T. Murai
Nagoya Math. J.
Vol. 85 (1982), 87-154

ON LACUNARY SERIES

TAKAFUMI MURAI

§ 1. Introduction

We are concerned with the behaviour of Fourier series in an interval $[0, 2\pi)$ and, in particular, interested in that of so-called lacunary series. The spectrum of a Taylor series $F(t) \sim \sum_{n=0}^{\infty} \hat{F}(n)e^{int}$ is defined by Spec $(F) = \{n \in Z^+; \hat{F}(n) \neq 0\}$, where Z^+ denotes the semi-group of positive integers.

Lacunary series are Taylor series whose spectra are sparse in Z^+. Let us define more precisely lacunary series. Let $h(x)$ be a positive increasing function in an interval $[1, +\infty)$. We say that a Taylor series $F(t)$ is h-lacunary if there exists a number $q > 1$ such that, for $n, m \in$ Spec (F) $(n > m)$, $h(n) \geq qh(m)$. We say that $F(t)$ is L-lacunary if it is $(\log x)$-lacunary. We say that $F(t)$ is Hadamard lacunary if it is x-lacunary.

According to J. P. Kahane [6], the history of lacunary series goes back to Weierstrass's example, which is a continuous and nowhere differentiable function: $\sum_{n=1}^{\infty} \xi^n \cos \lambda^n x$ (, where λ is an odd integer ≥ 3 and ξ a positive number such that $(1 + 3\pi/2)/\lambda < \xi < 1$) ([6]). The conception of "Hadamard lacunary" comes from the following classical theorem: A Taylor series $\sum_{k=1}^{\infty} a_k z^{n_k}$ satisfying $\limsup_{k\to\infty} |a_k|^{1/n_k} = 1$ has $\{z; |z| = 1\}$ as a natural boundary if there exists a number $q > 1$ such that, for any $k \geq 1$, $n_{k+1} \geq qn_k$ ([6]). Hadamard lacunary series are studied by many authors and many interesting properties are known. Various series having sparse spectra are also discussed by many authors but, as far as the author knows, L-lacunary series are first introduced in this paper.

There are many interesting properties of lacunary series which are reflection of properties of Steinhaus series ([14], p. 541) and so it is important to deal with lacunary series as series of almost independent random variables. From this point of view, the theory of lacunary series may be one of theories of sums of almost independent random variables.

The aim of this paper is to study the behaviour of Fourier series having sufficiently sparse spectra since it seems that such series have

Received July 24, 1976.

various interesting and important properties as stated above. We may expect that the behaviour of such series resembles that of Steinhaus series, which we shall confirm later. On the other hand, we shall also see some properties (of such series) which lose the meaning in Steinhaus series. However we know that some interesting properties of Hadamard lacunary series come from the fact that spectra of such series are sparse, many technical difficulties and mathematically essential something prevent us from doing parallel discussions with the probability theory and from discovering new interesting probabilitistic properties. So we introduce L-lacunary series. The conception of "L-lacunary" is nothing but a concrete representation of a vague and abstract conception of "sufficiently sparse" and a statement of "an L-lacunary series satisfies (P)" only signifies "a Fourier series having a sufficiently sparse spectrum satisfies (P)". However it is an interesting subject to study suitable conditions on spectra, we shall not discuss this subject in this paper.

Now we explain the content of this paper. § 2 is a chapter prepared for later applications. To avoid repeating the same discussion in the course of the proof, main lemmas are gathered in § 3.

§ 4: As stated above, the theory of lacunary series is in deep connection with the probability theory and so it is important to try to reorganize this theory from the point of view of the probability theory. We shall begin with the discussion on the 0–1 law. We shall also try to give some mathematical answer to the question why L-lacunary series behave like sums of independent random variables. To do this, we shall introduce the conception of "pseudo-independent", which is usual in the probability theory, and show that L-lacunary series are pseudo-independent. We shall see that Hadamard lacunary series are not generally pseudo-independent, (which attracts us) and which suggests that more deep investigations are necessary. This chapter is, in fact, incomplete and the author hopes to return to this subject at some time ([6], [16]).

§ 5: This is a main chapter. We shall study the behaviour of partial sums of L-lacunary series and shall show that simple conditions on coefficients determine the transience and the recurrence of L-lacunary series as seen in the probability theory. In 5.4, we shall show that, by our results on L-lacunary series, we can judge whether a given Steinhaus series is recurrent or transient. Even our results on Steinhaus series are new. ([7], [9])

§ 6: However we study, in § 5, the behaviour of L-lacunary series except sets of measure zero, the study of thin sets remains. We shall show that the volume of sets where L-lacunary series converge is simply determined. ([3], [9])

§ 7: The spectrum Spec (f) of an analytic function $f(z)$ in the open unit disk $\boldsymbol{D}$ is defined analogously as above. Then h-lacunary analytic functions are also defined. The study of lacunary analytic functions is classical. In this area, the following fact, which is called the Picard property of lacunary analytic functions, is well-known: There exists a positive number q (= about 100) such that an analytic function $f(z)$ attains every complex number infinitely often in $\boldsymbol{D}$ if $\sum_{n=0}^{\infty}|\hat{f}(n)| = +\infty$ and if, for any n, $m \in \mathrm{Spec}\,(f)$ $(n > m)$, $n \geq qm$. This fact shows that the value-distribution of lacunary analytic functions is, in a sense, uniform but this is not sufficient to know the quantitative uniformity of the behaviour of such functions.

There is a quantity δ, which is called the deficiency, as a quantitative representation of the value-distribution. The deficiency $\delta(\cdot) = \delta(\cdot, f)$ is a mapping associated with a given function $f(z)$ from the complex plane $\boldsymbol{C}$ to an interval [0, 1] and this plays an important role in the theory of the value-distribution. We remark that if $\delta(a, f) = 0$, then $f(z)$ attains a infinitely often in $\boldsymbol{D}$.

The above fact suggests that the deficiency $\delta(\cdot, f)$ of a lacunary analytic function $f(z)$ vanishes for all complex number. To check this fact, which may be called the Nevanlinna property of lacunary analytic functions, is to reconfirm that the value-distribution of such functions is uniform in the sense of the deficiency. We shall show the Nevanlinna property of L-lacunary analytic functions. ([5], [11], [18])

§ 8: In this section, we shall study more in detail ranges and cluster sets of L-lacunary analytic functions. Let $f(z)$ be an analytic function in $\boldsymbol{D}$ and U a subset of $\bar{\boldsymbol{D}}$. The range of $f(z)$ in U is defined by $R(U; f) = \{a \in \boldsymbol{C};\, {}^{\#}\{z \in U; f(z) = a\} = +\infty\}$, where ${}^{\#}\{\cdot\}$ denotes the cardinal number of $\{\cdot\}$. It is too difficult to study ranges of L-lacunary analytic functions in arbitrarily given subsets. We shall choose U suitably and shall show that simple conditions on coefficients give information on $R(U; \cdot)$.

However we only discuss, in § 5, the behaviour of partial sums of L-lacunary series, it is also interesting to study the behaviour of sums of such series by the Abel mean. This is to study the radial behaviour of

L-lacunary analytic functions. We shall discuss radial cluster sets. ([7], [12])

At last, the author would like to express the deep gratitude to Professor M. Itô for his encouragements.

§ 2. Preliminaries

2.1. Fourier series

An interval $[0, 2\pi)$ is identified with the unit circle $\boldsymbol{T}$ in the complex plane $\boldsymbol{C}$ by a mapping $t \in [0, 2\pi) \to e^{it} \in \boldsymbol{T}$. A topology on this interval is induced from $\boldsymbol{T}$. The distance between two elements s, t is $\min\{|s-t|, 2\pi - |s-t|\}$. We say that a subset U of $[0, 2\pi)$ is an interval if the image e^{iU} of U is an interval in $\boldsymbol{T}$. We denote by "m" the 1-dimensional Lebesgue measure.

Let $\boldsymbol{Z}$ denote the group of integers. For a sequence $(c_n)_{n\in\boldsymbol{Z}}$ of complex numbers, we consider a correspondence $F(t)$: $t \to (c_n e^{int})_{n\in\boldsymbol{Z}}$ and say that $F(t)$ is a (formal) Fourier series. We write $\hat{F}(n) = c_n$ $(n \in \boldsymbol{Z})$ and $F(t) \sim \sum_{n\in\boldsymbol{Z}} \hat{F}(n)e^{int}$. If $\lim_{N\to\infty} \sum_{|n|\leq N} \hat{F}(n)e^{int}$ exists almost everywhere, we write $F(t) = \sum_{n\in\boldsymbol{Z}} \hat{F}(n)e^{int}$. $A(0, 2\pi)$ denotes the totality of Fourier series $F(t)$ satisfying $\sum_{n\in\boldsymbol{Z}} |\hat{F}(n)| < +\infty$ and $L^2(0, 2\pi)$ denotes the totality of Fourier series $F(t)$ satisfying $\sum_{n\in\boldsymbol{Z}} |\hat{F}(n)|^2 < +\infty$.

For a Fourier series $F(t)$, we put:

$$(1)\qquad \begin{cases} \mathrm{Spec}\,(F) = \{n \in \boldsymbol{Z} - \{0\};\ \hat{F}(n) \neq 0\} \\ N_F = (\text{the cardinal number of } \mathrm{Spec}\,(F)) \qquad (\leq +\infty) \\ \deg\,(F) = \sup\,\{|n|;\ n \in \mathrm{Spec}\,(F) \cup \{0\}\} \qquad (\leq +\infty) \\ \nu(F) = \sup\,\{|\hat{F}(n)|;\ n \in \mathrm{Spec}\,(F)\} \qquad (\leq +\infty)\,. \end{cases}$$

We say that a Fourier series $F(t)$ is a (formal) Taylor series, if $\mathrm{Spec}\,(F)$ is a subset of the semi-group of positive integers $\boldsymbol{Z}^+$. For a Taylor series $F(t)$, we denote by $n_0(F) = 0$ and by $n_k(F)$ the k-th integer in $\mathrm{Spec}\,(F)$. We write simply $n_k = n_k(F)$ $(k \geq 0)$ when no ambiguity can arise. We also write $\tilde{F}(k) = \hat{F}(n_k)$ $(k \geq 0)$ and $F(t) \sim \sum_{k=0}^{\infty} \tilde{F}(k)e^{in_k t}$. For a Taylor series $F(t)$ and a positive integer m, we put:

$$(2)\qquad \begin{cases} F_m(t) = \sum_{k=0}^{m} \tilde{F}(k)e^{in_k t} \\ s(m; F) = \left(\sum_{k=1}^{m} |\tilde{F}(k)|^2\right)^{1/2} \\ w(m; F) = \left(\sum_{k=m}^{\infty} |\tilde{F}(k)|^2\right)^{1/2} \qquad (\leq +\infty)\,. \end{cases}$$

We say that a Fourier series $F(t)$ is a polynomial, if $N_F < +\infty$ and that $F(t)$ is a Taylor polynomial, if $N_F < +\infty$ and $\operatorname{Spec}(F) \subset \boldsymbol{Z}^+$.

Let $f(z)$ be an analytic function in the open unit disk $\boldsymbol{D}$. Putting $\hat{f}(n) = 1/2\pi \int_0^{2\pi} f(e^{it}/2)2^n e^{-int}dt$ $(n \geq 0)$, we have $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$. Using a Taylor series $\sum_{n=0}^{\infty} \hat{f}(n)e^{int}$, we define analogously: $\operatorname{Spec}(f)$, N_f, $\deg(f)$, $\nu(f)$, $n_k(f)$, $\tilde{f}(k)$, $f_m(z)$, $s(m;f)$, $w(m;f)$.

2.2. Lacunary series

Let $h(x)$ be a non-negative and strictly increasing function in an interval $[1, +\infty)$. We say that a subset E of $\boldsymbol{Z}^+$ is h-lacunary if there exists a number $q > 1$ such that, for any $n, m \in E$ $(n > m)$, $h(n) \geq qh(m)$. For example, $\boldsymbol{Z}^+$ itself is an e^x-lacunary set. The following four propositions evidently hold.

PROPOSITION 1. *Union of an h-lacunary set and a finite set is h-lacunary.*

PROPOSITION 2. *Let α be a positive number. Then a set is h-lacunary if and only if it is h^α-lacunary.*

PROPOSITION 3. *Suppose that $h(x)$ is continuously differentiable and that $\lim_{x\to\infty} h'(x)/h(x) = 0$. For an h-lacunary set E and a positive number α, there exists a positive number β such that, for any $\gamma \geq \beta$, an interval $[\gamma, \gamma + \alpha]$ contains at most one element of E. On the other hand, there exists an infinite h-lacunary set F such that $E \cup F$ is h-lacunary.*

PROPOSITION 4. *An h_1-lacunary set is h_2-lacunary if $h_j(x)$ $(j = 1, 2)$ are continuously differentiable and $h_1'(x)/h_1(x) \leq h_2'(x)/h_2(x)$.*

A strictly increasing sequence E in $\boldsymbol{Z}^+$ is h-lacunary if it is an h-lacunary set as a subset of $\boldsymbol{Z}^+$. A Taylor series $F(t)$ is h-lacunary if $\operatorname{Spec}(F)$ is h-lacunary. An analytic function $f(z)$ in $\boldsymbol{D}$ is h-lacunary if $\operatorname{Spec}(f)$ is h-lacunary. In this paper, we shall mainly discuss $(\log x)$-lacunary series. A $(\log x)$-lacunary series is called L-lacunary. An x-lacunary series is called Hadamard lacunary. Proposition 4 shows that an L-lacunary series is Hadamard lacunary.

For an L-lacunary sequence $E = (n_k)_{k=1}^N$ $(N < +\infty$ or $N = +\infty)$ in $\boldsymbol{Z}^+$, set

$$\gamma_E(m) = \sum_{k=m}^{N} n_k^{-1} \sum_{\ell=0}^{k-1} n_\ell, \quad \tilde{\gamma}_E(m) = \sum_{k=m}^{N} n_k^{-1} \tag{3}$$
$$(n_0 = 0) \ (1 \leq m \leq N \text{ if } N < +\infty;\ m \geq 1 \text{ if } N = +\infty)\,.$$

For a Taylor polynomial $Q(t)$, we write

(4) $\quad \gamma(m; Q) = \gamma_{\mathrm{Spec}\,(Q)}(m),\ \tilde{\gamma}(m; Q) = \tilde{\gamma}_{\mathrm{Spec}\,(Q)}(m) \qquad (1 \leq m \leq N_Q)$.

For an (infinite) L-lacunary series $E = (n_k)_{k=1}^{\infty}$ in Z^+, set

(5) $$\gamma_{E,j}(m) = \sum_{k=0}^{m-1} (n_k/n_m)^j + \sum_{k=m+1}^{\infty} (n_m/n_k)^j \qquad (n_0 = 0, j = 1, 2)\ .$$

LEMMA 5. *Let $E = (n_k)_{k=1}^{\infty}$ be an L-lacunary series. Then there exist two numbers $q(E) > 1$ and $\theta(E) > 0$ such that, for any $m \geq 1$,*

(6) $$\begin{cases} \log n_{m+1} \geq q(E) \log n_m,\ n_m^{-1} \leq \theta(E) \exp(-q(E)^m) \\ \gamma_E(m) \leq \theta(E) n_m^{-1+1/q(E)},\ \tilde{\gamma}_E(m) \leq \theta(E) n_m^{-1} \\ \gamma_{E,j}(m) \leq \theta(E) m^{-2} \qquad (j = 1, 2)\ . \end{cases}$$

Proof. Since E is L-lacunary, there exists a number $q > 1$ such that $\log n_{k+1} \geq q \log n_k$ (, that is, $n_{k+1}/n_k^q \geq 1$) ($k \geq 1$). Then there exist two numbers $\theta_1 > 0$ and $0 < \theta_2 < 1$ depending only on q such that $n_k^{-1} \leq \theta_1 \exp(-q^k)$ and $n_k/n_{k+1} \leq \theta_2$ ($k \geq 1$). We have, for $m \geq 1$,

$$\begin{aligned} \gamma_E(m) &= \sum_{k=m}^{\infty} (n_{k-1}/n_k) \sum_{\ell=0}^{k-1} (n_\ell/n_{k-1}) \leq \sum_{k=m}^{\infty} n_k^{-1+1/q} \sum_{\ell=0}^{k-1} \theta_2^\ell \\ &\leq \frac{1}{1-\theta_2} n_m^{-1+1/q} \sum_{k=m}^{\infty} (n_m/n_k)^{1-1/q} \leq \frac{1}{1-\theta_2} n_m^{-1+1/q} \sum_{\ell=0}^{\infty} \theta_2^{\ell(1-1/q)} \\ &= \frac{1}{(1-\theta_2)(1-\theta_2^{1-1/q})} n_m^{-1+1/q} \end{aligned}$$

and

$$\tilde{\gamma}_E(m) = n_m^{-1} \sum_{k=0}^{\infty} n_m/n_k \leq n_m^{-1} \sum_{\ell=0}^{\infty} \theta_2^\ell = \frac{1}{1-\theta_2} n_m^{-1}\ .$$

Since

$$\begin{aligned} \gamma_{E,j}(m) &= (n_{m-1}/n_m)^j \sum_{k=0}^{m-1} (n_k/n_{m-1})^j + (n_m/n_{m+1})^j \sum_{k=m+1}^{\infty} (n_{m-1}/n_k)^j \\ &\leq \sum_{\ell=0}^{\infty} \theta_2^{\ell j} \{(n_{m-1}/n_m)^j + (n_m/n_{m+1})^j\} \leq \frac{2}{1-\theta_2^j} n_m^{-j(1-1/q)} \\ &\leq \frac{2}{1-\theta_2^j} \theta_1^{j(1-1/q)} \exp\{-j(1-1/q)q^m\} \qquad (j = 1, 2)\ , \end{aligned}$$

there exists a positive number θ_3 depending only on q such that, for $m \geq 1$, $\gamma_{E,j}(m) \leq \theta_3 m^{-2}$ ($j = 1, 2$). Put $q(E) = q$ and

$$\theta(E) = \max\left\{\theta_1,\ \theta_3,\ \frac{1}{(1-\theta_2)(1-\theta_2^{1-1/q})}\right\}.$$

Then (6) holds. This completes the proof.

2.3. Bessel functions

Bessel functions of order 0 and 1 in an interval $[0, +\infty)$ are defined by

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin t - \nu t)dt \qquad (\nu = 0, 1).$$

Then:

(i) $|J_\nu(x)| \leq 1$ $(\nu = 0, 1)$

(ii) $1 - \frac{3}{8}x^2 \leq J_0(x) \leq 1 - \frac{1}{8}x^2$ $(0 \leq x \leq 1)$

(iii) $|J_0(x)| \leq \sqrt{\frac{2}{\pi}\frac{1}{x}}$

(iv) $J_1(x)/x \geq \frac{1}{4}$ $(0 < x \leq 1)$

(v) $\int_0^1 J_0(xr)rdr = J_1(x)/x$

(vi) $J_0(\xi)J_0(\eta) = \frac{1}{2\pi}\int_0^{2\pi} J_0(\sqrt{\xi^2 + \eta^2 - 2\xi\eta\cos t})dt$ $(\xi, \eta > 0)$

(vii) $\int_0^\infty J_0(\xi r)e^{-\eta^2 r^2}rdr = \frac{1}{2\eta^2}e^{-\xi^2/(4\eta^2)}$ $(\xi, \eta > 0)$.

Elementary calculus gives (i), (ii) and (iv). The formulas (v), (vi) and (vii) are well-known and seen in [15]. The inequality (iii) is not evident. For the proof, we use Hankel's formula ([15], p. 74):

$$J_0(x) = \sqrt{\frac{2}{\pi}\frac{1}{x}}\frac{H_0^{(1)}(x) + H_0^{(2)}(x)}{2},$$

where

$$H_0^{(1)}(x) = \frac{1}{\Gamma(1/2)}e^{i(x-\pi/4)}\int_0^\infty e^{-r}r^{-1/2}\left(1 + \frac{ir}{2x}\right)^{-1/2}dr$$

and $H_0^{(2)}(x) = \overline{H_0^{(1)}(x)}$. We have

$$|H_0^{(\nu)}(x)| \leq \frac{1}{\Gamma(1/2)}\int_0^\infty e^{-r}r^{-1/2}dr = 1 \qquad (\nu = 1, 2)$$

and hence (iii) holds.

For a finite sequence $\Xi = (c_k)_{k=1}^K$ of complex numbers, we define

(7) $$J(z; \Xi) = \prod_{k=1}^{K} J_0(|c_k z|) .$$

2.4. Fourier transform

For a Borel measure μ in C having finite total mass, the Fourier transform of μ is defined by $\hat{\mu}(z) = \int_C e^{-i \operatorname{Re} \bar{z}w} d\mu(w)$, where "$\operatorname{Re} \bar{z}w$" denotes the real part of $\bar{z}w$. (We shall denote by $\operatorname{Im} X$ the imaginary part of X.)

Let $\chi(z)$ denote the indicator function of D, that is, $\chi(z) = 1$ ($z \in D$) and $\chi(z) = 0$ ($z \in D^c$). Put $\lambda = \chi*\chi*\chi$, $\chi_\eta(z) = \chi(z/\eta)$ and $\lambda_\eta(z) = \lambda(z/\eta)$ ($\eta > 0$), where "$*$" denotes the convolution in C. Then we have:

(viii) $0 \leq \lambda_\eta(z) \leq \pi^2$

(ix) $\pi^2/64 \cdot \chi_{\eta/4}(z) \leq \lambda_\eta(z) \leq \pi^2 \chi_{3\eta}(z)$

(x) $|\hat{\lambda}_\eta(z)| \leq \pi^3 \eta^2$

(xi) $\hat{\lambda}_\eta(z) = (2\pi)^3 \eta^{-1} J_1(|\eta z|)^3 |z|^{-3}$

(xii) $\hat{\lambda}_\eta(z) \geq (2\pi)^3 10^{-2} \eta^2$ ($|z| \leq 1/\eta$).

Elementary calculus gives (viii) and (ix).

(x): Since $\hat{\lambda}(z) = \hat{\chi}(z)^3$ and $\hat{\chi}(0) = \pi$, we have

$$|\hat{\lambda}_\eta(z)| \leq \hat{\lambda}_\eta(0) = \hat{\lambda}(0)\eta^2 = \hat{\chi}(0)^3 \eta^2 = \pi^3 \eta^2 .$$

(xi): Since

$$\hat{\chi}(z) = 2\pi \int_0^1 J_0(|z|r) r \, dr = 2\pi J_1(|z|)/|z| ,$$

we have

$$\hat{\lambda}_\eta(z) = \hat{\lambda}(\eta z)\eta^2 = \hat{\chi}(\eta z)^3 \eta^2 = (2\pi)^3 \eta^{-1} J_1(|\eta z|)^3 |z|^{-3} .$$

(xii): By (iv), we have, for $|z| \leq 1/\eta$,

$$\hat{\lambda}_\eta(z) = \hat{\chi}(\eta z)^3 \eta^2 \geq (2\pi)^3 4^{-3} \eta^2 \geq (2\pi)^3 10^{-2} \eta^2 .$$

LEMMA 6. *For a polynomial $Q(t)$ and a Borel set U in $[0, 2\pi)$, set*

(8) $$\Phi[U; Q](z) = \int_U \exp\{-i \operatorname{Re} \bar{z} Q(t)\} dt .$$

Then we have

(9) $$\int_U \lambda_\eta(Q(t)) dt = (2\pi)^{-2} \int_C \hat{\lambda}_\eta(z) \Phi[U; Q](z) d\sigma(z) ,$$

where $d\sigma$ denotes the 2-dimensional Lebesgue measure.

Proof. Without loss of generality, we may assume that $\eta = 1$. For a function $g(z)$ in $\boldsymbol{C}$, we write $\check{g}(z) = g(-z)$. Now let us define a Borel measure $\mu_{U,Q}$ in $\boldsymbol{C}$ by: for every Borel set A in $\boldsymbol{C}$, set

$$\mu_{U,Q}(A) = m(\{t \in U;\ Q(t) \in A\}) .$$

Note that $\hat{\mu}_{U,Q}(z) = \Phi[U; Q](z)$. We easily see that $\chi_1 * \chi_1$ is square $d\sigma$-integrable. Since the support of $\mu_{U,Q}$ is compact, $\chi_1 * \mu_{U,Q}$ is also square $d\sigma$-integrable. By Parseval's formula, we have

$$\overset{\vee}{\overline{(\chi_1 * \chi_1)}} * (\chi_1 * \mu_{U,Q})(0) = (2\pi)^{-2} \overset{\vee}{\overline{\widehat{(\chi_1 * \chi_1)}}} * (\widehat{\chi_1 * \mu_{U,Q}})(0) .$$

Hence

$$\begin{aligned}\int_U \lambda_1(Q(t))dt &= \check{\lambda}_1 * \mu_{U,Q}(0) = \lambda_1 * \mu_{U,Q}(0) \\ &= (\chi_1 * \chi_1) * (\chi_1 * \mu_{U,Q})(0) = \overset{\vee}{\overline{(\chi_1 * \chi_1)}} * (\chi_1 * \mu_{U,Q})(0) \\ &= (2\pi)^{-2} \overset{\vee}{\overline{\widehat{(\chi_1 * \chi_1)}}} * (\widehat{\chi_1 * \mu_{U,Q}})(0) = (2\pi)^{-2} \overset{\vee}{\overline{(\hat{\chi}_1^2)}} * (\hat{\chi}_1 \hat{\mu}_{U,Q})(0) \\ &= (2\pi)^{-2} \int_C \hat{\chi}_1(z)^3 \hat{\mu}_{U,Q}(z) d\sigma(z) = (2\pi)^{-2} \int_C \hat{\lambda}_1(z) \Phi[U; Q](z) d\sigma(z) .\end{aligned}$$

2.5. Hausdorff dimensions

To discuss thin sets in $[0, 2\pi)$, we introduce Hausdorff dimensions. Let $0 < \alpha \le 1$, $\eta > 0$ and U a Borel set in $[0, 2\pi)$. We consider all coverings of U with a countable number of open intervals (α_ν, β_ν) satisfying $0 \le \alpha_\nu < \beta_\nu < 2\pi$ and $\beta_\nu - \alpha_\nu \le \eta$; and define $\Lambda_\alpha^\eta(U) = \inf \sum (\beta_\nu - \alpha_\nu)^\alpha$ for all such coverings. Since $\Lambda_\alpha^\eta(U)$ is increasing when $\eta \downarrow 0$, the limit $\Lambda_\alpha(U) = \lim_{\eta \to 0} \Lambda_\alpha^\eta(U)$ $(\le +\infty)$ exists and it is the α-dimensional Hausdorff measure. The Hausdorff dimension of U is defined by $\dim(U) = \sup\{\alpha;\ 0 < \alpha \le 1,\ \Lambda_\alpha(U) > 0\}$. We see that the 1-dimensional Hausdorff measure is the 1-dimensional Lebesgue measure and that the Haudorff dimension of a countable set is 0. We also note that, for a Borel set U satisfying $0 < \dim(U) < 1$, $\Lambda_\alpha(U) = +\infty$ and $\Lambda_\beta(U) = 0$ as long as $\alpha < \dim(U) < \beta$. (See [3].)

LEMMA 7. *Let k_0 be a positive integer, $E = (n_k)_{k=1}^\infty$ an L-lacunary series, $(\lambda_k)_{k=1}^\infty$ a decreasing sequence of positive numbers satisfying $\lambda_1 \le 1$ and $\lim_{k\to\infty} (\log 1/\lambda_k)/\log n_k = 0$ and let $(U_k)_{k=k_0}^\infty$ be a sequence of closed sets in $[0, 2\pi)$ satisfying the following four conditions*:

$$U_k \supset U_{k+1} \tag{10}$$

(11) *U_k is a finite union of closed intervals. (Let us write $U_k = \bigcup_{j=1}^{p(k)} \gamma_{k,j}$, where $\gamma_{k,j}$'s are mutually disjoint closed intervals.)*

(12) $$m(\gamma_{k,j}) = \lambda_k n_k^{-1} \qquad (j = 1, \cdots, p(k))$$

(13) *For every $1 \leq j \leq p(k)$, $\gamma_{k,j} - U_{k+1}$ is a union of open intervals of length $(2\pi - \lambda_{k+1})n_{k+1}^{-1}$ and at most two semi-open intervals of length $\leq 2\pi n_{k+1}^{-1}$.*

Then, with $U = \bigcap_{k=k_0}^{\infty} U_k$, $\dim(U) = 1$.

Proof. For a given $0 < \alpha < 1$, we shall show that $\Lambda_\alpha(U) = +\infty$. Put $\delta = 1/(1-\alpha)$ and

(14) $$\theta_\alpha = \max\{32\pi, (8\pi)^{1/(1-\alpha)}(1 - 2^{-1+\alpha})^{-1/(1-\alpha)}\}\ .$$

By (6) and $\lim_{k\to\infty} \log 1/\lambda_k/\log n_k = 0$, there exists an integer $k_\alpha \geq k_0$ such that, for $k \geq k_\alpha$,

(15) $$\begin{cases} \lambda_k n_k^{-1} \geq 8\pi \sum_{\ell=k+1}^{\infty} n_\ell^{-1} \\ \lambda_{k+1}^{\alpha} n_{k+1}^{1-\alpha} n_k^{-1+\alpha} \geq 2^{2+\alpha}\pi^\alpha(1 - 2^{-1+\alpha})^{-1} \\ \lambda_{k+1}^{\delta+1+\alpha} n_{k+1}^{1-\alpha} n_k^{-1+\alpha} \geq 2^{2+2\alpha}\pi^{1+\alpha}\theta_\alpha\ . \end{cases}$$

Set $q(k,j) = {}^{\#}\{\gamma_{k+1,\nu};\ \gamma_{k+1,\nu} \subset \gamma_{k,j},\ \nu = 1, \cdots, p(k+1)\}$ $(k \geq k_\alpha)$, where ${}^{\#}\{\cdot\}$ denotes the cardinal number of $\{\cdot\}$. Then

$$\begin{aligned} \lambda_k n_k^{-1} = m(\gamma_{k,j}) &\leq q(k,j)\lambda_{k+1} n_{k+1}^{-1} + (q(k,j) - 1)(2\pi - \lambda_{k+1})n_{k+1}^{-1} + 4\pi n_{k+1}^{-1} \\ &\leq 2\pi q(k,j) n_{k+1}^{-1} + 4\pi n_{k+1}^{-1}\ , \end{aligned}$$

and hence, from (15),

(16) $$q(k,j) \geq (2\pi)^{-1}\lambda_k n_{k+1} n_k^{-1} - 2 \geq (4\pi)^{-1}\lambda_k n_{k+1} n_k^{-1}\ .$$

We have also

(17) $$\begin{aligned} p(k) &= \sum_{j=1}^{p(k-1)} q(k-1, j) \geq (4\pi)^{-1}\lambda_{k-1} n_k n_{k-1}^{-1} p(k-1) \\ &\geq \cdots \geq (4\pi)^{-k+k_\alpha}\lambda_{k-1}\cdots\lambda_{k_\alpha} n_k n_{k_\alpha}^{-1} p(k_\alpha) \\ &\geq (4\pi)^{-k}\lambda_k \cdots \lambda_1 n_k n_{k_\alpha}^{-1} \qquad (k \geq k_\alpha)\ . \end{aligned}$$

For the proof of $\Lambda_\alpha(U) = +\infty$, we need the estimation of $\Lambda_\alpha^{\eta_k}(U_K)$ from below, where $k_\alpha \leq k \leq K$ and $\eta_k = \lambda_k n_k^{-1}$. For the estimation, we fix for a while three integers k, j and K $(k_\alpha \leq k \leq K,\ 1 \leq j \leq p(k))$.

For every $\gamma_{\mu,\nu}$ ($k \le \mu \le K$, $1 \le \nu \le p(\mu)$), we denote by $\tilde{\gamma}_{\mu,\nu}$ the smallest closed interval which contains $\cup \{\gamma_{K,\ell};\ \gamma_{K,\ell} \subset \gamma_{\mu,\nu},\ \ell = 1, \cdots, p(K)\}$. Put $V_\mu = \cup \{\tilde{\gamma}_{\mu,\nu};\ \gamma_{\mu,\nu} \subset \gamma_{k,j},\ \nu = 1, \cdots, p(\mu)\}$. We shall inductively show that

$$\Lambda_\alpha^{\eta_k}(V_\mu) = m(\tilde{\gamma}_{k,j})^\alpha \qquad (k \le \mu \le K)\ . \tag{18}$$

Since $V_k = \tilde{\gamma}_{k,j}$, (18) holds for $\mu = k$. Suppose that, for $k \le \mu < K$, (18) holds. Since $V_{\mu+1}$ is a finite union of closed intervals, there exists a finite covering $\{\Delta_\tau\}_{\tau=1}^N$ of $V_{\mu+1}$ by closed intervals of length $\le \eta_k$ such that

$$\Lambda_\alpha^{\eta_k}(V_{\mu+1}) = \sum_{\tau=1}^N m(\Delta_\tau)^\alpha\ . \tag{19}$$

Then $\Delta_\tau \cap \Delta_{\tau'} = \emptyset$ ($\tau \ne \tau'$) and

$$\begin{aligned} m(\Delta_\tau) &\ge \min\{m(\tilde{\gamma}_{\mu+1,\nu});\ \nu = 1, \cdots, p(\mu+1)\} \\ &\ge \lambda_{\mu+1} n_{\mu+1}^{-1} - 4\pi \sum_{\ell=\mu+2}^K n_\ell^{-1} \ge 2^{-1}\lambda_{\mu+1} n_{\mu+1}^{-1}\ , \end{aligned} \tag{20}$$

according to (15). For the proof of (18), it is sufficient to show that $\{\Delta_\tau\}_{\tau=1}^N$ is a covering of V_μ, since

$$m(\tilde{\gamma}_{k,j})^\alpha \ge \Lambda_\alpha^{\eta_k}(V_{\mu+1}) = \sum_{\tau=1}^N m(\Delta_\tau)^\alpha \ge \Lambda_\alpha^{\eta_k}(V_\mu) = m(\tilde{\gamma}_{k,j})^\alpha\ .$$

The following three cases are possible:

(a) $\{\Delta_\tau\}_{\tau=1}^N$ is a covering of V_μ.

