(A)} \Phi(tnt^{-1}tw)dn \\ &= \xi_{\delta(w)}(t) \int_{\underline{N}^w(A)} \Phi(ntw)dn \end{aligned}$$

where we have changed the variable once and used the fact that

$$d(tnt^{-1}) = \xi_{\delta(w)}(t)dn$$

with $\delta(w) = -\sum \alpha$ for $\alpha \in \Psi^+$ and $w^{-1}(\alpha) \notin \Psi^+$. Put ${}^w\chi(t) = \chi(w^{-1}tw)$. Using Lemma 5.1 (i), we get

$$\Phi(ntw) = \Phi(nw\, w^{-1}tw) = \Phi(nw)^{w}\chi(t)\xi_{-w\lambda-w\rho}(t)\ .$$

We also have

$$\rho = w\rho - \delta(w)\ .$$

Putting together these formulas, we get

$$c(w, \lambda)\Phi(t) = \left(\int_{\underline{N}^{w}(A)} \Phi(nw)dn\right)^{w}\chi(t)\xi_{-\rho-w\lambda}(t)\ . \tag{7.4}$$

Put $w_{\alpha j} = w_j$ and $\underline{N}_j = \underline{N}_{\alpha_j}$ for $\alpha_j \in \Delta$. Then $\underline{N}^{w}j = \underline{N}_j$. Choose a Haar measure $dx = \prod_p dx_p$ on A such that dx_∞ is the usual Euclidean measure on $\boldsymbol{R}$ and for all finite primes q, the volume of $\boldsymbol{Z}_q$ with respect to dx_q is 1. Identify the one parameter subgroup $\underline{N}_j(A)$ with A and give it the Haar measure dn induced from dx.

Let $\lambda_i \in \mathfrak{a}^*$ be defined by

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \tag{7.5}$$

where $\alpha_1, \cdots, \alpha_\ell$ are the simple roots (see § 1) and δ_{ij} is the Kronecker delta. Then $\lambda_1, \cdots, \lambda_\ell$ are the fundamental dominant weights and every $\lambda \in \mathfrak{a}_C^*$ can be written as $\sum s_i\lambda_i$, $s_i \in \boldsymbol{C}$. For $t \in T(A)$, we have

$$\xi_{-\lambda-\rho}(t) = \prod_{j=1}^{\ell} |\xi_{\lambda_j}(t)|_A^{-s_j-1}$$

where $|\cdot|_A$ is the adelic norm. (Note that $\rho = \sum \lambda_j$).

Now we can return to the integral in (7.4). For $w = w_j$ we have

$$\int_{\underline{N}_j(A)} \Phi(nw_j)dn = \operatorname*{Lim}_{\mathscr{S}} \int_{\underline{N}_j^{\mathscr{S}}} \Phi(nw_j)dn \tag{7.7}$$

where $\mathscr{S}$ is a finite set of primes including infinity and the number of elements of $\mathscr{S}$ goes to infinity. $\underline{N}_j^{\mathscr{S}}$ is the subgroup of $\underline{N}_j(A)$ consisting of those $n = (n_q)$ in which $n_q = 1$ for $q \notin \mathscr{S}$. This allows us to reduce the problem to the calculation of local factors. They are of three kinds. (In the following $\lambda = \sum s_i\lambda_i$ lies in the domain of convergence of the integral (7.7) and we apply Lemma 5.1)

The factor at infinity is

$$C_\infty = \int_{\underline{N}_j(\boldsymbol{R})} \phi_\lambda(n_\infty w_j)dn_\infty = \int_{\underline{N}_i(\boldsymbol{R})} \Lambda^m\ \mathrm{Ad}^*(k(n_\infty w_j))\phi(1) \prod_i |\xi_{\lambda_i}(a(n_\infty w_j))|^{-s_i-1}dn_\infty\ . \tag{7.8}$$

Here $n_\infty w_j = k(n_\infty w_j)a(n_\infty w_j)n(n_\infty w_j)$ (Iwasawa decomposition).

