T. Kimura
Nagoya Math. J.
Vol. 85 (1982), 1-80

THE *b*-FUNCTIONS AND HOLONOMY DIAGRAMS OF IRREDUCIBLE REGULAR PREHOMOGENEOUS VECTOR SPACES

TATSUO KIMURA

Introduction

The purpose of this paper is to investigate the micro-local structure and to calculate, by constructing the holonomy diagrams, the *b*-functions (See [2]) of irreducible regular prehomogeneous vector spaces (See [1]).

Since we know the relation of *b*-functions with respect to castling transformations (See § 12), it is enough to calculate them only when they are reduced. In this paper, we shall deal with twenty of all twenty nine reduced regular P.V.'s in the Table in [1]. Together with other articles, this completes the list of *b*-functions of irreducible reduced regular prehomogeneous vector spaces (See § 12) except $(SL(5)\times GL(4),\ \Lambda_2\otimes\Lambda_1,\ V(10)\otimes V(4))$ which is the most complicated case (See I. Ozeki [11]). This paper consists of the following twelve sections and one Appendix with I. Ozeki.

Received January 11, 1979
Revised September 7, 1979

Appendix with I. Ozeki. ($GL(1) \times \mathrm{Spin}\ (14)$, $\square \otimes$ half-spin rep., $V(1) \otimes V(64)$)

In § 1, we shall review the main results of [2] which will be used later. From § 2 to § 11, we do the classification of the orbits, construction of the holonomy diagrams and calculation of the b-functions. In § 12, we shall give the list of b-functions for irreducible reduced regular P.V.'s. Some of them have been already calculated by M. Sato and the author using the different method (See [7]). The holonomy diagrams in § 2, § 8 and § 10 are first obtained by M. Sato. The author would like to express his hearty thanks to Professors Mikio Sato and Masaki Kashiwara for their invaluable advice and encouragement.

§ 1. Preliminaries

Let (G, ρ, V) be an irreducible regular prehomogeneous vector space (abbrev. P.V.) with the singular set S. Then S is the zeros of the relative invariant $f(x): S = \{x \in V; f(x) = 0\}$, $f(\rho(g)x) = \chi(g)f(x)$ for all $g \in G$ and $x \in V$. We shall consider the micro-differential equations $\mathfrak{M} = \mathscr{E}f^s$ where $\mathscr{E}$ is the sheaf of micro-differential operators of finite order on the cotangent bundle $T^*V = V \times V^*$ (See [2]). Note that the group G acts on T^*V by $(x, y) \mapsto (\rho(g)x, \rho^*(g)y)$ for $x \in V$, $y \in V^*$ and $g \in G$ where ρ^* denotes the contragredient representation of ρ. Let Λ be the Zariski-closure of a conormal bundle of some G-orbit $\rho(G) \cdot x_0 (x_0 \in V)$. Since we consider only *the Zariski-closure* of a conormal bundle, we shall omit the word "the Zariski-closure" for simplicity. Assume that Λ is G-prehomogeneous and is contained in $W = \overline{\{(x, \mathrm{grad} \log f(x)^s); x \in V - S, s \in C\}}$. In this case, Λ is called a good holonomic variety. It is an irreducible component of the characteristic variety of $\mathfrak{M}$. We can show that there exists a local b-function $b_\Lambda(s)$ which is unique up to a constant multiple (See [2]). We have $b_{V \times \{0\}}(s) = 1$ and $b_{\{0\} \times V^*}(s) = b(s)$ where $b(s)$ denotes the b-function of this P.V. When two good holonomic varieties Λ_0 and Λ_1 intersect with codimension one, we have the relation between $b_{\Lambda_0}(s)$ and $b_{\Lambda_1}(s)$ as follows (See [2]).

THEOREM 1-1 ([2] Theorem 7-5). *Let Λ_0 and Λ_1 be good holonomic varieties whose intersection is of codimension one with the intersection exponent $(\mu : \nu)$. Assume that $\mathfrak{M} = \mathscr{E}f^s$ is a simple holonomic system with support $\Lambda_0 \cup \Lambda_1$ and $\Lambda_0 \cap \Lambda_1 \not\subset \overline{\mathrm{supp}\, \mathfrak{M} - (\Lambda_0 \cup \Lambda_1)}$. Assume that $m_0 > m_1$ where*

$\mathrm{ord}_{\Lambda_i} f^s = -m_i s - \mu_i/2$ $(i = 0, 1)$. *Then we have, up to a constant multiple,*

$$b_{\Lambda_0}(s)/b_{\Lambda_1}(s) = \prod_{k=0}^{\nu} \left[\frac{1}{\nu+1} (\mathrm{ord}_{\Lambda_1} f^s - \mathrm{ord}_{\Lambda_0} f^s) + \frac{\mu + 2k}{2(\nu+\mu)} \right]^{(m_0 - m_1)/(\nu+1)} \tag{1.1}$$

$$\text{where } [\alpha]^k = \alpha(\alpha+1)\cdots(\alpha+k-1) .$$

Here we denote by $\mathrm{ord}_\Lambda f^s$ the order of f^s at Λ (See [2]). Note that m_0 and m_1 are non-negative integers, and $(\mu : \nu) = (1 : 0)$ or $(\mu : \nu)$ is a pair of positive integers satisfying $\mu \geq 2$, $\nu \geq 1$, and $(m_0 - m_1)$ is a multiple of $(\nu + 1)$.