(b) There exist Δ_τ, $\tilde{\gamma}_{\mu,\nu_1}$ and $\tilde{\gamma}_{\mu,\nu_2}$ such that $\Delta_\tau \cap \tilde{\gamma}_{\mu,\nu_\ell} \ne \emptyset$ ($\ell = 1, 2$) and $\Delta_\tau \subsetneqq \tilde{\gamma}_{\mu,\nu_2}$.

(c) There exist a subset $\{\Delta_{\tau_m}\}_{m=1}^{N'}$ of $\{\Delta_\tau\}_{\tau=1}^N$ and $\tilde{\gamma}_{\mu,\nu_3}$ such that $\cup \{\tilde{\gamma}_{\mu+1,\nu};\ \tilde{\gamma}_{\mu+1,\nu} \subset \tilde{\gamma}_{\mu,\nu_3}\} \subset \bigcup_{m=1}^{N'} \Delta_{\tau_m}$.

Suppose that (b) exists and let Δ_τ the interval in (b). Since $\lambda_\mu \le 1$, $m(\Delta_\tau) \ge (2\pi - \lambda_\mu) n_\mu^{-1} \ge \pi n_\mu^{-1}$. Let $\Delta_{\tau'}$ be an adjacent interval. Then, denoting by γ' the open interval which connects Δ_τ and $\Delta_{\tau'}$, we have, from (15),

$$m(\gamma') \le (2\pi - \lambda_{\mu+1}) n_{\mu+1}^{-1} + 4\pi \sum_{\ell=\mu+2}^K n_\ell^{-1} \le 4\pi n_{\mu+1}^{-1}\ .$$

Note that $\xi^\alpha + (1 - \xi - \varepsilon)^\alpha > 1$ as long as $0 < \xi \le 1/2$, $0 \le \varepsilon \le 1/4$ and $\xi^\alpha \varepsilon^{-1} \ge (1 - 2^{-1+\alpha})^{-1}$. Replacing Δ_τ by $\Delta_{\tau'}$ if necessary, we may assume that $m(\Delta_{\tau'}) \le m(\Delta_\tau)$. Putting $\xi = m(\Delta_{\tau'}) m(\Delta_\tau \cup \gamma' \cup \Delta_{\tau'})^{-1}$ and $\varepsilon = m(\gamma') m(\Delta_\tau \cup \gamma' \cup \Delta_{\tau'})^{-1}$, we have, from (15),

$$\varepsilon \le m(\gamma') m(\Delta_\tau)^{-1} \le 4\pi n_{\mu+1}^{-1} \pi^{-1} n_\mu = 4 n_{\mu+1}^{-1} n_\mu \le 1/4$$

and

$$\begin{aligned}\xi^\alpha\varepsilon^{-1} &= m(\Delta_\tau \cup \gamma' \cup \Delta_{\tau'})^{1-\alpha} m(\Delta_{\tau'})^\alpha m(\gamma')^{-1} \\ &\geq m(\Delta_\tau)^{1-\alpha} m(\Delta_{\tau'})^\alpha m(\gamma')^{-1} \geq \pi^{1-\alpha} n_\mu^{-1+\alpha} 2^{-\alpha} \lambda_{\mu+1}^\alpha n_{\mu+1}^{-\alpha} (4\pi)^{-1} n_{\mu+1} \\ &= 2^{-2-\alpha}\pi^{-\alpha}\lambda_{\mu+1}^\alpha n_{\mu+1}^{1-\alpha} n_\mu^{-1+\alpha} \geq (1-2^{-1+\alpha})^{-1} ,\end{aligned}$$

and hence

$$\{m(\Delta_{\tau'})^\alpha + m(\Delta_\tau)^\alpha\} m(\Delta_\tau \cup \gamma' \cup \Delta_{\tau'})^{-\alpha} = \xi^\alpha + (1-\xi-\varepsilon)^\alpha > 1 ,$$

that is, $m(\Delta_{\tau'})^\alpha + m(\Delta_\tau)^\alpha > m(\Delta_\tau \cup \gamma' \cup \Delta_{\tau'})^\alpha$, which contradicts (19).

Suppose that (c) exists and let $\{\Delta_{\tau_m}\}_{m=1}^{N'}$ be the set in (c). Then, for all Δ_{τ_m}, ${}^\#\{\tilde{\gamma}_{\mu+1,\nu};\ \tilde{\gamma}_{\mu+1,\nu} \subset \Delta_{\tau_m}\} \leq \theta_\alpha \lambda_{\mu+1}^{-\delta}$ ($\delta = 1/(1-\alpha)$). If this does not hold, there exists $\Delta_{\tau_{m'}}$ such that ${}^\#\{\tilde{\gamma}_{\mu+1,\nu};\ \tilde{\gamma}_{\mu+1,\nu} \subset \Delta_{\tau_{m'}}\} > \theta_\alpha \lambda_{\mu+1}^{-\delta}$. Let $\Delta_{\tau_{m''}}$ be an adjacent interval and γ'' the open interval which connects $\Delta_{\tau_{m'}}$ and $\Delta_{\tau_{m''}}$. Then $m(\gamma'') \leq 4\pi n_{\mu+1}^{-1}$. Put $\xi = \kappa m(\Delta_{\tau_{m'}} \cup \gamma'' \cup \Delta_{\tau_{m''}})^{-1}$ and $\varepsilon = m(\gamma'') m(\Delta_{\tau_{m'}} \cup \gamma'' \cup \Delta_{\tau_{m''}})^{-1}$, where $\kappa = \min\{m(\Delta_{\tau_{m'}}), m(\Delta_{\tau_{m''}})\}$. By (14), we have

$$\varepsilon \leq 4\pi n_{\mu+1}^{-1} 2\lambda_{\mu+1}^{-1} n_{\mu+1} \theta_\alpha^{-1} \lambda_{\mu+1}^\delta \leq 8\pi\theta_\alpha^{-1} \leq 1/4$$

and

$$\begin{aligned}\xi^\alpha\varepsilon^{-1} &\geq m(\Delta_{\tau_{m'}})^{1-\alpha}\kappa^\alpha m(\gamma'')^{-1} \\ &\geq \theta_\alpha^{1-\alpha}\lambda_{\mu+1}^{-\delta(1-\alpha)} 2^{-1+\alpha}\lambda_{\mu+1}^{1-\alpha} n_{\mu+1}^{-1+\alpha} 2^{-\alpha}\lambda_{\mu+1}^\alpha n_{\mu+1}^{-\alpha}(4\pi)^{-1} n_{\mu+1} \\ &= (8\pi)^{-1}\theta_\alpha^{1-\alpha} \geq (1-2^{-1+\alpha})^{-1} ,\end{aligned}$$

and hence $m(\Delta_{\tau_{m'}})^\alpha + m(\Delta_{\tau_{m''}})^\alpha > m(\Delta_{\tau_{m'}} \cup \gamma'' \cup \Delta_{\tau_{m''}})^\alpha$, which contradicts (19).

Hence, for all Δ_{τ_m}, ${}^\#\{\tilde{\gamma}_{\mu+1,\nu};\ \tilde{\gamma}_{\mu+1,\nu} \subset \Delta_{\tau_m}\} \leq \theta_\alpha\lambda_{\mu+1}^{-\delta}$. Let $\cup\{\tilde{\gamma}_{\mu+1,\nu};\ \tilde{\gamma}_{\mu+1,\nu} \subset \tilde{\gamma}_{\mu,\nu_3}\} \subset \bigcup_{m=1}^{N'} \Delta_{\tau_m}$. Then we have $N'\theta_\alpha\lambda_{\mu+1}^{-\delta} \geq q(\mu, \nu_3) \geq (4\pi)^{-1}\lambda_\mu n_{\mu+1} n_\mu^{-1}$ and hence $N' \geq (4\pi)^{-1}\theta_\alpha^{-1}\lambda_{\mu+1}^{\delta+1} n_{\mu+1} n_\mu^{-1}$. By (15), we have

$$\begin{aligned}\sum_{m=1}^{N'} m(\Delta_{\tau_m})^\alpha &\geq N' 2^{-\alpha}\lambda_{\mu+1}^\alpha n_{\mu+1}^{-\alpha} \geq 2^{-2-\alpha}\pi^{-1}\theta_\alpha^{-1}\lambda_{\mu+1}^{\delta+1+\alpha} n_{\mu+1}^{1-\alpha} n_\mu^{-1} \\ &= \{2^{-2-2\alpha}\pi^{-1-\alpha}\theta_\alpha^{-1}\lambda_{\mu+1}^{\delta+1+\alpha} n_{\mu+1}^{1-\alpha} n_\mu^{-1+\alpha}\}(2\pi n_\mu^{-1})^\alpha \geq (2\pi n_\mu^{-1})^\alpha > m(\tilde{\gamma}_{\mu,\nu_3})^\alpha ,\end{aligned}$$

which contradicts (19). This shows that (18) holds for $\mu + 1$. Consequently, (18) holds for $\mu = k, \cdots, K$.

In particular, $\Lambda_\alpha^{\eta_k}(V_K) = m(\tilde{\gamma}_{k,j})^\alpha$. Hence we have, from (17),

$$\begin{aligned}\Lambda_\alpha^{\eta_k}(U_K) &= \sum_{j=1}^{p(k)} m(\tilde{\gamma}_{k,j})^\alpha \geq p(k)\min\{m(\tilde{\gamma}_{k,j})^\alpha;\ j = 1, \cdots, p(k)\} \\ &\geq (4\pi)^{-k}\lambda_k \cdots \lambda_1 n_k n_{k_\alpha}^{-1} 2^{-\alpha}\lambda_k^\alpha n_k^\alpha\ (= 2^{-\alpha} n_{k_\alpha}^{-1} A_\alpha(k),\ say) .\end{aligned}$$

Since the last term is independent of K, we have $\Lambda_\alpha^{\eta_k}(U) = \lim_{K\to\infty}\Lambda_\alpha^{\eta_k}(U_K)$

$\geq 2^{-\alpha} n_{k_\alpha}^{-1} A_\alpha(k)$ $(k \geq k_\alpha)$. Hence, to prove $\Lambda_\alpha(U) = +\infty$, it is sufficient to show that $\lim_{k\to\infty} \log A_\alpha(k) = +\infty$.

First we note that

$$\begin{aligned}\log A_\alpha(k) &= (1-\alpha)\log n_k - \alpha \log 1/\lambda_k - \sum_{j=1}^{k} \log 1/\lambda_j - k \log 4\pi \\ &\geq (1-\alpha)\log n_k - 2\sum_{j=1}^{k} \log 1/\lambda_j - 10k \qquad (k \geq k_\alpha).\end{aligned}$$

Let us remember the notation $q(E)$ in Lemma 5. We have $\log n_{k+1} \geq q(E) \log n_k$ $(k \geq 1)$. Set $\varepsilon_{\alpha,E} = ((1-\alpha)/4)(q(E)-1)/q(E)$. Since $\lim_{k\to\infty} \log 1/\lambda_k / \log n_k = 0$, there exists a positive number θ_4 such that $\log 1/\lambda_k \leq \varepsilon_{\alpha,E} \log n_k + \theta_4$ $(k \geq 1)$. We have, for any $k \geq k_\alpha$,

$$\begin{aligned}2\sum_{j=1}^{k} \log 1/\lambda_j &\leq 2\varepsilon_{\alpha,E} \sum_{j=1}^{k} \log n_j + 2\theta_4 k \\ &\leq 2\varepsilon_{\alpha,E} \sum_{j=1}^{k} q(E)^{-k+j} \log n_k + 2\theta_4 k \leq \frac{1-\alpha}{2} \log n_k + 2\theta_4 k,\end{aligned}$$

and hence

$$\log A_\alpha(k) \geq \frac{1-\alpha}{2} \log n_k - 2(\theta_4 + 5)k.$$

Since $\lim_{k\to\infty} \log n_k / k = +\infty$, we have $\lim_{k\to\infty} \log A_\alpha(k) = +\infty$. This completes the proof.

§ 3. Main lemmas

In this chapter, we prepare some lemmas in which Lemma 18, 20 and Corollary 22 will play important roles throughout this paper.

LEMMA 8. ([19], p. 216 Lemma (8.26)) *Let $(\Omega, \mathscr{B}, \mathscr{P})$ be a probability space and $X(\omega)$ a non-negative Borel function in Ω. Then, for any $0 \leq \eta < 1$,*

$$\mathscr{P}\Big(\Big\{\omega \in \Omega;\, X(\omega) > \eta \int X(\omega) d\mathscr{P}(\omega)\Big\}\Big) \geq (1-\eta)^2 \Big(\int X(\omega) d\mathscr{P}(\omega)\Big)^2 \Big(\int X(\omega)^2 d\mathscr{P}(\omega)\Big)^{-1}, \tag{21}$$

COROLLARY 9. *Let $X(t)$ be a non-negative Borel function in $[0, 2\pi)$ and U a Borel set in $[0, 2\pi)$. Then*

$$m(\{t \in U;\, X(t) > 0\}) \geq \Big(\int_U X(t)dt\Big)^2 \Big(\int_U X(t)^2 dt\Big)^{-1}. \tag{22}$$

LEMMA 10. *Let $Q(t)$ be a non-constant Taylor polynomial. Then, for any $\rho > 0$, a set $\{t \in [0, 2\pi); |Q(t)| < \rho\}$ is a finite union of at most* $\deg(Q)$ *open intervals.*

Proof. Set $N = \deg(Q)$. It is evident that a set $U = \{t \in [0, 2\pi); |Q(t)| < \rho\} = \{t \in [0, 2\pi); |Q(t)|^2 < \rho^2\}$ is a finite union of open intervals. Let us write $U = \bigcup_{\mu=1}^{\nu} (\alpha_\mu, \beta_\mu)$ and $U^c = \bigcup_{\mu=1}^{\nu} [\alpha'_\mu, \beta'_\mu]$, where (α_μ, β_μ), $[\alpha'_\mu, \beta'_\mu]$ $(1 \leq \mu \leq \nu)$ are mutually disjoint intervals. There exist $t_\mu \in (\alpha_\mu, \beta_\mu)$, $t'_\mu \in [\alpha'_\mu, \beta'_\mu]$ $(1 \leq \mu \leq \nu)$ such that $d/dt \cdot |Q(t_\mu)|^2 = d/dt \cdot |Q(t'_\mu)|^2 = 0$. Since $|Q(t)|^2$ is a real-valued polynomial of degree $\leq N$, we have $^{\#}\{t \in [0, 2\pi); d/dt \cdot |Q(t)|^2 = 0\} \leq 2N$ ([2], p. 192) and hence $2\nu \leq 2N$, that is, $\nu \leq N$.

LEMMA 11. *Let n be a positive integer, $y > 0$ and let $P(t)$ an infinitely differentiable real-valued function in a neighborhood of a closed interval $[\alpha, \beta]$. Suppose that, for any $t \in [\alpha, \beta]$, $|P^{(n)}(t)| \geq y$.* Then

$$m(\{t \in [\alpha, \beta]; |P(t)| \leq \rho\}) \leq 4^n \rho^{1/n} y^{-1/n} \qquad (\rho > 0) . \tag{23}$$

Proof. We show inductively (23). For a fixed $\rho > 0$, set $M(\alpha', \beta') = m(\{t \in [\alpha', \beta']; |P(t)| \leq \rho\})$. Since $P^{(n)}(t) \neq 0$ $(t \in [\alpha, \beta])$, $M(\gamma, \gamma) = 0$ $(\gamma \in [\alpha, \beta])$. In the case of $n = 1$, we have $\{t \in [\alpha, \beta]; |P(t)| \leq \rho\} \subset [\gamma_0 - \rho/y, \gamma_0 + \rho/y]$, where $|P(\gamma_0)| = \min_{\alpha \leq t \leq \beta} |P(t)|$, and hence $M(\alpha, \beta) \leq 2\rho/y \leq 4\rho y^{-1}$.

Suppose that, for $n-1$, (23) holds. We show that, for n, (23) holds. Without loss of generality, we may assume that $P^{(n)}(t) \geq y$ $(t \in [\alpha, \beta])$. Then $P^{(n-1)}(t)$ is increasing. Let γ_0 $(\alpha \leq \gamma_0 \leq \beta)$ be a number satisfying $|P^{(n-1)}(\gamma_0)| = \min_{\alpha \leq t \leq \beta} |P^{(n-1)}(t)|$. We choose two numbers γ_1 and γ_2 so that $\gamma_1 < \gamma_0 < \gamma_2$ and that $4^{n-1} \rho^{1/(n-1)} y^{-1/(n-1)} |\gamma_0 - \gamma_j|^{-1/(n-1)} = |\gamma_0 - \gamma_j|$ $(j = 1, 2)$. Set $\gamma'_1 = \max\{\alpha, \gamma_1\}$ and $\gamma'_2 = \min\{\beta, \gamma_2\}$. In the case of $\alpha < \gamma'_1$, we have $P^{(n-1)}(t) \leq -y(\gamma_0 - \gamma'_1)$ on $[\alpha, \gamma'_1]$. By the assumption,

$$M(\alpha, \gamma'_1) \leq 4^{n-1} \rho^{1/(n-1)} y^{-1/(n-1)} (\gamma_0 - \gamma'_1)^{-1/(n-1)} .$$

In the case of $\alpha = \gamma'_1$, this inequality evidently holds since $M(\alpha, \gamma'_1) = 0$.

On the other hand, $M(\gamma'_1, \gamma_0) \leq \gamma_0 - \gamma'_1 \leq \gamma_0 - \gamma_1$. Since $\gamma_0 - \gamma_1 = 4^{(n-1)^2/n} \rho^{1/n} y^{-1/n}$, we have

$$M(\alpha, \gamma_0) = M(\alpha, \gamma'_1) + M(\gamma'_1, \gamma_0) \leq 2 \cdot 4^{(n-1)^2/n} \rho^{1/n} y^{-1/n} .$$

Analogously, $M(\gamma_0, \beta) \leq 2 \cdot 4^{(n-1)^2/n} \rho^{1/n} y^{-1/n}$. Hence

$$M(\alpha, \beta) = M(\alpha, \gamma_0) + M(\gamma_0, \beta) \leq 4^n \rho^{1/n} y^{-1/n} .$$

Elementary calculus gives the following

LEMMA 12. *Let $\phi(t)$ and $\psi(t)$ be two continuously differentiable functions in $[0, 2\pi)$. Then, for any positive integer n and any interval I in $[0, 2\pi)$,*

$$\begin{aligned}(24)\quad &\left|\frac{1}{2\pi}\int_I \phi(t)\psi(nt)dt - \frac{1}{2\pi}\int_I \phi(t)dt\frac{1}{2\pi}\int_0^{2\pi}\psi(t)dt\right| \\ &\leq \frac{2m(I)}{n}\sup_{t\in[0,2\pi)}|\phi'(t)|\sup_{t\in[0,2\pi)}|\psi(t)| + \frac{8\pi}{n}\sup_{t\in[0,2\pi)}|\phi(t)|\sup_{t\in[0,2\pi)}|\psi'(t)| .\end{aligned}$$

LEMMA 13. *Let $\Xi = (u_k)_{k=1}^K$ be a decreasing sequence of positive numbers such that $K \geq 64$ and $u_1 \leq 1/4\cdot\|\Xi\| = (1/4)(\sum_{k=1}^K c_k^2)^{1/2}$. Then*

$$(25)\qquad v(\eta;\Xi) = \int_{\eta\leq|z|}|J(z;\Xi)|d\sigma(z) \leq 10^6\eta^{-2}\|\Xi\|^{-4} \qquad (\eta > 0)$$

and

$$(26)\qquad v(\Xi) = \int_C |J(z;\Xi)|d\sigma(z) \leq 10^7\|\Xi\|^{-2} .$$

Proof. We first show (25). Putting $d_k = u_k\|\Xi\|^{-1}$ $(1 \leq k \leq K)$ and $\Xi' = (d_k)_{k=1}^K$, we have $v(\eta;\Xi) = \|\Xi\|^{-2}v(\eta\|\Xi\|;\Xi')$ and hence it is sufficient to show that

$$(27)\qquad (2\pi)^{-1}v(\eta';\Xi') = \int_{\eta'}^\infty |J(r;\Xi')|r\,dr \leq 10^5\eta'^{-2} \qquad (\eta' > 0) .$$

Let us note that $\sum_{k=1}^K d_k^2 = 1$ and $\sum_{k=1}^8 d_k^2 \leq 8d_1^2 \leq 8u_1^2\|\Xi\|^{-2} \leq 1/2$. Set $j = \min\{\ell; \sum_{k=9}^\ell d_k^2 \geq 1/4\}$. First suppose that $d_j^{-1} \leq \eta'$. Since Ξ' is decreasing, we have, for $1 \leq k \leq 8$,

$$d_k \geq d_8 \geq \frac{1}{j-8}\sum_{\ell=9}^j d_\ell \geq \frac{1}{j-8}\left(\sum_{\ell=9}^j d_\ell^2\right)^{1/2} \geq \frac{1}{2(j-8)} \geq \frac{1}{2j} ,$$

and hence

$$\begin{aligned}(2\pi)^{-1}v(\eta';\Xi') &\leq \int_{\eta'}^\infty \prod_{k=1}^j |J_0(d_k r)|r\,dr \leq \left(\frac{2}{\pi}\right)^{j/2}\prod_{k=1}^j d_k^{-1/2}\int_{\eta'}^\infty r^{-j/2+1}dr \\ &= \frac{2}{j-4}\left(\frac{2}{\pi}\right)^{j/2}\prod_{k=1}^j d_k^{-1/2}\eta'^{-j/2+2} \leq \frac{2}{j-4}\left(\frac{2}{\pi}\right)^{j/2}\prod_{k=1}^8 d_k^{-1/2}\eta'^{-2} \\ &\leq \frac{2}{j-4}(2j)^4\left(\frac{2}{\pi}\right)^{j/2}\eta'^{-2} \leq 2^6 j^3\left(\frac{2}{\pi}\right)^{j/2}\eta'^{-2} \\ &\leq 2^6 e^{-3}\left(\frac{6}{\log\pi/2}\right)^3\eta'^{-2} \leq 10^4\eta'^{-2} ,\end{aligned}$$

which shows that (27) holds in the case of $d_j^{-1} \leq \eta'$. Next suppose that

$d_j^{-1} > \eta'$. Since

$$\sum_{k=j}^{K} d_k^2 = 1 - \sum_{k=1}^{j-1} d_k^2 \geq 1 - \frac{3}{4} = \frac{1}{4},$$

we have

$$\begin{aligned}\int_{\eta'}^{1/d_j} \prod_{k=1}^{K} |J_0(d_k r)| r\,dr &\leq \int_{\eta'}^{1/d_j} \prod_{k=j}^{K} |J_0(d_k r)| r\,dr \\ &\leq \int_{\eta'}^{1/d_j} \prod_{k=j}^{K} \left(1 - \frac{1}{8} d_k^2 r^2\right) r\,dr \leq \int_{\eta'}^{1/d_j} \exp\left(-\frac{1}{8}\sum_{k=j}^{K} d_k^2 r^2\right) r\,dr \\ &\leq \int_{\eta'}^{\infty} \exp\left(-\frac{1}{32} r^2\right) r\,dr = 16 \exp\left(-\frac{1}{32}\eta'^2\right) \leq 10^3 \eta'^{-2}.\end{aligned}$$

By the same method as above, we have

$$\int_{1/d_j}^{\infty} \prod_{k=1}^{K} |J_0(d_k r)| r\,dr \leq \int_{1/d_j}^{\infty} \prod_{k=1}^{j} |J_0(d_k r)| r\,dr \leq 10^4 (1/d_j)^{-2}.$$

Hence

$$(2\pi)^{-1} v(\eta'; \Xi') = \int_{\eta'}^{1/d_j} + \int_{1/d_j}^{\infty} \leq 10^3 \eta'^{-2} + 10^4 (1/d_j)^{-2} \leq 10^5 \eta'^{-2},$$

which shows that (27) holds in the case of $d_j^{-1} > \eta'$. Consequently, we have (25).

Inequality (26) is an immediate result of (25) since

$$\begin{aligned} v(\Xi) &= 2\pi \int_0^{\|\Xi\|^{-1}} |J(r; \Xi)| r\,dr + v(\|\Xi\|^{-1}; \Xi) \\ &\leq 2\pi \int_0^{\|\Xi\|^{-1}} r\,dr + 10^6 \|\Xi\|^{-2} \leq 10^7 \|\Xi\|^{-2}.\end{aligned}$$

LEMMA 14. *For a sequence* $\Xi = (u_k)_{k=1}^{K}$ $(K \geq 5)$ *of positive numbers, we define*

$$u_2(t) = (u_2^2 + u_1^2 - 2u_2 u_1 \cos t)^{1/2} \qquad (t \in [0, 2\pi))$$

and

$$u_k(t_1, \cdots, t_{k-1}) = (u_k^2 + u_{k-1}(t_1, \cdots, t_{k-2})^2 - 2u_k u_{k-1}(t_1, \cdots, t_{k-2}) \cos t_{k-1})^{1/2}$$
$$(t_j \in [0, 2\pi)\ (1 \leq j \leq k-1),\ 3 \leq k \leq K).$$

Then

(28) $$m_{K-1}(\{(t_1, \cdots, t_{K-1}) \in [0, 2\pi)^{K-1};\ u_K(t_1, \cdots, t_{K-1})^2 > \tfrac{1}{2}\|\Xi\|^2\}) \geq (2\pi)^{K-1}/8,$$

where m_{K-1} *denotes the* $(K-1)$*-dimensional Lebesgue measure.*

Proof. We shall use Lemma 8 for $\Omega = (0, 2\pi)^{K-1}$, $X(\omega) = u_K(t_1, \cdots, t_{K-1})^2$ $(\omega = (t_1, \cdots, t_{K-1}))$ and $\eta = 1/2$. We have

$$(2\pi)^{-K+1}\int_0^{2\pi}\cdots\int_0^{2\pi} u_K(t_1, \cdots, t_{K-1})^2 dt_1 \cdots dt_{K-1}$$
$$= u_K^2 + (2\pi)^{-K+2}\int_0^{2\pi}\cdots\int_0^{2\pi} u_{K-1}(t_1, \cdots, t_{K-2})^2 dt_1 \cdots dt_{K-2} = \cdots = \|\Xi\|^2 .$$

Since

$$u_K(t_1, \cdots, t_{K-1})^4 = \{u_K^2 + u_{K-1}(t_1, \cdots, t_{K-2})^2 - 2u_K u_{K-1}(t_1, \cdots, t_{K-2}) \cos t_{K-1}\}^2$$
$$= u_K^4 + u_{K-1}(t_1, \cdots, t_{K-2})^4 + 4u_K^2 u_{K-1}(t_1, \cdots, t_{K-2})^2 \cos^2 t_{K-1}$$
$$+ 2u_K^2 u_{K-1}(t_1, \cdots, t_{K-2})^2 - 4u_K^3 u_{K-1}(t_1, \cdots, t_{K-2}) \cos t_{K-1}$$
$$- 4u_K u_{K-1}(t_1, \cdots, t_{K-2})^3 \cos t_{K-1} ,$$

we have

$$(2\pi)^{-K+1}\int_0^{2\pi}\cdots\int_0^{2\pi} u_K(t_1, \cdots, t_{K-1})^4 dt_1 \cdots dt_{K-1}$$
$$= u_K^4 + (2\pi)^{-K+2}\int_0^{2\pi}\cdots\int_0^{2\pi} u_{K-1}(t_1, \cdots, t_{K-2})^4 dt_1 \cdots dt_{K-2}$$
$$+ 4u_K^2(2\pi)^{-K+2}\int_0^{2\pi}\cdots\int_0^{2\pi} u_{K-1}(t_1, \cdots, t_{K-2})^2 dt_1 \cdots dt_{K-2}$$
$$= u_K^4 + 4u_K^2\sum_{k=1}^{K-1} u_k^2 + (2\pi)^{-K+2}\int_0^{2\pi}\cdots\int_0^{2\pi} u_{K-1}(t_1, \cdots, t_{K-2})^4 dt_1 \cdots dt_{K-2}$$
$$= \sum_{k=1}^{K} u_K^4 + 4\sum_{k=2}^{K} u_k^2 \sum_{j=1}^{k} u_j^2 \le 2\|\Xi\|^4 .$$

Hence

$$(2\pi)^{-K+1} m_{K-1}\Big(\Big\{(t_1, \cdots, t_{K-1}) \in [0, 2\pi)^{K-1};\ u_K(t_1, \cdots, t_{K-1})^2 > \frac{1}{2}\|\Xi\|^2\Big\}\Big)$$
$$\ge \frac{1}{4}\Big\{(2\pi)^{-K+1}\int_0^{2\pi}\cdots\int_0^{2\pi} u_K(t_1, \cdots, t_{K-1})^2 dt_1 \cdots dt_{K-1}\Big\}^2$$
$$\times \Big\{(2\pi)^{-K+1}\int_0^{2\pi}\cdots\int_0^{2\pi} u_K(t_1, \cdots, t_{K-1})^4 dt_1 \cdots dt_{K-1}\Big\}^{-1} \ge \frac{1}{8} .$$

LEMMA 15. *Let* $\Xi = (u_k)_{k=1}^K$ *be the sequence in Lemma* 13. *Then*

$$v_0(\Xi) = \int_C J(z; \Xi) d\sigma(z) \ge 10^{-1}\|\Xi\|^{-2} . \tag{29}$$

Proof. For a positive number $\eta > 0$, set $v_\eta(\Xi) = \int_C J(z; \Xi) e^{-\eta^2|z|^2}\, d\sigma(z)$. By (vi) and (vii), we have

$$v_\eta(\Xi) = (2\pi)^{-K+2}\int_0^{2\pi}\cdots\int_0^{2\pi} dt_1\cdots dt_{K-1}\int_0^\infty J_0(u_K(t_1,\cdots,t_{K-1})r)e^{-\eta^2r^2}r\,dr$$
$$\left(= (2\pi)^{-K+2}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{1}{2\eta^2}\exp\left(-\frac{1}{4\eta^2}u_K(t_1,\cdots,t_{K-1})^2\right)dt_1\cdots dt_{K-1}\right).$$

Since $J(z;\Xi)$ is $d\sigma$-integrable,

$$v_0(\Xi) = \lim_{\eta\to 0} v_\eta(\Xi) \geq 2(2\pi)^{-K+1}\int_0^{2\pi}\cdots\int_0^{2\pi} u_K(t_1,\cdots,t_{K-1})^{-2}dt_1\cdots dt_{K-1}.$$

Putting $\Omega' = \{(t_1,\cdots,t_{K-1})\in[0,2\pi)^{K-1};\ u_K(t_1,\cdots,t_{K-1})^2 > 1/2\cdot\|\Xi\|^{-2}\}$, we have, from Lemma 14, $v_0(\Xi) \geq (2\pi)^{-K+1}\|\Xi\|^{-2}m_{K-1}(\Omega') \geq 1/8\cdot\|\Xi\|^{-2} \geq 10^{-1}\|\Xi\|^{-2}$.