The factor at a finite prime $q \neq p$ is

$$C_q = \int_{\underline{N}_j(Q_q)} \chi(t(n_q w_j)) \prod_i |\xi_{\lambda_i}(t(n_q w_j))|_q^{-s_i-1} dn_q \,. \tag{7.9}$$

Here $n_q w_j = k(n_q w_j)t(n_q w_j)n(n_q w_j)$ (Iwasawa decomposition) and $|\cdot|_q$ is the valuation of $\boldsymbol{Q}_q$ such that $|\tilde{\omega}_q|_q = q^{-1}$ if $\tilde{\omega}_q$ is the uniformizing element of $\boldsymbol{Q}_q$. And finally the factor at p is

$$C_p = \int_{\underline{N}_j(Q_p)} (L(k(n_p w_j)f_\chi)\chi(t(n_p w_j)) \prod_i |\xi_{\lambda_i}(t(n_p w_j))|_p^{-s_i-1} dn_p \tag{7.10}$$

where $n_p w_j = k(n_p w_j)t(n_p w_j)n(n_p w_j)$ is the Iwasawa decomposition of $n_p w_j$ in $\underline{G}(\boldsymbol{Q}_p)$.

§ 8.

To calculate the factor C_∞ we need to know the explicit action of Λ^* Ad.

We number the rows and columns of $2\ell \times 2\ell$ matrices by

$$\{1, 2, \cdots, \ell, -1, -2, \cdots, -\ell\} \,.$$

Let e_{ij} be the matrix which is 1 at $(i,j)^{\text{th}}$ entry and 0 elsewhere. For $\alpha_j \in \Delta$ $(1 \leq i \leq \ell - 1)$ the Lie algebra $\mathfrak{a}_j$ of A_{α_j} is spanned by $e_j = e_{jj} - e_{j+1,j+1} - e_{-j,-j} + e_{-j-1,-j-1}$. and the Lie algebra $\mathfrak{a}_\ell$ of A_{α_ℓ} is spanned by $e_{\ell,\ell} - e_{-\ell,-\ell}$. We write

$$e_\alpha = \begin{cases} e_{ij} - e_{-j,-i} \\ e_{ji} - e_{-i,-j} \\ e_{i,-j} + e_{j,-i} \\ e_{-i,j} + e_{-j,i} \\ e_{i,-i} \\ e_{-i,i} \end{cases} \quad \text{if } \alpha = \begin{cases} \varepsilon_i - \varepsilon_j \\ \varepsilon_j - \varepsilon_i \\ \varepsilon_i + \varepsilon_j \\ -\varepsilon_i - \varepsilon_j \\ 2\varepsilon_i \\ -2\varepsilon_i \end{cases} \quad (i < j) \,. \tag{8.1}$$

Then e_α is basis vector of $\mathfrak{g}_\alpha$. Let $\mathfrak{m}_\alpha$ be the space spanned by $e_\alpha + e_{-\alpha}$ and $\mathfrak{m} = \sum_{\alpha \in \Psi^+} \mathfrak{m}_\alpha$. Then $\mathfrak{b}$ can be embedded in the space

$$\mathfrak{p} = \textstyle\sum_{\alpha_j \in \Delta} \mathfrak{a}_j \oplus \sum_{\alpha \in \Psi^+} \mathfrak{m}_\alpha \,.$$

If $1 \leq j \leq \ell - 1$, we write

$$(8.2)\qquad k_j(\theta)=\left(\begin{array}{ccccccc|ccccccc}
1 &&&&&&&&&&&&& \\
& \ddots &&&&&&&&&&&& \\
&& 1 &&&&&&&&&&& \\
&&& \cos\theta & \sin\theta &&&&&&&&& \\
&&& -\sin\theta & \cos\theta &&&&&&&&& \\
&&&&& 1 &&&&&&&& \\
&&&&&& \ddots &&&&&&& \\
&&&&&&& 1 &&&&&& \\
\hline
&&&&&&&& 1 &&&&& \\
&&&&&&&&& \ddots &&&& \\
&&&&&&&&&& 1 &&& \\
&&&&&&&&&&& \cos\theta & \sin\theta & \\
&&&&&&&&&&& -\sin\theta & \cos\theta & \\
&&&&&&&&&&&&& 1 \\
&&&&&&&&&&&&& \ddots \\
&&&&&&&&&&&&& 1
\end{array}\right)\begin{array}{l} \\ \\ \\ \longleftarrow j^{\text{th}} \\ \\ \\ \\ \\ \\ \\ \\ \longleftarrow -j^{\text{th}} \\ \\ \\ \\ \\ \end{array}$$

and

$$k_\ell(\theta)=\left(\begin{array}{cc|cc}
I & & & \\
& \cos\theta & & \sin\theta \\
\hline
& & I & \\
& -\sin\theta & & \cos\theta
\end{array}\right),$$

where I is the $(\ell-1)\times(\ell-1)$ identity matrix.