COROLLARY 1–2 ([2] Corollary 7–6). *If Λ_0 and Λ_1 intersect regularly, i.e., $\mu = 1$ and $\nu = 0$, we have*

$$b_{\Lambda_0}(s)/b_{\Lambda_1}(s) = \prod_{k=1}^{m_0 - m_1} \left((m_0 - m_1)s + \frac{\mu_0 - \mu_1 - 1}{2} + k \right) \tag{1.2}$$

$$\textit{where } \mathrm{ord}_{\Lambda_i} f^s = -m_i s - \frac{\mu_i}{2} \quad (i = 0, 1) .$$

Let Λ be a good holonomic variety. Then $\Lambda = \overline{G(x_0, y_0)}$ for some $x_0 \in V$, $y_0 \in V^*$ where $G(x_0, y_0) = \{(\rho(g)x_0, \rho^*(g)y_0); g \in G\}$. In this case, we can calculate the order $\mathrm{ord}_\Lambda f^s$ by the following proposition.

PROPOSITION 1–3 ([2] Proposition 4–14). *Let A_0 be an element of the Lie algebra $\mathfrak{g}$ of G satisfying $d\rho(A_0)x_0 = 0$ and $d\rho^*(A_0)y_0 = y_0$. Then we have*

$$\mathrm{ord}_\Lambda f^s = s\delta\chi(A_0) - \mathrm{tr}_{V_{x_0}^*}\, d\rho_{x_0}(A_0) + \tfrac{1}{2} \dim V_{x_0}^* \tag{1.3}$$

where $V_{x_0}^*$ denotes the conormal vector space $(d\rho(\mathfrak{g}) \cdot x_0)^\perp$, and $d\rho_{x_0}$ denotes the representation of $\mathfrak{g}_{x_0} = \{A \in \mathfrak{g};\ d\rho(A)x_0 = 0\}$ induced by $d\rho^*$.

Now let $\Lambda_0 = \overline{G(x_0, y_0)}$ and $\Lambda_1 = \overline{G(x_1, y_1)}$ be good holonomic varieties such that $(x_0, y_1) \in \Lambda_0 \cap \Lambda_1$ and $\dim G(x_0, y_1) = \dim V - 1$. In this case, the intersection exponent $(\mu : \nu)$ is given by the following proposition.

PROPOSITION 1–4 ([2] Proposition 6–5). *Let A_1 be an element of $\mathfrak{g}$ satisfying $d\rho(A_1)x_0 = 0$ and $d\rho^*(A_1)y_1 = y_1$. Then A_1 acts on the one-dimensional vector space $\tilde{V} = V_{x_0}^*$ modulo $d\rho_{x_0}(\mathfrak{g}_{x_0})y_1$. Let β be its eigenvalue, i.e., $\beta = \mathrm{tr}_{\tilde{V}}\, A_1$. Then μ and ν are given by $\beta = \mu/(\mu + \nu)$, $(\mu, \nu) = 1$. If β is not determined uniquely, i.e., β depends on A_1, then we have $\mu = 1$, $\nu = 0$, and Λ_0, Λ_1 intersect regularly.*

Let $\Lambda = \overline{T(\rho(G)x_0)}^{\perp}$ be a conormal bundle of a G-orbit $\rho(G)x_0$. Then G acts on Λ prehomogeneously if and only if the colocalization $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ at x_0 is a P.V. We shall consider some sufficient conditions that $\Lambda \subset W$, i.e., Λ is a good holonomic variety.

PROPOSITION 1–5 ([2] Proposition 6–6). *Let* Λ_0 *and* Λ_1 *be two conormal bundles of some G-orbits. Assume that* $\dim \mathfrak{g}_0 \cdot p = \dim V - 1$ *for some* $p \in \Lambda_0 \cap \Lambda_1$ *where* $\mathfrak{g}_0 = \{A \in \mathfrak{g}; \delta\chi(A) = 0\}$. *Assume that* Λ_0 *(or* Λ_1*)* $\subset W$. *Then we have* $\Lambda_0 \cup \Lambda_1 \subset W$. *Moreover W is non-singular and* $W = \{(x, y) \in V \times V^*; \langle d\rho(A)x, y\rangle = 0$ *for all* $A \in \mathfrak{g}_0\}$ *near* p.

Let $V_{x_0} = V \bmod d\rho(\mathfrak{g})x_0$ be the normal vector space. Then the isotropy subgroup G_{x_0} acts on V_{x_0}. We denote this action by $\tilde{\rho}_{x_0}$. Let f_{x_0} be the localization of $f(x)$ at x_0 (See [2]). This is a relative invariant of $(G_{x_0}, \tilde{\rho}_{x_0}, V_{x_0})$ corresponding to $\chi | G_{x_0}$. Let S_{x_0} be the singular set of $(G_{x_0}, \tilde{\rho}_{x_0}, V_{x_0})$.

PROPOSITION 1–6 ([2] Proposition 6–9). *If* $\operatorname{grad}\log f_{x_0} : V_{x_0} - S_{x_0} \rightarrow V_{x_0}^*$ *is generically surjective, then* $\Lambda_0 = \overline{T(\rho(G)x_0)}^{\perp} \subset W$, *i.e.,* Λ_0 *is a good holonomic variety.*

COROLLARY 1–7 ([2] Corollary 6–10). *Assume that the colocalization* $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ *of* (G, ρ, V) *at* x_0 $(\in V)$ *is a regular P.V. If* $\delta\chi | \mathfrak{g}_{x_0}$ *is a non-degenerate element, then the conormal bundle* $\Lambda_0 = \overline{T(\rho(G)x_0)}^{\perp}$ *of the G-orbit* $\rho(G)x_0$ *is a good holonomic variety.*

COROLLARY 1–8 ([2] Corollary 6–11). *Assume that the colocalization* $(G_{x_0}, \rho_{x_0}, V_{x_0}^*)$ *of* (G, ρ, V) *at* $x_0(\in V)$ *is an irreducible regular P.V. Then the conormal bundle* $\Lambda_0 = \overline{T(\rho(G)x_0)}^{\perp}$ *of the orbit* $\rho(G)x_0$ *is a good holonomic variety.*

PROPOSITION 1–9 ([1] Proposition 14 in § 4).