LEMMA 16. *Let $Q(t)$ be a Taylor polynomial such that $N_Q \geq 64$ and $4\nu(Q)\leq w(1;Q)$ and let J a positive integer such that $J\leq N_Q - 63$ and $4\nu(Q)\leq w(J;Q)$. Then, for any $\eta>0$, $M>0$ and any interval I in $[0,2\pi)$,*

$$\int_I \lambda_\eta(Q(t))dt \leq 10^8\{m(I)\eta^2 w(J;Q)^{-2} + (m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q))\eta^2\nu(Q)M^6 + m(I)\eta^{-1}M^{-2}\}. \tag{30}$$

Proof. We write simply $N = N_Q$, $n_k = n_k(Q)$, $\Phi_k(z) = \Phi[I;Q_k](z)$ $(0\leq k\leq N)$ and $\Phi(z) = \Phi_N(z)$. Using Lemma 6, we write $L = \int_I \lambda_\eta(Q(t))\,dt$ in the following form:

$$L = (2\pi)^{-2}\int_C \hat{\lambda}_\eta(z)\Phi(z)d\sigma(z)$$
$$= (2\pi)^{-2}\left\{\int_{|z|\leq M^2} + \int_{M^2<|z|}\right\} = (2\pi)^{-2}\{L_1 + L_2\}.$$

Since $|\Phi(z)|\leq m(I)$ and $|\hat{\lambda}_\eta(z)|\leq(2\pi)^3\eta^{-1}|z|^{-3}$ ((xi)), we have

$$|L_2| \leq m(I)(2\pi)^3\eta^{-1}\int_{M^2<|z|}|z|^{-3}d\sigma(z) \leq (2\pi)^4 m(I)\eta^{-1}M^{-2} \leq 10^9 m(I)\eta^{-1}M^{-2}. \tag{31}$$

Put $\Xi = (\tilde{Q}(k))_{k=J}^N$. By Lemma 12, we have, for $J-1\leq k\leq N-1$,

$$\frac{1}{2\pi}|\Phi_{k+1}(z) - \Phi_k(z)J_0(|\tilde{Q}(k+1)z|)|$$
$$= \left|\frac{1}{2\pi}\int_I \exp\{-i\operatorname{Re}\bar{z}Q_k(t)\}\exp\{-i\operatorname{Re}\bar{z}\tilde{Q}(k+1)e^{in_{k+1}t}\}dt\right.$$
$$\left. - \frac{1}{2\pi}\int_I \exp\{-i\operatorname{Re}\bar{z}Q_k(t)\}dt\frac{1}{2\pi}\int_0^{2\pi}\exp\{-i\operatorname{Re}\bar{z}\tilde{Q}(k+1)e^{it}\}dt\right|$$

$$\leq 2m(I)n_{k+1}^{-1} \sup_{t\in[0,2\pi)} \left|\frac{d}{dt}\exp\{-i\operatorname{Re}\bar{z}Q_k(t)\}\right| \sup_{t\in[0,2\pi)} |\exp\{-i\operatorname{Re}\bar{z}\tilde{Q}(k+1)e^{it}\}|$$

$$+ 8\pi n_{k+1}^{-1} \sup_{t\in[0,2\pi)} |\exp\{-i\operatorname{Re}\bar{z}Q_k(t)\}| \sup_{t\in[0,2\pi)} \left|\frac{d}{dt}\exp\{-i\operatorname{Re}\bar{z}\tilde{Q}(k+1)e^{it}\}\right|$$

$$\leq \left\{2m(I)n_{k+1}^{-1}\sum_{j=1}^{k}|\tilde{Q}(j)|\,n_j + 8\pi n_{k+1}^{-1}|\tilde{Q}(k+1)|\right\}|z|$$

$$\leq 10^2\Big(m(I)n_{k+1}^{-1}\sum_{j=1}^{k} n_j + n_{k+1}^{-1}\Big)\nu(Q)\,|z|\,,$$

and hence

$$(32)\qquad \begin{aligned} &|\Phi(z) - \Phi_{J-1}(z)J(z;\Xi)| \\ &\quad\leq |\Phi_N(z) - \Phi_{N-1}(z)J_0(|\tilde{Q}(N)z|)| \\ &\qquad + \sum_{k=J-1}^{N-2}\left|\Phi_{k+1}(z)\prod_{j=k+2}^{N}J_0(|\tilde{Q}(j)z|) - \Phi_k(z)\prod_{j=k+1}^{N}J_0(|\tilde{Q}(j)z|)\right| \\ &\quad\leq \sum_{k=J-1}^{N-1}|\Phi_{k+1}(z) - \Phi_k(z)J_0(|\tilde{Q}(k+1)z|)| \\ &\quad\leq 2\pi 10^2\{m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q)\}\nu(Q)\,|z| \\ &\quad\leq 10^3\{m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q)\}\nu(Q)\,|z|\,. \end{aligned}$$

Since $|\Phi_{J-1}(z)| \leq m(I)$ and $|\hat{\lambda}_\eta(z)| \leq \pi^3\eta^2 \leq 10^2\eta^2$ ((x)), we have, from (32),

$$\begin{aligned} |L_1| &\leq \int_{|z|\leq M^2}|\hat{\lambda}_\eta(z)\Phi(z)|\,d\sigma(z) \\ &\leq 10^2\eta^2\int_{|z|\leq M^2}\{|\Phi_{J-1}(z)J(z;\Xi)| + 10^3(m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q))\nu(Q)\,|z|\}d\sigma(z) \\ &\leq 10^2 m(I)\eta^2\int_{|z|\leq M^2}|J(z;\Xi)|\,d\sigma(z) \\ &\quad + 10^5\{m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q)\}\eta^2\nu(Q)\int_{|z|\leq M^2}|z|\,d\sigma(z) \\ &\leq 10^2 m(I)\eta^2 v(\Xi) + 2\pi/3\cdot 10^5\{m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q)\}\eta^2\nu(Q)M^6\,. \end{aligned}$$

Since $N - J + 1 \geq 64$ and $4\nu(Q) \leq w(J;Q)$, we have, from (23), $v(\Xi) \leq 10^7\|\Xi\|^{-2} = 10^7 w(J;Q)^{-2}$, and hence

$$(33)\qquad \begin{aligned} |L_1| &\leq 10^9 m(I)\eta^2 w(J;Q)^{-2} + 2\pi/3\cdot 10^5(m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q))\eta^2\nu(Q)M^6 \\ &\leq 10^9\{m(I)\eta^2 w(J;Q)^{-2} + (m(I)\gamma(J;Q) + \tilde{\gamma}(J;Q))\eta^2\nu(Q)M^6\}\,. \end{aligned}$$

Since $L = (2\pi)^{-2}\{L_1 + L_2\} \leq 10^{-1}\{|L_1| + |L_2|\}$, the required inequality (30) follows from (31) and (33).

LEMMA 17. *Let $Q(t)$ be a non-constant Taylor polynomial. Let us write* *simply* $N=N_Q$. *Then, for any* $0<\eta\le|\tilde{Q}(N)|/3$, $M>0$ *and any interval* I *in* $[0,2\pi)$,

$$\int_I \lambda_\eta(Q(t))dt \le 10^6\{m(I)\eta\,|\tilde{Q}(N)|^{-1}+(m(I)\gamma(N;Q)+\tilde{\gamma}(N;Q))\eta^2\nu(Q)M^6 + m(I)\eta^{-1}M^{-2}\}\,. \tag{34}$$

Proof. First we remark the following inequality:

$$1/2\pi\int_0^{2\pi}\chi_{\eta'}(\rho' e^{is}+a)ds\le\eta'/\rho'\ ,$$

where $0<\eta'\le\rho'$ and $a\in\boldsymbol{C}$. We use the notation L, L_1 and L_2 in the preceding lemma. We have

$$|L_2|\le(2\pi)^4 m(I)\eta^{-1}M^{-2}\le 1/2\cdot 10^6 m(I)\eta^{-1}M^{-2}\,. \tag{35}$$

Set $\rho=|\tilde{Q}(N)|$, $R(t)=Q(t)-\tilde{Q}(N)e^{iNt}$ and $X_t(s)=\rho e^{is}+R(t)$. We have

$$\begin{aligned}
L_3 &= \int_C \hat{\lambda}_\eta(z)\Phi[I;R](z)J_0(|\rho z|)d\sigma(z)\\
&= \int_C \hat{\lambda}_\eta(z)d\sigma(z)\int_I dt\frac{1}{2\pi}\int_0^{2\pi}\exp\{-i\operatorname{Re}\bar{z}(\rho e^{is}+R(t))\}ds\\
&= \int_C \hat{\lambda}_\eta(z)d\sigma(z)\int_I (2\pi)\Phi[(0,2\pi);X_t](z)dt\\
&= 2\pi\int_I dt\int_C \hat{\lambda}_\eta(z)\Phi[(0,2\pi);X_t](z)d\sigma(z)\\
&= (2\pi)^3\int_I dt\int_0^{2\pi}\lambda_\eta(X_t(s))ds=(2\pi)^4\int_I dt\frac{1}{2\pi}\int_0^{2\pi}\lambda_\eta(\rho e^{is}+R(t))ds\\
&\le (2\pi)^4\pi^2\int_I dt\frac{1}{2\pi}\int_0^{2\pi}\chi_{3\eta}(\rho e^{is}+R(t))ds\\
&\le (2\pi)^4 3\pi^2 m(I)\eta/\rho\le 10^6 m(I)\eta/\rho\,.
\end{aligned}$$

By (32), we have

$$|\Phi[I;Q](z)-\Phi[I;R](z)J_0(|\rho z|)|\le 10^3\{m(I)\gamma(N;Q)+\tilde{\gamma}(N;Q)\}\nu(Q)\,|z|\,.$$

Taking care of $|\Phi[I;R](z)|\le m(I)$ and $|\hat{\lambda}_\eta(z)|\le 10^2\eta^2$, we have

$$\begin{aligned}
|L_1| &\le \int_{|z|\le M^2}\hat{\lambda}_\eta(z)\Phi[I;R](z)J_0(|\rho z|)d\sigma(z)\\
&\quad + 10^3\{m(I)\gamma(N;Q)+\tilde{\gamma}(N;Q)\}\nu(Q)\int_{|z|\le M^2}|\hat{\lambda}_\eta(z)z|\,d\sigma(z)
\end{aligned} \tag{36}$$

$$\leq L_3 + \int_{M^2<|z|} |\hat{\lambda}_\eta(z)\Phi[I; R](z)J_0(|\rho z|)|\, d\sigma(z)$$
$$+ 2\pi/3\cdot 10^5\{m(I)\gamma(N; Q) + \tilde{\gamma}(N; Q)\}\eta^2\nu(Q)M^6$$
$$\leq 10^6 m(I)\eta/\rho + 1/2\cdot 10^6 m(I)\eta^{-1}M^{-2}$$
$$+ 2\pi/3\cdot 10^5\{m(I)\gamma(N; Q) + \tilde{\gamma}(N; Q)\}\eta^2\nu(Q)M^6$$
$$\leq 10^6\{m(I)\eta/\rho + 1/2\cdot m(I)\eta^{-1}M^{-2} + [m(I)\gamma(N; Q) + \tilde{\gamma}(N; Q)]\eta^2\nu(Q)M^6\}\,.$$

Since $L \leq 10^{-1}\{|L_1| + |L_2|\}$, the required inequality (34) follows from (35) and (36).

LEMMA 18. *Let $Q(t)$ and $R(t)$ be two Taylor polynomials such that $N_Q \geq 64$, $N_R \geq 64$, $4\nu(Q) \leq w(1; Q)$ and $4\nu(R) \leq w(1; R)$ and let J, J' be two positive integers such that $J \leq N_Q - 63$, $J' \leq N_R - 63$, $4\nu(Q) \leq w(J; Q)$ and $4\nu(R) \leq w(J'; R)$. Then, for any η, η', M, $M' > 0$ and any interval I in $[0, 2\pi)$,*

$$\int_I \lambda_\eta(Q(t))\lambda_{\eta'}(R(t))\, dt$$
$$\leq 10^{20}\{m(I)\eta^2 w(J; Q)^{-2} + (m(I) + 1)\eta^2\nu(Q)\tilde{\gamma}(J; Q)M^6 + m(I)\eta^{-1}M^{-2}\} \tag{37}$$
$$\times \{\eta'^2 w(J'; R)^{-2} + \eta'^2\nu(R)\gamma(J'; R)M'^6 + \eta'^{-1}M'^{-2}\}$$
$$+ 10^{10}\eta'^2\nu(R)\tilde{\gamma}(J'; R)M'^6 \deg(Q)\,.$$

Proof. Set $\deg(Q) = W$. Since $\lambda_\eta \leq \pi^2\chi_{3\eta}$, we have

$$L = \int_I \lambda_\eta(Q(t))\lambda_{\eta'}(R(t))dt \leq \pi^2 \int_I \chi_{3\eta}(Q(t))\lambda_{\eta'}(R(t))dt = \pi^2 L'\,.$$

Since a set $U = \{t \in [0, 2\pi); |Q(t)| < 3\eta\}$ is a finite union of at most W open intervals, we can write $U \cap I = \bigcup_{\mu=1}^{\nu} I_\mu$ ($\nu \leq W + 1$), where I_μ's are mutually disjoint intervals. For a fixed $1 \leq \mu \leq \nu$, we estimate $L'_\mu = \int_{I_\mu} \lambda_{\eta'}(R(t))\, dt$. Since $J' \leq N_R - 63$ and $4\nu(R) \leq w(J'; R)$, we have, from Lemma 16,

$$L'_\mu \leq 10^8 m(I_\mu)\{\eta'^2 w(J'; R)^{-2} + \eta'^2\nu(R)\gamma(J'; R)M'^6 + \eta'^{-1}M'^{-2}\}$$
$$+ 10^8\eta'^2\nu(R)\tilde{\gamma}(J'; R)M'^6\,.$$

Let us remark that $\tilde{\gamma}(J; Q) \leq \gamma(J; Q)$. Since $J \leq N_Q - 63$ and $4\nu(Q) \leq w(J; Q)$, we have, from Lemma 16,

$$64\int_I \lambda_{12\eta}(Q(t))dt$$
$$\leq 64\cdot 10^8\{m(I)(12\eta)^2 w(J; Q)^{-2} + [m(I)\gamma(J; Q) + \tilde{\gamma}(J; Q)](12\eta)^2\nu(Q)M^6$$
$$+ m(I)(12\eta)^{-1}M^{-2}\}$$

$$\leq 10^{12}\{m(I)\eta^2 w(J;\,Q)^{-2} + (m(I)+1)\eta^2\nu(Q)\gamma(J;\,Q)M^6 + m(I)\eta^{-1}M^{-2}\}\,.$$

Note that $\pi^2\chi_{3\eta} \leq 64\lambda_{12\eta}$. We have

$$\begin{aligned}
L &\leq \pi^2 L' = \pi^2 \sum_{\mu=1}^{\nu} L'_\mu \\
&\leq \pi^2 10^8 \sum_{\mu=1}^{\nu} m(I_\mu)\{\eta'^2 w(J';\,R)^{-2} + \eta'^2\nu(R)\gamma(J';\,R)M'^6 + \eta'^{-1}M'^{-2}\} \\
&\quad + \pi^2 10^8 \nu\eta'^2\nu(R)\tilde{\gamma}(J';\,R)M'^6 \\
&\leq \pi^2 10^8 \int_I \chi_{3\eta}(Q(t))dt\{\eta'^2 w(J';\,R)^{-2} + \eta'^2\nu(R)\gamma(J';\,R)M'^6 + \eta'^{-1}M'^{-2} \\
&\quad + 2\pi^2 10^8 \eta'^2\nu(R)\tilde{\gamma}(J';\,R)M'^6 \deg(Q) \\
&\leq 64\cdot 10^8 \int_I \lambda_{12\eta}(Q(t))dt\{\eta'^2 w(J';\,R)^{-2} + \eta'^2\nu(R)\gamma(J';\,R)M'^6 + \eta'^{-1}M'^{-2}\} \\
&\quad + 10^{10}\eta'^2\nu(R)\tilde{\gamma}(J';\,R)M'^6 \deg(Q) \\
&\leq 10^{20}\{m(I)\eta^2 w(J;\,Q)^{-2} + (m(I)+1)\eta^2\nu(Q)\gamma(J;\,Q)M^6 + m(I)\eta^{-1}M^{-2}\} \\
&\quad \times\{\eta'^2 w(J';\,R)^{-2} + \eta'^2\nu(R)\gamma(J';\,R)M'^6 + \eta'^{-1}M'^{-2}\} \\
&\quad + 10^{10}\eta'^2\nu(R)\tilde{\gamma}(J';\,R)M'^6 \deg(Q)\,.
\end{aligned}$$

The following lemma is proved analogously as in the preceding lemma. (Use Lemma 16 and 17.)

LEMMA 19. *Let $Q(t)$ be a Taylor polynomial such that $N_Q \geq 64$ and $4\nu(Q) \leq w(1;\,Q)$ and J a positive integer such that $J \leq N_Q - 63$ and $4\nu(Q) \leq w(J;\,Q)$ and let $R(t)$ be a non-constant Taylor polynomial. Then, for any η, M, $M' > 0$, $0 < \eta' \leq |\tilde{R}(N_R)|/3$ and any interval I in $[0, 2\pi)$,*

$$\begin{aligned}
&\int_I \lambda_\eta(Q(t))\lambda_{\eta'}(R(t))dt \\
&\leq 10^{17}\{m(I)\eta^2 w(J;\,Q)^{-2} + (m(I)+1)\eta^2\nu(Q)\gamma(J;\,Q)M^6 + m(I)\eta^{-1}M^{-2}\} \\
&\quad \times\{\eta'|\tilde{R}(N_R)|^{-1} + \eta'^2\nu(R)\gamma(N_R,\,R)M'^6 + \eta'^{-1}M'^{-2}\} \\
&\quad + 10^7\eta'^2\nu(R)\tilde{\gamma}(N_R,\,R)M'^6 \deg(Q)\,.
\end{aligned} \tag{38}$$

LEMMA 20. *Let $Q(t)$ be a Taylor polynomial such that $N_Q \geq 64$, $\nu(Q) \geq 1$ and $4\nu(Q) \leq w(1;\,Q)$ and let J a positive integer such that $J \leq N_Q - 63$ and $4\nu(Q) \leq w(J;\,Q)$. Then, for any $0 < \eta \leq \nu(Q)$, $M > 0$ and any interval I in $[0, 2\pi)$,*

$$\begin{aligned}
\int_I \lambda_\eta(Q(t))dt &\geq 10^{-3}m(I)\eta^2 w(1;\,Q)^{-2} \\
&\quad - 10^6[m(I)\eta^2 w(1;\,Q)^{-2}\{\nu(Q)w(1;\,Q)^{-1}\}^{2/5} \\
&\quad + m(I)\eta^2|\tilde{Q}(0)|w(1;\,Q)^{-2}w(1;\,Q)^{-2/5}
\end{aligned} \tag{39}$$

$$+ (m(I)+1)\eta^2\gamma(1;Q)w(1;Q)^{-2}\{\nu(Q)w(1;Q)^{-1}\}^{2/5}$$
$$+ m(I)\eta^2 w(1;Q)^{-2}\{\nu(Q)w(1;Q)^9 w(J;Q)^{-10}\}^{2/5}$$
$$+ (m(I)+1)\eta^2\nu(Q)\gamma(J;Q)M^6 + m(I)\eta^{-1}M^{-2}] .$$

Proof. Set $\Phi(z)=\Phi[I;Q](z)$ and $\xi=\nu(Q)^{-1/5}w(1;Q)^{-4/5}$. Using Lemma 6, we express $L=\int_I \lambda_\eta(Q(t))dt$ in the following form:

$$L=(2\pi)^{-2}\int_C \hat{\lambda}_\eta(z)\Phi(z)d\sigma(z)$$
$$=(2\pi)^{-2}\left\{\int_{|z|\le\xi}+\int_{\xi<|z|\le M^2}+\int_{M^2<|z|}\right\}=(2\pi)^{-2}\{L_1+L_2+L_3\} .$$

We have

$$|L_3|\le\int_{M^2<|z|}|\hat{\lambda}_\eta(z)\Phi(z)|\,d\sigma(z)\le 10^8 m(I)\eta^{-1}M^{-2} . \tag{40}$$

Writing $\Xi=(\tilde{Q}(k))_{k=J}^{N_Q}$, we have, from (32),

$$|\Phi(z)|\le m(I)J(z;\Xi)+10^3\{m(I)\gamma(J;Q)+\tilde{\gamma}(J;Q)\}\nu(Q)|z|$$
$$\le m(I)J(z;\Xi)+10^3(m(I)+1)\nu(Q)\gamma(J;Q)|z| .$$

Since $v(\xi,\Xi)\le 10^6\xi^{-2}\|\Xi\|^{-4}=10^6\nu(Q)^{2/5}w(1;Q)^{8/5}w(J;Q)^{-4}$, we have

$$\begin{aligned}|L_2|&\le\int_{\xi\le|z|<M^2}|\hat{\lambda}_\eta(z)|\{m(I)\,|J(z;\Xi)|+10^3(m(I)+1)\nu(Q)\gamma(J;Q)\,|z|\}d\sigma(z)\\&\le 10^2 m(I)\eta^2\int_{\xi\le|z|<M^2}|J(z;\Xi)|\,d\sigma(z)\\&\qquad+10^5(m(I)+1)\eta^2\nu(Q)\gamma(J;Q)\int_{\xi\le|z|<M^2}|z|\,d\sigma(z)\\&\le 10^2 m(I)\eta^2 v(\xi;\Xi)+2\pi/3\cdot 10^5(m(I)+1)\eta^2\nu(Q)\gamma(1;Q)M^6\\&\le 10^8[m(I)\eta^2 w(1;Q)^{-2}\{\nu(Q)w(1;Q)^9 w(J;Q)^{-10}\}^{2/5}\\&\qquad\qquad+(m(I)+1)\eta^2\nu(Q)\gamma(J;Q)M^6] .\end{aligned} \tag{41}$$

Putting $\Xi'=(\tilde{Q}(k))_{k=1}^{N_Q}$, we have, from (32),

$$\Phi(z)\ge m(I)\exp\{-i\,\mathrm{Re}\,\bar{z}\tilde{Q}(0)\}J(z;\Xi')-10^3\{m(I)\gamma(1;Q)+\tilde{\gamma}(1;Q)\}\nu(Q)\,|z|$$
$$\ge m(I)J(z;\Xi')-m(I)\,|\tilde{Q}(0)z|-10^3(m(I)+1)\nu(Q)\gamma(1;Q)\,|z| .$$

Note that $J(z;\Xi')\ge 0$ ($|z|\le 1/\nu(Q)$) and $\eta^2\le\hat{\lambda}_\eta(z)\le 10^2\eta^2$ ($|z|\le 1/\eta$) ((x), (xii)). Since $\xi\le 1/\nu(Q)\le 1/\eta$, these inequalities hold in $|z|\le\xi$. We have

$$\begin{aligned}L_1\ge\int_{|z|\le\xi}\hat{\lambda}_\eta(z)\{&m(I)J(z;\Xi')-m(I)\,|\tilde{Q}(0)z|\\&-10^3(m(I)+1)\nu(Q)\gamma(1;Q)\,|z|\}d\sigma(z)\end{aligned} \tag{42}$$

$$\geq m(I)\eta^2 \int_{|z|\leq\xi} J(z; \Xi')d\sigma(z)$$

$$-10^2\eta^2\{m(I)\,|\tilde{Q}(0)| + 10^3(m(I)+1)\nu(Q)\gamma(1; Q)\} \int_{|z|\leq\xi} |z|\, d\sigma(z)$$

$$\geq m(I)\eta^2\{v_0(\Xi') - v(\xi; \Xi')\}$$

$$-2\pi/3\cdot 10^2\eta^2\xi^3\{m(I)\,|\tilde{Q}(0)| + 10^3(m(I)+1)\nu(Q)\gamma(1; Q)\}$$

$$\geq m(I)\eta^2[10^{-1}w(1; Q)^{-2} - 10^6 w(1; Q)^{-2}\{\nu(Q)w(1; Q)^{-1}\}^{2/5}]$$

$$-\ 10^8[m(I)\eta^2\,|\tilde{Q}(0)|\, w(1; Q)^{-2}\{\nu(Q)^{-3/5}w(1; Q)^{-2/5}\}$$

$$+\ (m(I)+1)\eta^2\gamma(1; Q)w(1; Q)^{-2}\{\nu(Q)w(1; Q)^{-1}\}^{2/5}]$$

$$\geq 10^{-1}m(I)\eta^2 w(1; Q)^{-2} - 10^8[m(I)\eta^2 w(1; Q)^{-2}\{\nu(Q)w(1; Q)^{-1}\}^{2/5}$$

$$+\ m(I)\eta^2\,|\tilde{Q}(0)|\, w(1; Q)^{-2}w(1; Q)^{-2/5}$$

$$+\ (m(I)+1)\eta^2\gamma(1; Q)w(1; Q)^{-2}\{\nu(Q)w(1; Q)^{-1}\}^{2/5}\ .$$

Since $L = (2\pi)^{-2}\{L_1 + L_2 + L_3\} \geq 10^{-2}\{L_1 - |L_2| - |L_3|\}$, the required inequality (39) follows from (40), (41) and (42).

LEMMA 21. *Let $F(t)$ be a Taylor series and W a positive integer such that*

(43) $$\Upsilon(W, F) = \sum_{n\neq W} (n/W + n^2/W^2)\,|\hat{F}(n)| \geq |\hat{F}(W)|/4\ .$$

Then

(44) $$M_F(\rho) = m(\{t\in[0, 2\pi); |F(t)| \leq \rho\}) \leq 10^2 p_{F,W}\rho^{1/2}\quad (0 < \rho \leq 1)\ ,$$

where $p_{F,W} = \max\{|\hat{F}(W)|^{-1}, |\hat{F}(W)|^{-1/2}\}$.

Proof. Set $P(t) = \mathrm{Re}\,F(t)$ and $U = \{t\in[0, 2\pi); |\sin Wt| \geq 1/\sqrt{2}\}$. Without loss of generality, we may assume that $\hat{F}(W) > 0$. We have

$$|P'(t)| \geq \hat{F}(W)W|\sin Wt| - \sum_{n\neq W} n\,|\hat{F}(n)| \geq \hat{F}(W)W|\sin Wt| - \Upsilon(W, F)W$$

$$\geq \hat{F}(W)W\{|\sin Wt| - 1/4\} \geq \tfrac{1}{4}\hat{F}(W)W\ .$$

Since U is a union of $2W$ intervals, let us write $U = \bigcup_{\mu=1}^{2W} I_\mu$, where I_μ's are mutually disjoint intervals. For every $1 \leq \mu \leq 2W$, we have, from Lemma 11,

$$m(\{t\in I_\mu; |P(t)| \leq \rho\}) \leq 4(\tfrac{1}{4}\hat{F}(W)W)^{-1}\rho \leq 16p_{F,W}W^{-1}\rho^{1/2}\ ,$$

and hence

$$m(\{t\in U; |P(t)| \leq \rho\}) \leq 2W\,16p_{F,W}W^{-1}\rho^{1/2} = 32p_{F,W}\rho^{1/2}\ .$$

Note that $U^c = \{t\in[0, 2\pi); |\cos Wt| > 1/\sqrt{2}\}$. We have, for $t\in U^c$,

$$|P''(t)| \geq \hat{F}(W)W^2|\cos Wt| - \sum_{n\neq W} n^2|\hat{F}(n)| \geq \hat{F}(W)W^2|\cos Wt| - \Upsilon(W, F)W^2$$
$$\geq \hat{F}(W)W^2\{|\cos Wt| - 1/4\} \geq \tfrac{1}{4}\hat{F}(W)W^2 .$$

Let us write $U^c = \bigcup_{\mu=1}^{2W} I'_\mu$, where I'_μ's are mutually disjoint intervals. For every $1 \leq \mu \leq 2W$, we have, from Lemma 11,

$$m(\{t \in I'_\mu; |P(t)| \leq \rho\}) \leq 4^2(\tfrac{1}{4}\hat{F}(W)W^2)^{-1/2}\rho^{1/2} \leq 32p_{F,W}W^{-1}\rho^{1/2} ,$$

and hence

$$m(\{t \in U^c; |P(t)| \leq \rho\}) \leq 2W\,32p_{F,W}W^{-1}\rho^{1/2} = 64p_{F,W}\rho^{1/2} .$$

Consequently,

$$\begin{aligned} M_F(\rho) &\leq m(\{t \in [0, 2\pi); |P(t)| \leq \rho\}) \\ &= m(\{t \in U; |P(t)| \leq \rho\}) + m(\{t \in U^c; |P(t)| \leq \rho\}) \\ &\leq 96p_{F,W}\rho^{1/2} \leq 10^2p_{F,W}\rho^{1/2} . \end{aligned}$$

COROLLARY 22. *Let $Q(t)$ be a Taylor polynomial such that*

$$8\nu(Q)N_Q n_{N_Q-1}(Q) \leq |\tilde{Q}(N_Q)|\, n_{N_Q}(Q) .$$

Then $M_Q(\rho) \leq 10^2p_Q\rho^{1/2}$ $(0 < \rho \leq 1)$, *where* $p_Q = p_{Q,N_Q}$.

Proof. We write simply $N = N_Q$ and $n_k = n_k(Q)$ $(k \geq 0)$. We have

$$\begin{aligned} \Upsilon(n_N, Q) &= \sum_{k=0}^{N-1}(n_k/n_N + n_k^2/n_N^2)|\tilde{Q}(k)| \leq 2\nu(Q)Nn_{N-1}n_N^{-1} \\ &\leq |\tilde{Q}(N)/4 = |\hat{Q}(n_N)|/4 . \end{aligned}$$

By Lemma 21, we have the required inequality.

§ 4. Subsequences of $(e^{int})_{n\in Z}$

An interval $[0, 2\pi)$ is a probability space having a probability measure $m/2\pi$. Then $(e^{int})_{n\in Z}$ is a sequence of random variables having mean 0 and variance 1. In this chapter, we shall study three probabilistic properties of subsequences of $(e^{int})_{n\in Z}$: the $0-1$ law, pseudo-independence (mixing), the law of large numbers. However we shall not show, in this paper, direct applications of our results, they play important role in the theory of lacunary series. We shall also note well-known results about the central limit theorem and the law of iterated logarithm of subsequences of $(e^{int})_{n\in Z}$. The recurrence property of L-lacunary series will be studied in detail in Chapter 5.

Our results are elementary and incomplete. But it is no doubt that various problems in this area are interesting.

4.1. The $0-1$ law

A countable product C_∞ of C is a measurable space, where Borel sets are induced from cylinder sets. Let $X=(X_k(t))_{k=1}^\infty$ be a sequence of Borel measurable functions in $[0, 2\pi)$. We say that a Borel set U in $[0, 2\pi)$ is a tail set defined by X, if, for every positive integer k, there exists a Borel set B_k in C_∞ such that $U=\{t\in[0, 2\pi); (X_k(t), X_{k+1}(t), \cdots)\in B_k\}$. We say that X satisfies the $0-1$ law, if, for any tail set U defined by X, $m(U)=0$ or 2π. We shall show the following

PROPOSITION 23. *Let $E=(n_k)_{k=1}^\infty$ be a strictly increasing sequence of positive integers.*

(45) *If, for every $k\geq 1$, n_{k+1} is a multiple of n_k, then $(e^{int})_{n\in E}$ satisfies the $0-1$ law.*

(46) *If there exist two strictly increasing sequences $(k_j)_{j=1}^\infty$ and $(k'_j)_{j=1}^\infty$ of positive integers such that $\sup_j \langle n_{k_j}, n_{k'_j}\rangle<+\infty$, then $(e^{int})_{n\in E}$ does not satisfy the $0-1$ law, where $\langle\cdot,\cdot\rangle$ denotes the greatest common divisor.*

COROLLARY 24. *Let E be a strictly increasing sequence of positive integers such that $(e^{int})_{n\in E}$ satisfies the $0-1$ law. Then E is Hadamard lacunary.*

For the proof, we prepare the following

LEMMA 25. *Let I and U be an open interval and a Borel set in $[0, 2\pi)$, respectively. Writing $U(n)=\{t\in[0, 2\pi); nt\in U \pmod{2\pi}\}$, we have*

$$|m(I\cap U(n))-(2\pi)^{-1}m(I)m(U)|\leq 8\pi/n\,. \tag{47}$$

Proof. Without loss of generality, we may assume that U is open and that $m(I)\geq 4\pi/n$. Set $U_0=U(n)\cap[0, 2\pi/n)$ and $U_j=U_0+2\pi j/n$ ($j=1, \cdots, n-1$). Then $m(U_j)=m(U)/n$ and $U(n)=\bigcup_{j=0}^{n-1}U_j$. Let N be the cardinal number of a set $\{U_j; U_j\subset I, j=0, \cdots, n-1\}$ of Borel sets. Then we have $|m(I\cap U(n))-Nm(U)/n|\leq 4\pi/n$ and $|m(I)-2\pi N/n|\leq 4\pi/n$. Hence

$$\begin{aligned}&|m(I\cap U(n))-(2\pi)^{-1}m(I)m(U)|\\&\quad\leq|m(I\cap U(n))-Nm(U)/n|+(2\pi)^{-1}m(U)\,|m(I)-2\pi N/n|\leq 8\pi/n\,.\end{aligned}$$

This completes the proof.

Proof of Proposition 23. (45): Let $E=(n_k)_{k=1}^{\infty}$ be the sequence in this proposition and let U be a tail set in $[0, 2\pi)$ defined by $(e^{int})_{n\in E}$ such that $m(U)>0$. It is sufficient to show that $m(U)=2\pi$. For a given positive number ε, there exists a finite union of open intervals J such that $m((U-J)\cup(J-U))\leq\varepsilon$. Let us write $J=\bigcup_{j=1}^{M} I_j$, where I_j's are open intervals. For a given positive integer k, there exists a Borel set B_k in C_∞ such that $U=\{t\in[0,2\pi); (e^{in_k t}, e^{in_{k+1}t}, \cdots)\in B_k\}$. Set

$V=\{t\in[0,2\pi); (e^{it}, e^{i(n_{k+1}/n_k)t}, e^{i(n_{k+2}/n_k)t}, \cdots)\in B_k\}$. Since, for any $j\geq k$, n_j is a multiple of n_k, we have $U=V(n_k)$ and $m(U)=m(V)$. By Lemma 25, we have, for $j=1,\cdots,M$,

$$\begin{aligned}&|m(I_j\cap U)-(2\pi)^{-1}m(I_j)m(U)|\\&\quad=|m(I_j\cap V(n_k))-(2\pi)^{-1}m(I_j)m(V)|\leq 8\pi/n_k\,,\end{aligned}$$

and hence $|m(J\cap U)-(2\pi)^{-1}m(J)m(U)|\leq 8\pi M/n_k$. Since $m((U-J)\cup(J-U))\leq\varepsilon$, we have

$$\begin{aligned}|m(U)-(2\pi)^{-1}m(U)^2|&\leq|m(U)-m(J\cap U)|\\&\quad+|m(J\cap U)-(2\pi)^{-1}m(J)m(U)|\\&\quad+(2\pi)^{-1}m(U)\,|m(J)-m(U)|\leq 2\varepsilon+8\pi M/n_k\,.\end{aligned}$$

Letting $k\to\infty$, $|1-(2\pi)^{-1}m(U)|\leq 2\varepsilon m(U)^{-1}$. Since ε is arbitrary, we have $m(U)=2\pi$.

(46): Let $(k_j)_{j=1}^{\infty}$ and $(k'_j)_{j=1}^{\infty}$ be two sequences in (46). Choosing subsequences if necessary, we may assume that there exists a positive integer p such that, for all j, $\langle n_{k_j}, n_{k'_j}\rangle=p$. We show that a set $U_p=\bigcup_{j=1}^{p}(2(j-1)\pi/p, (2j-1)\pi/p)$ having measure π is a tail set defined by $X=(e^{int})_{n\in E}$.

Set $B_j=\{(\exp in_{k_j}t, \exp in_{k'_j}t); t\in U_p\}$ $(j\geq 1)$. Since $\langle n_{k_j}, n_{k'_j}\rangle=p$, we have $U_p=\{t\in[0,2\pi); (\exp in_{k_j}t, \exp in_{k'_j}t)\in B_j\}$. For every integer k, we arbitrarily choose an integer j such that $k\leq k_j$, $k\leq k'_j$ and set $B'_k=\{(c_k, c_{k+1},\cdots);$ "$c_k=C$ if $k\neq k_j, k'_j$", $(c_{k_j}, c_{k'_j})\in B_j\}$. Then we have, for all k, $U_p=\{t\in[0,2\pi); (e^{in_k t}, e^{in_{k+1}t},\cdots)\in B'_k\}$ and hence it is a tail set defined by X having measure π. This shows that X does not satisfy the $0-1$ law.