LEMMA 8.1. *For $\alpha_j\in\Delta$, let* Ad_j *denotes the restriction to* K_{α_j} *of the adjoin action of* K *on* $\mathfrak{p}$. *Then we list below the* Ad_j *invariant subspaces together with the matrix of* $\mathrm{Ad}_j\, k_j(\theta)$ *with respect to the above basis.*

(A) *The case* $\alpha_j=\varepsilon_j-\varepsilon_{j+1}\qquad(1\le j\le\ell)$

(1) $\mathfrak{a}_j+\mathfrak{m}_{\alpha_j}$; *the matrix is*

$$\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix}$$

(2) $\mathfrak{m}_{2\varepsilon_j}+\mathfrak{m}_{2\varepsilon_{j+1}}+\mathfrak{m}_{\varepsilon_j+\varepsilon_{j+1}}$; *the matrix is*

$$\begin{pmatrix}\cos^2\theta & \sin^2\theta & \sin 2\theta\\ \sin^2\theta & \cos^2\theta & -\sin 2\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos 2\theta\end{pmatrix}$$

(3) $\mathfrak{m}_{\varepsilon_j-\varepsilon_h}+\mathfrak{m}_{\varepsilon_{j+1}-\varepsilon_h}\qquad$ *for* $j+2\le h\le\ell$

(4) $\mathfrak{m}_{\varepsilon_j+\varepsilon_h} + \mathfrak{m}_{\varepsilon_{j+1}+\varepsilon_h}$ *for* $1 \leq h \leq j-1$

(5) $\mathfrak{m}_{\varepsilon_j+\varepsilon_h} + \mathfrak{m}_{\varepsilon_{j+1}+\varepsilon_h}$ *for* $j+2 \leq h \leq \ell$

(6) $\mathfrak{m}_{\varepsilon_h-\varepsilon_j} + \mathfrak{m}_{\varepsilon_h-\varepsilon_{j+1}}$ *for* $1 \leq h \leq j-1$.

The matrix for the cases (3) *to* (6) *is*

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(7) $\mathfrak{m}_{\varepsilon_i \mp \varepsilon_h}$ *for all pairs* (i, h) *satisfying* $i < h$ *and either*

$$\begin{array}{lll} & 1 \leq i < j, & 2 \leq h < j\ ; \\ \text{or} & 1 \leq i < j, & j+1 < h \leq \ell\ ; \\ \text{or} & j+1 < i < \ell, & j+1 < h \leq \ell\ . \end{array}$$

(8) $\mathfrak{m}_{2\varepsilon_h}$ *for* $1 \leq h \leq \ell$ *and* $h \neq j,\ j+1$

$\mathrm{Ad}_j\, k_j(\theta)$ *acts trivially on each of the spaces in* (7) *and* (8).

(B) *The case of* α_ℓ

(1) $\mathfrak{a}_\ell + \mathfrak{m}_{\alpha_\ell}$; *the matrix is*

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

(2) $\mathfrak{m}_{\varepsilon_h-\varepsilon_\ell} + \mathfrak{m}_{\varepsilon_h+\varepsilon_\ell}$ *for* $1 \leq h < \ell$; *the matrix is*

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

(3) $\mathfrak{m}_{\varepsilon_i \mp \varepsilon_h}$ *for all pairs* (i, j) *satisfying* $i < h$ *and* $1 \leq i \leq \ell-2,\ 2 \leq h \leq \ell-1$

(4) $\mathfrak{m}_{2\varepsilon_h}$ *for* $1 \leq h \leq \ell-1$

$\mathrm{Ad}_j\, k_j(\theta)$ *acts trivially on each of the spaces in* (3) *and* (4).

COROLLARY 8.2. *Write* $m = \dim \mathfrak{n}$ *and* $\mathfrak{e} = \Lambda_{\alpha\in\psi^+,\alpha\neq\alpha_j}(e_\alpha + e_{-\alpha})$. *Then* $\Lambda^m\, \mathrm{Ad}_j\, k_j(\theta)$ *acts on the* 2 *dimensional space*

$$\Lambda^m(\mathfrak{a}_j \oplus \textstyle\sum_{\alpha\in\psi^+,\alpha\neq\alpha_j}\mathfrak{m}_\alpha) \oplus \Lambda^m\mathfrak{m}$$

by the matrix

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

with respect to the basis $\{e_j \wedge \mathfrak{e},\ (e_{\alpha_j} + e_{-\alpha_j}) \wedge \mathfrak{e}\}$.