(1) *For* $d = \deg f$ *and* $n = \dim V$, *we have* $d | 2n$ *and* $\chi(g)^{2n/d} = \det_V \rho(g)^2$ *for* $g \in G$.

(2) $\delta\chi(A) = (d/n)\operatorname{tr} d\rho(A)$ *for* $A \in \mathfrak{g}$.

Remark 1–10. Let (G, ρ, V) be an irreducible regular P.V. with finitely many orbits. Let $\mathscr{L} = \{\Lambda, \Lambda', \cdots, \Lambda''\}$ be the set of all conormal bundles in W, of some G-orbits in V. The holonomy diagram is, by definition, given as follows.

If $\dim \Lambda \cap \Lambda' = \dim V - 1$, and $\Lambda \cap \Lambda' \not\subset \Lambda''$ for any other Λ'' in $\mathscr{L}$, then we write the diagram as in Figure 1–1. Moreover, if Λ and Λ' are good

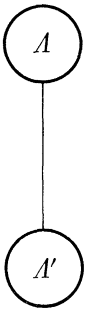


Figure 1-1.

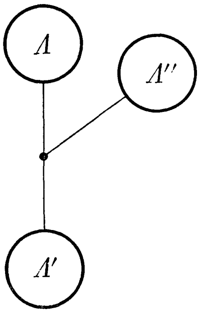


Figure 1-2.

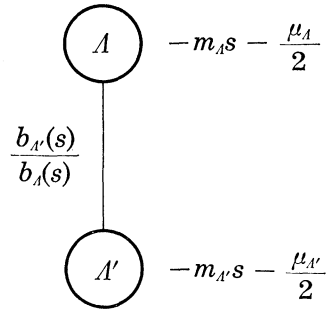
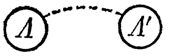
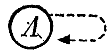


Figure 1-3. $(m_{\Lambda'} > m_\Lambda)$

holonomic varieties, we write the orders $\mathrm{ord}_\Lambda f^s = -m_\Lambda s - \mu_\Lambda/2$ for Λ and Λ', and the ratio of the b-functions as in Figure 1-3. If $\dim \Lambda \cap \Lambda' = \dim V - 1$ and $\Lambda \cap \Lambda' \subset \Lambda''$ for some Λ'', then we write the diagram as in Figure 1-2 (e.g. Figure 11-1). Although some general theory for such cases has been established, it is not published yet and hence in this paper we avoid to argue this case. Actually, only in § 11, such case will appear and to calculate the b-function in § 11, we can use another part of the holonomy diagram. Although usually we do not write the conormal bundles outside W (e.g. Figure 3-2), sometimes we write them (e.g. Figure 4-1, Figure 11-1). Since G is reductive, we have $(G, \rho, V) \cong (G, \rho^*, V^*)$ and we identify them.

We sometimes write as  when T and T' are the dual orbits of each other (See § 11) where Λ and Λ' are the conormal bundles of T and T' respectively. If $T = T'$, we write as  (e.g. Figure 4-1 and Figure 11-1).

§ 2. Regular P.V.'s related with $GL(n)$

We shall use the same notations as in [1].

2-1. $(\tilde{G} \times GL(m), \tilde{\rho} \otimes \Lambda_1, V(m) \otimes V(m))$ where $\tilde{\rho}: \tilde{G} \to GL(V(m))$ is an m-dimensional irreducible representation of a connected semi-simple algebraic group $\tilde{G}$(or $\tilde{G} = \{1\}$ and $m = 1$)

The representation space $V = V(m) \otimes V(m)$ can be identified with the totality of $m \times m$ matrices $M(m, C)$. Then the action $\rho = \tilde{\rho} \otimes \Lambda_1$ is given by $\rho(g)X = \tilde{\rho}(g_1)X{}^tg_2$ for $g = (g_1, g_2) \in G = \tilde{G} \times GL(m)$, $X \in M(m, C)$. The relative invariant $f(X)$ is given by the determinant: $f(X) = \det X$. Since we are concerned with relative invariants, we may assume that $\tilde{G} = SL(m)$ and $\tilde{\rho} = \Lambda_1$. It is well-known that there exist $(m + 1)$-orbits

$$\rho(G)X_\mu = \{X \in M(m, C);\ \mathrm{rank}\, X = \mu\}$$

$$\text{where} \quad X_\mu = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots_{\mu} & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{bmatrix} \quad \text{for } \mu = 0, 1, \cdots, m\,.$$

We identify the dual V^* of V with $V = M(m, C)$ by $\langle X, Y\rangle = \mathrm{tr}\, {}^tXY$ for $X, Y \in M(m, C)$. Then the dual ρ^* of ρ is given by $\rho^*(g)Y = {}^tg_1^{-1}Yg_2^{-1}$ for $g = (g_1, g_2) \in G = SL(m) \times GL(m)$, $Y \in M(m, C)$.