Proof of Corollary 24. Let $E=(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers and suppose that it is not Hadamard lacunary. Then there exists a sequence $(m_j)_{j=1}^{\infty}$ tending to infinity such that $1<n_{m_j+1}/n_{m_j}<2$. Put $k_j=m_{j+1}$ and $k'_j=m_j$ $(j\geq 1)$. Then $\langle n_{k_j}, n_{k'_j}\rangle=1$ for all j. By (46), $(e^{int})_{n\in E}$ does not satisfy the $0-1$ law.

4.2. Pseudo-independence (Mixing)

We say that a sequence $(X_k(t))_{k=1}^{\infty}$ of Borel functions in $[0, 2\pi)$ is identically distributed if, for any Borel set A in $\boldsymbol{C}$,

$$m(\{t \in [0, 2\pi); X_k(t) \in A\}) = m(\{t \in [0, 2\pi); X_1(t) \in A\}) \qquad (k \geq 1).$$

We say that a sequence $(X_k(t))_{k=1}^{\infty}$ of identically distributed functions is pseudo-independent (or we say that it satisfies the property of mixing), if, for any positive integer M and Borel sets $\{A_j\}_{j=1}^{M}$ in $\boldsymbol{C}$,

$$\begin{aligned}\lim_{(k_1,\cdots,k_M)} & m(\{t \in [0, 2\pi); X_{k_j}(t) \in A_j \ (j = 1, \cdots, M)\}) \\ &= (2\pi)^{-M+1} \prod_{j=1}^{M} m(\{t \in [0, 2\pi); X_1(t) \in A_j\}) ,\end{aligned}$$

where $\lim_{(k_1,\cdots,k_M)}$ denotes the limit when k_j's diverge satisfying $k_j \neq k_{j'}$ $(j \neq j')$. It is evident that any subsequence of a pseudo-independent sequence is pseudo-independent. We remark also that, for any strictly increasing sequence E of positive integers, $(e^{int})_{n \in E}$ is identically distributed. We shall show the following

PROPOSITION 26. *Let $E = (n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers such that the limit $\lim_{k \to \infty} n_{k+1}/n_k = \sigma(\leq +\infty)$ exists. If $\sigma = +\infty$ or it is a transcendental number, then $(e^{int})_{n \in E}$ is pseudo-independent.*

In the case where σ is an algebraic number, it seems difficult to discuss the pseudo-independence of $(e^{int})_{n \in E}$. The following example shows that there are non-pseudo-independent Hadamard lacunary series.

EXAMPLE 27. An algebraic integer $\sigma > 1$ is called a Pisot number if it is a (rational) integer or all its conjugates (not σ itself) have moduli strictly less than 1 ([13]). Let σ be a Pisot number. Then $(e^{i\langle \sigma^k \rangle t})_{k=1}^{\infty}$ is not pseudo-independent, where $\langle x \rangle$ denotes the nearest integer to x.

COROLLARY 28. *Let E be an L-lacunary series. Then $(e^{int})_{n \in E}$ is pseudo-independent. There exists an Hadamard lacunary series E such that $(e^{int})_{n \in E}$ is not pseudo-independent.*

For the proof, we prepare two lemmas. Throughout the proof of this proposition, $E = (n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers.

LEMMA 29. *$(e^{int})_{n \in E}$ is pseudo-independent if and only if*:

(48) *for any positive integer M and functions $\{\phi_j(t)\}_{j=1}^{M}$ in $A(0, 2\pi)$,*

$$\lim_{(k_1,\cdots,k_M)} \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=1}^{M} \phi_j(n_{k_j} t) dt = \prod_{j=1}^{M} \hat{\phi}_j(0) .$$

Proof. We say that a Borel function $\phi(t)$ in $[0, 2\pi)$ is an indicator function if $\phi(t) = 0$ or 1. From the definition, the following property evidently holds: $(e^{int})_{n\in E}$ is pseudo-independent if and only if, for any positive integer M and indicator functions $\{\phi_j(t)\}_{j=1}^M$,

$$\lim_{(k_1,\cdots,k_M)} \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \phi_j(n_{k_j}t)dt = \prod_{j=1}^M \hat{\phi}_j(0) .$$

For any $\phi \in A(0, 2\pi)$ and $\varepsilon > 0$, there exists "a finite sum of indicator functions" ψ such that $\sup_{t\in[0,2\pi)} |\phi(t) - \psi(t)| \leq \varepsilon$. This shows that the above property holds if and only if (48).

LEMMA 30. *$(e^{int})_{n\in E}$ is pseudo-independent if and only if:*

(49) *for any positive integer M and integers $\{m_j\}_{j=1}^M$ (all m_j is not 0), there exists a positive integer K such that, for M integers $\{k_j\}_{j=1}^M$ satisfying $k_j \geq K$ $(j = 1, \cdots, M)$ and $k_j \neq k_{j'}$ $(j \neq j')$, $\sum_{j=1}^M m_j n_{k_j} \neq 0$.*

Proof. First we prove the "if" part. Suppose that (49) holds. Let M be a positive integer and let $\{\phi_j(t)\}_{j=1}^M$ be functions in $A(0, 2\pi)$. Then we have

$$\begin{aligned}
&\lim_{(k_1,\cdots,k_M)} \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \phi_j(n_{k_j}t)dt \\
&= \lim_{(k_1,\cdots,k_M)} \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \sum_{m\in Z} \hat{\phi}_j(m) \exp(imn_{k_j}t)dt \\
&= \lim_{(k_1,\cdots,k_M)} \sum_{m_1\in Z} \cdots \sum_{m_M\in Z} \hat{\phi}_1(m_1) \cdots \hat{\phi}_M(m_M) \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \exp(im_jn_{k_j}t)dt \\
&= \sum_{m_1\in Z} \cdots \sum_{m_M\in Z} \hat{\phi}_1(m_1) \cdots \hat{\phi}_M(m_M) \lim_{(k_1,\cdots,k_M)} \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \exp(im_jn_{k_j}t)dt .
\end{aligned}$$

By the hypothesis, the last part in the equalities is $\prod_{j=1}^M \hat{\phi}_j(0)$ and hence (48) holds. By Lemma 29, $(e^{int})_{n\in E}$ is pseudo-independent.

Next we prove the "only if" part. Suppose that $(e^{int})_{n\in E}$ is pseudo-independent. Then (48) holds. Let M be a positive integer and let $\{m_j\}_{j=1}^M$ be integers such that all m_j is not 0. By (48), we have

$$\lim_{(k_1,\cdots,k_M)} \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=1}^M \exp(im_jn_{k_j}t)dt = (2\pi)^{-M+1} \prod_{j=1}^M \int_0^{2\pi} \exp(im_jt)dt = 0 .$$

Hence $\sum_{j=1}^M m_jn_{k_j} \neq 0$ for all sufficiently large mutually distinct M positive integers $\{k_j\}_{j=1}^M$. This signifies (49).

Proof of Proposition 26. We use (49). For mutually distinct M positive integers $\{k_j\}_{j=1}^M$, set $K(k_1, \cdots, k_M) = \max\{k_j; j = 1, \cdots, M\}$. First suppose that $\sigma = +\infty$. Let $\{m_j\}_{j=1}^M$ be integers such that all m_j is not 0. Then we have

$$\lim_{(k_1,\cdots,k_M)} \left|\sum_{j=1}^{M} m_j n_{k_j}\right| \geq \lim_{(k_1,\cdots,k_M)} \left\{n_{K(k_1,\cdots,k_M)} - n_{K(k_1,\cdots,k_M)-1} \sum_{j=1}^{M} |m_j|\right\} = +\infty .$$

By (49), $(e^{int})_{n\in E}$ is pseudo-independent.

Next suppose that σ is a transcendental number and let $\{m_j\}_{j=1}^M$ be the same as above. Since σ is a transcendental number with > 1, we have $\eta = \inf|\sum_{j=1}^M m_j \sigma^{n'_j}| > 0$, where "inf" signifies the infimum over all M non-negative integers $\{n'_j\}_{j=1}^M$ satisfying $n'_j \neq n'_{j'}$ $(j \neq j')$ and $\min_{1\leq j\leq M} n'_j = 0$. Hence

$$\lim_{(k_1,\cdots,k_M)} \left|\sum_{j=1}^{M} m_j n_{k_j}\right| \geq \lim_{(k_1,\cdots,k_M)} n_{K(k_1,\cdots,k_M)}\left\{\left|\sum_{j=1}^{M} m_j \sigma^{k_j - K(k_1,\cdots,k_M)}\right| + o(1)^{\dagger)}\right\} \leq \lim_{(k_1,\cdots,k_M)} \eta n_{K(k_1,\cdots,k_M)} = +\infty .$$

By (49), $(e^{int})_{n\in E}$ is pseudo-independent.

Now we give the proof of the statement in Example 27: If σ is a Pisot number, then $X = (e^{i\langle\sigma^k\rangle t})_{k=1}^\infty$ is not pseudo-independent.

Set $n_k = \langle\sigma^k\rangle$ $(k \geq 1)$. If σ is an integer, then $\sigma n_k - n_{k+1} = 0$ for all k. By Lemma 30, X is not pseudo-independent. Suppose that σ is not an integer. Let $\omega_1, \cdots, \omega_{M-1}$ denote all its conjugates. Since σ is an algebraic integer, $\sigma^k + \omega_1^k + \cdots + \omega_{M-1}^k$ is an integer and hence $\sigma^k - \langle\sigma^k\rangle \leq (M-1)\max_j|\omega_j|^k$ $(k \geq 1)$ ([13]). Since $\max_j|\omega_j| < 1$, we have $\lim_{k\to\infty}(\sigma^k - \langle\sigma^k\rangle) = 0$. Let $\sum_{j=0}^M m_j z^j = 0$ be the algebraic equation whose roots are σ, $\omega_1, \cdots, \omega_{M-1}$. Then

$$\lim_{k\to\infty} \sum_{j=0}^{M} m_j n_{k+j} = \lim_{k\to\infty} \sum_{j=0}^{M} m_j(\langle\sigma^{k+j}\rangle - \sigma^{k+j}) = 0 .$$

Since $\sum_{j=0}^M m_j n_{k+j}$ is an integer for all k, this shows that $\sum_{j=0}^M m_j n_{k+j} = 0$ for all sufficiently large k. By Lemma 30, X is not pseudo-independent.

Corollary 28 is an immediate result of Proposition 26 and Example 27.

---

†) Throughout this paper, we use the symbols $O(g(x))$ and $o(g(x))$. See [19], p. 14.

4.3. The law of large numbers

In the theory of uniform distribution, H. Weyl proved the following theorem: Let $E = (n_k)_{k=1}^{\infty}$ be a sequence of positive integers. Then $m(\{t \in [0, 2\pi); \lim_{K\to\infty} |\sum_{k=1}^{K} e^{in_k t}|/K = 0\}) = 0$. This theorem corresponds to the strong law of large numbers in the probability theory. On the other hand, R. Salem estimated the exceptional set $\{t \in [0, 2\pi); \limsup_{K\to\infty} |\sum_{k=1}^{K} e^{in_k t}|/K > 0\}$ in the case where $n_k = O(k^p)$ (p: a given positive integer) ([14], p. 494).

We shall estimate the exceptional set in the case where E is L-lacunary.

Let $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ denote the one-point compactification of $\boldsymbol{C}$. For a strictly increasing sequence $E = (n_k)_{k=1}^{\infty}$ of positive integers, $B(t; E)$ denotes the totality of cluster points of a sequence $(\sum_{k=1}^{K} e^{in_k t}/K)_{K=1}^{\infty}$ in the space $\hat{\boldsymbol{C}}$. For a compact set A in $\hat{\boldsymbol{C}}$, set $B^{-1}(A; E) = \{t \in [0, 2\pi); B(t; E) = A\}$. We shall show the following

PROPOSITION 29. *Let E be an L-lacunary series. Then, for any compact set A in $\bar{\boldsymbol{D}}$,* $\dim(B^{-1}(A; E)) = 1$.

Proof. Let $E = (n_k)_{k=1}^{\infty}$ be an L-lacunary series and let A a compact set in $\bar{\boldsymbol{D}}$. There exists a sequence $(c_k)_{k=1}^{\infty}$ of complex numbers such that $|c_k| \le 1$ $(k \ge 1)$ and $A = \{c \in \boldsymbol{C}; \liminf_{K\to\infty} |\sum_{k=1}^{K} c_k/K - c| = 0\}$. There exists a sequence $(\phi_k)_{k=1}^{\infty}$ in $[0, 2\pi)$ such that $\exp(i\phi_{2k-1}) + \exp(i\phi_{2k}) = 2c_k$ $(k \ge 1)$. Then $A = \{c \in \boldsymbol{C}; \liminf_{K\to\infty} |\sum_{k=1}^{K} e^{i\phi_k}/K - c| = 0\}$. Set $\lambda_k = k^{-2}$ $(k \ge 1)$. Let k_0 be a positive integer such that, for $k \ge k_0$, $\lambda_k n_k^{-1} \ge 2\pi n_{k+1}^{-1}$. Set $\gamma'_{k,j} = [(2\pi j + \phi_k - 2^{-1}\lambda_k)/n_k, (2\pi j + \phi_k + 2^{-1}\lambda_k)/n_k]$ $(j = 1, \cdots, n_k, k \ge k_0)$. and $U'_k = \bigcup_{j=1}^{n_k} \gamma'_{k,j}$ $(k \ge k_0)$. Then $m(\gamma'_{k,j}) = \lambda_k n_k^{-1}$, the distance between $\gamma'_{k,j}$ and $\gamma'_{k,j+1}$ is $(2\pi - \lambda_k)/n_k$ and $\gamma'_{k,j}$ contains at least one interval in $\{\gamma'_{k+1,\ell}\}_{\ell=1}^{n_{k+1}}$. Define inductively $(U_k)_{k=k_0}^{\infty}$ by $U_{k_0} = U'_{k_0}$ and $U_{k+1} = \cup \{\gamma'_{k+1,j}; \gamma'_{k+1,j} \subset U_k, j = 1, \cdots, n_{k+1}\}$ $(k \ge k_0)$. Then $(\lambda_k)_{k=1}^{\infty}$ and $(U_k)_{k=k_0}^{\infty}$ satisfy the conditions in Lemma 7. Hence, with $U = \bigcap_{k=k_0}^{\infty} U_k$, $\dim(U) = 1$. For every $t_0 \in U$, we have

$$\sum_{k=1}^{K} e^{in_k t_0}/K = \sum_{k=1}^{K} e^{i\phi_k}/K + O\Big(\sum_{k=1}^{K} \lambda_k\Big)/K = \sum_{k=1}^{K} e^{i\phi_k} + o(1) ,$$

and hence $B(t_0; E) = A$. Since $B^{-1}(A; E) \supset U$, $\dim(B^{-1}(A; E)) = 1$. This completes the proof.

4.4. Known theorems

There are various interesting theorems in this area. Let us dote the following three theorems: Let $E=(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers.

(50) If $\varepsilon>0$, then
$m(\{t\in[0,2\pi);\ \lim_{K\to\infty}|\sum_{k=1}^{K}\cos n_k t|/K^{1/2+\varepsilon}=0\})=2\pi$ (The strong law of large numbers, [4]).

(51) If $\varepsilon>0$ and $n_{k+1}/n_k\geq 1+k^{-1/2-\varepsilon}$ $(k\geq 1)$, then, for $0<\xi<\eta$, $\lim_{K\to\infty} m(\{t\in[0,2\pi);\ \xi\leq|\sum_{k=1}^{K}\cos n_k t/\sqrt{K}|\leq\eta\})=2\pi\left(\frac{1}{\sqrt{\pi}}\int_{\xi}^{\eta}e^{-x^2}dx\right)$ (The central limit theorem, [6]).

(52) If E satisfies the condition in (51), then
$m(\{t\in[0,2\pi);\ \limsup_{K\to\infty}|\sum_{k=1}^{K}\cos n_k t|/\sqrt{K\log\log K}=1\})=2\pi$ (The law of iterated logarithm, [16]).

§ 5. The behaviour of partial sums of L-lacunary series

5.1. If E is L-lacunary, $(e^{int})_{n\in E}$ is pseudo-independent. Hence it seems that $(e^{int})_{n\in E}$ behaves like an independent sequence. In this paper, we shall study the behaviour of partial sums of L-lacunary series from the point of view of the recurrence and the transience.

Let $\hat{\boldsymbol{C}}=\boldsymbol{C}\cup\{\infty\}$ and $\hat{\boldsymbol{R}}=\boldsymbol{R}\cup\{\infty\}$ be the one-point compactifications of $\boldsymbol{C}$ and $\boldsymbol{R}$, respectively. Let $F(t)$ be a Taylor series. $C(t;F)$ denotes the totality of cluster points of a sequence $(F_m(t))_{m=1}^{\infty}$ in $\hat{\boldsymbol{C}}$. $C(t;\operatorname{Re}F)$ (respectively, $C(t;\operatorname{Im}F)$) denotes the totality of cluster points of a sequence $(\operatorname{Re}F_m(t))_{m=1}^{\infty}$ (resp. $(\operatorname{Im}F_m(t))_{m=1}^{\infty}$) in $\hat{\boldsymbol{R}}$. For a compact set A in $\hat{\boldsymbol{C}}$, set $C^{-1}(A;F)=\{t\in[0,2\pi);\ C(t;F)=A\}$. For a compact set B in $\hat{\boldsymbol{R}}$, set $C^{-1}(B;\operatorname{Re}F)=\{t\in[0,2\pi);\ C(t;\operatorname{Re}F)=B\}$. $C^{-1}(B;\operatorname{Im}F)$ is analogously defined. We say that:

$F(t)$ is recurrent in $\boldsymbol{C}$ if $m(C^{-1}(\hat{\boldsymbol{C}};F))=2\pi$.

$F(t)$ is transient in $\boldsymbol{C}$ if $m(C^{-1}(\{\infty\};F))=2\pi$.

$F(t)$ is recurrent in $\boldsymbol{R}$ if $m(C^{-1}(\hat{\boldsymbol{R}};\operatorname{Re}F)\cap C^{-1}(\hat{\boldsymbol{R}};\operatorname{Im}F))=2\pi$.

$F(t)$ is transient in $\boldsymbol{R}$ if $m(C^{-1}(\{\infty\};\operatorname{Re}F)\cap C^{-1}(\{\infty\};\operatorname{Im}F))=2\pi$.

We shall show the following

THEOREM 30. *Let $F(t)$ be an L-lacunary series such that $(|\tilde{F}(k)|)_{k=1}^{\infty}$ is increasing.*

(A) If $\sum_{m=1}^{\infty} s(m;F)^{-2}<+\infty$, *then $F(t)$ is transient in $\boldsymbol{C}$.*

(B) *If* $\sum_{m=1}^{\infty} s(m;F)^{-2} = +\infty$, *then* $F(t)$ *is recurrent in* $\boldsymbol{C}$.

(C) *If* $\sum_{m=1}^{\infty} s(m;F)^{-1} < +\infty$, *then* $F(t)$ *is transient in* $\boldsymbol{R}$.]

(*D*) *If* $\sum_{m=1}^{\infty} s(m;F)^{-1} = +\infty$, *then* $F(t)$ *is recurrent in* $\boldsymbol{R}$.

THEOREM 31. *Let* $F(t)$ *be an L-lacunary series such that* $F \notin L^2(0, 2\pi)$ *and* $\nu(F) < +\infty$. *Then* $F(t)$ *is recurrent in* $\boldsymbol{C}$.

COROLLARY 32. *Let* α *be a real number and let* $F_\alpha(t)$ *be an L-lacunary series such that* $|\tilde{F}_\alpha(k)| = k^\alpha$ $(k \geq 1)$.

(53) *If* $\alpha > 0$, *then* $F_\alpha(t)$ *is transient in* $\boldsymbol{C}$.

(54) *If* $-1/2 \leq \alpha \leq 0$, *then* $F_\alpha(t)$ *is recurrent in* $\boldsymbol{C}$.

(55) *If* $\alpha > 1/2$, *then* $F_\alpha(t)$ *is transient in* $\boldsymbol{R}$.

(56) *If* $-1/2 \leq \alpha \leq 1/2$, *then* $F_\alpha(t)$ *is recurrent in* $\boldsymbol{R}$.

5.2. Proof of (A) and (B)

We prepare some notation and lemmas. Let $F(t)$ be an L-lacunary series such that $(|\tilde{F}(k)|)_{k=1}^{\infty}$ is increasing. Set:

$$(57)\quad \begin{cases} \tau(k) = (\text{the integral part of } k/2) \\ n_k = n_k(F),\ a_k = |\tilde{F}(k)| \\ s_m = s(m;F),\ w(m, M) = \left(\sum_{k=m}^{M} a_k^2\right)^{1/2} \\ T_M = \sum_{m=1}^{M} s_m^{-2},\ (T_0 = 0) \\ U_M = \sum_{m=1}^{M} s_m^{-2}(a_m s_m^{-1})^{2/5},\ V_M = \sum_{m=1}^{M} s_m^{-2} s_m^{-2/5} \\ \gamma(m, M) = \sum_{k=m}^{M} n_k^{-1} \sum_{j=0}^{k-1} n_j,\ \gamma(m) = \gamma(m;F) = \lim_{M\to\infty} \gamma(m, M) \\ \tilde{\gamma}(m, M) = \sum_{k=m}^{M} n_k^{-1},\ \tilde{\gamma}(m) = \tilde{\gamma}(m;F) = \lim_{M\to\infty} \tilde{\gamma}(m, M) \\ \Lambda = \{m \in \boldsymbol{Z}^+;\ m \geq 128,\ 4a_m \leq w(m-63, m),\ a_m \leq 2^m\},\ \Lambda^c = \boldsymbol{Z}^+ - \Lambda, \end{cases}$$

where k is a non-negative integer and m, M are positive integers satisfying $m \leq M$.

For $\eta > 0$, two positive integers K, M $(K < M)$ and a Borel set U in $[0, 2\pi)$, we put

$$(58)\qquad \begin{cases} h(K, M; U, \eta) = \int_U \sum_{K \le m \le M, m \in \Lambda} \lambda_\eta(F_m(t))dt \\ H(K, M; U, \eta) = \int_U \Big\{ \sum_{K \le m \le M, m \in \Lambda} \lambda_\eta(F_m(t)) \Big\}^2 dt \,. \end{cases}$$

LEMMA 33. $C_1 = \sum_{m=1}^\infty 2^m m^6 \gamma(\tau(m)) < +\infty$,
$C_2 = \sum_{m=1}^\infty n_m \sum_{j=m+2}^\infty 2^j j^6 \gamma(\tau(m+j)) < +\infty$.

Proof. Let us remember the notation $q(\cdot)$ and $\theta(\cdot)$ in Lemma 5. Set $q = q(\mathrm{Spec}\,(F))$ and $\theta = \theta(\mathrm{Spec}\,(F))$. By Lemma 5, we have

$$C_1 \le \theta \sum_{m=1}^\infty 2^m m^6 n_{\tau(m)}^{-1+1/q} \le \theta^2 \sum_{m=1}^\infty 2^m m^6 \exp\{-q^{\tau(m)}(1 - 1/q)\} < +\infty$$

and

$$C_2 \le \theta \sum_{m=1}^\infty \sum_{j=m+2}^\infty 2^j j^6 n_m n_{\tau(m+j)}^{-1} \le \theta \sum_{m=1}^\infty \sum_{j=m+2}^\infty 2^j j^6 \gamma(\tau(m+j))$$
$$\le \theta^2 \sum_{m=1}^\infty \sum_{j=m+2}^\infty 2^j j^6 \exp\{-q^{\tau(m+j)}(1 - 1/q)\} < +\infty \,.$$

LEMMA 34. $\sum_{m \in \Lambda^c} a_m^{-1/2} < +\infty$, $C_3 = \sum_{m \in \Lambda^c} s_m^{-2} < +\infty$.

Proof. Since $C_3 \le \sum_{m \in \Lambda^c} a_m^{-2} \le a_1^{-3/2} \sum_{m \in \Lambda^c} a_m^{-1/2}$, it is sufficient to show the first inequality. Set $\Lambda_1' = \{1, 2, \cdots, 127\}$, $\Lambda_2' = \{m \in \Lambda^c - \Lambda_1';\, a_m > 2^m\}$, $\Lambda_3'(\mu) = \{m \in \Lambda^c - \Lambda_1' - \Lambda_2';\, 64 \,|\, (m - \mu)\}$ $(1 \le \mu \le 64)$. The required inequality follows from 64 inequalities "$\sum_{m \in \Lambda_3'(\mu)} a_m^{-1/2} \le 4a_1$ $(1 \le \mu \le 64)$" since

$$\sum_{m \in \Lambda^c} a_m^{-1/2} = \Big(\sum_{m \in \Lambda_1'} + \sum_{m \in \Lambda_2'} + \sum_{\ell=1}^{64} \sum_{m \in \Lambda_3'(\mu)}\Big) a_m^{-1/2}$$
$$\le 127 a_1^{-1/2} + \sum_{m=1}^\infty 2^{-m/2} + 256 a_1^{-1/2} < +\infty \,.$$

Hence we show that these 64 inequalities hold. Suppose that there exists an integer $1 \le \mu \le 64$ such that the cardinal number of $\Lambda_3'(\mu)$ is less than 2 and let μ be one of such integers. Then we have evidently $\sum_{m \in \Lambda_3'(\mu)} a_m^{-1/2} \le 4a_1^{-1/2}$. Suppose that there exists an integer $1 \le \mu \le 64$ such that the cardinal number of $\Lambda_3'(\mu)$ is larger than 1 and let μ be one of such integers. Then, for two integers $m, j \in \Lambda_3'(\mu)$ $(m < j)$, we have $4a_j \ge w(j - 63, j) \ge (64 a_{j-63}^2)^{1/2} \ge 8a_m$, and hence $2a_m \le a_j$. This shows that $\sum_{m \in \Lambda_3'(\mu)} a_m^{-1/2} \le a_1^{-1/2} \sum_{k=0}^\infty 2^{-k/2} \le 4a_1^{-1/2}$.

LEMMA 35. *If* $\lim_{M\to\infty} T_M = +\infty$, *then* $\lim_{M\to\infty} U_M/T_M = \lim_{M\to\infty} V_M/T_M = 0$.

Proof. Since $\lim_{m\to\infty} s_m = +\infty$, we have evidently $\lim_{M\to\infty} V_M/T_M = 0$.

For a fixed number $0 < \varepsilon < 1$, we put $N_\varepsilon = \{n \in \boldsymbol{Z}^+; n\varepsilon^2 \geq 2\}$, $\Gamma_\varepsilon = \{m \in \boldsymbol{Z}^+; a_m s_m^{-1} \geq \varepsilon\}$, $\Gamma_\varepsilon^c = \boldsymbol{Z}^+ - \Gamma_\varepsilon$ and $\Gamma_\varepsilon(\mu) = \{m \in \Gamma_\varepsilon; N_\varepsilon | (m-\mu)\}$ $(1 \leq \mu \leq N_\varepsilon)$. For a fixed integer $1 \leq \mu \leq N_\varepsilon$, we shall show that $W_{\varepsilon,\mu} = \sum_{m \in \Gamma_\varepsilon(\mu)} a_m^{-2} \leq 2a_1^{-2}$. If $\Gamma_\varepsilon(\mu) = \emptyset$, then this inequality evidently holds. If the cardinal number of $\Gamma_\varepsilon(\mu)$ is 1, we have, from $a_1 \leq a_m$ $(m \geq 1)$, $W_{\varepsilon,\mu} \leq a_1^{-2} \leq 2a_1^{-2}$. If the cardinal number of $\Gamma_\varepsilon(\mu)$ is larger than 1, we have for two integers m, $j \in \Gamma_\varepsilon(\mu)$ $(m < j)$,

$$a_j^2 \geq \varepsilon^2 s_j^2 \geq \varepsilon^2 w(j - N_\varepsilon + 1, j)^2 \geq N_\varepsilon \varepsilon^2 a_{j-N_\varepsilon+1}^2 \geq N_\varepsilon \varepsilon^2 a_m^2 \geq 2a_m^2 ,$$

and hence $W_{\varepsilon,\mu} \leq a_1^{-2} \sum_{k=0}^{\infty} 2^{-k} = 2a_1^{-2}$. We have

$$\sum_{m \in \Gamma_\varepsilon} a_m^{-2} = \sum_{\mu=1}^{N} W_{\varepsilon,\mu} \leq 2N_\varepsilon a_1^{-2} ,$$

and hence

$$\begin{aligned} U_M &\leq \varepsilon^{2/5} \sum_{1 \leq m \leq M, m \in \Gamma_\varepsilon^c} s_m^{-2} + \sum_{1 \leq m \leq M, m \in \Gamma_\varepsilon} s_m^{-2} \\ &\leq \varepsilon^{2/5} T_M + \sum_{m \in \Gamma_\varepsilon} a_m^{-2} \leq \varepsilon^{2/5} T_M + 2N_\varepsilon a_1^{-2} . \end{aligned}$$

Therefore, $\limsup_{M\to\infty} U_M/T_M \leq \varepsilon^{2/5}$. Letting $\varepsilon \to 0$, we obtain $\lim_{M\to\infty} U_M/T_M = 0$.

Now we give the proof of (A). Suppose that $\sum_{m=1}^{\infty} s_m^{-2} < +\infty$. Since Spec (F) is L-lacunary, there exists a positive integer m_1' such that, for $m \geq m_1'$, $8mn_{m-1} \leq a_m n_m$. Note that $N_{F_m} = a_m$ and $\nu(F_m) = a_m$. We have, from Corollary 22,

$$\begin{aligned} \overline{m}_1 &= \sum_{m \in \Lambda^c} m\left(\left\{t \in [0, 2\pi); |F_m(t)| \leq \frac{1}{4}\right\}\right) = \sum_{m \in \Lambda^c, m < m_1} + \sum_{m \in \Lambda^c, m_1 \leq m} \\ &\leq 2\pi(m_1 - 1) + 10^2 4^{-1/2} \sum_{m \in \Lambda^c, m_1 \leq m} p_{F_m} \\ &\leq 2\pi m_1 + 10^2 \max\{a_1^{-1/2}, 1\} \sum_{m \in \Lambda^c} a_m^{-1/2} < +\infty . \end{aligned}$$

Let $m \in \Lambda$. We shall use Lemma 16 for $Q(t) = F_m(t)$, $J = \tau(m)$, $\eta = 1$, $M = m$ and $I = [0, 2\pi)$. First we note that $w(\tau(m), m)^{-2} \leq 2s_m^{-2}$ and $\gamma(\tau(m), m) \leq \gamma(\tau(m))$. Since $N_{F_m} = m \geq 64$, $J = \tau(m) \leq m - 63 = N_{F_m}$ and $4\nu(F_m) = 4a_m \leq w(m-63, m) \leq w(\tau(m), m) = w(\tau(m); F_m)$, we have, from Lemma 16,

$$\begin{aligned} \int_0^{2\pi} \lambda_1(F_m(t))dt &\leq 10^8[w(\tau(m), m)^{-2} + \{2\pi\gamma(\tau(m), m) + \tilde{\gamma}(\tau(m), m)\}a_m m^6 + 2\pi m^{-2}] \\ &\leq 10^8\{2s_m^{-2} + (2\pi + 1)2^m m^6 \gamma(\tau(m)) + 2\pi m^{-2}\} \\ &\leq 10^9(s_m^{-2} + 2^m m^6 \gamma(\tau(m)) + m^{-2}) . \end{aligned}$$

Hence

$$\overline{m}_2 = \lim_{M\to\infty} h(1, M; [0, 2\pi), 1) = \int_0^{2\pi} \sum_{m\in\Lambda} \lambda_1(F_m(t))dt$$
$$\leq 10^9\Big\{\sum_{m\in\Lambda} s_m^{-2} + \sum_{m\in\Lambda} 2^m m^6\gamma(\tau(m)) + \sum_{m\in\Lambda} m^{-2}\Big\}$$
$$\leq 10^9\Big(\sum_{m=1}^{\infty} s_m^{-2} + C_1 + 2\Big) < +\infty .$$

Consequently, (using $\chi_{1/4} \leq 64\pi^{-2}\lambda_1$,)

$$\sum_{m=1}^{\infty} m(\{t\in[0, 2\pi); |F_m(t)| < 1/4\}) \tag{59}$$
$$= \sum_{m\in\Lambda^c} + \sum_{m\in\Lambda} \leq \overline{m}_1 + 64\pi^{-2}\overline{m}_2 < +\infty.$$

Choosing a countable dense set Σ in $\boldsymbol{C}$, set $U = \bigcup_{a\in\Sigma} U_a$, where $U_a = \{t\in[0, 2\pi); \liminf_{m\to\infty} |F_m(t) - a| < 1/4\}$. By (59), we have

$$m(U_0) \leq \lim_{\mu\to\infty} \sum_{m=\mu}^{\infty} m(\{t\in[0, 2\pi); |F_m(t)| < 1/4\}) = 0 .$$

Considering $F(t) - a$, we have $m(U_a) = 0$ and hence $m(U) \leq \sum_{a\in\Sigma} m(U_a) = 0$, that is, $m(U^c) = 2\pi$.

Let $t_0 \in U^c$. For every $b\in\boldsymbol{C}$, there exists $a\in\Sigma$ such that $|b - a| \leq 1/8$. Then

$$\liminf_{m\to\infty} |F_m(t_0) - b| \geq \liminf_{m\to\infty} |F_m(t_0) - a| - 1/8 \geq 1/8 .$$

Since $b\in\boldsymbol{C}$ is arbitrary, we have $\lim_{m\to\infty} |F_m(t_0)| = +\infty$. This completes the proof of (A).

For the proof of (B), we prepare some more lemmas.