The lemma is proved by means of a simple matrix calculation which will be omitted here. The corollary follows trivially from the lemma.

If $\alpha_j = \varepsilon_j - \varepsilon_{j+1}$, $1 \leq j \leq \ell - 1$, then every element of $\underline{G}_{\alpha_j}(\boldsymbol{R})$ can be written as

$$k_j(\theta) a_t n_x$$

for some θ, $x \in \boldsymbol{R}$, $t \in \boldsymbol{R}^{\times}_{+}$; where $k_j(\theta)$ is given in § 8, $n_x = I + x e_{\alpha_j}$ and

$$(8.4) \qquad a_t = \left[\begin{array}{ccccccc|ccccccc}
1 & & & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & \\
 & & & t & & & & & & & & & & \\
 & & & & t^{-1} & & & & & & & & & \\
 & & & & & 1 & & & & & & & & \\
 & & & & & & \ddots & & & & & & & \\
 & & & & & & 1 & & & & & & & \\
\hline
 & & & & & & & 1 & & & & & & \\
 & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & t^{-1} & & & \\
 & & & & & & & & & & & t & & \\
 & & & & & & & & & & & & 1 & \\
 & & & & & & & & & & & & & \ddots \\
 & & & & & & & & & & & & & 1
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \\ \longleftarrow j^{\text{th}} \\ \\ \\ \\ \\ \\ \\ \longleftarrow -j^{\text{th}} \\ \\ \\ \\ \\
\end{array} .$$

And in the case of α_ℓ, we have

$$(8.5) \qquad k_\ell(\theta) a_t n_x .$$

Here $n_x = (I + x e_{\alpha_\ell})$, $k(\theta)$ is given in § 8 and

$$(8.6) \qquad a_t = \left[\begin{array}{cccc|cccc}
1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & 1 & & & & & \\
 & & & t & & & & \\
\hline
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & 1 & \\
 & & & & & & & t^{-1}
\end{array}\right]
\begin{array}{l}
\\ \\ \\ \longleftarrow \ell^{\text{th}} \\ \\ \\ \\ \longleftarrow -\ell^{\text{th}}
\end{array} .$$

Moreover we put

$$(8.7)\qquad w_j = \left(\begin{array}{cccccc|cccccc}
1 &&&&&&&&&&& \\
& \ddots &&&&&&&&&& \\
&& 1 &&&&&&&&& \\
&&& 0 & -1 &&&&&&& \\
&&& 1 & 0 &&&&&&& \\
&&&&& 1 &&&&&& \\
&&&&&& \ddots &&&&& \\
&&&&&&& 1 &&&& \\
\hline
&&&&&&&& 1 &&& \\
&&&&&&&&& \ddots && \\
&&&&&&&&&& 1 & \\
&&&&&&&&&&& 0 \quad -1 \\
&&&&&&&&&&& 1 \quad 0 \\
&&&&&&&&&&& \quad 1 \\
&&&&&&&&&&& \quad\ddots \\
&&&&&&&&&&& \quad\quad 1
\end{array}\right) \quad 1 \le j \le \ell - 1$$

and

$$(8.8)\qquad w_\ell = \left(\begin{array}{cccc|cccc}
1 &&&&&&& \\
& \ddots &&&&&& \\
&& 1 &&&&& \\
&&& 0 &&&& 1 \\
\hline
&&&& 1 &&& \\
&&&&& \ddots && \\
&&&&&& 1 & \\
&&& 1 &&&& 0
\end{array}\right).$$

For $1 \le j \le \ell$, if the Iwasawa decomposition of $n_x w_j$ is $k_j(\theta) \cdot a_t \cdot n_{x'}$, then

$$(8.9)\qquad x' = -\frac{x}{1+x^2}, \quad t = \sqrt{1+x^2} \quad \text{and} \quad \sin\theta = \frac{-1}{\sqrt{1+x^2}}$$

so that

$$\prod_i |\xi_{\lambda_i}(a(n_x w_j))|^{-s_i-1} = t^{-s_j-1}.$$

By using Corollary 8.2, we see immediately from (7.8) that

$$\begin{aligned} C_\infty &= \left(\int_R \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} t^{-s_j-1} dx\right)\phi(1) \\ &= \left(\int_R \begin{pmatrix} x^2-1 & -2x \\ 2x & x^2-1 \end{pmatrix} (x^2+1)^{(-s_j-3)/2} dx\right)\phi(1). \end{aligned}$$

Evaluating the integrals in terms of Γ-functions we get

$$C_\infty = \left(\frac{1-s_j}{1+s_j}\right)\frac{\Gamma(1/2)\Gamma(s_j/2)}{\Gamma((1+s_j)/2)}\phi(1) . \tag{8.10}$$

§ 9.