Since $d\rho(\tilde{A})X_\mu = AX_\mu + X_\mu{}^tB = \left(\begin{array}{c|c} A_1 + {}^tB_1 & {}^tB_3 \\ \hline A_3 & 0 \end{array}\right)$ for $\tilde{A} = (A, B) \in \mathfrak{g}$ with $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, the conormal vector space $V^*_{X_\mu} = (d\rho(\mathfrak{g})X_\mu)^\perp$ is given by $V^*_{X_\mu} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & Y_\mu \end{pmatrix}; Y_\mu \in M(m - \mu, C)\right\}$. The isotropy subalgebra $\mathfrak{g}_{X_\mu} = \{\tilde{A} \in \mathfrak{g};\ d\rho(\tilde{A})X_\mu = 0\}$ is given by

$$\begin{aligned} \mathfrak{g}_{X_\mu} = \Bigg\{\left(\begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}, \begin{pmatrix} -{}^tA_1 & B_2 \\ 0 & B_4 \end{pmatrix}\right) \in \mathfrak{g};\ A_1 \in M(\mu, C),\ A_2,\ B_2 \\ \in M(\mu, m - \mu, C),\ A_4, B_4 \in M(m - \mu, C)\Bigg\}\,. \end{aligned} \tag{2.1}$$

This $\mathfrak{g}_{X_\mu}$ acts on $V^*_{X_\mu}$ as $d\rho_{X_\mu}(\tilde{A})Y_\mu = -{}^tA_4Y_\mu - Y_\mu B_4$ for $\tilde{A} \in \mathfrak{g}_{X_\mu}$. Therefore we have $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu}) \cong (SL(m - \mu) \times GL(m - \mu),\ \Lambda_1 \otimes \Lambda_1,\ V(m - \mu) \otimes V(m - \mu))$. Put $Y_\mu = \begin{pmatrix} 0 & 0 \\ 0 & I_{m-\mu} \end{pmatrix}$ where $I_{m-\mu}$ denotes the unit matrix of size $m - \mu (\mu = 0, 1, \cdots, m)$. Then Y_μ is a generic point of the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$, and $Y_{\mu+1}$ is a point of the one-codimensional orbit $(\mu \leq m - 1)$. We denote by Λ_μ the conormal bundle of $\rho(G)X_\mu$ $(0 \leq \mu \leq m)$. Then, we have $\dim \Lambda_\mu \cap \Lambda_{\mu+1} = \dim V - 1$. Note that the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$ $(\mu = 0, 1, \cdots, m)$ has finitely many orbits with the unique one-codimensional orbit, it is clear that we have obtained all one-codimensional intersections among Λ_μ $(\mu = 0, 1, \cdots, m)$. Since $\mathfrak{g}_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(m)$, we have $\dim \mathfrak{g}_0(X_\mu, Y_{\mu+1}) = m^2 - 1$ for $\mu = 0, 1, \cdots, m - 1$, and hence by Proposition 1–5, we have $\Lambda_\mu \subset W$, i.e., Λ_μ is a good holonomic variety $(0 \leqq \mu \leqq m)$. Note that $\Lambda_m = V \times \{0\}$ is always a good holonomic variety. We shall calculate the intersection exponent $(\tilde{\mu} : \tilde{\nu})$ of Λ_μ and $\Lambda_{\mu+1}$ by using Proposition 1–4. For any $\beta \in C$, put $A^\beta_\mu = \left((0), \left(\begin{array}{c|cc} 0 & 0 & \\ \hline 0 & -\beta & \\ & & -I_{m-\mu-1} \end{array}\right)\right) \in \mathfrak{g}$. Then we have $d\rho(A^\beta_\mu)X_\mu = 0$, $d\rho^*(A^\beta_\mu)Y_{\mu+1} = Y_{\mu+1}$, and $\beta = \mathrm{tr}\, A^\beta_\mu$ where tr denotes the trace of A_μ^β on $V_{X_\mu}^*$ modulo $d\rho_{X_\mu}(\mathfrak{g}_{X_\mu})Y_{\mu+1}$ since $V_{X_\mu}^*$ modulo $d\rho_{X_\mu}(\mathfrak{g}_{X_\mu})Y_{\mu+1}$ $\cong \{yE_{\mu+1,\mu+1} \in M(m, \boldsymbol{C}); y \in \boldsymbol{C}\}$ where E_{ij} denotes the matrix unit. Therefore we have $\tilde{\mu} = 1$ and $\tilde{\nu} = 0$, i.e., they intersect regularly. Now by Proposition 1–3, we shall calculate the order $\mathrm{ord}_{\Lambda_\mu} f^s$ of $\mathfrak{m} = \mathscr{E} f^s$ at Λ_μ where $f(X) = \det X$.

Put $A_\mu = \left((0), \begin{pmatrix} 0 & 0 \\ 0 & -I_{m-\mu} \end{pmatrix}\right) \in \mathfrak{g}$. Then $d\rho(A_\mu)X_\mu = 0$ and $d\rho^*(A_\mu)Y_\mu = Y_\mu$ $(0 \leq \mu \leq m)$. The character $\delta\chi$ corresponding to $f(X) = \det X$ is given by $\delta\chi(\tilde{A}) = \mathrm{tr}\, B$ for $\tilde{A} = (A, B) \in \mathfrak{g} = \mathfrak{sl}(m) \oplus \mathfrak{gl}(m)$. Since $\dim V_{X_\mu}^* = (m - \mu)^2$ and $\mathrm{tr}_{V_{X_\mu}^*}\, d\rho_{X_\mu}(A_\mu) = (m - \mu)^2$, we have $\mathrm{ord}_{\Lambda_\mu} f^s = s\delta\chi(A_\mu) - \mathrm{tr}_{V_{X_\mu}^*}\, d\rho_{X_\mu}(A_\mu) + (1/2)\dim V_{X_\mu}^* = -(m - \mu)s - ((m - \mu)^2/2)$. Thus we obtain the holonomy diagram (Figure 2–1).