LEMMA 36. *Let $\eta > 0$, K a positive integer, U a finite union of intervals in $[0, 2\pi)$ and $C_1 = \sum_{m=1}^{\infty} 2^m m^6\gamma(\tau(m))$. Then there exists a positive number $A_1(\eta, U, C_1)$ depending only on η, U and C_1 such that $h(K, M; U, \eta) \leq 10^{10}m(U)\eta^2 T_M$ as long as $T_M \geq A_1(\eta, U, C_1)$.*

Proof. Without loss of generality, we may assume that $m(U) \neq 0$. Let us write $U = \bigcup_{\mu=1}^{\nu_U} I_\mu$, where I_μ's are mutually disjoint intervals. Set $\kappa_U = 1 + \max\{m(I_\mu)^{-1}; m(I_\mu) \neq 0, 1 \leq \mu \leq \nu_U\}$.

Let $m\in\Lambda$. We shall use Lemma 16 for $Q(t) = F_m(t)$, $J = \tau(m)$, $M = m$ and $I = I_\mu$. Since $N_{F_m} = m \geq 64$, $J = \tau(m) \leq m - 63 = N_{F_m} - 63$ and $4\nu(F_m) = 4a_m \leq w(m - 63, m) \leq w(\tau(m), m) = w(\tau(m); F_m)$, we have, for $1 \leq \mu \leq \nu_U$ satisfying $m(I_u) \neq 0$,

$$\begin{aligned}&\int_{I_\mu}\lambda_\eta(F_m(t))dt\\&\quad\le 10^8\{m(I_\mu)\eta^2 w(\tau(m), m)^{-2}+(m(I_\mu)+1)\eta^2 a_m\gamma(\tau(m), m)m^6+m(I_\mu)\eta^{-1}m^{-2}\}\\&\quad\le 10^9 m(I_\mu)\eta^2(s_m^{-2}+\kappa_U 2^m m^6\gamma(\tau(m))+\eta^{-3}m^{-2}) .\end{aligned}$$

Hence

$$\begin{aligned}h(K, M; U, \eta) &= \sum_{m(I_\mu)\ne 0}\int_{I_\mu}\sum_{K\le m\le M, m\in\Lambda}\lambda_\eta(F_m(t))dt\\&\le 10^9 m(U)\eta^2\sum_{K\le m\le M, m\in\Lambda}(s_m^{-2}+\kappa_U 2^m m^6\gamma(\tau(m))+\eta^{-3}m^{-2})\\&\le 10^9 m(U)\eta^2(T_M+\kappa_U C_1+2\eta^{-3}) .\end{aligned}$$

Putting $A_1(\eta, U, C_1)=\kappa_U C_1+2\eta^{-3}$, we obtain the required inequality.

LEMMA 37. *Let η, K, U, C_1 be the same as in Lemma 36 and let $C_2 = \sum_{m=1}^{\infty} n_m \sum_{j=m+2}^{\infty} 2^j j^6 \tilde{\gamma}(\tau(m+j))$. Then there exists a positive number $A_2(\eta, U, C_1, C_2)$ depending only on η, U, C_1, C_2 such that $H(K, M; U, \eta)\le 10^{30} m(U)\eta^4 T_M^2$ as long as $T_M \ge A_2(\eta, U, C_1, C_2)$.*

Proof. We use the same notation ν_U, I_μ $(1\le\mu\le\nu_U)$ and κ_U as in Lemma 36. Putting

$$\begin{cases}\Lambda(\ell)=\{m\in\Lambda;\, 128\,|\,(m-\ell)\}\\ H_\ell(K, M; I_\mu, \eta)=\int_{I_\mu}\Big\{\sum_{K\le m\le M, m\in\Lambda(\ell)}\lambda_\eta(F_m(t))\Big\}^2 dt\\ H'_\ell(K, M; I_\mu, \eta)=\int_{I_\mu}\sum_{K\le m\le M, m\in\Lambda(\ell)}\ \sum_{m<j\le M, j\in\Lambda(l)}\lambda_\eta(F_m(t))\lambda_\eta(F_j(t))dt\\ \qquad\qquad (1\le\ell\le 128, 1\le\mu\le\nu_U) ,\end{cases}$$

we have

$$\begin{aligned}H(K, M; U, \eta)&\le 128^2\sum_{\ell=1}^{128}\sum_{m(I_\mu)\ne 0}H_\ell(K, M; I_\mu, \eta)\\&=128^2\Big\{\int_U\sum_{K\le m\le M, m\in\Lambda}\lambda_\eta(F_m(t))^2 dt+2\sum_{\ell=1}^{128}\sum_{m(I_\mu)\ne 0}H'_\ell(K, M; I_\mu, \eta)\Big\} \qquad (60)\\&=10^5\Big\{\pi^2 h(K, M; U, \eta)+\sum_{\ell=1}^{128}\sum_{m(I_\mu)\ne 0}H'_\ell(K, M; I_\mu, \eta)\Big\} .\end{aligned}$$

Now we estimate $H'_\ell(K, M; I_\mu, \eta)$ for fixed ℓ $(1\le\ell\le 128)$ and μ $(1\le\mu\le\nu_U, m(I_\mu)\ne 0)$. Let m and j be two integers in $\Lambda(\ell)$ such that $m<j$. We use Lemma 18 for $Q(t)=F_m(t)$, $R(t)=F_j(t)$, $J=\tau(m)$, $J'=\tau(m+j)$ $\eta=\eta'$, $M=m$, $M'=j$ and $I=I_\mu$. Note that

$$\begin{cases} w(\tau(m), m)^2 \geq s_m^2/2, \ w(\tau(m+j), m)^2 \geq w(m, j)^2/2 \geq s_{j-m+1}^2/2 \\ a_m \leq 2^m, \ a_j \leq 2^j \\ \gamma(\tau(m), m) \leq \gamma(\tau(m)), \ \tilde{\gamma}(\tau(m), m) \leq \tilde{\gamma}(\tau(m)) . \end{cases}$$

Since

$$\begin{cases} N_{F_m} = m \geq 64, \ N_{F_j} = j \geq 64 \\ J = \tau(m) \leq m - 63 = N_{F_m} - 63, \\ J' = \tau(m-j) \leq \dfrac{m+j}{2} \leq \dfrac{2j-127}{2} \leq j - 63 = N_{F_j} - 63 \\ 4\nu(F_m) = 4a_m \leq w(m-63, m) \leq w(\tau(m), m) = w(\tau(m); F_m) \\ 4\nu(F_j) = 4a_j \leq w(j-63, j) \leq w(\tau(m+j), j) = w(\tau(m+j); F_j) , \end{cases}$$

we have, from Lemma 18,

$$\begin{aligned} &\int_{I_\mu} \lambda_\eta(F_m(t))\lambda_\eta(F_j(t))dt \\ &\quad \leq 10^{20}\{m(I_\mu)\eta^2 w(\tau(m), m)^{-2} + (m(I_\mu)+1)\eta^2 a_m \gamma(\tau(m), m)m^6 + m(I_\mu)\eta^{-1}m^{-2}\} \\ &\qquad \times \{\eta^2 w(\tau(m+j), j)^{-2} + \eta^{-1}m^{-2}\} + 10^{10}\eta^2 a_j \tilde{\gamma}(\tau(m+j), j) j^6 n_m \\ &\quad \leq 10^{21} m(I_\mu)\eta^4 (s_m^{-2} + \kappa_U 2^m m^6 \gamma(\tau(m)) + \eta^{-3}m^{-2})(s_{j-m+1}^{-2} + \eta^{-3}j^{-2}) \\ &\qquad + 10^{10}\eta^2 2^j j^6 \tilde{\gamma}(\tau(m+j))n_m , \end{aligned}$$

and hence

$$\begin{aligned} &H'_\ell(K, M; I_\mu, \eta) \\ &\quad \leq 10^{21} m(I_\mu)\eta^4 \sum_{K \leq m \leq M, m \in \Lambda(\ell)} \ \sum_{m < j \leq M, j \in \Lambda(\ell)} (s_m^{-2} + \kappa_U 2^m m^6 \gamma(\tau(m)) + \eta^{-3}m^{-2}) \\ &\qquad \times (s_{j-m+1}^{-2} + \eta^{-3}j^{-2}) + 10^{10}\eta^2 \sum_{K \leq m \leq M, m \in \Lambda(\ell)} \ \sum_{m < j \leq M, j \in \Lambda(\ell)} 2^j j^6 \tilde{\gamma}(\tau(m+j))n_m \\ &\quad \leq 10^{21} m(I_\mu)\eta^4 \sum_{m=1}^{M} (s_m^{-2} + \kappa_U 2^m m^6 \gamma(\tau(m)) + \mu^{-3}m^{-2}) \\ &\qquad\qquad \times \sum_{j=m+2}^{M} (s_{j-m+1}^{-2} + \eta^{-3}j^{-2}) + 10^{10}\eta^2 C_2 \\ &\quad \leq 10^{21} m(I_\mu)\eta^4 (T_M + \kappa_U C_1 + 2\eta^{-3})(T_M + 2\eta^{-3}) + 10^{10}\eta^2 C_2 . \end{aligned}$$

By (60), we have

$$\begin{aligned} &H(K, M; U, \eta) \leq 10^5 \pi^2 h(K, M; U, \eta) \\ &\qquad + 10^{26} 128 m(U)\eta^4 (T_M + \kappa_U C_1 + 2\eta^{-3})(T_M + 2\eta^{-3}) + 10^{15} 128 \nu_U \eta^2 C_2 . \end{aligned}$$

Taking account of this inequality, choose a positive number $A_2(\eta, U, C_1, C_2)$ sufficiently large. Then the required inequality follows from Lemma 36.

LEMMA 38. *Let K, U, C_1 be the same as in Lemma 36. Let $0 < \eta \leq a_1$*

and $C_3=\sum_{m\in\Lambda^c}s_m^{-2}$. *Then there exists a positive constant* $A_3=A_3(T_{K-1}, \eta, U, a_0, C_1, C_3, \gamma(1))$ *such that* $h(K, M; U, \eta)\geq 10^{-4}m(U)\eta^2T_M$ *as long as* $\min\{T_M, T_M/U_M, T_M/V_M\}\geq A_3$.

Proof. We use the same notation ν_U, I_μ $(1\leq\mu\leq\nu_U)$, κ_U as in Lemma 36. Let $m\in\Lambda$. We use Lemma 20 for $Q(t)=F_m(t)$, $J=\tau(m)$, $M=m$ and $I=I_\mu$. We have, for $1\leq\mu\leq\nu_U$ satisfying $m(I_\mu)\neq 0$,

$$\begin{aligned}\int_{I_\mu}\lambda_\eta(F_m(t))dt &\geq 10^{-3}m(I_\mu)\eta^2w(1, m)^{-2}\\ &\quad -10^6[m(I_\mu)\eta^2w(1, m)^{-2}\{a_mw(1, m)^{-1}\}^{2/5}+m(I_\mu)\eta^2a_0w(1, m)^{-2}w(1, m)^{-2/5}\\ &\quad +(m(I_\mu)+1)\eta^2\gamma(1, m)w(1, m)^{-2}\{a_mw(1, m)^{-1}\}^{2/5}\\ &\quad +m(I_\mu)\eta^2w(1, m)^{-2}\{a_mw(1, m)^9w(\tau(m), m)^{-10}\}^{2/5}\\ &\quad +(m(I_\mu)+1)\eta^2a_m\gamma(\tau(m), m)m^6+m(I_\mu)\eta^{-1}m^{-2}]\\ &\geq 10^{-3}m(I_\mu)\eta^2s_m^{-2}-10^6m(I_\mu)\eta^2[s_m^{-2}(a_ms_m^{-1})^{2/5}+a_0s_m^{-2}s_m^{-2/5}\\ &\quad +\kappa_U\gamma(1)s_m^{-2}(a_ms_m^{-1})^{2/5}+s_m^{-2}(2^{10}a_ms_m^{-1})^{2/5}+\kappa_U2^mm^6\gamma(\tau(m))+\eta^{-3}m^{-2}]\,,\end{aligned}$$

and hence

$$\begin{aligned}h(K, M; U, \eta) &=\sum_{m(I_\mu)\neq 0}\int_{I_\mu}\sum_{K\leq m\leq M, m\in\Lambda}\lambda_\eta(F_m(t))dt\\ &\geq 10^{-3}m(U)\eta^2(T_M-T_{K-1}-C_3)\\ &\quad -10^6m(U)\eta^2[(1+\kappa_U\gamma(1)+2^4)U_M+a_0V_M+\kappa_UC_1+2\eta^{-3}]\,.\end{aligned}$$

Taking account of this inequality, choose a positive constant A_3 sufficiently large. Then we obtain the required inequality.

Now we give the proof of (B). For $0<\eta\leq a_1$, $a\in\boldsymbol{C}$ and two positive integers μ, K $(\mu\leq K)$, we put $U(\mu, K, a, \eta)=\bigcup_{m=\mu}^K\{t\in[0, 2\pi); |F_m(t)-a|<\eta\}$ and $U(\mu, a, \eta)=\bigcup_{K=\mu}^\infty U(\mu, K, a, \eta)$. Note that $U(\mu, K, a, \eta)^c$ is a finite union of intervals. By Lemma 37 and 38, there exists a positive integer M $(K<M)$ such that $h(K+1, M; U(\mu, K, 0, \eta)^c, \eta/3)\geq 10^{-4}m(U(\mu, K, 0, \eta)^c)(\eta/3)^2T_M$, $H(K+1, M; U(\mu, K, 0, \eta)^c, \eta/3)\leq 10^{30}m(U(\mu, K, 0, \eta)^c)(\eta/3)^4T_M^2$. Then we have, from Corollary 9,

$$\begin{aligned}&m(U(\mu, K, 0, \eta)^c\cap U(K+1, M, 0, \eta))\\ &\quad =m\Big(\Big\{t\in U(\mu, K, 0, \eta)^c;\ \sum_{m=K+1}^M\chi_\eta(F_m(t))>0\Big\}\Big)\\ &\quad \geq m\Big(\Big\{t\in U(\mu, K, 0, \eta)^c;\ \sum_{m=K+1}^M\lambda_{\eta/3}(F_m(t))>0\Big\}\Big)\\ &\quad \geq h(K+1, M; U(\mu, K, 0, \eta)^c, \eta/3)^2H(K+1, M; U(\mu, K, 0, \eta)^c, \eta/3)^{-1}\\ &\quad \geq 10^{-38}m(U(\mu, K, 0, \eta)^c)\,,\end{aligned}$$

and hence

$$\begin{aligned} m(U(\mu, M, 0, \eta)^c) &= m(U(\mu, K, 0, \eta)^c) - m(U(\mu, K, 0, \eta)^c \cap U(K+1, M, 0, \eta)) \\ &\le (1 - 10^{-38}) m(U(\mu, K, 0, \eta)^c) . \end{aligned}$$

Repeating this discussion, we have $m(U(\mu, 0, \eta)^c) = 0$, that is, $m(U(\mu, 0, \eta)) = 2\pi$. Considering $F(t) - a$, we have $m(U(\mu, a, \eta)) = 2\pi$. Choosing a countable dense set Σ in $\boldsymbol{C}$, we put $U = \bigcap_{a \in \Sigma} \bigcap_{\ell=1}^{\infty} \bigcap_{\mu=1}^{\infty} U(\mu, a, a_1/\ell)$. Then $m(U) = 2\pi$. Hence it is sufficient to show that, for any $t \in U$, $C(t; F) = \hat{\boldsymbol{C}}$. Let $t_0 \in U$. We have, for every $a \in \Sigma$, $\liminf_{m \to \infty} |F_m(t_0) - a| = 0$ and hence $C(t_0; F) \supset \Sigma$. Since $C(t_0; F)$ is closed, $C(t_0; F) = \hat{\boldsymbol{C}}$. This completes the proof of (A).

5.3. Proof of Theorem 31

Let $F(t)$ be an L-lacunary series such that $F \notin L^2(0, 2\pi)$ and $\nu(F) < +\infty$. Set:

$$\text{(61)} \qquad \begin{cases} n_k = n_k(F),\ a_k = |\tilde{F}(k)| & (k \ge 0) \\ w(m, M) = \left(\sum_{k=m}^{M} a_k^2 \right)^{1/2} & (1 \le m \le M) \\ \gamma(m) = \gamma(m; F),\ \tilde{\gamma}(m) = \tilde{\gamma}(m; F) & (m \ge 1) . \end{cases}$$

Without loss of generality, we may assume that $a_m \le 1$ $(m \ge 1)$. We define inductively a sequence $(\Gamma(m))_{m=0}^{\infty}$ of non-negative integers by $\Gamma(0) = 0$ and by $\Gamma(m) = \min \{k \in \boldsymbol{Z}^+; w(\Gamma(m-1)+1, k) \ge 8\}$. Let us note that $\Gamma(m) - \Gamma(m-1) \ge 64$ $(m \ge 1)$. For $\eta > 0$, two positive integers K, M $(K < M)$ and a Borel set U in $[0, 2\pi)$, we put

$$\text{(62)} \qquad \begin{cases} \tilde{h}(K, M; U, \eta) = \int_U \sum_{m=K}^{M} \lambda_\eta(F_{\Gamma(m)}(t)) dt \\ \tilde{H}(K, M; U, \eta) = \int_U \left\{ \sum_{m=K}^{M} \lambda_\eta(F_{\Gamma(m)}(t)) \right\}^2 dt . \end{cases}$$

LEMMA 39. *Putting $\tilde{\Gamma}(m) = \Gamma(\tau(m))$ $(m \ge 1)$, we have, for $k \ge 1$, $M \ge k+1$,*

$$\text{(63)} \qquad \begin{cases} \sum_{m=k+1}^{M} w(\tilde{\Gamma}(m+k), \Gamma(m))^{-2} \le \log(M-k) + 1 \\ \sum_{m=1}^{M} w(1, \Gamma(m))^{-2} \ge 10^{-2} \log M . \end{cases}$$

Proof. We have $64 \le w(\Gamma(m)+1, \Gamma(m+1))^2 \le 65$ $(m \ge 1)$. Inequalities (63) follows from these inequalities.

The following Lemmas 40, 41, 42 and 43 play the analogous role as Lemmas 33 36, 37 and 38, respectively.

LEMMA 40. $\tilde{C}_1 = \sum_{m=1}^{\infty} m^6 \gamma(\tilde{\Gamma}(m) + 1) < +\infty$,
$\tilde{C}_2 = \sum_{m=1}^{\infty} n_{\Gamma(m)} \sum_{j=m+1}^{\infty} j^6 \tilde{\gamma}(\tilde{\Gamma}(m+j) + 1) < +\infty$.

Proof. Use Lemma 5.

LEMMA 41. *Let $\eta > 0$, K a positive integer and let U be a finite union of intervals in $[0, 2\pi)$. Then there exists a positive number $\tilde{A}_1(\eta, U, \tilde{C}_1)$ depending only on η, U, $\tilde{C}_1$ such that $\tilde{h}(K, M; U, \eta) \leq 10^9 m(U)\eta^2 \log M$ as long as $\log M \geq \tilde{A}_1(\eta, U, \tilde{C}_1)$.*

Proof. Without loss of generality, we may assume that U is an interval satisfying $m(U) \neq 0$. Set $\kappa_U = 1 + m(U)^{-1}$. The estimation of $\tilde{L}_m = \int_U \lambda_\eta(F_{\Gamma(m)}(t))dt$ $(m \geq 1)$ is essential, for which we use Lemma 16. Put $Q(t) = F_{\Gamma(m)}(t)$, $J = \tilde{\Gamma}(m) + 1$, $M = m$ and $I = U$. Then we have, from Lemma 16,

$$\tilde{L}_m \leq 10^8 m(U)\eta^2\{w(\tilde{\Gamma}(m) + 1, \Gamma(m))^{-2} + \kappa_U m^6 \gamma(\tilde{\Gamma}(m) + 1) + \eta^{-3}m^{-2}\} .$$

Hence, by (63), we have, for $K < M$,

$$\tilde{h}(K, M; U, \eta) \leq 10^8 m(U)\eta^2(\log M + 1 + \kappa_U \tilde{C}_1 + 2\eta^{-3}) .$$

Put $\tilde{A}_2(\eta, U, \tilde{C}_1) = 1 + \kappa_U \tilde{C}_1 + 2\eta^{-3}$. Then we obtain the required inequality.

LEMMA 42. *Let η, K, U be the same as in Lemma 41. Then there exists a positive number $\tilde{A}_2(\eta, U, \tilde{C}_1, \tilde{C}_2)$ depending only on η, U, $\tilde{C}_1$, $\tilde{C}_2$ such that $\tilde{H}(K, M; U, \eta) \leq 10^{21} m(U)\eta^4(\log M)^2$ as long as $\log M \geq \tilde{A}_2(\eta, U, \tilde{C}_1, \tilde{C}_2)$.*

Proof. We may assume that U is an interval satisfying $m(U) \neq 0$. Set $\kappa_U = 1 + m(U)^{-1}$. The estimation of $\tilde{L}_{m,j} = \int_U \lambda_\eta(F_{\Gamma(m)}(t))\lambda_\eta(F_{\Gamma(j)}(t))dt$ $(1 \leq m < j)$ is essential, for which we use Lemma 18. Put $Q(t) = F_{\Gamma(m)}(t)$, $R(t) = F_{\Gamma(j)}(t)$, $J = \tilde{\Gamma}(m) + 1$, $J' = \tilde{\Gamma}(m+j) + 1$, $\eta = \eta'$, $M = m$, $M' = j$ and $I = U$. Then we have, from Lemma 18,

$$\tilde{L}_{m,j} \leq 10^{20} m(U)\eta^4\{w(\tilde{\Gamma}(m) + 1, \Gamma(m))^{-2} + \kappa_U m^6 \gamma(\tilde{\Gamma}(m) + 1) + \eta^{-3}m^{-2}\}$$
$$\times \{w(\tilde{\Gamma}(m+j) + 1, \Gamma(j))^{-2} + \eta^{-3}j^{-2}\} + 10^{10}\eta^2 j^6 \tilde{\gamma}(\tilde{\Gamma}(m+j) + 1)n_{\Gamma(m)} .$$

Hence we have, from (63),

$$\tilde{H}(K, M; U, \eta) \leq \pi^2 \tilde{h}(K, M; U, \eta) + 2 \sum_{m=K}^{M} \sum_{j=m+1}^{M} \tilde{L}_{m,j}$$

$$\leq \pi^2 \tilde{h}(K, M; U, \eta) + 2\cdot 10^{20} m(U)\eta^4(\log M + 1 + \kappa_U \tilde{C}_1 + 2\eta^{-3}) \\ \times (\log M + 1 + 2\eta^{-3}) + 10^{10}\eta^2 \tilde{C}_2 .$$

Taking account of the last term and Lemma 41, choose a positive number $\tilde{A}_2(\eta, U, \tilde{C}_1, C_2)$ sufficiently large. Then we obtain the required inequality.

LEMMA 43. *Let K and U be the same as in Lemma 41 and let $0 < \eta \leq a_1$. Then there exists a positive constant $\tilde{A}_3 = \tilde{A}_3(K, \eta, U, a_0, \tilde{C}_1, \gamma(1))$ such that $\tilde{h}(K, M; U, \eta) \geq 10^{-6} m(U)\eta^2 \log M$ as long as $\log M \geq \tilde{A}_3$.*

Proof. We may assume that U is an interval satisfying $m(U) \neq 0$. The estimation of $\tilde{L}_m = \int_U \lambda_\eta(F_{\Gamma(m)}(t))dt$ $(m \geq 1)$ is essential, for which we use Lemma 20. Put $Q(t) = F_{\Gamma(m)}(t)$, $J = \tilde{\Gamma}(m) + 1$, $M = m$, $I = U$. Then, we have, from Lemma 20, $\tilde{h}(K, M; U, \eta) \geq 10^{-3} m(U)\eta^2 w(1, \Gamma(m))^{-2} - B_m$, where B_m is determined by (39). Note that $\sum_{m=1}^{M} B_m = o(\log M)$. We have, from (61),

$$\tilde{h}(K, M; U, \eta) \geq 10^{-5} m(U)\eta^2 \log M + o(\log M)$$

Taking account of this inequality, choose a positive constant $\tilde{A}_3$ sufficiently large. Then we obtain the required inequality.

Now we give the proof of Theorem 31. For $0 < \eta \leq a_1$, $a \in \boldsymbol{C}$ and two positive integers μ, $K(\mu \leq K)$, we put $\tilde{U}(\mu, K, a, \eta) = \bigcup_{m=\mu}^{K} \{t \in [0, 2\pi);\ |F_{\Gamma(m)}(t) - a| < \eta\}$ and $\tilde{U}(\mu, a, \eta) = \bigcup_{K=\mu}^{\infty} \tilde{U}(\mu, K, a, \eta)$. By Lemma 42 and 43 there exists a positive integer M $(K < M)$ such that $\tilde{h}(K + 1, M; \tilde{U}', \eta/3) \geq 10^{-6} m(\tilde{U}')(\eta/3)^2 \log M$ and $\tilde{H}(K + 1, M; \tilde{U}', \eta/3) \leq 10^{21} m(\tilde{U}')(\eta/3)^4(\log M)^2$, where $\tilde{U}' = \tilde{U}(\mu, K, 0, \eta)^c$. Then we have, from Corollary 9, $m(\tilde{U}(\mu, M, 0, \eta)^c) \leq (1-10^{-33}) m(\tilde{U}(\mu, K, 0, \eta)^c)$. Repeating this discussion, we have $m(\tilde{U}(\mu, 0, \eta)^c) = 0$, that is, $m(\tilde{U}(\mu, 0, \eta)) = 2\pi$. Considering $F(t) - a$, we have $m(\tilde{U}(\mu, a, \eta)) = 2\pi$. Choosing a countable dense set Σ in $\boldsymbol{C}$, set $\tilde{U} = \bigcap_{a\in\Sigma} \bigcap_{\ell=1}^{\infty} \bigcap_{\mu=1}^{\infty} \tilde{U}(\mu, a, a_1/\ell)$. Then $m(\tilde{U}) = 2\pi$ and, for any $t \in \tilde{U}$, $C(t; F) = \hat{\boldsymbol{C}}$. This completes the proof.

5.4. Proof of (C) and (D)

The method of the proof of (C) and (D) is analogous as in (A) and (B). Hence we only give the sketch of the proof. Let $F(t)$ be an L-lacunary series such that $(a_k)_{k=1}^{\infty}$ is increasing, where $a_k = |\tilde{F}(k)|$ $(k \geq 1)$. Set $s_m = (\sum_{k=1}^{m} a_k^2)^{1/2}$ and $\overset{*}{T}_m = \sum_{k=1}^{m} s_k^{-1}$ $(m \geq 1)$. It is sufficient to show our assertion with respect to cluster sets of $P(t) = \operatorname{Re} F(t)$. Using Corollary 22, we choose suitably a set $\overset{*}{\Lambda}$ in $\boldsymbol{Z}^+$ so that $\sum_{m\in\overset{*}{\Lambda}^c} s_m^{-1} < +\infty$ and that, for any $\rho > 0$, $\sum_{m \in \mathring{\Lambda}^c} m(\{t \in [0, 2\pi); |P_m(t)| < \rho\}) < +\infty$, where $\mathring{\Lambda}^c = \boldsymbol{Z}^+ - \mathring{\Lambda}$. For $0 < \eta \leq a_1$, m, $j \in \mathring{\Lambda}$ $(m < j)$ and a finite union of intervals U satisfying $m(U) \neq 0$, set

$$\text{(64)} \qquad \begin{cases} \mathring{L}_m(U, \eta) = \int_U \chi_\eta(P_m(t))dt \qquad (P_m(t) = \operatorname{Re} F_m(t)) \\ \mathring{L}_{m,j}(U, \eta) = \int_U \chi_\eta(P_m(t))\chi_\eta(P_j(t))dt \\ \mathring{\ell}_m(U, \eta) = \dfrac{4}{\pi} m(U) \displaystyle\int_0^\infty \frac{\sin \eta r}{r} \prod_{k=1}^m J_0(a_k r)dr\,. \end{cases}$$

Then we have, for sufficiently large m and j satisfying m, $j \in \mathring{\Lambda}$, $m + 128 \leq j$,

$$\text{(65)} \qquad \begin{cases} \theta_5 m(U)\eta s_m^{-1} \leq \mathring{\ell}_m(U, \eta) \leq \theta_6 m(U)\eta s_m^{-1} \\ |\mathring{L}_m(U, \eta) - \mathring{\ell}_m(U, \eta)| \leq o(1)/s_m \\ |\mathring{L}_{m,j}(U, \eta)| \leq \theta_7 m(U)\eta^2 s_m^{-1} s_{j-m+1}^{-1}\,, \end{cases}$$

where θ_5, θ_6 and θ_7 are positive constants which are independent of η and U. This shows that $\sum_{m=1}^\infty \mathring{L}_m(U, \eta)$ converges if and only if $\sum_{m=1}^\infty s_m^{-1}$ converges. The property (C) follows from this fact.

To prove (D), suppose that $\sum_{m=1}^\infty s_m^{-1} = +\infty$. For $0 < \eta \leq a_1$ two positive integers K, M $(K < M)$ and a finite union of intervals U, set

$$\text{(66)} \qquad \begin{cases} \mathring{h}(K, M;\, U, \eta) = \displaystyle\int_U \sum_{K \leq m \leq M, m \in \mathring{\Lambda}} \chi_\eta(P_m(t))\, dt \\ \mathring{H}(K, M;\, U, \eta) = \displaystyle\int_U \Big\{ \sum_{K \leq m \leq M, m \in \mathring{\Lambda}} \chi_\eta(P_m(t)) \Big\}^2 dt\,. \end{cases}$$

Note that

$$\mathring{H}(K, M;\, U, \eta) \leq 128^2 \mathring{h}(K, M;\, U, \eta) + 2 \cdot 128^2 \sum_{\ell=1}^{128} \sum\nolimits_{m,j}^{(\ell)} \mathring{L}_{m,j}(U, \eta)\,,$$

where $\sum_{m,j}^{(\ell)}$ denotes the summation over all m, $j \in \mathring{\Lambda}$ satisfying $K \leq m < j \leq M$, $m = \ell$ (mod 128) and $j - m = 0$ (mod 128). Using (65), we see that, for given η, K and U $(m(U) \neq 0)$, there exists a positive integer M $(K<M)$ such that $\mathring{h}(K, M; U, \eta) \geq 2^{-1}\theta_5 m(U)\eta \mathring{T}_M$ and $\mathring{H}(K, M; U, \eta) \leq 4 \cdot 128^3 \theta_7 m(U)\eta^2 \mathring{T}_M^2$. Using these two inequalities, we have $m(\mathring{U}(\mu, 0, \eta)) = 2\pi$, where $\mathring{U}(\mu, 0, \eta) = \bigcup_{m=\mu}^\infty \{t \in [0, 2\pi); |P_m(t)| < \eta\})$. Choosing a countable dense set $\mathring{\Sigma}$ in $\hat{\boldsymbol{R}}$, set $\mathring{U}(\mu, \xi, \eta) = \bigcup_{m=\mu}^\infty \{t \in [0, 2\pi); |P_m(t) - \xi| < \eta\})$ $(\xi \in \mathring{\Sigma})$ and $\mathring{U} = \bigcap_{\xi \in \mathring{\Sigma}} \bigcap_{\mu=1}^\infty \bigcap_{\ell=1}^\infty \mathring{U}(\mu, \xi, a_1/\ell)$.

Considering $P(t)-\xi$, we have $m(\overset{*}{U}(\mu, \xi, \eta)) = 2\pi$ and hence $m(\overset{*}{U}) = 2\pi$. For any $t \in \overset{*}{U}$, $C(t; P) = \hat{\boldsymbol{R}}$. This shows that (D) holds.

We note that if $F(t)$ is recurrent in $\boldsymbol{C}$, then it is recurrent in $\boldsymbol{R}$. Hence Corollary 32 is an immediate consequence of Theorem 30 and 31.

5.5. Application to the theory of cluster sets of Steinhaus series

Let Ω be a countable product of an interval $[0, 1)$. Element of Ω is denoted by a small letter ω or a sequence $(\phi_1, \phi_2, \cdots)$, where $\phi_j \in [0, 1)$. A probability measure in Ω (, which is denoted by "Pr",) is defined by a countable product of the 1-dimensional Lebesgue measure.

For a sequence $(c_k)_{k=1}^{\infty}$ of complex numbers, a (formal) complex Steinhaus series is defined by $\Psi(\omega) \sim \sum_{k=1}^{\infty} c_k \exp(2\pi i \phi_k)$ $(\omega = (\phi_1, \phi_2, \cdots))$. We write $\tilde{\Psi}(k) = c_k$ and $s(k; \Psi) = (\sum_{\ell=1}^{k} |c_\ell|^2)^{1/2}$ $(k \geq 1)$. $C(\omega; \Psi)$ denotes the totality of cluster points of a sequence $(\Psi_m(\omega))_{m=1}^{\infty}$ in $\hat{\boldsymbol{C}}$, where $\Psi_m(\omega) = \sum_{k=1}^{m} c_k \exp(2\pi i \phi_k)$ $(\omega = (\phi_1, \phi_2, \cdots))$. For a compact set A in $\hat{\boldsymbol{C}}$, set $C^{-1}(A; \Psi) = \{\omega \in \Omega; C(\omega; \Psi) = A\}$.

For a sequence $(\xi_k)_{k=1}^{\infty}$ of real numbers, a (formal) real Steinhaus series is defined by $\psi(\omega) \sim \sum_{k=1}^{\infty} \xi_k \cos 2\pi\phi_k$ $(\omega = (\phi_1, \phi_2, \cdots))$. We define analogously as above: $\tilde{\psi}(k)$, $s(k; \psi)$, $C(\cdot\,; \psi)$, $C^{-1}(\cdot\,; \psi)$.