We compute in this section the factors at the finite primes. Put

$$k_j(x) = \left(\begin{array}{ccccc|ccccc} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & 0 & & & & & & \\ & & x^{-1} & 1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & x^{-1} & \\ & & & & & & & 0 & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{array}\right) \qquad 1 \le j \le \ell - 1$$

and

$$k_\ell(x) = \left(\begin{array}{cc|cc} I & & & \\ & 1 & & 0 \\ \hline & & I & \\ & x^{-1} & & 1 \end{array}\right)$$

where I is the $(\ell-1)\times(\ell-1)$ identity matrix.

For a given $\alpha \in \Delta$, if $x \in \boldsymbol{Q}_q$ write $n_x = I + xe_\alpha$; if $t \in \boldsymbol{Q}_q^\times$, let a_t be given by the matrix (8.4) (resp. (8.6)) in case $\alpha = \alpha_j$, $1 \le j \le \ell - 1$ (resp. $\alpha = \alpha_\ell$), then $T_{\alpha_j}(\boldsymbol{Q}_q)$ is just the set of matrices a_t. Let χ_j be the restriction of χ to $T_{\alpha_j}(\boldsymbol{Q}_q)$. Write $\chi_j(\tilde{\omega}_q)$ for $\chi_j(a_{\tilde{\omega}_q})$. In this way we can regard χ_j as a character of $\boldsymbol{Q}_q$. We use the same w_j as given in § 8. Finally, if $x \in \boldsymbol{Q}_q^\times$, $\alpha = \alpha_j$, then

$$n_x w_j = k_j(x) a_x n_{-x^{-1}} . \tag{9.1}$$

First let us handle the case of C_q, $q \neq p$. Since $n_x w_j \in \underline{G}_{\alpha_j}(\boldsymbol{Z}_q)$ if $x \in \boldsymbol{Z}_q$, we have

$$\prod_i |\xi_{\lambda_i}(t(n_x w_j))|_q^{-s_i-1} = \begin{cases} 1 & \text{if } x \in \boldsymbol{Z}_q \\ |x|_q^{-s_j-1} & \text{if } x \notin \boldsymbol{Z}_q \end{cases} .$$

By the formula (7.9) C_q equals to

$$\int_{Z_q} dx + \sum_{m=1}^{\infty} \int_{\tilde{\omega}_q^{-m} Z_q \setminus \tilde{\omega}_q^{-(m-1)} Z_q} \chi_j(a_x)|x|_q^{-s_j-1} dx = \frac{1 - \bar{\chi}_j(\tilde{\omega}_q)^{-(s_j+1)}}{1 - \bar{\chi}_j(\tilde{\omega}_q) q^{-s_j}} .$$

Let

$$L(s, \chi) = \prod_q (1 - \chi(\tilde{\omega}_q) q^{-s})^{-1}$$

be the Hecke L-function (the product is taken over all the finite primes q where χ is unramified).

Let $\mathscr{S}_j$ be the set of finite primes at which χ_j is ramified. Let

$$\kappa_j = \begin{cases} \prod_{q \in \mathscr{S}_j \setminus \{p\}} C_q & \text{if } p \in \mathscr{S}_j \\ (\prod_{q \in \mathscr{S}_j} C_q) \dfrac{1 - \bar{\chi}_j(\tilde{\omega}_p) p^{-s_j}}{1 - \bar{\chi}_j(\tilde{\omega}_p) p^{-(1+s_j)}} & \text{if } p \notin \mathscr{S}_j \end{cases} .$$

Then we get

$$\prod_{q \neq p} C_q = \kappa_j \frac{L(s_j, \bar{\chi}_j)}{L(1 + s_j, \bar{\chi}_j)} . \tag{9.2}$$

(Note that the above formula was obtained under the assumption that λ lies the domain of convergence of the integral (7.7). We can extend the formula to all λ by the principle of analytic continuation.)