By Corollary 1–2, we have $b_{\Lambda_\mu}(s)/b_{\Lambda_{\mu+1}}(s) = s + (m - \mu)\ (0 \leq \mu \leq m - 1)$. Hence

$$\begin{aligned} b(s) = b_{\Lambda_0}(s) &= b_{\Lambda_m}(s) \cdot \prod_{\mu=0}^{m-1} b_{\Lambda_\mu}(s)/b_{\Lambda_{\mu+1}}(s) \\ &= \prod_{\mu=0}^{m-1} (s + m - \mu) = (s + 1)(s + 2)\cdots(s + m). \end{aligned}$$

Note that $b_{\Lambda_m}(s) = 1$.

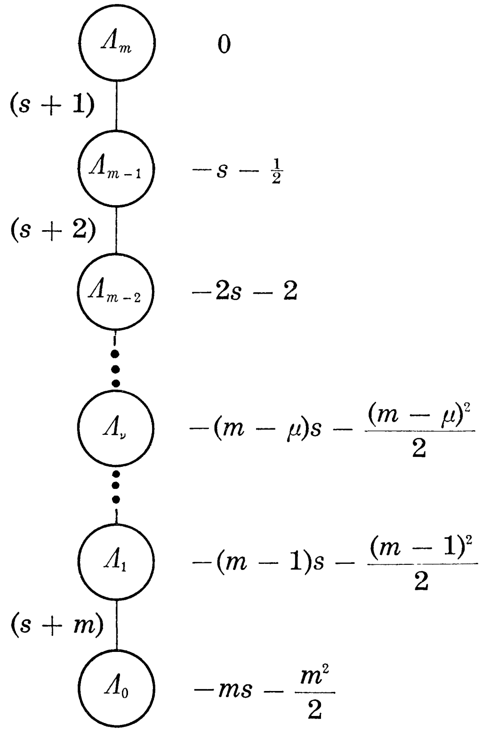


Figure 2–1. Holonomy diagram of $(SL(m) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(m))$

Remark 2–1. The b-function of **2–1** is classically known by using Capelli's identity (See H. Weyl: Classical Groups).

2–2. $(GL(n), 2\Lambda_1, V(\frac{1}{2}n(n+1)))$ $(n \geq 2)$

The representation space can be identified with the totality of $n \times n$ symmetric matrices $V = \{X \in M(n, C); {}^tX = X\}$. Then the action $\rho = 2\Lambda_1$ of $GL(n)$ on V is given by $\rho(g)X = gX{}^tg$ for $g \in GL(n)$, $X \in V$. It is well-known that there exist $(n+1)$-orbits $\rho(G)X_\nu = \{X \in V; \operatorname{rank} X = \nu\}$ where

$$X_\nu = \begin{bmatrix} \overset{\nu}{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}} & 0 \\ 0 & \begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix} \end{bmatrix} \quad \text{for } \nu = 0, 1, \cdots, n\,.$$

The relative invariant is given by the determinant $f(X) = \det X$. If we identify the dual V^* of V with V by $\langle X, Y\rangle = \operatorname{tr} XY$, we have $\rho^*(g)Y = {}^tg^{-1}Yg^{-1}$ for $g \in GL(n)$. We have

$$\begin{aligned} d\rho(A)X_\nu &= AX_\nu + X_\nu{}^tA = \left(\begin{array}{c|c} A_1 + {}^tA_1 & {}^tA_3 \\ \hline A_3 & 0 \end{array}\right) \\ \text{for } A &= \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathfrak{gl}(n)\,. \end{aligned} \tag{2.2}$$

Therefore we have

$$\begin{aligned} \mathfrak{g}_{X_\nu} = \Big\{ &\begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}; {}^tA_1 = -A_1, A_1 \in M(\nu), A_2 \in M(\nu, n-\nu), \\ &A_4 \in M(n-\nu) \Big\}\,. \end{aligned} \tag{2.3}$$

Since $\dim \mathfrak{g}_{X_\nu} = n(n-\nu) + (\nu(\nu-1)/2)$, we have $\dim \rho(G)X_\nu = n\nu - (\nu(\nu-1)/2)$. The conormal vector space $V^*_{X_\nu}$ is given by

$$V^*_{X_\nu} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & W_\nu \end{pmatrix}; {}^tW_\nu = W_\nu \in M(n-\nu) \right\}\,. \tag{2.4}$$

Since $d\rho^*\left(\begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & W_\nu \end{pmatrix} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -{}^tA_4W - WA_4 \end{array}\right)$, the colocalization $(G_{X_\nu}, \rho_{X_\nu}, V^*_{X_\nu})$ at X_ν is isomorphic to $(GL(n-\nu), 2\Lambda_1, V(\frac{1}{2}(n-\nu)(n-\nu+1)))$. Put $Y_\nu = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-\nu} \end{pmatrix}$ $(\nu = 0, 1, \cdots, n)$. Then Y_ν is a generic point of the colocalization at X_ν and $Y_{\nu+1}$ is a point of the unique one-codimensional orbit. Thus we have $\dim \Lambda_\nu \cap \Lambda_{\nu+1} = \dim V - 1$, where Λ_ν denotes the conormal bundle of $\rho(G)X_\nu$. Since $\dim d\rho^*(\mathfrak{g}_{X_\nu} \cap \mathfrak{g}_0)Y_{\nu+1} = \dim d\rho^*(\mathfrak{g}_{X_\nu})Y_{\nu+1}$, we have $\dim \mathfrak{g}_0(X_\nu, Y_{\nu+1}) = (n(n+1)/2) - 1$, and hence Λ_ν is a good holonomic variety by Proposition 1-5 ($\nu = 0, 1, \cdots, n$).