We say that:

$$\Psi(\omega) \text{ is recurrent in } \boldsymbol{C} \text{ if } m(C^{-1}(\hat{\boldsymbol{C}}; \Psi)) = 2\pi .$$
$$\Psi(\omega) \text{ is transient in } \boldsymbol{C} \text{ if } m(C^{-1}(\{\infty\}; \Psi)) = 2\pi .$$
$$\psi(\omega) \text{ is recurrent in } \boldsymbol{R} \text{ if } m(C^{-1}(\hat{\boldsymbol{R}}; \psi)) = 2\pi .$$
$$\psi(\omega) \text{ is transient in } \boldsymbol{R} \text{ if } m(C^{-1}(\{\infty\}; \psi)) = 2\pi .$$

Then we have the following

THEOREM 44. *Let $\Psi(\omega)$ and $\psi(\omega)$ be a complex Steinhaus series and a real Steinhaus series such that $(|\tilde{\Psi}(k)|)_{k=1}^{\infty}$ and $(|\tilde{\psi}(k)|)_{k=1}^{\infty}$ are increasing, respectively.*

(A)′ *If* $\sum_{m=1}^{\infty} s(m; \Psi)^{-2} < +\infty$, *then $\Psi(\omega)$ is transient in $\boldsymbol{C}$.*

(B)′ *If* $\sum_{m=1}^{\infty} s(m; \Psi)^{-2} = +\infty$, *then $\Psi(\omega)$ is recurrent in $\boldsymbol{C}$.*

(C)′ *If* $\sum_{m=1}^{\infty} s(m; \psi)^{-1} < +\infty$, *then $\psi(\omega)$ is transient in $\boldsymbol{R}$.*

(D)′ *If* $\sum_{m=1}^{\infty} s(m; \psi)^{-1} = +\infty$, *then $\psi(\omega)$ is recurrent in $\boldsymbol{R}$.*

THEOREM 45. *Let $\Psi(\omega)$ be a complex Steinhaus series such that* $\sup_k |\tilde{\Psi}(k)| < +\infty$ *and* $\sum_{k=1}^{\infty} |\tilde{\Psi}(k)|^2 = +\infty$. *Then it is recurrent in* $\boldsymbol{C}$.

Proof of Theorem 44 *and* 45. Let $\Psi(\omega)$ be a complex Steinhaus series such that $(|\tilde{\Psi}(k)|)_{k=1}^{\infty}$ is increasing. For every $\omega = (\phi_1, \phi_2, \cdots) \in \Omega$, we consider an L-lacunary series $F_\omega(t) \sim \sum_{k=1}^{\infty} \tilde{\Psi}(k) \exp(2\pi i \phi_k) \exp(i n_k t)$, where $n_k = 2^{2^k}$ $(k \geq 1)$ and, for every $t \in [0, 2\pi)$, we consider a complex Steinhaus series $\Psi_t(\omega) = F_\omega(t)$. Then $s(m; F_\omega) = s(m; \Psi_t) = s(m; \Psi)$ $(m \geq 1)$. We easily see the following implications:

"$F_\omega(t)$ is transient in $\boldsymbol{C}$ for all $\omega \in \Omega$"
$\Rightarrow$ "$m \times \Pr(\{t \in [0, 2\pi), \omega \in \Omega;\ C(t; F_\omega) = \{\infty\}\}) = 2\pi$"
$\Rightarrow$ "$\Pr(\{\omega \in \Omega;\ C(t; F_\omega) = \{\infty\}\}) = 1$ for almost all $t \in [0, 2\pi)$"
$\Rightarrow$ "$\Psi_t(\omega)$ is transient in $\boldsymbol{C}$ for almost all $t \in [0, 2\pi)$".

Now, to prove (A)$'$, suppose that $\sum_{m=1}^{\infty} s(m; \Psi)^{-2} < +\infty$. Then $F_\omega(t)$ is transient in $\boldsymbol{C}$ for all $\omega \in \Omega$ and hence $\Psi_t(\omega)$ is transient in $\boldsymbol{C}$ for almost all $t \in [0, 2\pi)$. There exists $t_0 \in [0, 2\pi)$ such that $\Psi_{t_0}(\omega)$ is transient in $\boldsymbol{C}$. Since a mapping $(\phi_1, \phi_2, \cdots) \in \Omega \to (\phi_1 + n_1 t_0/2\pi (\mathrm{mod}\ 1), \phi_2 + n_2 t_0/2\pi (\mathrm{mod}\ 1), \cdots) \in \Omega$ is bijective and preserves the measure "Pr", $\Psi(\omega)$ is also transient in $\boldsymbol{C}$.

Since the proofs of other properties are analogous as in (A)$'$, we omit the proof.

§ 6. Convergence of L-lacunary series

For a Taylor series $F(t)$, $a \in \hat{\boldsymbol{C}}$ and $\xi \in \hat{\boldsymbol{R}}$, we write simply

$$(67)\quad \begin{cases} C^{-1}(a; F) = C^{-1}(\{a\}; F) = \{t \in [0, 2\pi);\ C(t; F) = \{a\}\} \\ C^{-1}(\xi; \mathrm{Re}\, F) = C^{-1}(\{\xi\}; \mathrm{Re}\, F) = \{t \in [0, 2\pi);\ C(t; \mathrm{Re}\, F) = \{\xi\}\}\,. \end{cases}$$

We write $\log^+ x = \max\{\log x, 0\}$ $(x > 0)$. In this chapter, we shall show the following

THEOREM 46. *Let $F(t)$ be an L-lacunary series such that $F \notin A(0, 2\pi)$.*

(68) $\dim(C^{-1}(\infty; \mathrm{Re}\, F)) = \dim(C^{-1}(\infty; F)) = 1$.

(69) *If* $\lim_{k\to\infty} \log^+ |\tilde{F}(k)|/\log n_k(F) = 0$, *then, for any* $\xi \in \boldsymbol{R}$, $\dim(C^{-1}(\xi; \mathrm{Re}\, F)) = 1$.

(70) *If* $\limsup_{k\to\infty} \log^+ |\tilde{F}(k)|/\log n_k(F) > 0$, *then, for any* $\xi \in \boldsymbol{R}$, $\dim(C^{-1}(\xi; \mathrm{Re}\, F)) < 1$.

(71) *If* $\lim_{k\to\infty}|\tilde{F}(k)| = 0$, *then, for any* $a \in \boldsymbol{C}$, $\dim(C^{-1}(a; F)) = 1$.

(72) *If* $\limsup_{k\to\infty}|\tilde{F}(k)| > 0$, *then, for any* $a \in \boldsymbol{C}$, $C^{-1}(a; F) = \emptyset$.

Proof. We write simply $P(t) = \mathrm{Re}\, F(t)$, $P_k(t) = \mathrm{Re}\, F_k(t)$, $a_k = |\tilde{F}(k)|$, $n_k = n_k(F)$ $(k \geq 0)$. There exists a sequence $(\phi_k)_{k=0}^{\infty}$ in $[0, 2\pi)$ such that $P(t) \sim \sum_{k=0}^{\infty} a_k \cos(n_k t + \phi_k)$. For the sake of simplicity, we give the proof in the case of $\phi_k = 0$ $(k \geq 0)$. (The proof in the general case is analogously given.) Note that $\sum_{k=0}^{\infty} a_k = +\infty$.

(68): Let k_0' be a positive integer such that, for $k \geq k_0'$, $(\pi/4)n_k^{-1} \geq 2\pi n_{k+1}^{-1}$. Set $\gamma_{k,j}' = [(2\pi j - \pi/4)/n_k, (2\pi j + \pi/4)/n_k]$ $(j = 1, \cdots, n_k; k \geq k_0')$. We define inductively $(U_k)_{k=k_0'}^{\infty}$ by $U_{k_0'} = \bigcup_{j=1}^{n'} \gamma_{k_0',j}'$ $(n' = n_{k_0'})$ and by $U_k = \bigcup \{\gamma_{k,j}'; \gamma_{k,j}' \subset U_{k-1}, j = 1, \cdots, n_k\}$. Set $\lambda_k = \pi/4$ $(k \geq 1)$. Then we see that $(U_k)_{k=k_0'}^{\infty}$ and $(\lambda_k)_{k=1}^{\infty}$ satisfy the four conditions in Lemma 7. Hence, writing $U = \bigcap_{k=k_0'}^{\infty} U_k$, we have $\dim(U) = 1$. For every $t \in U$, $\liminf_{m\to\infty} P_m(t) \geq 1/\sqrt{2} \cdot \liminf_{m\to\infty} \sum_{k=0}^{m} a_k = +\infty$, and hence $C^{-1}(\infty; P) \supset U$. Consequently, $\dim(C^{-1}(\infty; P)) = 1$. Since $C^{-1}(\infty; F) \supset C^{-1}(\infty; P)$, $\dim(C^{-1}(\infty; F)) = 1$.

(69): Suppose that $\lim_{k\to\infty} \log^+ a_k/\log n_k = 0$. Considering $P(t) - \xi$ if necessary, it is sufficient to show that $\dim(C^{-1}(0; P)) = 1$.

There exists a decreasing sequence $(\lambda_k)_{k=1}^{\infty}$ of positive numbers such that $\lambda_1 \leq 1$, $\lim_{k\to\infty}(\log 1/\lambda_k)/\log n_k = 0$ and $\sum_{k=1}^{\infty} a_k\lambda_k < +\infty$. There exists a sequence $(s_k)_{k=0}^{\infty}$ in $[0, 2\pi)$ such that $0 \leq s_k < 2\pi/n_k$, $a_k \cos n_k s_k \geq 0$ $(k \geq 0)$, $\lim_{k\to\infty} a_k \cos n_k s_k = 0$ and $\sum_{k=1}^{\infty} a_k \cos n_k s_k = +\infty$. Set $b_k = \sup_{\ell \geq k} a_\ell \cos n_\ell s_\ell$ and $\eta(k) = b_k + \sum_{\ell=0}^{k-1} a_\ell n_\ell/n_k + \sum_{\ell=k}^{\infty} a_\ell \lambda_\ell$ $(k \geq 1)$. Then $\lim_{k\to\infty} \eta(k) = 0$.

We say that an interval γ in $[0, 2\pi)$ is λ-interval, if there exists a positive integer k such that $m(\gamma) = \lambda_k n_k^{-1}$. Then such an integer is uniquely determined and denoted by $k(\gamma)$. Set $k'(\gamma) = k(\gamma) + 1$. We denote by t_γ the middle point of γ and write $\xi_\gamma = \sum_{k=0}^{k(\gamma)} a_k \cos n_k s_k$. For every λ-interval γ, we shall define a positive integer $m(\gamma)$ and two finite sets $\Delta(\gamma)$, $\nabla(\gamma)$ of λ-intervals.

$\langle$Definition of $m(\gamma)$, $\Delta(\gamma)$ and $\nabla(\gamma)\rangle$: Let γ be a λ-interval and suppose that $\xi_\gamma \neq 0$. We define inductively a sequence $(\varepsilon_k(\gamma))_{k=k'(\gamma)}^{\infty}$ by $\varepsilon_{k'(\gamma)}(\gamma) = -\mathrm{sign}_0(\xi_\gamma)$ and $\varepsilon_k(\gamma) = -\mathrm{sign}_0(\xi_\gamma + \sum_{\ell=k'(\gamma)}^{k-1} \varepsilon_\ell(\gamma) a_\ell \cos n_\ell s_\ell)$, where $\mathrm{sign}_0\, x = 1$ $(x > 0)$, $= 0$ $(x = 0)$ and $= -1$ $(x < 0)$. Set

$$m(\gamma) = \min\left\{m; \left|\xi_\gamma + \sum_{k=k'(\gamma)}^{m} \varepsilon_k(\gamma) \mathrm{a}_k \cos n_k s_k\right| \leq |\xi_\gamma|/2\right\}.$$

(Since $\lim_{m\to\infty}(\xi_\gamma + \sum_{k=k'(\gamma)}^{m} \varepsilon_k(\gamma) a_k \cos n_k s_k) = 0$, it is defined.) For every

$k'(\gamma) \leq k \leq m(\gamma)$, set $\Delta'_k = \{\gamma'_{k,j}; \gamma'_{k,j} \subset \gamma, j = 1, \cdots, n_k\}$, where $\gamma'_{k,1} = [s_k + \pi/4\cdot(1 - \mathrm{sign}_0\, \varepsilon_k(\gamma))n_k^{-1} - (1/2)\lambda_k n_k^{-1},\ s_k + \pi/4\cdot(1 - \mathrm{sign}_0\, \varepsilon_k(\gamma))n_k^{-1} + (1/2)\lambda_k n_k^{-1}]$ and $\gamma'_{k,j} = \gamma'_{k,1} + 2\pi(j-1)n_k^{-1}$ $(2 \leq j \leq n_k)$. We define inductively $(\Delta_k)_{k=k'(\gamma)}^{m(\gamma)}$ by $\Delta_{k'(\gamma)} = \Delta'_{k'(\gamma)}$ and $\Delta_k = \{\tau; \tau \in \Delta'_k$, there exists $\gamma' \in \Delta_{k-1}$ such that $\tau \subset \gamma'\}$. Now we put $\Delta(\gamma) = \Delta_{m(\gamma)}$ and $\nabla(\gamma) = \{\tau; \tau \in \Delta_k, k = k'(\gamma), \cdots, m(\gamma)\}$.

Suppose that $\xi_\gamma = 0$. Then we put $m(\gamma) = k'(\gamma) = k(\gamma) + 1$ and $\Delta(\gamma) = \nabla(\gamma) = \{[s_{m(\gamma)} - (1/2)\lambda_{m(\gamma)}n_{m(\gamma)}^{-1}, s_{m(\gamma)} + (1/2)\lambda_{m(\gamma)}n_{m(\gamma)}^{-1}] + 2\pi j n_{m(\gamma)}^{-1}; j = 1, \cdots, n_{m(\gamma)}\}$.

We denote by $\tau \prec \gamma$, if $\tau \in \Delta(\gamma)$ and by $\tau \,[\, \gamma$, if $\tau \in \nabla(\gamma)$. Now we show the following

LEMMA 47. *Let γ and τ be two λ-intervals.*

(73) *If $\tau \prec \gamma$, then* $|\xi_\tau| \leq 1/2\cdot|\xi_\gamma| + \eta(k(\gamma))$.

(74) *If $\tau \,[\, \gamma$, then, for any $t \in \tau$,* $|P_{k(\tau)}(t)| \leq |\xi_\gamma| + \eta(k(\gamma))$.

Proof. (73): Suppose that $\xi_\gamma \neq 0$. Then $|\xi_\gamma + \sum_{k=k'(\gamma)}^{m(\gamma)} \varepsilon_k(\gamma)a_k \cos n_k s_k| \leq 1/2\cdot|\xi_\gamma|$. Note that $k(\tau) = m(\gamma)$ and $|t_\tau - t_\gamma| \leq \lambda_{k(\gamma)}n_{k(\gamma)}^{-1}$. There exists a finite sequence $(t_k)_{k=k'(\gamma)}^{m(\gamma)}$ in $[0, 2\pi)$ such that $\varepsilon_k(\gamma)\cos n_k s_k = \cos n_k t_k$ and $|t_\tau - t_k| \leq \lambda_k n_k^{-1}$ $(k'(\gamma) \leq k \leq m(\gamma))$. Hence we have

$$\begin{aligned}|\xi_\tau| &= \Big|\xi_\gamma + \sum_{k=k'(\gamma)}^{k(\tau)} a_k \cos n_k t_\tau + \sum_{k=0}^{k(\gamma)} a_k(\cos n_k t_\tau - \cos n_k t_\gamma)\Big| \\ &= \Big|\xi_\gamma + \sum_{k=k'(\gamma)}^{k(\tau)} \varepsilon_k(\gamma)a_k \cos n_k s_k + \sum_{k=k'(\gamma)}^{k(\tau)} a_k(\cos n_k t_\tau - \cos n_k t_k) \\ &\qquad + \sum_{k=0}^{k(\gamma)} a_k(\cos n_k t_\tau - \cos n_k t_\gamma)\Big| \\ &\leq 1/2\cdot|\xi_\gamma| + \sum_{k=k'(\gamma)}^{k(\tau)} a_k\lambda_k + \sum_{k=1}^{k(\gamma)} a_k n_k \lambda_{k(\gamma)} n_{k(\gamma)}^{-1} \leq 1/2\cdot|\xi_\gamma| + \eta(k(\gamma)) .\end{aligned}$$

Suppose that $\xi_\gamma = 0$. Then $k'(\gamma) = m(\gamma) = k(\tau)$. There exists $t' \in [0, 2\pi)$ such that $\cos n_{k(\tau)}s_{k(\tau)} = \cos n_{k(\tau)}t'$ and $|t_\tau - t'| \leq \lambda_{k(\tau)}n_{k(\tau)}^{-1}$. Hence we have

$$\begin{aligned}|\xi_\tau| &= \Big|\xi_\gamma + a_{k(\tau)} \cos n_{k(\tau)}t' + a_{k(\tau)}(\cos n_{k(\tau)}t_\tau - \cos n_{k(\tau)}t') \\ &\qquad + \sum_{k=0}^{k(\gamma)} a_k(\cos n_k t_\tau - \cos n_k t_\gamma)\Big| \\ &\leq b_{k(\tau)} + a_{k(\tau)}\lambda_{k(\tau)} + \sum_{k=1}^{k(\gamma)} a_k n_k \lambda_{k(\gamma)} n_{k(\gamma)}^{-1} \leq \eta(k(\gamma)) = 1/2\cdot|\xi_\gamma| + \eta(k(\gamma)) .\end{aligned}$$

(74): Using

$$\Big|\xi_\gamma + \sum_{k=k'(\gamma)}^{k(\tau)} \varepsilon_k(\gamma)a_k \cos n_k s_k\Big| \leq |\xi_\gamma| + b_{k(\gamma)} ,$$

we see analogously as in (73) the required inequality. This completes the proof of Lemma 47.

Now we return to the proof of (69). Considering $P(t)-\xi$ if necessary, it is sufficient to show that $\dim(C^{-1}(0;P))=1$. First we choose a sequence $(U_k)_{k=k_0''}^{\infty}$ which satisfies the conditions in Lemma 7. Let k_0'' be a positive integer such that, for $k\geq k_0''$, $\lambda_k n_k^{-1}\geq 2\pi n_{k+1}^{-1}$. We define inductively a sequence $(\varDelta(m))_{m=1}^{\infty}$ by $\varDelta(1)=\{\gamma_{k_0''}''+2\pi j/n_{k_0''};\, j=1,\cdots,n_{k_0''}\}$, where $\gamma_{k_0''}''=[s_{k_0''}-(1/2)\lambda_{k_0''}n_{k_0''}^{-1},\, s_{k_0''}+(1/2)\lambda_{k_0''}n_{k_0''}^{-1}]$ and by $\varDelta(m)=\{\tau;\tau\in\varDelta(\gamma),\gamma\in\varDelta(m-1)\}$. Then $\varDelta(m)$ $(m\geq 1)$ is a finite set of λ-intervals. Set $\varDelta=\{\gamma;\gamma\in\varDelta(m), m\geq 1\}$ and $\nabla=\{\tau;\tau\in\nabla(\gamma),\gamma\in\varDelta\}\cup\varDelta(1)$. Then they are infinite sets of λ-intervals. Now we define $(U_k)_{k=k_0''}^{\infty}$ by $U_k=\bigcup\{\gamma; m(\gamma)=\lambda_k n_k^{-1},\gamma\in\nabla\}$ $(k\geq k_0'')$. Then it satisfies the four conditions in Lemma 7 and hence, putting $U=\bigcap_{k=k_0''}^{\infty}U_k$, we have $\dim(U)=1$.

Next we show that, for any $t\in U$, $\lim_{m\to\infty}P_m(t)=0$. Let $t_0\in U$. There exists a sequence $(\gamma_\ell)_{\ell=1}^{\infty}\subset\varDelta$ such that $\gamma_\ell\succ\gamma_{\ell+1}$ and $\gamma_\ell\ni t_0$ $(\ell\geq 1)$. By (73), $|\xi_{\gamma_{\ell+1}}|\leq 1/2\cdot|\xi_{\gamma_\ell}|+\eta(k(\gamma_\ell))$ $(\ell\geq 1)$. Hence $\lim_{\ell\to\infty}\xi_{\gamma_\ell}=0$. There exists a sequence $(\tau_m)_{m=k_0''}^{\infty}\subset\nabla$ such that $\tau_m\supset\tau_{m+1}$, $m(\tau_m)=\lambda_m n_m^{-1}$ and $\tau_m\ni t_0$ $(m\geq 1)$. For every $m\geq 1$, there exists a positive integer $\ell(m)$ such that $\tau_m\,[\,\gamma_{\ell(m)}$. By (74), $|P_m(t_0)|\leq|\xi_{\gamma_{\ell(m)}}|+\eta(k(\gamma_{\ell(m)}))$. Hence $\lim_{m\to\infty}P_m(t_0)=0$. This completes the proof.

(70): Suppose that $\limsup_{k\to\infty}\log^+ a_k/\log n_k>0$. Let us show that $\dim(C^{-1}(0;P))<1$. There exists a positive number α and a strictly increasing sequence $(k_j)_{j=1}^{\infty}$ of positive integers such that $a_{k_j}\geq n_{k_j}^{\alpha}$ $(j\geq 1)$. Choose a number β so that $1/(1+\alpha)<\beta<1$. Note that $C^{-1}(0;P)\subset\bigcap_{\mu=1}^{\infty}\bigcup_{j=\mu}^{\infty}V_j$, where $V_j=\{t\in[0,2\pi);\, a_{k_j}|\cos n_{k_j}t|\leq 1\}$ $(j\geq 1)$. We have, for any $j\geq 1$, $V_j=\bigcup_{\sigma=\pm 1}\bigcup_{\ell=1}^{n''}\tau_{k_j,\ell}^{(\sigma)}$, where $n''=n_{k_j}$, $\tau_{k_j,\ell}^{(\sigma)}=\tau_{k_j}+(2\pi\ell+(\pi/2)\,\mathrm{sign}\,\sigma)/n_{k_j}$ $(\ell=1,\cdots,n_{k_j};\sigma=\pm 1)$ and $\tau_{k_j}=[-(\pi/2)a_{k_j}^{-1}n_{k_j}^{-1},(\pi/2)a_{k_j}^{-1}n_{k_j}^{-1}]$. Hence

$$\begin{aligned}\varLambda_\beta(C^{-1}(0;P)) &\leq \lim_{\mu\to\infty}\sum_{j=\mu}^{\infty}\varLambda_\beta(V_j)\leq\lim_{\mu\to\infty}\sum_{j=\mu}^{\infty}2n_{k_j}m(\tau_{k_j})^\beta\\ &=2\pi^\beta\lim_{\mu\to\infty}\sum_{j=\mu}^{\infty}a_{k_j}^{-\beta}n_{k_j}^{1-\beta}\leq 2\pi^\beta\lim_{\mu\to\infty}\sum_{j=\mu}^{\infty}n_{k_j}^{1-(1+\beta)}=0\,.\end{aligned}$$

Consequently, $\dim(C^{-1}(0;P))\leq\beta<1$.

(71): Set $\lambda_k=1$ $(k\geq 1)$. Then λ-intervals are defined. Set $s_k=0$ $(k\geq 0)$. Then, for every λ-interval γ, $m(\gamma)$, $\varDelta(\gamma)$ and $\nabla(\gamma)$ are defined. Then the analogous argument as (69) is applicable. Hence we omit the proof.

(72): Evident.

§ 7. The deficiency of L-lacunary analytic functions

Let $g(z)$ be an analytic function in $\boldsymbol{D}$. The characteristic function of $g(z)$ is defined by $T(r, g) = 1/2\pi \int_0^{2\pi} \log^+ |g(re^{it})|\, dt$ $(0 < r < 1)$, where $\log^+ x = \max\{\log x, 0\}$ $(x > 0)$. The counting function of $g(z)$ is defined by $N(a, r, g) = \int_0^r n(a, s, g)/s ds$ $(a \in \boldsymbol{C}, 0 < r < 1)$, where $n(a, s, g)$ denotes the cardinal number of $\{z; 0 < |z| < s, g(z) = a\}$. We say that an analytic function $g(z)$ in $\boldsymbol{D}$ is of unbounded type if $\lim_{r \to 1} T(r, g) = +\infty$. For an analytic function $g(z)$ of unbounded type, the deficiency $\delta(a, g)$ $(a \in \boldsymbol{C})$ is defined by $\delta(a, g) = 1 - \limsup_{r \to 1} N(a, r, g)/T(r, g)$. Note that, if $\delta(a, g) = 0$, then $g(z)$ attains $a \in \boldsymbol{C}$ infinitely often in $\boldsymbol{D}$. In the theory of value-distribution, the deficiency plays an important role and we know the following theorem: The deficiency of an analytic function (of unbounded type) vanishes except a set of the logarithmic capacity zero, where the logarithmic capacity is a potential theoretic outer measure ([3]).

On the other hand, various properties for value-distribution of lacunary analytic functions are known. Let us note the following two theorems:

(75) There exists a positive number $\theta > 1$ such that an analytic function $g(z)$ in $\boldsymbol{D}$ attains every complex number infinitely often in $\boldsymbol{D}$ if $\sum_{k=0}^{\infty} |\tilde{g}(k)| = +\infty$ and $n_{k+1}(g)/n_k(g) \geq \theta$ $(k \geq 1)$ ([17]).

(76) Let $g(z)$ be an analytic function in $\boldsymbol{D}$ such that $\limsup_{k \to \infty} |\tilde{g}(k)| > 0$ and that $\mathrm{Spec}(g)$ is a finite union of Hadamard lacunary series. Then $\delta(a, g) = 0$ for all $a \in \boldsymbol{C}$ ([10]).

These two theorems suggest that the deficiency of an analytic function $g(z)$ of unbounded type vanishes for all complex number if $\mathrm{Spec}(g)$ is sufficiently thin. We shall show the following

THEOREM 48. *Let $f(z)$ be an L-lacunary analytic function of unbounded type. Then $\delta(a, f) = 0$ for all $a \in \boldsymbol{C}$.*

It is natural to define δ-thin sets: A subset E in $\boldsymbol{Z}^+$ is δ-thin, if, for any analytic function $g(z)$ of unbounded type satisfying $\mathrm{Spec}(g) \subset E$, $\delta(a, g) = 0$ for all $a \in \boldsymbol{C}$. Theorem 48 shows that there exist δ-thin sets and that L-lacunary sets are δ-thin. But it seems difficult to determine

δ-thin sets.

For the proof of the theorem, we prepare some lemmas; in which the first lemma is given by elementary calculus.

LEMMA 49. *For any* $a,\ b \in \boldsymbol{C}$ $(b \neq 0)$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ 1/|a + be^{it}|\, dt \leq \min\{\log^+ 1/|a|, \log^+ 1/|b|\} + 1\ .$$

LEMMA 50. *Let* $Q(t)$ *be a non-constant Taylor polynomial such that* $|\tilde{Q}(N_Q)| \geq 1$. *Then*

$$M_Q(\rho) = m(\{t \in [0, 2\pi); |Q(t)| \leq \rho\}) \leq 32 N_Q \rho^{1/N_Q} \qquad (\rho > 0)\ . \tag{77}$$

Proof. We prove (77) by an induction for N_Q. In the case of $N_Q = 1$, we have

$$\begin{aligned} M_Q(\rho) &\leq m(\{t \in [0, 2\pi); |\tilde{Q}(1) \sin n_1(Q)t| \leq \rho\}) \\ &= m(\{t \in [0, 2\pi); |\tilde{Q}(1) \sin t| \leq \rho\}) \leq 4m\Big(\Big\{t \in \Big[0, \frac{\pi}{2}\Big);\ \Big|\tilde{Q}(1)\frac{2}{\pi}t\Big| \leq \rho\Big\}\Big) \\ &\leq \frac{16}{\pi}\rho\,|\tilde{Q}(1)|^{-1} \leq 32\rho\ . \end{aligned}$$

Suppose that (77) holds for all Taylor polynomial $R(t)$ satisfying $N_R = k$ and $|\tilde{R}(k)| \geq 1$ and let $Q(t)$ be a Taylor polynomial such that $N_Q = k + 1$ and $|\tilde{Q}(k + 1)| \geq 1$. Now let us show that $M_Q(\rho) \leq 32(k + 1)\rho^{1/(k+1)}$ $(\rho > 0)$. In the case of $\rho \geq 1$, this inequality evidently holds. Next we fix for a while a number $0 < \rho < 1$. Set $R(t) = Q'(t)e^{-in_1t}n_{k+1}^{-1}$, where $n_1 = n_1(Q)$ and $n_{k+1} = n_{k+1}(Q)$. Since $N_R = k$ and $|\tilde{R}(k)| = |\tilde{Q}(k + 1)| \geq 1$, we have $M_R(\rho^{k/(k+1)}) \leq 32k(\rho^{k/(k+1)})^{1/k} = 32k\rho^{1/(k+1)}$. Set

$$\begin{cases} P_1(t) = \operatorname{Re} Q(t), \qquad P_2(t) = \operatorname{Im} Q(t) \\ U_1 = \{t \in [0, 2\pi); |\operatorname{Re} Q'(t)| \geq 1/\sqrt{2} \cdot n_{k+1}\rho^{k/(k+1)}\} \\ U_2 = \{t \in [0,2\pi); |\operatorname{Im} Q'(t)| \geq 1/\sqrt{2} \cdot n_{k+1}\rho^{k/(k+1)}\}\ . \end{cases}$$

We have, for every $t \in U_1$, $|P_1'(t)| = |\operatorname{Re} Q'(t)| \geq 1/\sqrt{2} \cdot n_{k+1}\rho^{k/(k+1)}$ and hence, for any interval I in U_1,

$$m(\{t \in I; |P_1(t)| \leq x\}) \leq 4\sqrt{2} \cdot n_{k+1}^{-1}\rho^{-k/(k+1)}x\ .$$

Since U_1 is a finite union of at most $2n_{k+1}$ intervals,

$$m(\{t \in U_1; |P_1(t)| \leq x\}) \leq 8\sqrt{2} \cdot \rho^{-k/(k+1)}x\ .$$

Analogously,

$$m(\{t \in U_2; |P_2(t)| \leq x\}) \leq 8\sqrt{2} \cdot \rho^{-k/(k+1)} x \ .$$

Hence

$$\overline{m}_1 = m(\{t \in U_1 \cup U_2; |Q(t)| \leq x\}) \leq 16\sqrt{2} \cdot \rho^{-k/(k+1)} x \ .$$

On the other hand,

$$\begin{aligned} \overline{m}_2 = m(U_1^c \cap U_2^c) &\leq m(\{t \in [0, 2\pi); |Q'(t) n_{k+1}^{-1}| \leq \rho^{k/(k+1)}\}) \\ &= M_R(\rho^{k/(k+1)}) \leq 32 k \rho^{1/(k+1)} \ . \end{aligned}$$

Consequently,

$$M_Q(x) \leq \overline{m}_1 + \overline{m}_2 \leq 16\sqrt{2} \cdot \rho^{-k/(k+1)} x + 32 k \rho^{1/(k+1)} \ .$$

Choosing $x = \rho$, we obtain $M_Q(\rho) \leq 32(k+1)\rho^{1/(k+1)}$.