Next we consider the local factor C_p. According to Lemma 5.1, it is sufficient to study the integral

$$\int_{\underline{N}_j(\boldsymbol{Q}_p)} |(\xi_{-\lambda-\rho}(t_p(n w_j))| dn . \tag{9.3}$$

An easy calculation as above shows that (9.3) is equal to

$$\frac{1 - q^{-(\sigma+1)}}{1 - q^{-\sigma}}, \qquad \text{if } s_j = \sigma + \sqrt{-1}\tau .$$

As a consequence we have the following

Lemma 9.1. *C_p is holomorphic in $\lambda = \sum s_i \lambda_i$ if* $\operatorname{Re} s_j > 0$.

§ 10.

Suppose that under the map (3.3) the element ω of $\oplus_{P\in D} H^m(e'(P), \boldsymbol{C})$ corresponds to $[\phi] \otimes \sum_\chi f_\chi$. If for each χ that appears the corresponding $\chi_j \neq 1$, then we say that ω is a regular class. Now we can state the theorem.

THEOREM. *If $\omega \in \oplus_{P\in D} H^m(e'(P), \boldsymbol{C})$ is a regular class, then there exists a $\tilde{\omega} \in H^*(\overline{X}/\Gamma, \boldsymbol{C})$ such that $r(\tilde{\omega}) = \omega$ (r is the restriction map* (1.1)). *Moreover $\tilde{\omega}$ can be represented by a harmonic form.*

We first put together the results of the previous three sections on the constant term $E^B(g, \Phi)$ (6.9) of the Eisenstein series. We have

$$E^B(g, \Phi) = \sum_{w\in W} c(w, \lambda)\Phi(g) \ . \tag{7.2}$$

For $1 \leq j \leq \ell$ and $\lambda = \sum s_i\lambda_i$, $c(w_j, \lambda)\Phi$ is the product of

$$(\phi(1) \otimes C_P)\chi^{w_j}\xi_{-\rho-w_j\lambda} \tag{10.1}$$

and a constant

$$\frac{1-s_j}{1+s_j}\,\frac{\Gamma(1/2)\Gamma(s_j/2)}{\Gamma((1+s_j)/2)}\,\frac{L(s_j, \bar{\chi}_j)}{L(1+s_j, \bar{\chi}_j)}\kappa_j \tag{10.2}$$

(cf: (7.4), (8.10) and (9.2)).

Now if χ is such that $\bar{\chi}_j$ is not the trivial character for all j, and $\lambda = \sum \lambda_i = \rho$, then since $c(w_j, \lambda)$ is a linear transformation on function space (and is independent of g) it is clear from (10.2) that $c(w_j, \lambda)\phi$ is zero for all j. Moreover, as $c(w, \lambda)\Phi$ is holomorphic at ρ, the general properties of the transformations $c(w, \lambda)$ (cf: [8]) implies that the same is true for $c(w, \lambda)\Phi$. In fact the functional equation (7.3) implies that $c(w, \rho)\Phi$ is zero for all Weyl group elements w which is not the identity. Thus at $\lambda = \rho$, we have

$$E^B(g, \Phi) = c(\mathrm{Id}, \lambda)\Phi(g) = \Phi(g). \tag{10.3}$$

An easy calculation (using formula (2.3) of [5]) shows that

$$d\Phi(H, X_1, \cdots, X_m) = (-\mu - \rho - \lambda)H\Phi(X_1, \cdots, X_m)$$

for $H \in \mathfrak{a}$, $X_1, \cdots, X_m \in \mathfrak{n}$. According to (6.6), $\mu = -2\rho$.

Hence at $\lambda = \rho$, we have

$$dE^B(g, \Phi) = 0 \tag{10.4}$$

and by (6.7),

$$\Delta E(g, \phi) = 0 . \tag{10.5}$$

Suppose $[\phi] \otimes \sum_\chi f_\chi$ corresponds to the element ω in $\oplus_{P\in D} H^*(e'(P), C)$. If ω is regular then $\chi_j \neq 1$ and by Lemma 4.1 of [8] $E(g, \Phi_\chi)$ is holomorphic at $\lambda = \rho$. Therefore by Lemma 6.2, (10.4), (10.5) the value of $E(g, \omega)$ at the special point $\lambda = \rho$ is an harmonic form representing a cohomology class $\tilde{\omega}$ and the restriction $r(\tilde{\omega})$ of $\tilde{\omega}$ to the boundary $\partial(\overline{X}/\Gamma)$ can be represented by $E^B(g, \omega)$. Hence by (10.3), $r(\tilde{\omega}) = \omega$.

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