Put $A^\beta_\nu = \begin{array}{c} \left(\begin{array}{c|c|c} 0 & & \\ \hline & -\beta & \\ \hline & & -\frac{1}{2}I_{n-\nu-1} \end{array}\right) \\ \underbrace{\quad}_{\nu}\;\underbrace{\quad}_{1}\;\underbrace{\qquad\quad}_{n-\nu-1} \end{array} \in \mathfrak{gl}(n)$. Then we have $d\rho(A^\beta_\nu)X_\nu = 0$, $d\rho^*(A^\beta_\nu)Y_{\nu+1} = Y_{\nu+1}$ and $2\beta = \operatorname{tr} A^\beta_\nu$ where tr denotes the trace of A^β_ν on $V^*_{X_\nu}$ modulo $d\rho_{X_\nu}(\mathfrak{g}_{X_\nu})Y_{\nu+1}$. Hence, Λ_ν and $\Lambda_{\nu+1}$ intersect regularly, i.e., the intersection exponent of Λ_ν and $\Lambda_{\nu+1}$ equals $(1:0)$. We shall calculate the order $\operatorname{ord}_{\Lambda_\nu} f^s$ by Proposition 1-3. Put $A_\nu = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2}I_{n-\nu} \end{pmatrix}$ $(0 \leq \nu \leq n)$. Then $d\rho(A_\nu)X_\nu = 0$ and $d\rho^*(A_\nu)Y_\nu = Y_\nu$. Since $\delta\chi(A_\nu) = 2\operatorname{tr} A_\nu = -(n-\nu)$, and $\operatorname{tr}_{V^*_{X_\nu}} d\rho_{X_\nu}(A_\nu) = \dim V^*_{X_\nu} = \frac{1}{2}(n-\nu)(n-\nu+1)$, we have $\operatorname{ord}_{\Lambda_\nu} f^s = s\delta\chi(A_\nu) - \operatorname{tr}_{V^*_{X_\nu}} d\rho_{X_\nu}(A_\nu) + \frac{1}{2}\dim V^*_{X_\nu} = -(n-\nu)s - \frac{1}{4}(n-\nu)(n-\nu+1)$.

Thus we obtain the holonomy diagram (Figure 2-2). By Corollary

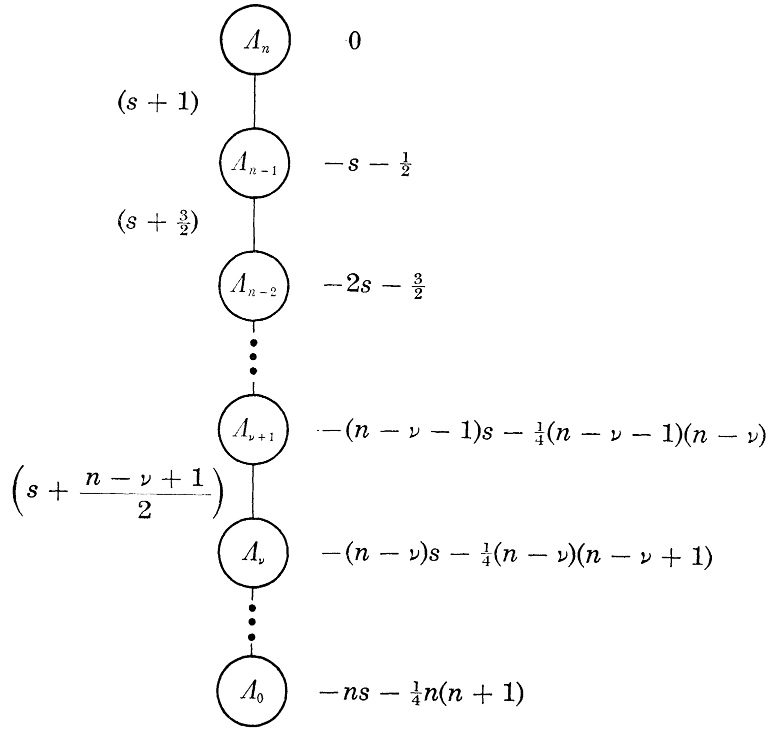


Figure 2-2. Holonomy diagram of $(GL(n),\ 2\Lambda_1,\ V(\frac{1}{2}n(n+1)))$ $(n \geq 2)$

1–2, we have $b_{A_\nu}(s)/b_{A_{\nu+1}}(s) = s + ((n - \nu + 1)/2)$ $(0 \leq \nu \leq n - 1)$, and hence $b(s) = b_{A_0}(s) = \prod_{\nu=1}^{n}(s + (\nu + 1)/2)$.

Remark. The b-function of **2–2** is also already known. It can be obtained by using Capelli's identity or by a direct calculation of the Fourier transform of $f(x)^s$.