LEMMA 51. *Let $Q(t)$ be a non-constant Taylor polynomial such that $\nu(Q) \geq 1$. Then*

$$M_Q(\rho) \leq 32 N_Q \rho^{1/N_Q^2} \qquad (\rho > 0) \ . \tag{78}$$

Proof. We prove (78) by an induction for N_Q. In the case of $N_Q = 1$, we easily see (78) since $\nu(Q) = |\tilde{Q}(N_Q)| \geq 1$. Suppose that (78) holds for all Taylor polynomial $R(t)$ satisfying $N_R = k$ and $\nu(R) \geq 1$, and let $Q(t)$ be a Taylor polynomial such that $N_Q = k+1$ and $\nu(Q) \geq 1$. Now we show $M_Q(\rho) \leq 32(k+1)\rho^{1/(k+1)^2}$ $(\rho > 0)$. In the case of $\rho \geq 1$, this inequality evidently holds. For a fixed number $0 < \rho < 1$, the following two cases are possible:

$$\text{(d)} \quad |\tilde{Q}(k+1)| \geq \rho^{k2/(k+1)^2}, \qquad \text{(e)} \quad |\tilde{Q}(k+1)| < \rho^{k2/(k+1)^2} \ .$$

In the case of (d), we consider $R(t) = \rho^{-k2/(k+1)2} Q(t)$. Since $N_R = k+1$ and $|\tilde{R}(k+1)| \geq 1$, we have, from Lemma 50,

$$\begin{aligned} M_Q(\rho) = M_R(\rho^{1-k2/(k+1)^2}) &\leq 32(k+1)(\rho^{1-k2/(k+1)^2})^{1/(k+1)} \\ &= 32(k+1)\rho^{(2k+1)/(k+1)^3} \leq 32(k+1)\rho^{1/(k+1)^2} \ . \end{aligned}$$

In the case of (e), we consider $R(t) = Q(t) - \tilde{Q}(k+1) e^{i n_{K+1}(Q) t}$. Since $N_R = k$ and $\nu(R) \geq 1$, we have, from the assumption,

$$\begin{aligned} M_Q(\rho) \leq M_R(\rho + \rho^{k2/(k+1)^2}) &\leq M_R(2\rho^{k2/(k+1)^2}) \leq 32k(2\rho^{k2/(k+1)^2})^{1/k2} \\ &\leq 32(k+1)\rho^{1/(k+1)^2} \ . \end{aligned}$$

For a function $P(t)$ in $[0, 2\pi)$, we denote by

$$h(P) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|P(t)|}\,dt\,.$$

LEMMA 52. *Let $Q(t)$ be a non-constant Taylor polynomial such that $\nu(Q) \geq 1$ and let n be a positive integer such that $4\pi\nu(Q)Nn_N e^{(N+1)^4} \leq n$, where $N = N_Q$ and $n_N = n_{N_Q}(Q)$. Then, with $R_{a,n}(t) = Q(t) + ae^{int}$,*

$$h(R_{a,n}) \leq \frac{1}{2\pi}\int_0^{2\pi} h(Q + ae^{is})ds + 6(N+1)^4 e^{-N}\,.$$

Proof. Set

$$\begin{cases} R(t) = R_{a,n}(t) \\ U = \{t \in [0, 2\pi); |R(t)| \leq \tau\} \qquad (\tau = e^{-(N+1)^3}) \\ L = \dfrac{1}{2\pi}\displaystyle\int_U \log^+ \frac{1}{|R(t)|}\,dt \\ t_\nu = 2\pi\nu/n, \qquad U_\nu = \{t \in [0, 2\pi); t_\nu + t/n \in U^c\} \\ L_\nu = \dfrac{1}{2\pi}\displaystyle\int_{U_\nu} \log^+ \frac{1}{|R(t_\nu + t/n)|}\,dt \qquad (\nu = 1, \cdots, n)\,. \end{cases}$$

Note that $h(R) = (1/n)\sum_{\nu=1}^n L_\nu + L$. By Lemma 51, we have

$$\begin{aligned} L &= \frac{1}{2\pi}\int_0^\tau \log\frac{1}{\rho} M_R(\rho)d\rho \\ &= \frac{1}{2\pi}\left\{\int_0^\tau \frac{1}{\rho} M_R(\rho)d\rho + M_R(\tau)\log\frac{1}{\tau} - \lim_{\varepsilon\to 0} M_R(\varepsilon)\log\frac{1}{\varepsilon}\right\} \\ &= \frac{16}{\pi}(N+1)\int_0^\tau \rho^{1/(N+1)^2-1}d\rho + \frac{16}{\pi}(N+1)^4 e^{-N-1} \leq 5(N+1)^4 e^{-N}\,. \end{aligned}$$

We have, for any $t \in U_\nu$, $s \in [0, 2\pi)$,

$$\begin{aligned} |Q(t_\nu + s/n) + ae^{it}| &\geq |R(t_\nu + t/n)| - |Q(t_\nu + s/n) - Q(t_\nu + t/n)| \\ &\geq \tau - 2\pi\sum_{k=1}^N |\tilde{Q}(k)|\, n_k(Q)n^{-1} \geq \tau - 2\pi\nu(Q)Nn_N n^{-1} \\ &\geq \tau - \frac{1}{2}e^{-(N+1)^4} \geq \tau/2\,, \end{aligned}$$

and hence

$$\begin{aligned} &\left|\log^+ \frac{1}{|Q(t_\nu + s/n) + ae^{it}|} - \log^+ \frac{1}{|R(t_\nu + t/n)|}\right| \\ &\quad\leq \max\left\{\frac{1}{|Q(t_\nu + s/n) + ae^{it}|}, \frac{1}{|R(t_\nu + t/n)|}\right\}|Q(t_\nu + s/n) - Q(t_\nu + t/n)| \end{aligned}$$

$$\leq 2\tau\cdot 2\pi \sum_{k=1}^{N} |\tilde{Q}(k)|\, n_k(Q)n^{-1} \leq 4\pi\tau\nu(Q)Nn_N n^{-1} \leq \tau e^{-(N+1)^4} \leq e^{-N} .$$

We have, for any $s \in [0, 2\pi)$,

$$L_\nu \leq \frac{1}{2\pi}\int_{U_\nu} \log^+ \frac{1}{|Q(t_\nu + s/n) + ae^{it}|} dt + e^{-N} .$$

Integrating each term by $ds/2\pi$ in the above inequality, we have

$$\begin{aligned} L_\nu &\leq (2\pi)^{-2}\int_{U_\nu} dt \int_0^{2\pi} \log^+ \frac{1}{|Q(t_\nu + s/n) + ae^{it}|} ds + e^{-N} \\ &\leq (2\pi)^{-2}\int_0^{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|Q(t_\nu + s/n) + ae^{it}|} dt\, ds + e^{-N} . \end{aligned}$$

Hence

$$h(R) = \frac{1}{n}\sum_{\nu=1}^{n} L_\nu + L \leq \frac{1}{2\pi}\int_0^{2\pi} h(Q + ae^{is})ds + e^{-N} + 5(N+1)^4 e^{-N} .$$

This shows that the required inequality holds.

LEMMA 53. *Let $F(t)$ be a Taylor series such that there exists a positive integer W such that $\Upsilon(W, F) \leq |\hat{F}(W)|/4$. (See the notation in Lemma 21.) Then $\lim_{m\to\infty} h(F_m) = h(F)$. If $F(t)$ satisfies also $|\hat{F}(W)| \geq 1$, then $h(F) \leq 10^2$.*

Proof. By Lemma 21, we see that, for any $0 < \rho \leq 1$, $M_F(\rho)/\rho$ and $M_{F_m}(\rho)/\rho$ $(m \geq W)$ are less than $10^2 p_{F,W}\rho^{1/2}$. The Lebesgue dominated convergence theorem shows that the first equality holds. If $|\hat{F}(W)| \geq 1$, then $p_{F,W} \leq 1$, and hence

$$h(F) = \frac{1}{2\pi}\int_0^1 \frac{1}{\rho} M_F(\rho)d\rho \leq \frac{1}{2\pi} 10^2 p_{F,W}\int_0^1 \rho^{-1/2}d\rho \leq 10^2 .$$

LEMMA 54. *Let $g(z)$ be an analytic function of unbounded type. Then $\sum_{n=0}^{\infty} |\hat{g}(n)|^2 = +\infty$.*

Proof. Since $\log x$ is concave,

$$T(r, g) \leq \frac{1}{2}\log\left(1 + \frac{1}{2\pi}\int_0^{2\pi} |g(re^{it})|^2 dt\right) \leq \frac{1}{2}\log\left(1 + \sum_{n=0}^{\infty} |\hat{g}(n)|^2\right) .$$

Now we give the proof of Theorem 48. Let $f(z)$ be the function in the theorem. Without loss of generality, it is sufficient to show that $\delta(0, f) = 0$. Writing $h(r, f) = 1/2\pi \int_0^{2\pi} \log^+ 1/|f(re^{it})|\, dt$ $(0 < r < 1)$, we have $T(r, f) = N(0, r, f) + h(r, f) + O(1)$ ("the first fundamental theorem" [11] p. 166) and hence it is sufficient to show that $\liminf_{r\to 1} h(r, f) < +\infty$. Set $E = (n_k)_{k=1}^\infty$, where $n_k = n_k(f)$ $(k \geq 1)$. Let us remember the notation $\theta(E)$ and $q(E)$. Since E is L-lacunary, there exists an integer $m_2' \geq 2$ such that, for $m \geq m_2'$, $m \geq 192\theta(E)$ and $4\pi\theta(E)^2 m e^{m^4} \exp\{-q(E)^m(1 - q(E)^{-1})\} \leq 1$.

The proof in the case of $\limsup_{k\to\infty} |\tilde{f}(k)| = +\infty$:

For each number $0 < \eta < 1$, set

$$\begin{cases} m(\eta) = \max\{m \in \boldsymbol{Z}^+ ; |\tilde{f}(m)|\, \eta^{n_m} = \max\{|\tilde{f}(k)|\, \eta^{n_k}; k \in \boldsymbol{Z}^+\}\} \\ W(\eta) = n_{m(\eta)},\ \mu(\eta) = |\tilde{f}(m(\eta))|\, \eta^{W(\eta)},\ r(\eta) = \eta(1 - W(\eta)^{-1})\,. \end{cases}$$

We easily see that $\lim_{\eta\to 1} m(\eta) = \lim_{\eta\to 1} \mu(\eta) = +\infty$ and $\lim_{\eta\to 1} r(\eta) = 1$. (See [5].) There exists a number $0 < \eta_0 < 1$ such that, for $\eta_0 \leq \eta < 1$, $m(\eta) \geq m_2'$ and $\mu(\eta) \geq 4$. For a fixed $\eta_0 \leq \eta < 1$, we consider a Taylor series $F_\eta(t) = \sum_{k=0}^\infty \tilde{f}(k) r(\eta)^{n_k} e^{i n_k t}$ $(n_0 = 0)$. Then $h(r(\eta), f) = h(F_\eta)$. We shall prove $h(F_\eta) \leq 10^2$. For the proof, we use Lemma 21 for $F(t) = F_\eta(t)$ and $W = W(\eta)$. Since

$$|\hat{F}_\eta(W(\eta))| = \mu(\eta)(1 - W(\eta)^{-1})^{W(\eta)} \geq 4(1 - W(\eta)^{-1})^{W(\eta)} \geq 1$$

and

$$\begin{aligned} \Upsilon(W(\eta), F_\eta) &= \sum_{n\neq W(\eta)} \{n/W(\eta) + n^2/W(\eta)^2\}\, |\hat{F}_\eta(n)| \\ &= \sum_{k\neq m(\eta)} \{n_k/n_{m(\eta)} + n_k^2/n_{m(\eta)}^2\}\, |\tilde{f}(k)|\, r(k)^{n_k} \\ &\leq \mu(\eta) \sum_{k\neq m(\eta)} \{n_k/n_{m(\eta)} + n_k^2/n_{m(\eta)}^2\}(1 - W(\eta)^{-1})^{n_k} \\ &\leq \mu(\eta)\Bigg[\sum_{k=1}^{m(\eta)-1} \{n_k/n_{m(\eta)} + n_k^2/n_{m(\eta)}^2\} \\ &\quad + 6 \sum_{k=m(\eta)+1}^\infty \{n_k/n_{m(\eta)} + n_k^2/n_{m(\eta)}^2\} n_{m(\eta)}^3/n_k^3\Bigg] \\ &\leq 6\mu(\eta)\{\gamma_{E,1}(m(\eta)) + \gamma_{E,2}(m(\eta))\} \leq 12\theta(E) m(\eta)^{-2}\mu(\eta) \\ &\leq 16^{-1}\mu(\eta) \leq |\hat{F}_\eta(W(\eta))|/4\,, \end{aligned}$$

we have, from Lemma 53, $h(F_\eta) \leq 10^2$. Hence $\liminf_{r\to 1} h(r, f) \leq \liminf_{\eta\to 1} h(F_\eta) \leq 10^2$.

The proof in the case of $\limsup_{k\to\infty} |\tilde{f}(k)| < +\infty$:

Without loss of generality, we may assume that $\nu(f) \leq 1$. Since $\sum_{k=0}^\infty |\tilde{f}(k)|^2 = +\infty$, there exists a strictly increasing sequence $(m(\ell))_{\ell=1}^\infty$ of positive integers such that $m(\ell) \geq m_2'$ and $|\tilde{f}(m(\ell))| \geq m(\ell)^{-1}$. Set $W(\ell) = n_{m(\ell)}$ and $r(\ell) = 1 - W(\ell)^{-1}$. For a fixed integer $\ell \geq 1$, we consider a Taylor series $F^\ell(t) = \sum_{k=0}^{\infty} \tilde{f}(k) r(\ell)^{n_k} e^{in_k t}$. Set $F_m^\ell(t) = \sum_{k=0}^{m} \tilde{f}(k) r(\ell)^{n_k} e^{in_k t}$ $(m \geq 1)$. We use Lemma 52 for $F(t) = F_{m-1}^\ell(t)$, $n = n_m$ and $a = \tilde{f}(m) r(\ell)^{n_m}$. Since

$$\begin{aligned} &4\pi\nu(F_{m-1})(m-1)n_{m-1}e^{m^4} \leq 4\pi m n_m \gamma_E(m) e^{m^4} \\ &\leq 4\pi\theta(E)^2 m e^{m^4} \exp\{-q(E)^m(1 - q(E)^{-1})\} \leq 1\,, \end{aligned}$$

we have, from Lemma 52,

$$h(F_m) \leq \frac{1}{2\pi}\int_0^{2\pi} h(F_{m-1} + \tilde{f}(m) r(m)^{n_m} e^{is}) ds + 6m^4 e^{-m+1}\,,$$

and hence

$$\begin{aligned} h(F_m) &\leq (2\pi)^{-m+m_2'} \int_0^{2\pi} \cdots \int_0^{2\pi} h\Big(F_{m_2'-1} + \sum_{k=m_2'}^{m} \tilde{f}(k) r(\ell)^{n_k} e^{is_k}\Big) ds_{m_2'} \cdots ds_m \\ &\leq 6e \sum_{k=m_2'}^{m} k^4 e^{-k} \leq \log^+ 1/|\tilde{f}(m_2') r(\ell)^{n_{m_2'}}| + 1 + 10^5. \qquad \text{(Lemma 49)}\,. \end{aligned}$$

Since

$$\begin{aligned} \Upsilon(W(\ell), F^\ell) &= \sum_{n \neq W(\ell)} \{n/W(\ell) + n^2/W(\ell)^2\} |\hat{F}^\ell(n)| \\ &\leq \sum_{k \neq m(\ell)} \{n_k/n_{m(\ell)} + n_k^2/n_{m(\ell)}^2\} r(\ell)^{n_k} \\ &\leq 6\{\gamma_{E,1}(m(\ell)) + \gamma_{E,2}(m(\ell))\} \leq 12\theta(E) m(\ell)^{-2} \leq 16^{-1} m(\ell)^{-1} \\ &\leq 4^{-1} |\tilde{f}(m(\ell))|\, r(\ell)^{W(\ell)} = |\hat{F}^\ell(W(\ell))|/4\,, \end{aligned}$$

we have, from Lemma 53, $\lim_{m\to\infty} h(F_m^\ell) = h(F^\ell)$. Hence

$$h(F^\ell) \leq \log^+ 1/|\tilde{f}(m_2') r(\ell)^{n_{m_2'}}| + 1 + 10^5\,.$$

Consequently,

$$\liminf_{r\to 1} h(r, f) \leq \liminf_{\ell\to\infty} h(F^\ell) \leq \log^+ 1/|\tilde{f}(m_2')| + 1 + 10^5\,.$$

§ 8. Ranges and cluster sets of L-lacunary analytic functions

8.1. In § 7, we showed that an L-lacunary analytic function of unbounded type attains any complex number infinitely often in $\boldsymbol{D}$. In this chapter, we shall study in detail the value-distribution of L-lacunary analytic functions. Let $g(z)$ be an analytic function in $\boldsymbol{D}$ and U a subset of $\boldsymbol{D}$ such that $\overline{U} \cap \partial\boldsymbol{D} \neq \emptyset$, where $\partial\boldsymbol{D}$ is the boundary of $\boldsymbol{D}$. The range of $g(z)$ in U is defined by $R(U; g) = \{a \in \boldsymbol{C};\, {}^\#\{z \in U;\, g(z) = a\} = +\infty\}$. We denote by $M(U; g)$ the totality of cluster points of a set $(g(z))_{z\in U}$ in $\hat{C}$ when $|z|$ tends to 1. We shall study ranges and cluster sets of L-lacunary analytic functions in some set U such that $\bar{U} \cap \partial D$ is a singleton.

We denote by $D(a, \rho)$ $(a \in C, \rho > 0)$ the open disk with center a and radius ρ. For every $t \in [0, 2\pi)$, set

$$\begin{cases} R(t; g) = \bigcap_{0<\varepsilon\leq 1/2} R(\Gamma_\varepsilon(t); g) & \text{(the non-tangential range of } g(z) \text{ at } t) \\ M(t; g) = \bigcap_{0<\varepsilon\leq 1/2} M(\Gamma_\varepsilon(t); g) & \text{(the non-tangential cluster set of } g(z) \text{ at } t) \\ M_\rightarrow(t; g) = M(\{re^{it}\}_{0\leq r<1}; g) & \text{(the radial cluster set of } g(z) \text{ at } t)\,, \end{cases}$$

where $\Gamma_\varepsilon(t) = \bigcup_{0\leq r<1} D(re^{it}, \varepsilon(1-r))$. We say that $t \in [0, 2\pi)$ is a Borel direction of $g(z)$ if $R(t; g) = C$. We say that $t \in [0, 2\pi)$ is a dense direction of $g(z)$ if $M(t; g) = \hat{C}$. For a compact set A in $\hat{C}$, set $M_\rightarrow^{-1}(A; g) = \{t \in [0, 2\pi); M_\rightarrow(t; g) = A\}$. For a strictly increasing sequence $W = (r_k)_{k=1}^\infty$ of positive numbers tending to 1, set $M_W(t; g) = M(\{r_k e^{it}\}_{k=1}^\infty; g)$, $R_W(t; g) = \bigcap_{0<\varepsilon\leq 1/2} R(\bigcup_{k=1}^\infty D(r_k e^{it}, \varepsilon(1-r_k)))$. We say that:

W is a covering sequence (by $g(z)$) for $t \in [0, 2\pi)$, if $R_W(t; g) = C$.

W is a void sequence for t, if $R_W(t; g) = \emptyset$.

W is of pit type at t, if, for any compact set A in C, $g(D(r_k e^{it}, (1-r_k))$ contains A for infinitely many k.

W is of recurrent type at t, if it is a covering sequence of non-pit type at t.

We write $W(g) = (r_k(g))_{k=1}^\infty$, where $r_k(g) = 1 - n_k(g)^{-1}$ $(k \geq 1)$. We shall show the following

THEOREM 55. *Let $f(z)$ be an L-lacunary analytic function in D such that $(|\tilde{f}(m)|)_{m=1}^\infty$ is increasing and $\lim_{m\to\infty} |\tilde{f}(m)|/s(m; f) = 0$. Then $W(f)$ is a covering sequence for almost all $t \in [0, 2\pi)$ (a.a.t). Almost all directions are Borel directions of $f(z)$.*

(79) *If $\nu(f) < +\infty$, then $W(f)$ is of recurrent type for a.a.t.*

(80) *If $\sum_{m=1}^{\infty} |\tilde{f}(m)|\, s(m; f)^{-2} < +\infty$, then $W(f)$ is of pit type for a.a.t.*

COROLLARY 56. *Let $f_\alpha(z)$ $(\alpha \in R)$ be an L-lacunary analytic function such that $|\tilde{f}_\alpha(m)| = m^\alpha$ $(m \geq 1)$.*

(81) *If $\alpha = 0$, then $W(f_\alpha)$ is of recurrent type for a.a.t.*

(82) *If $\alpha > 0$, then $W(f_\alpha)$ is pit type for a.a.t.*

THEOREM 57. *Let $f(z)$ be the function in Theorem 55. Then almost all directions are dense directions of $f(z)$.*

(83) *If* $\sum_{m=1}^{\infty} s(m;f)^{-2} = +\infty$, *then* $m(M_{\to}^{-1}(\hat{C};f)) = 2\pi$.

(84) *If* $\sum_{m=1}^{\infty} |\tilde{f}(m)|\, s(m;f)^{-2} < +\infty$, *then* $m(M_{\to}^{-1}(\{\infty\};f)) = 2\pi$.

THEOREM 58. *Let $f(z)$ be an L-lacunary analytic function in $\mathbf{D}$ such that $\nu(f) < +\infty$ and $\sum_{k=1}^{\infty} |\tilde{f}(k)|^2 = +\infty$. Then $m(M_{\to}^{-1}(\hat{C};f)) = 2\pi$.*

COROLLARY 59. *Let $f_\alpha(z)$ be the function in Corollary 56.*

(85) *If* $-1/2 \leq \alpha \leq 0$, *then* $m(M_{\to}^{-1}(\hat{C};f_\alpha)) = 2\pi$.

(86) *If* $\alpha > 0$, *then* $m(M_{\to}^{-1}(\{\infty\};f_\alpha)) = 2\pi$.

THEOREM 60. *Let $f_\alpha(z)$ be the function in Corollary 56 and let $W = (r_{k_j}'(f_\alpha))_{j=1}^{\infty}$ a subsequence of $W(f_\alpha)$.*

(87) *If* $\sum_{j=1}^{\infty} 1/k_j < +\infty$, *then W is a void sequence for a.a.t.*

(88) *If* $\sum_{j=1}^{\infty} 1/k_j = +\infty$ *and* $(k_{j+1} - k_j)_{j=1}^{\infty}$ *is increasing, then W is a covering sequence for a.a.t.*

Remark 61. In Theorem 55, we cannot replace "almost all" by "all" since, for any L-lacunary series $(n_k)_{k=1}^{\infty}$, $0 \in [0, 2\pi)$ is not a Borel direction of an analytic function $\sum_{k=1}^{\infty} z^{n_k}$. The condition "$\lim_{m\to\infty} |\tilde{f}(m)|/s(m;f) = 0$" is natural in the theory of lacunary series ([14], p. 396) and it is necessary in this theorem since we see that, if an L-lacunary analytic function $g(z)$ satisfies $|\tilde{g}(m)|/s(m;g) > \sqrt{1 - 1/m}$ $(m \geq 1)$ and $\mathrm{Ord}\,(g) = \limsup_{m\to\infty} \log\log s(m;g)/\log m < +\infty$, then $W(g)$ is a void sequence for all t. At last we note that the statements in (87) and (88) are independent of α.

8.2. Lemmas

The following lemma is essential in our discussion.

LEMMA 62 ([5]). *Let p be a positive integer and $g(z)$ an analytic function in $D(0,r)$ such that $|g^{(p)}(0)| \geq y_1$ and $|g^{(p)}(z)| \leq y_2$ $(|z| < r)$. Then $g(z)$ attains all values w satisfying $|w - g(0)| \leq \lambda(p) r^p y_1^{p+1} y_2^{-p}$, where $\lambda(p)$ is a positive constant depending only on p.*

LEMMA 63. *Let* $m \geq 2$ *and let* $(u_k)_{k=1}^m$ *be a finite sequence of positive numbers. Then, with* $v_k = \sum_{\ell=1}^k u_\ell$ $(1 \leq k \leq m)$, *we have* $\sum_{k=2}^m u_k/v_k \leq \log(v_m/u_1)$. *If* $2u_k \leq v_k$ $(2 \leq k \leq m)$, *then* $\sum_{k=2}^m u_k/v_k \geq 1/2 \cdot \log(v_m/u_1)$.

Proof. Let us define a function $h(x)$ in $[1, m)$ by $h(x) = v_{k-1} + u_k(x - k + 1)$ $(k - 1 \leq x < k, k \geq 2)$. Then $u_k/v_k \leq h'(x)/h(x) \leq u_k/v_{k-1}$ $(k - 1 < x < k, k \geq 2)$. Hence

$$\sum_{k=2}^m u_k/v_k \leq \int_1^m h'(x)/h(x)dx = \log(v_m/u_1) .$$

If $2u_k \leq v_k$ $(2 \leq k \leq m)$, then $v_k \leq 2v_{k-1}$ $(k \geq 2)$. Hence

$$\sum_{k=2}^m u_k/v_k \geq \frac{1}{2} \sum_{k=2}^m u_k/v_{k-1} \geq \frac{1}{2} \int_1^m h'(x)/h(x)dx = \frac{1}{2} \log(v_m/u_1) .$$

Throughout 8.2, $g(z)$ is an L-lacunary analytic function in $\boldsymbol{D}$ such that $\mathrm{Ord}(g) = \limsup_{m\to\infty} \log\log s(m; g)/\log m < +\infty$. We put:

$$(89)\quad \begin{cases} n_0 = 0, \quad n_k = n_k(g), \quad r_k = r_k(g) = 1 - n_k(g)^{-1} \quad (k \geq 1) . \\ C(t; g) = \text{the totality of cluster points of a sequence } (g_m(e^{it}))_{m=1}^\infty \text{ in} \\ \hat{C}, \text{ where } g_m(z) = \sum_{k=0}^m \tilde{g}(k) z^{n_k} . \\ F_{g,m}(t) = \sum_{k=0}^{m-1} \tilde{g}(k) e^{in_k t} + e^{-1}\tilde{g}(m)e^{in_m t} \qquad (m \geq 1) . \\ A_{g,m} = \sum_{k=0}^{m-1} |\tilde{g}(k)| n_k/n_m + \sum_{k=m+1}^\infty |\tilde{g}(k)| (n_k/n_m)\left(\frac{1 + r_m}{2}\right)^{n_k - 1} \qquad (m \geq 1) . \end{cases}$$

LEMMA 64. *Let* $0 < \varepsilon \leq 1/2$, *m a positive integer and* $t \in [0, 2\pi)$. *Then*:

$$(90)\qquad \lim_{k\to\infty} A_{g,k} = 0 .$$

$$(91)\qquad |g(r_m e^{it}) - F_{g,m}(t)| \leq |\tilde{g}(m)|/n_m + A_{g,m} .$$

$$(92)\quad |g(r_m e^{it}) - g(z)| \leq \varepsilon|\tilde{g}(m)| + \varepsilon A_{g,m} \qquad (z \in D_m(t, \varepsilon) = D(r_m e^{it}, \varepsilon(1 - r_m))) .$$

(93) *If* $|g(r_m e^{it}) - a| \leq 2^{-7}\lambda(1)\varepsilon|\tilde{g}(m)|$ *and* $A_{g,m} \leq |\tilde{g}(m)|/8$, *then* $g(D_m(t, \varepsilon)) \ni a$.

Proof. (90): Since $\mathrm{Ord}(g) < +\infty$, there exists a positive number M such that $|\tilde{g}(k)| \leq e^{k^M}$ $(k \geq 1)$. Denoting by $E = (n_k)_{k=1}^\infty$, we have

$$\begin{aligned} A_{g,m} &\leq \sum_{k=0}^{m-1} e^{k^M} n_k/n_m + 2 \sum_{k=m+1}^\infty e^{k^M}(n_k/n_m)\exp(-n_k/n_m) \\ &\leq e^{m^M}\gamma_E(m) + 2(3!\,2^3) \sum_{k=m+1}^\infty e^{k^M} n_m^2/n_k^2 \end{aligned}$$

$$\leq e^{mM}\gamma_E(m) + 10^2\gamma_{E,1}(m) \sup_{k\geq m+1} (e^{kM}\gamma_E(k)) .$$

Hence Lemma 5 shows that $\lim_{m\to\infty} A_{g,m} = 0$.

(91): We have

$$|g(r_m e^{it}) - F_{g,m}(t)| \leq \sum_{k=0}^{m-1} |\tilde{g}(k)| (1 - r_m^{n_k}) + |\tilde{g}(m)| (e^{-1} - r_m^{n_m})$$
$$+ \sum_{k=m+1}^{\infty} |\tilde{g}(k)| r_m^{n_k} \leq |\tilde{g}(m)|/n_m + A_{g,m} .$$

(92): We have, for any $z \in D_m(t, \varepsilon)$,

$$|g(r_m e^{it}) - g(z)| \leq |r_m e^{it} - z| \sum_{k=1}^{\infty} |\tilde{g}(k)| n_k \left(\frac{1 + r_m}{2}\right)^{n_k - 1}$$
$$\leq (1 - r_m)n_m(|\tilde{g}(m)| + A_{g,m}) \leq \varepsilon |\tilde{g}(m)| + \varepsilon A_{g,m} .$$

(93): We have, for any $z \in D_m(t, \varepsilon)$,

$$|g'(r_m e^{it})| \geq |\tilde{g}(m)| n_m r_m^{n_m - 1} - \sum_{k\neq m} |\tilde{g}(k)| n_k r_m^{n_k - 1} \geq \frac{1}{4} |\tilde{g}(m)| n_m - A_{g,m} n_m$$
$$\geq |\tilde{g}(m)| n_m \left(\frac{1}{4} - A_{g,m}/|\tilde{g}(m)|\right) \geq \frac{1}{8} |\tilde{g}(m)| n_m$$

and

$$|g'(z)| \leq |\tilde{g}(m)| n_m + \sum_{k\neq m} |\tilde{g}(k)| n_k r_m^{n_k - 1}$$
$$\leq |\tilde{g}(m)| n_m(1 + A_{g,m}/|\tilde{g}(m)|) \leq 2 |\tilde{g}(m)| n_m .$$

Hence Lemma 62 shows that $g(z)$ attains, in $D_m(t, \varepsilon)$, all values w satisfying $|w - g(r_m e^{it})| \leq \lambda(1)\varepsilon n_m^{-1}(1/8 \cdot |\tilde{g}(m)| n_m)^2(2 |\tilde{g}(m)| n_m)^{-1} = 2^{-7}\lambda(1)\varepsilon |\tilde{g}(m)|$.

LEMMA 65. *There exists a strictly increasing sequence W of positive numbers tending to 1 such that, for any $t \in [0, 2\pi)$, $M_W(t; g) = C(t; g)$.*

Proof. We have $|\tilde{g}(k)| \leq e^{kM}$ $(k \geq 1)$. Putting $W = (r'_k)_{k=1}^{\infty}$ $(r'_k = 1 - e^{-2kM}n_k^{-1})$, we show that W is a required sequence. There exists a positive integer m'_3 such that, for $m \geq m'_3$, $\gamma_{\mathrm{Spec}(g)}(m) \leq 2e^{-2mM}$. Then we have, for $m \geq m'_3$ and $t \in [0, 2\pi)$,

$$|g(r'_m e^{it}) - g_m(e^{it})| \leq (1 - r'_m) \sum_{k=0}^{m} |\tilde{g}(k)| n_k + \sum_{k=m+1}^{\infty} |\tilde{g}(k)| r'^{n_k}_m$$
$$\leq |\tilde{g}(m)| e^{-2mM} + |\tilde{g}(m + 1)| r'^{n_{m+1}}_m + A_{g,m+1}$$
$$\leq e^{-mM} + e^{(m+1)M} \exp\{e^{2mM} - n_{m+1}/n_m\} + A_{g,m+1} .$$

Lemma 5 and (90) show that the last term tends to 0 when $m \to \infty$. Hence $M_W(t; g) = C(t; g)$ for any $t \in [0, 2\pi)$.

LEMMA 66. *If* $\liminf_{k\to\infty} |\tilde{g}(k)| > 0$, *then, for any* $t \in [0, 2\pi)$ *and any subsequence* W *of* $W(g)$, $R_W(t; g)$ *is closed.*

Proof. Since $\liminf_{k\to\infty} |\tilde{g}(k)| > 0$, $K_g = \inf_{k\geq 1} |\tilde{g}(k)| > 0$. Let $a \in \boldsymbol{C}$ and Σ a set in $R_W(t; g)$ such that $\bar{\Sigma} \ni a$. Given $0 < \varepsilon \leq 1/2$, we put $\varepsilon_0 = 2^{-9}\lambda(1)\varepsilon$. There exists $b \in \Sigma$ such that $|b - a| \leq \varepsilon_0 K_g$. Since $\Sigma \subset R(\Gamma_{\varepsilon_0, W}(t), g)$, there exists a strictly increasing sequence $(m_j)_{j=1}^{\infty}$ of positive integers such that $g(D_{m_j}(t, \varepsilon_0)) \ni b$ $(j \geq 1)$. By (92), $|g(r_{m_j}e^{it}) - b| \leq \varepsilon_0 |\tilde{g}(m_j)| + \varepsilon_0 A_{g,m_j}$ and hence $|g(r_{m_j}e^{it}) - a| \leq \varepsilon_0 |\tilde{g}(m_j)| + \varepsilon_0 A_{g,m_j} + \varepsilon_0 K_g$ $(j \geq 1)$. Since $\lim_{j\to\infty} A_{g,m_j} = 0$, we have, from (93), $g(D_{m_j}(t, \varepsilon)) \ni a$ for all sufficiently large j. Hence $R(\Gamma_{\varepsilon, W}(t); g) \ni a$. Since $0 < \varepsilon \leq 1/2$ is arbitrary, we have $R_W(t; g) \ni a$.

LEMMA 67. *Let* γ *be a rectifiable curve in* $\boldsymbol{D}$ *such that, for any* $0 < r < 1$, $\gamma \cap D(0, r)^c$ *is connected. Set* $\Lambda_\eta(\gamma; g) = \int_\gamma |g'(z)|\, \lambda_\eta(g(z)) ds_\gamma(z)$ $(\eta > 0)$, *where* ds_γ *is the element of the curvilinear integral. If* $\Lambda_\eta(\gamma; g) < +\infty$ *for all* $\eta > 0$, *then* $M(\gamma; g)$ *is a singleton in* $\hat{\boldsymbol{C}}$.

Proof. Suppose that $M(\gamma; g)$ is not a singleton. Then $M(\gamma; g) \cap \boldsymbol{C} \neq \emptyset$. Since γ is connected, $M(\gamma; g)$ is also connected and hence $M(\gamma; g) \cap \boldsymbol{C}$ contains at least two points. Let a, b $(a \neq b)$ be such two points. Set $\varepsilon_0' = |a - b|/3$ and $\eta_0 = |a| + 2\varepsilon_0'$. Since $\gamma \cap D(0, r)^c$ is connected for all $0 < r < 1$, we can choose inductively two sequences $(z_j)_{j=1}^{\infty}$, $(z_j')_{j=1}^{\infty}$ in γ and a sequence $(\gamma_j)_{j=1}^{\infty}$ of subcurves of γ so that $|g(z_j) - a| \leq \varepsilon_0'$, $|g(z_j) - g(z_j')| = 2\varepsilon_0'$, $g(z) \in \overline{D(a, 2\varepsilon_j')}$ $(z \in \bigcup_{j=1}^{\infty} \gamma_j)$ and $\gamma_j \cap \gamma_{j'} = \emptyset$ $(j \neq j')$. Then

$$
\begin{aligned}
2\varepsilon_0' = |g(z_j) - g(z_j')| &\leq \int_{\gamma_j} |g'(z)|\, ds_\gamma(z) = \int_{\gamma_j} |g'(z)|\, \chi_{\eta_0}(g(z)) ds_\gamma(z) \\
&\leq 64\pi^{-2} \int_{\gamma_j} |g'(z)|\, \lambda_{4\eta_0}(g(z)) ds_\gamma(z) .
\end{aligned}
$$

Since $\Lambda_{4\eta_0}(\gamma; g) < +\infty$, the last term tends to 0 when $j \to \infty$. This is a contradiction.