2–3. $(GL(2m), \Lambda_2, V(m(2m - 1)))$ $(m \geq 3)$

The representation space can be identified with $V_m = \{X \in M(2m, C) \mid {}^tX = -X\}$. Then the action $\rho = \Lambda_2$ is given by $\rho(g)X = gX{}^tg$ for $g \in GL(2m)$, $X \in V_m$. The relative invariant $f(X)$ is the Pfaffian of X. It is well-known that there exists $(m + 1)$-orbits $\rho(G)X_\mu = \{X \in V_m \,;\, \text{rank}\, X = 2\mu\}$ where
$$X_\mu = \begin{bmatrix} 0 & 0 & I_\mu & 0 \\ 0 & 0 & 0 & 0 \\ -I_\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (0 \leq \mu \leq m).$$
By simple calculation, we have

$$d\rho(\tilde{A})X_\mu = \left[\begin{array}{c|c|c|c} \overbrace{\qquad}^{\mu} & \overbrace{\quad}^{m-\mu} & \overbrace{\qquad}^{\mu} & \overbrace{\quad}^{m-\mu} \\ {}^tB_1 - B_1 & {}^tB_3 & A_1 + {}^tD_1 & {}^tD_3 \\ \hline -B_3 & & A_3 & \\ \hline -D_1 - {}^tA_1 & -{}^tA_3 & C_1 - {}^tC_1 & -{}^tC_3 \\ \hline -D_3 & & C_3 & \end{array}\right] \tag{2.5}$$

$$\text{where } \tilde{A} = \left[\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ \hline C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{array}\right]$$

and hence,

$$\mathfrak{g}_{X_\mu} = \left\{ \left[\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ 0 & A_4 & 0 & B_4 \\ \hline C_1 & C_2 & -{}^tA_1 & D_2 \\ 0 & C_4 & 0 & D_4 \end{array}\right] ; {}^tB_1 = B_1, {}^tC_1 = C_1 \right\} \tag{2.6}$$

$$V^*_{X_\mu} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X & 0 & Y \\ 0 & 0 & 0 & 0 \\ 0 & -{}^tY & 0 & Z \end{bmatrix} ; \begin{matrix} {}^tX = -X \\ {}^tZ = -Z \end{matrix} \right\} \cong \left\{ \left(\begin{array}{c|c} X & Y \\ \hline {}^tY & Z \end{array} \right) ; \begin{matrix} {}^tX = -X \\ {}^tZ = -Z \end{matrix} \right\} = V_{m-\mu} .$$

Since $\mathfrak{g}_{X_\mu}$ acts on $V^*_{X,\mu}$ as $\tilde{X} \mapsto -{}^t\tilde{A}_4\tilde{X} - \tilde{X}\tilde{A}_4$ where $\tilde{A}_4 = \left(\begin{array}{c|c} A_4 & B_4 \\ \hline C_4 & D_4 \end{array} \right)$ and

$\tilde{X} = \left(\begin{array}{c|c} X & Y \\ \hline -{}^tY & Z \end{array}\right)$ with ${}^tX = -X$, ${}^tZ = -Z$, the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$ at X_μ is isomorphic to $(GL(2m - 2\mu), \Lambda_2, V((m - \mu)(2m - 2\mu - 1)))$. Here we identified the dual V^*_m of V_m with V_m by $\langle X, Y\rangle = \operatorname{tr} XY$.

$$\text{Put } Y_\mu = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{m-\mu} \\ 0 & 0 & 0 & 0 \\ 0 & -I_{m-\mu} & 0 & 0 \end{bmatrix} \quad (0 \leq \mu \leq m)\,.$$

Then Y_μ is a generic point of the colocalization $(G_{X_\mu}, \rho_{X_\mu}, V^*_{X_\mu})$ at X_μ, and $Y_{\mu+1}$ is a point of the one-dimensional orbit and hence we have $\dim \Lambda_\mu \cap \Lambda_{\mu+1} = \dim V - 1$ $(0 \leq \mu \leq m - 1)$ where Λ_μ denotes the conormal bundle of $\rho(G)X_\mu$.

By (2.6), we have $\mathfrak{g}_{X_\mu} \not\subset \mathfrak{g}_0$ for $\mu \neq m$, and hence $\dim d\rho(\mathfrak{g})X_\mu = \dim d\rho(\mathfrak{g}_0)X_\mu$ for $\mu \neq m$. Applying this fact to the colocalization at X_μ, we have $\dim \mathfrak{g}_0(X_\mu, Y_{\mu+1}) = \dim \mathfrak{g}(X_\mu, Y_{\mu+1}) = m(2m - 1) - 1$. This implies that Λ_μ is a good holonomic variety by Proposition 1–5.

$$\text{Put } A^\beta_\mu = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \begin{matrix} -\beta & \\ & -\frac{1}{2}I_{m-\mu-1} \end{matrix} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \begin{matrix} -\beta & \\ & -\frac{1}{2}I_{m-\mu-1} \end{matrix} \end{array}\right) \qquad \text{for } \beta \in \boldsymbol{C}\,.$$

Then we have $d\rho(A^\beta_\mu)X_\mu = 0$, $d\rho^*(A^\beta_\mu)Y_{\mu+1} = Y_{\mu+1}$ and $\operatorname{tr} A^\beta_\mu = 2\beta$ where tr denotes the trace of A^β_μ on $V^*_{X_\mu}$ modulo $d\rho(\mathfrak{g}_{X_\mu})Y_{\mu+1}$, and hence by Proposition 1–4, Λ_μ and $\Lambda_{\mu+1}$ intersect regularly, i.e., the intersection exponent of Λ_μ and $\Lambda_{\mu+1}$ equals $(1:0)$. We shall calculate the order $\operatorname{ord}_{\Lambda_\mu} f^s$.

$$\text{Put } A_0 = \left(\begin{array}{c|c|c|c} & & & \\ \hline & -\frac{1}{2}I_{m-\mu} & & \\ \hline & & & \\ \hline & & & -\frac{1}{2}I_{m-\mu} \end{array}\right).$$

Then we have $d\rho(A_0)X_\mu = 0$ and $d\rho^*(A_0)Y_\mu = Y_\mu$. Since $\delta\chi(A_0) = -(m - \mu)$, $\operatorname{tr}_{V^*_{X_\mu}} A_0 = \dim V^*_{X_\mu} = (m - \mu)(2m - 2\mu - 1)$, we have $\operatorname{ord}_{\Lambda_\mu} f^s = s\delta\chi(A_0) - \operatorname{tr}_{V^*_{X_\mu}} A_0 + \frac{1}{2}\dim V^*_{X_\mu} = -(m - \mu)s - \frac{1}{2}(m - \mu)(2m - 2\mu - 1)$.