8.3. Proof of Theorem 55

Let $f(z)$ be an L-lacunary analytic function in $\boldsymbol{D}$ such that $(|\tilde{f}(m)|)_{m=1}^{\infty}$ is increasing and $\lim_{m\to\infty} |\tilde{f}(m)|/s(m; f) = 0$. Without loss of generality, we may assume that $|\tilde{f}(1)| = 1$. We use the notation $F_{\cdot,m}(t)$ and $A_{\cdot,m}$ in the preceding paragraph. Set:

$$(94)\quad \begin{cases} n_0 = 0, \quad n_k = n_k(f), \quad r_k = r_k(f) = 1 - n_k(f)^{-1} \quad (k \geq 1) \\ A_f = \sup_{m \geq 1} A_{f,m}, \qquad F_m(t) = F_{f,m}(t) = \sum_{k=0}^{m-1} \tilde{f}(k) e^{i n_k t} + e^{-1} \tilde{f}(m) e^{i n_m t} \\ a_0 = |\tilde{f}(0)|, \quad a_m = |\tilde{f}(m)|, \quad s_m = s(m; f), \quad \gamma(m) = \gamma_{\mathrm{Spec}(f)}(m) \\ w(m, M) = \left(\sum_{k=m}^{M} a_k^2 \right)^{1/2} \\ \hat{T}_M = \log s_M \qquad (k \geq 1, 1 \leq m \leq M) . \end{cases}$$

For $0 < \varepsilon \leq 1/3e$, two positive integers K, M $(K < M)$ and a Borel set U in $[0, 2\pi)$, we put:

$$(95)\quad \begin{cases} \hat{h}(K, M; U, \varepsilon) = \int_U \sum_{m=K}^{M} \lambda_{\varepsilon a_m} (F_m(t)) dt \\ \hat{H}(K, M; U, \varepsilon) = \int_U \left\{ \sum_{m=K}^{M} \lambda_{\varepsilon a_m} (F_m(t)) \right\}^2 dt. \end{cases}$$

The following two lemmas play an analogous role as in Theorem 30.

LEMMA 68. *Let $0 < \varepsilon \leq 1/3e$, K a positive integer and let U be a finite union of intervals in $[0, 2\pi)$. Then there exists a positive integer $\hat{M}_1$ such that, for $M \geq \hat{M}_1$, $10^{-4} m(U) \varepsilon^2 \hat{T}_M \leq \hat{h}(K, M; U, \varepsilon) \leq 10^{10} m(U) \varepsilon^2 \hat{T}_M$. (Compare with Lemma 36 and 38.)*

Proof. Without loss of generality, we may assume that U is an interval satisfying $m(U) \neq 0$. Since $\lim_{m \to \infty} a_m / s_m = 0$, there exists a positive integer $m_4' \geq 128$ such that, for $m \geq m_4'$, $a_m \leq 2^{m/3}$ and $4\nu(F_m) \leq w(\tau(m); F_m)$. For every $m \geq m_4'$, we use Lemma 16 and 20 for $Q(t) = F_m(t)$, $J = \tau(m)$, $\eta = \varepsilon a_m$, $M = m$ and $I = U$. Then

$$\int_U \lambda_{\varepsilon a_m} (F_m(t)) dt \leq 10^8 m(U) \varepsilon^2 a_m^2 w(\tau(m); F_m)^{-2} + O(m^{-2})$$

and

$$\int_U \lambda_{\varepsilon a_m} (F_m(t)) dt \geq 10^{-3} m(U) \varepsilon^2 a_m^2 w(1; F_m)^{-2} - o(a_m^2 s_m^{-2}) - O(m^{-2}) .$$

Note that $s_m^{-2} \leq w(1; F_m)^{-2}$ and $w(\tau(m); F_m)^{-2} \leq 2e^2 s_m^{-2}$. Hence

$$\begin{aligned} 10^{-3}(1 - o(1)) m(U) \varepsilon^2 \sum_{m=K}^{M} a_m^2 s_m^{-2} &\leq \hat{h}(K, M; U, \varepsilon) \\ &\leq 10^8 (2e^2 + o(1)) m(U) \varepsilon^2 \sum_{m=K}^{M} a_m^2 s_m^{-2} . \end{aligned}$$

Since $\lim_{m \to \infty} a_m / s_m = 0$, we can apply Lemma 63 for all sufficiently large M. Then we easily see the required property.

LEMMA 69. *Let ε, K and U be the same as in the preceding lemma. Then there exists a positive integer $\hat{M}_2$ such that, for $M \geq \hat{M}_2$, $\hat{H}(K, M; U, \varepsilon) \leq 10^{31}\varepsilon^3 m(U)\hat{T}_M^2$. (Compare with Lemma 37.)*

Proof. Without loss of generality, we may assume that U is an interval satisfying $m(U) \neq 0$. We have

$$\hat{H}(K, M; U, \varepsilon) \leq 10^5\left\{\pi^2\hat{h}(K, M; U, \varepsilon) + \sum_{\ell=1}^{128}\hat{H}_\ell(K, M; U, \varepsilon)\right\}, \tag{96}$$

where $\hat{H}_\ell(K, M; U, \varepsilon) = \int_U \Sigma_\ell \lambda_{\varepsilon a_m}(F_m(t))\lambda_{\varepsilon a_j}(F_j(t))dt$ and Σ_ℓ is the summation over all m, j satisfying $K \leq m < j \leq M$, $m = \ell$ (mod 128) and $j = \ell$ (mod 128). (See (60).)

For the estimation of $\hat{H}_\ell(K, M; U, \varepsilon)$ $(1 \leq \ell \leq 128)$, we estimate $\hat{L}_{m,j} = \int_U \lambda_{\varepsilon a_m}(F_m(t))\lambda_{\varepsilon a_j}(F_j(t))dt$ $(j - 127 \geq m \geq m_4')$, where m_4' is the integer chosen in the preceding lemma. If $4\nu(F_j) \leq w(\tau(m+j); F_j)$, we use Lemma 18 for $Q(t) = F_m(t)$, $R(t) = F_j(t)$, $J = \tau(m)$, $J' = \tau(m+j)$, $\eta = \varepsilon a_m$, $\eta' = \varepsilon a_j$, $M = m$, $M' = j$ and $I = U$. Since $w(\tau(m); F_m)^{-2} \leq 2e^2 s_m^{-2}$ and $w(\tau(m+j); F_j)^{-2} \leq 2e^2 w(m, j)^{-2}$, we have

$$\begin{aligned}\hat{L}_{m,j} &\leq 10^{20}m(U)\{\varepsilon^2 a_m^2 w(\tau(m); F_m)^{-2} + O(m^{-2})\}\{\varepsilon^2 a_j^2 w(\tau(m+j); F_j)^{-2} + O(j^{-2})\} \\ &\quad + O(m^{-2}j^{-2}) \\ &\leq 2^2e^4 10^{20}m(U)\varepsilon^4\{a_m^2 s_m^{-2} + O(m^{-2})\}\{a_j^2 w(m, j)^{-2} + O(j^{-2})\} + O(j^{-2}m^{-2}).\end{aligned}$$

If $4\nu(F_j) \geq w(\tau(m+j); F_j)$, then $a_j^2 w(m, j)^{-2} \geq 2^{-1}e^{-2}\nu(F_j)^2 w(\tau(m+j); F_j)^{-2} \geq 2^{-5}e^{-2}$. In this case, we use Lemma 19 for $Q(t) = F_m(t)$, $R(t) = F_j(t)$, $J = \tau(m)$, $\eta = \varepsilon a_m$, $\eta' = \varepsilon a_j$ $(\leq |\tilde{F}_j(j)|/3)$, $M = m$, $M' = j$ and $I = U$. We have

$$\begin{aligned}\hat{L}_{m,j} &\leq 10^{17}m(U)\{\varepsilon^2 a_m^2 w(\tau(m); F_m)^{-2} + O(m^{-2})\}\{\varepsilon a_j |\tilde{F}_j(j)|^{-1} + O(j^{-2})\} \\ &\quad + O(m^{-2}j^{-2}) \\ &\leq 2e^2 10^{17}m(U)\varepsilon^3\{a_m^2 s_m^{-2} + O(m^{-2})\}\{e + O(j^{-2})\} + O(m^{-2}j^{-2}) \\ &\leq 2^6e^5 10^{17}m(U)\varepsilon^3\{a_m^2 s_m^{-2} + O(m^{-2})\}\{a_m^2 w(m, j)^{-2} + O(j^{-2})\} + O(m^{-2}j^{-2}).\end{aligned}$$

In any case, we have

$$\hat{L}_{m,j} \leq 10^{23}m(U)\varepsilon^3\{a_m^2 s_m^{-2} + O(m^{-2})\}\{a_j^2 w(m, j)^{-2} + O(j^{-2})\} + O(m^{-2}j^{-2}).$$

By (96), we have

$$\begin{aligned}\hat{H}(K, M; U, \varepsilon) &\leq 10^5\pi^2\hat{h}(K, M; U, \varepsilon) \\ &\quad + 10^{28}128m(U)\varepsilon^3\left\{\sum_{m=K}^{M} a_m^2 s_m^{-2} + O(1)\right\}\left\{\sum_{j=m+1}^{M} a_j^2 w(m, j)^{-2} + O(1)\right\} + O(1).\end{aligned}$$

Since $\sum_{m=K}^{M} a_m^2 s_m^{-2} \sum_{j=m+1}^{M} a_j^2 w(m, j)^{-2} \leq 4\hat{T}_M(\log w(m, M) - \log a_m) \leq 4\hat{T}_M^2$, we obtain the required property.

For $0 < \varepsilon \leq 1/3e$, $a \in \boldsymbol{C}$ and a positive integer μ, set $\hat{U}(\mu, a, \varepsilon) = \bigcup_{m=\mu}^{\infty} \{t \in [0, 2\pi); |F_m(t) - a| < \varepsilon a_m\}$. Using Lemma 68, 69 and Corollary 9, we have $m(\hat{U}(\mu, 0, \varepsilon)) = 2\pi$. Considering $f(z) - a$, we have $m(\hat{U}(\mu, a, \varepsilon)) = 2\pi$. Choosing a countable dense set Σ in $\boldsymbol{C}$, we put $\hat{U} = \bigcap_{a \in \Sigma} \bigcap_{\ell=10}^{\infty} \bigcap_{\mu=1}^{\infty} \hat{U}(\mu, a, 1/\ell)$. Then $m(\hat{U}) = 2\pi$. Now we show that, for any $t \in \hat{U}$, $R_{W(f)}(t; f) = \boldsymbol{C}$. Let $t_0 \in \hat{U}$, $a \in \Sigma$ and $0 < \varepsilon \leq 1/3e$. Choose a positive integer ℓ_0 so that $1/\ell_0 \leq 2^{-8}\lambda(1)\varepsilon$. Since $t_0 \in \bigcap_{\mu=1}^{\infty} \hat{U}(\mu, a, 1/\ell_0)$, there exists a strictly increasing sequence $(m_j)_{j=1}^{\infty}$ of positive integers such that $|F_{m_j}(t_0) - a| \leq a_{m_j}/\ell_0$. Then

$$\begin{aligned} |f(r_{m_j}e^{it_0}) - a| &\leq |f(r_{m_j}e^{it_0}) - F_{m_j}(t_0)| + |F_{m_j}(t_0) - a| \\ &\leq a_{m_j}/n_{m_j} + A_{f,m_j} + a_{m_j}/\ell_0 \qquad (j \geq 1)\,. \end{aligned}$$

Since $\lim_{j\to\infty} A_{f,m_j} = 0$, we have, from (93), $f(D_{m_j}(t_0, \varepsilon)) \ni a$ for all sufficiently large j. Hence $R(\Gamma_{\varepsilon,W(f)}(t_0); f) \ni a$. Since $0 < \varepsilon \leq 1/3e$ is arbitrary, $R_{W(f)}(t_0; f) \ni a$. Since $a \in \Sigma$ is arbitrary and $R_{W(f)}(t_0; f)$ is closed, $R_{W(f)}(t_0; f) = \boldsymbol{C}$. Hence $W(f)$ is a covering sequence for a.a.t. As an immediate consequence, we know that almost all directions are Borel directions of $f(z)$.

(79): Suppose that $\nu(f) < +\infty$. By (92), $f(D_m(t, 1)) \subset D(f(r_m e^{it}), \nu(f)+A_f)$. Hence $f(D_m(t, 1))$ $(m \geq 1, t \in [0, 2\pi))$ does not contain any open disk having radius $\nu(f) + A_f + 1$.

(80): Suppose that $\sum_{m=1}^{\infty} a_m s_m^{-2} < +\infty$. For each $0 < \varepsilon \leq 1/30e$, we shall define a sequence $(\hat{D}_m(t, \varepsilon))_{m=1}^{\infty}$ $(t \in [0, 2\pi))$ of domains in $\boldsymbol{D}$ such that $D_m(t, \varepsilon) \subset \hat{D}_m(t, \varepsilon) \subset D_m(t, 10\varepsilon)$ for all $m \geq 1$ and all $t \in [0, 2\pi)$ and that $C(\hat{\gamma}(t, \varepsilon); f) = \{\infty\}$ for a.a.t, where $\hat{\gamma}(t, \varepsilon) = \partial(\bigcup_{m=1}^{\infty} \hat{D}_m(t, \varepsilon))$.

Then our assertion immediately follows from Rouche's theorem. In the following lemma, we define such a sequence.

LEMMA 70. *For $\eta > 0$, $0 < \varepsilon \leq 1/30e$, a positive integer m and $t \in [0, 2\pi)$, set:*

$$\begin{cases} \gamma_{\rightarrow,m}(t, \varepsilon) = \{re^{it - i\varepsilon n_m}; r_m - \varepsilon/n_m \leq r \leq r_m + \varepsilon/n_m\} \\ \gamma_{\leftarrow,m}(t, \varepsilon) = \{re^{it + i\varepsilon/n_m}; r_m - \varepsilon/n_m \leq r \leq r_m + \varepsilon/n_m\} \\ \gamma_{\uparrow,m}(t, \varepsilon) = \{(r_m + \varepsilon/n_m)e^{it+is}; -\varepsilon/n_m \leq s \leq \varepsilon/n_m\} \\ \gamma_{\downarrow,m}(t, \varepsilon) = \{(r_m - \varepsilon/n_m)e^{it+is}; -\varepsilon/n_m \leq s \leq \varepsilon/n_m\} \end{cases}$$

$$\begin{cases} \gamma_m(t,\varepsilon) = \cup\,\{\gamma_{\zeta,m}(t,\varepsilon);\ \zeta = \rightarrow, \leftarrow, \uparrow, \downarrow\} \\ \hat{D}_m(t,\varepsilon) = \textit{the interior of } \gamma_m(t,\varepsilon) \\ \hat{\gamma}(t,\varepsilon) = \partial\Big(\bigcup_{m=1}^{\infty} \hat{D}_m(t,\varepsilon)\Big) \\ \gamma_{\rightarrow}(t) = \{re^{it};\ 0 \le r < 1\} \\ \gamma(t,\varepsilon) = \hat{\gamma}(t,\varepsilon) \cup \gamma_{\rightarrow}(t) \\ \Lambda_{\zeta,\eta}(t,\varepsilon) = \sum_{m=1}^{\infty}\int_{\gamma_{\zeta,m}(t,\varepsilon)} |f'(z)|\,\lambda_\eta(f(z))ds_{\gamma(t,\varepsilon)}(z) \qquad (\zeta = \rightarrow, \leftarrow, \uparrow, \downarrow) \\ \Lambda_{\rightarrow,\eta}(t) = \int_{\gamma_{\rightarrow}(t)} |f'(z)|\,\lambda_\eta(f(z))ds_{\gamma(t,\varepsilon)}(z) \\ \Lambda_\eta(t,\varepsilon) = \int_{\gamma(t,\varepsilon)} |f'(z)|\,\lambda_\eta(f(z))ds_{\gamma(t,\varepsilon)}(z)\,. \end{cases}$$

Then, for any $0 < \varepsilon \le 1/30e$, "$\{\Lambda_\eta(t,\varepsilon) < +\infty$ *for all* $\eta > 0\}$ *for* a.a.t".

Proof. First we show that, for any $\eta > 0$, $\Lambda_{\rightarrow,\eta} = \int_0^{2\pi} \Lambda_{\rightarrow,\eta}(t)dt < +\infty$. Denoting by $L_{m,\eta} = \int_{r_{m-1}}^{r_m} \sum_{k=1}^{\infty} a_k n_k r^{n_k-1}\Big\{\int_0^{2\pi} \lambda_\eta(f(re^{it}))dt\Big\}dr$ $(m \ge 1,\ r_0 = 0)$, we have

$$\Lambda_{\rightarrow,\eta} \le \int_0^1 \sum_{k=1}^{\infty} a_k n_k r^{n_k-1}\Big\{\int_0^{2\pi} \lambda_\eta(f(re^{it}))dt\Big\}dr = \sum_{m=1}^{\infty} L_{m,\eta}\,.$$

Hence it is essential to estimate $L_{m,\eta}$ for all sufficiently large m.

Since $(a_m)_{m=1}^{\infty}$ is increasing and $\lim_{m\to\infty} a_m/s_m = 0$, there exists a positive integer $m_5' \ge 128$ such that, for $m \ge m_5'$, $8a_m \le s_{m-2}$, $s_m \le \sqrt{2}\, s_{m-2}$ and $a_m \le 2^{m/10}$. For a fixed r satisfying $r_{m-1} \le r < r_m$ $(m \ge m_5')$, we have $|f(re^{it})| \le |F_{r,m}(t)| + A_f$, where $F_{r,m}(t) = \sum_{k=0}^{m-2} \tilde{f}(k)e^{in_kt} + \tilde{f}(m-1)r^{n_{m-1}}e^{in_{m-1}t} + \tilde{f}(m)r^{n_m}e^{in_mt}$. Hence $\lambda_\eta(f(re^{it})) \le \lambda_{\eta_0}(F_{r,m}(t))$, where $\eta_0 = \eta + A_f$. Note that $m - 63 \ge \tau(m)$, $4\nu(F_{r,m}) \le w(\tau(m); F_{r,m})$ and $w(\tau(m); F_{r,m})^{-2} \le 2e^2 s_m^{-2}$. We use Lemma 18 for $Q(t) = F_{r,m}(t)$, $J = \tau(m)$, $M = 2^{m/10}$ and $I = [0, 2\pi)$. Then we have

$$\begin{aligned} \int_0^{2\pi} \lambda_\eta(f(re^{it}))dt &\le \int_0^{2\pi} \lambda_{\eta_0}(F_{r,m}(t))dt \\ &\le 10^3\{(2\pi)\eta_0^2 w(\tau(m); F_{r,m})^{-2} + (2\pi + 1)\gamma(\tau(m))\eta_0^2 2^{m/10}2^{6m/10} + \eta_0^{-1}2^{-2m/10}\} \\ &\le 10^{10}\eta_0^2 s_m^{-2} + O(2^{7m/10}\gamma(\tau(m))) + O(2^{-2m/10})\,. \end{aligned}$$

We have also, for any $m \ge m_5'$,

$$\begin{aligned} \int_{r_{m-1}}^{r_m} \sum_{k=1}^{\infty} a_k n_k r^{n_k-1} dr &\le \int_{r_{m-1}}^{r_m} \Big(\sum_{k=1}^{m-1} a_k n_k + \sum_{k=m}^{\infty} a_k n_k r^{n_k-1}\Big)dr \\ &\le \sum_{k=1}^{m-1} a_k n_k/n_{m-1} + \sum_{k=m}^{\infty} a_k r_m^{n_k} \le A_{f,m-1} + a_{m-1} + a_m + A_{f,m} \le 2(a_m + A_f)\,. \end{aligned}$$

Hence

$$L_{m,\eta} \le \int_{r_{m-1}}^{r_m} \sum_{k=1}^{\infty} a_k n_k r^{n_k-1}\{10^{10}\eta_0^2 s_m^{-2} + O(2^{7m/10}\gamma(\tau(m))) + O(2^{-2m/10})\}dr$$
$$\le 2\cdot 10^{10}\eta_0^2(a_m + A_f)s_m^{-2} + O(2^{8m/10}\gamma(\tau(m))) + O(2^{-m/10}) .$$

Consequently,

$$\Lambda_{\to,\eta} = \sum_{m=1}^{\infty} L_{m,\eta} \le 2\cdot 10^{10}\eta_0^2 \sum_{m=m_5'}^{\infty} (a_m + A_f)s_m^{-2} + O(1) < +\infty .$$

We have analogously, for any $\eta > 0$, $0 < \varepsilon \le 1/30e$, $\int_0^{2\pi} \Lambda_{\zeta,\eta}(t, \varepsilon)dt < +\infty$ ($\zeta = \leftarrow, \uparrow, \downarrow$) and hence $\int_0^{2\pi} \Lambda_\eta(t, \varepsilon)dt < +\infty$. Putting $\ddot{U}(\varepsilon) = \bigcap_{\ell=1}^{\infty}\{t \in [0, 2\pi);$ $\Lambda_\ell(t, \varepsilon) < +\infty\}$ $(0 < \varepsilon \le 1/30e)$, we have $m(\ddot{U}(\varepsilon)) = 2\pi$ and, for any $t \in \ddot{U}(\varepsilon)$, "$\Lambda_\eta(t, \varepsilon) < +\infty$ for all $\eta > 0$". This completes the proof of this lemma.

Since $\sum_{m=1}^{\infty} s_m^{-2} < \infty$, $C(t; f) = \{\infty\}$ for a.a.t. Since $\mathrm{Ord}(f) < +\infty$, there exists a sequence W such that $M_W(t; f) = C(t; f)$ for a.t. Lemma 67 and 70 show that "$M(\hat{\gamma}(t, \varepsilon); f) = \{\infty\}$ for a.a.t" for any $0 < \varepsilon \le 1/30e$. Putting $\ddot{U} = \bigcap_{\ell=100}^{\infty}\{t \in [0, 2\pi); R_{W(f)}(t; f) = \boldsymbol{C},\ M(\hat{\gamma}(t, \varepsilon); f) = \{\infty\}\}$, we have $m(\ddot{U}) = 2\pi$. Rouche's theorem shows that, for any $t \in \ddot{U}$, $W(f)$ is a pit sequence for t. This completes the proof of (80).

8.4. Proof of Corollary 56, 59 and Theorem 57, 58

Corollary 56 is an immediate consequence of Theorem 55. We show Theorem 57. Let $f(z)$ be the function in this theorem. We use the notation a_m, s_m, $\Lambda_{\to,\eta}(t)$ in 8.3. Then almost all directions are Borel directions of $f(z)$ and hence almost all directions are dense directions of $f(z)$.

(83): Suppose that $\sum_{m=1}^{\infty} s_m^{-2} = +\infty$. Since $(a_m)_{m=1}^{\infty}$ is increasing, (D) in Theorem 30 shows that $m(C^{-1}(\hat{C}; f)) = 2\pi$. Since $\mathrm{Ord}(f) < +\infty$, there exists a sequence W such that $M_W(t; f) = C(t; f)$ for a.t. Since $M_{\to}(t; f) \supset M_W(t; f)$ for a.t, $m(M_{\to}^{-1}(\hat{C}; f)) = 2\pi$.

(84): Suppose that $\sum_{m=1}^{\infty} a_m s_m^{-2} < +\infty$. By (C) in Theorem 30, $m(C^{-1}(\{\infty\}; f)) = 2\pi$. There exists a sequence W such that $M_W(t; f) = C(t; f)$ for a.t. We have also "$\Lambda_{\to,\eta}(t) < +\infty$ for all $\eta > 0$" for a.a.t. By Lemma 65, $m(M_{\to}^{-1}(\{\infty\}; f)) = 2\pi$.

Theorem 58 is an immediate consequence of Theorem 31. Corollary 59 is an immediate consequence of Theorem 57 and 58.

8.5. Proof of Theorem 60

Let $f_\alpha(z)$ be the function in this theorem. We write simply $f(z) = f_\alpha(z)$

and use the notation in 7.3: a_m, s_m, $w(m, M)$, n_m, r_m, $F_m(t)$, $A_{f,m}$. Let $(m_j)_{j=1}^{\infty}$ be a strictly increasing sequence of positive integers. Note that $a_{m_j} = m_j^{\alpha}$ and $s_{m_j} = \{(1 + o(1))/(2\alpha + 1)\}m_j^{\alpha+1/2}$ $(j \geq 1)$. If $\sum_{j=1}^{\infty} 1/m_j < +\infty$, then

$$\sum_{j=1}^{\infty} a_{m_j}^2 s_{m_j}^{-2} = O\Big(\sum_{j=1}^{\infty} \frac{1}{m_j}\Big) < +\infty .$$

If $\sum_{j=1}^{\infty} 1/m_j = +\infty$ and $(m_{j+1} - m_j)_{j=1}^{\infty}$ is increasing, then

$$\sum_{j=1}^{M} a_{m_j}^2 s_{m_j}^{-2} = (2\alpha + 1)(1 + o(1)) \sum_{j=1}^{\infty} \frac{1}{m_j} \tag{97}$$

and

$$\begin{aligned}
&\sum_{j=1}^{M} a_{m_j}^2 s_{m_j}^{-2} \sum_{k=j+1}^{M} a_{m_k}^2 w(m_j, m_k)^{-2} \\
&\quad = (2a + 1)^2(1 + o(1)) \sum_{j=1}^{M} \frac{1}{m_j} \sum_{k=j+1}^{M} \frac{1}{m_k - m_j(m_j/m_k)^{2\alpha}} \\
&\quad \leq (2\alpha + 1)^2(1 + o(1)) \sum_{j=1}^{M} \frac{1}{m_j} \sum_{k=j+1}^{M} \frac{1}{m_k - m_j} \\
&\quad \leq (2\alpha + 1)^2(1 + o(1)) \sum_{j=1}^{M} \frac{1}{m_j} \sum_{k=2}^{M-j+1} \frac{1}{m_k - m_1} \\
&\quad \leq (2\alpha + 1)^2(1 + o(1))\theta(m_1)\Big(\sum_{j=1}^{M} \frac{1}{m_j}\Big)^2 ,
\end{aligned} \tag{98}$$

where $\theta(m_1)$ is a positive constant depending only on m_1.

For $0 < \varepsilon \leq 1/3e$ and $a \in \boldsymbol{C}$, set $Y(a, \varepsilon) = \{t \in [0, 2\pi);$ "$|F_{m_j}(t) - a| < \varepsilon a_{m_j}$" holds for infinitely many $j\}$. Choosing a countable dense set Σ in $\boldsymbol{C}$, set $Y(\varepsilon) = \bigcap_{a \in \Sigma} Y(a, \varepsilon)$, $Y = \bigcap_{\ell=30}^{\infty} Y(1/\ell)$ and $Y_c = \bigcap_{a \in \Sigma} Y(a, 1/40)^c$.

(87): Suppose that $\sum_{j=1}^{\infty} 1/m_j < +\infty$. First we remark the following implication: Let $t \in [0, 2\pi)$ and $a \in \boldsymbol{C}$. Then

$$\text{"}\liminf_{j\to\infty} |F_{m_j}(t) - a|/a_{m_j} \geq 1/50\text{"} \Rightarrow \text{"}R_{W(f)}(t; f)^c \supset D(a, 1/100)\text{"} .$$

In fact, we have, for $z \in D_{m_j}(t, 1/80)$ $(j \geq 1)$,

$$\begin{aligned}
|f(z) - a| &\geq |F_{m_j}(t) - a| - |f(r_{m_j}e^{it}) - F_{m_j}(t)| - |f(z) - f(r_{m_j}e^{it})| \\
&\geq |F_{m_j}(t) - a| - \Big(\frac{1}{80} a_{m_j} + \frac{1}{80} A_{f, m_j}\Big) - (a_{m_j}/n_{m_j} + A_{f,m_j}) ,
\end{aligned}$$

and hence

$$\liminf_{j\to\infty} \inf\{|f(z) - a|; z \in D_{m_j}(t, 1/80)\}$$

$$\geq \liminf_{j\to\infty}\left\{|F_{m_j}(t)-a|-\frac{1}{80}a_{m_j}\right\}\geq\frac{3}{200},$$

which shows that $R_{W(f)}(t;f)^c\supset D(a,1/100)$.

Using Lemma 68, we have

$$\int_0^{2\pi}\sum_{j=1}^{\infty}\lambda_{a_{m_j}/10}(F_{m_j}(t))\,dt=O\Big(\sum_{j=1}^{\infty}a_{m_j}^2 s_{m_j}^2\Big)<+\infty$$

and hence $\int_0^{2\pi}\sum_{j=1}^{\infty}\chi_{a_{m_j}/40}(F_{m_j}(t))dt<+\infty$, which shows that $m(Y(0,1/50)^c)=2\pi$. Considering $f(z)-a$, we have $m(Y(a,1/50)^c)=2\pi$ and hence $m(Y_c)=2\pi$. For $t\in Y_c$, we have $R_{W(f)}(t;f)^c\supset\bigcup_{a\in\Sigma}D(a,1/100)=\boldsymbol{C}$ that is, $R_{W(f)}(t;f)=\emptyset$.

(88): Suppose that $\sum_{j=1}^{\infty}1/m_j=+\infty$ and $(m_{j+1}-m_j)_{j=1}^{\infty}$ is increasing. For $0<\varepsilon\leq 1/3e$, two positive integers K, M $(K<M)$ and a Borel set U in $[0,2\pi)$, we define

$$\text{(99)}\qquad\begin{cases}\ddot{h}(K,M;U,\varepsilon)=\displaystyle\int_U\sum_{j=K}^{M}\lambda_{\varepsilon a_{m_j}}(F_{m_j}(t))dt\\ \ddot{H}(K,M;U,\varepsilon)=\displaystyle\int_U\Big\{\sum_{j=K}^{M}\lambda_{\varepsilon a_{m_j}}(F_{m_j}(t))\Big\}^2dt\,.\end{cases}$$

Then we see analogously as in Theorem 55 that, for any ε, K and U $(m(U)\neq 0)$, there exists a positive integer $\dot{M}_1$ such that, for $M\geq\dot{M}_1$, $\ddot{h}(K,M;U,\varepsilon)\geq 10^{-4}\varepsilon^2m(U)\sum_{j=1}^{M}a_{m_j}^2s_{m_j}^{-2}$ and

$$\ddot{H}(K,M;U,\varepsilon)\leq 10^{31}\varepsilon^3m(U)\sum_{j=1}^{M}a_{m_j}^2s_{m_j}^{-2}\sum_{k=j+1}^{M}a_{m_k}^2w(m_j,m_k)^{-2}\,.$$

By (97) and (98), there exists a positive integer $\ddot{M}_2$ such that, for $M\geq\ddot{M}_2$, $\ddot{h}(K,M;U,\varepsilon)^2\ddot{H}(K,M;U,\varepsilon)^{-1}\geq 10^{-40}\varepsilon^{-1}\theta(m_1)^{-1}m(U)$. From this fact, we obtain $m(Y(0,\varepsilon/3))=2\pi$. Considering $f(z)-a$, we have $m(Y(a,\varepsilon/3))=2\pi$ and hence $m(Y(\varepsilon/3))=2\pi$. Consequently, $m(Y)=2\pi$. By (92), (93) and Lemma 66, we obtain $R_{W(f)}(t;f)=\boldsymbol{C}$ $(t\in Y)$.

Remark 71. We see more in detail the following proposition: Let $f(z)$ be an L-lacunary analytic function such that $(|\tilde{f}(m)|)_{m=1}^{\infty}$ is increasing and that $(|\tilde{f}(m)/\tilde{f}(m+1)|)_{m=1}^{\infty}$ is decreasing and let $W=(r_{k_j}(f))_{j=1}^{\infty}$ be a subsequence of $W(f)$.

(100) If $\sum_{j=1}^{\infty}|\tilde{f}(k_j)|^2s(k_j;f)^{-2}<+\infty$, then W is a void sequence a.a.t.

(101) If $\sum_{j=1}^{\infty}|\tilde{f}(k_j)|^2s(k_j;f)^{-2}=+\infty$ and $(k_{j+1}-k_j)_{j=1}^{\infty}$ is increasing, then W is a covering sequence a.a.t.

Remark 72. It is an interesting problem to determine analytic functions in D having Borel directions. In this area, the following question is natural: For a given positive continuous function $h(r)$ in $[0, 1)$ satisfying $\lim_{r\to 1} h(r) = +\infty$, is there exist an analytic function $g(z)$ such that $T(r, g) \leq h(r)$ $(0 \leq r < 1)$ and that almost all directions are Borel directions of $g(z)$?

We can answer, in this paper, this question by using Theorem 55. In fact, given such a function $h(r)$, we can define an L-lacunary analytic function $f_0(z) = \sum_{k=1}^{\infty} z^{n_k}$ such that $T(r, f_0) \leq h(r)$ $(0 \leq r < 1)$. Then Theorem 55 shows that it is a required function.

References

[1] Bari, N., Trigonometric series, Moskva, 1961.
[2] Benedetto, J., Harmonic analysis on totally disconnected sets, Lecture note, **202**, springer, 1971.
[3] Carleson, L., Selected problems on exceptional sets, Van Nostrand, 1967.
[4] ——, On the convergence and growth of partial sums of Fourier series, Acta Math., **116** (1967), 135–157.
[5] Fuchs, W. H. J., On the zeros of power series with Hadamard gaps, Nagoya Math. J., **29** (1967), 167–174.
[6] Kahane, J. P., Lacunary Taylor and Fourier series, Bull. Amer. Math. Soc., **70** (1964), 199–213.
[7] ——, Some random series of functions, D.C. Heath and Co., Boston Mass., 1968.
[8] ——, Séries de Fourier absolument convergentes, Springer, 1970.
[9] Kahane, J. P. and Salem, R., Ensembles parfaits et séries trigonométriques, Harmann, 1962.
[10] Murai, T., Sur la distribution des valeurs des séries lacunaries, J. London M.S. (2) **21** (1980), 93–110.
[11] Nevanlinna, R., Analytic functions, Springer, 1970.
[12] Noshiro, K., Cluster sets, Springer, 1960.
[13] Salem, R., Algebraic numbers and Fourier analysis, D.C. Heath and Co., Boston Mass., 1963.
[14] ——, Oeuvres mathématiques, Hermann, 1967.
[15] Watson, G. N., Theory of Bessel functions, Cambridge, 1922.
[16] Weiss, M., The law of the iterated logarithm for lacunary trigonometric series, Trans. Amer. Math. Soc., **99** (1959), 444–469.
[17] ——, Concerning a theorem of Paley on lacunary power series, Acta Math., **102** (1959), 225–245.
[18] Weiss, M. and Weiss, G., On the Picard property of lacunary power series, Studia Math., **22** (1963), 221–245.
[19] Zygmund, A., Trigonometric series I, Cambridge, 1959.

Department of Mathematics
Faculty of Science
Nagoya University
Chikusa-ku, Nagoya, 464, Japan