By Corollary 1–2, we have $b_{\Lambda_\mu}(s)/b_{\Lambda_{\mu+1}}(s) = s + 2(m - \mu) - 1$ $(0 \leq \mu \leq m - 1)$. Hence we obtain the holonomy diagram (Figure 2–3) and b-function $b(s) = \prod_{\mu=0}^{m-1}(s + 2(m - \mu) - 1) = \prod_{k=1}^{m}(s + 2k - 1)$.

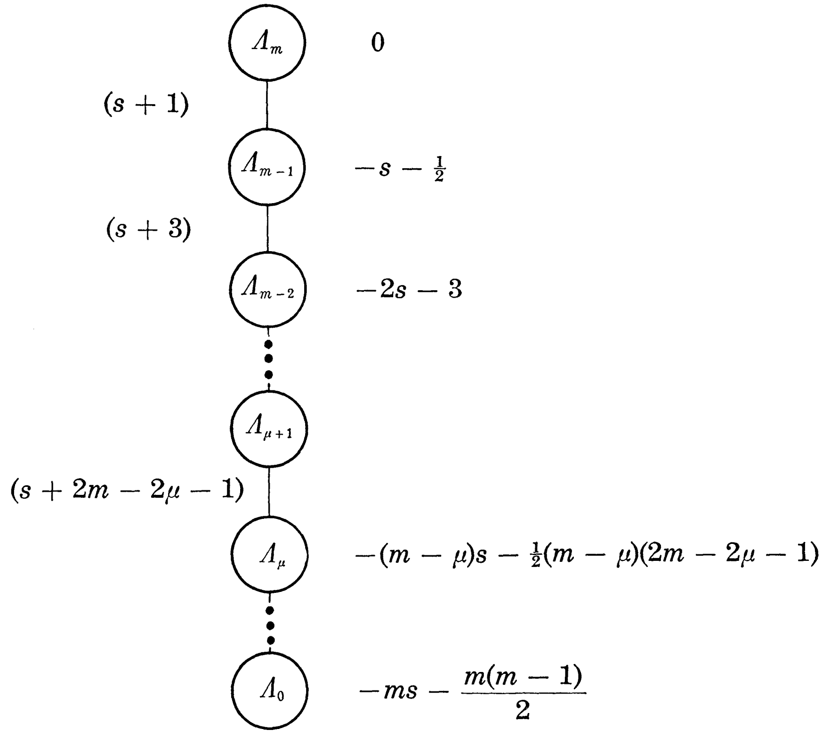
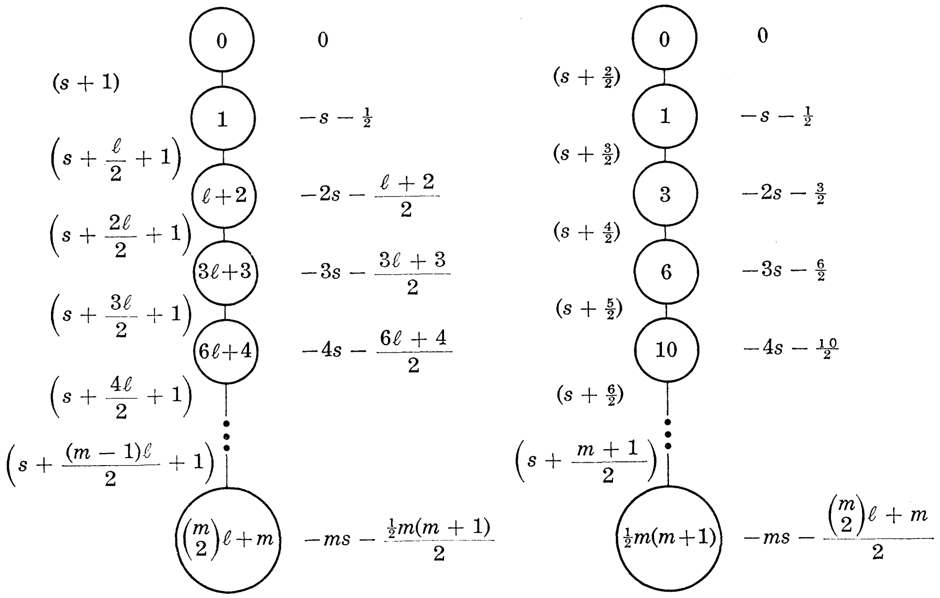


Figure 2-3. Holonomy diagram of $(GL(2m), \Lambda_2, V(m(2m-1)))$ $(m \geq 3)$.

Remark. These three P.V.'s have many common properties: (1) $\left(GL(m), 2\Lambda_1, V\left(\binom{m}{2}\ell + m\right)\right)$ with $\ell = 1$ (2) $\left(SL(m) \times GL(m), \Lambda_1 \otimes \Lambda_1, V\left(\binom{m}{2}\ell + m\right)\right)$ with $\ell = 2$ (3) $\left(GL(2m), \Lambda_2, V\left(\binom{m}{2}\ell + m\right)\right)$ with $\ell = 4$. They have $(m+1)$-orbits and their relative invariants are of degree m of $\binom{m}{2}\ell + m$ variables. We denote $\textcircled{\Lambda}$ by $\textcircled{\mu}$ if Λ is the conormal bundle of a μ-codimensional orbit. Then their holonomy diagrams are as in Figure 2-4.

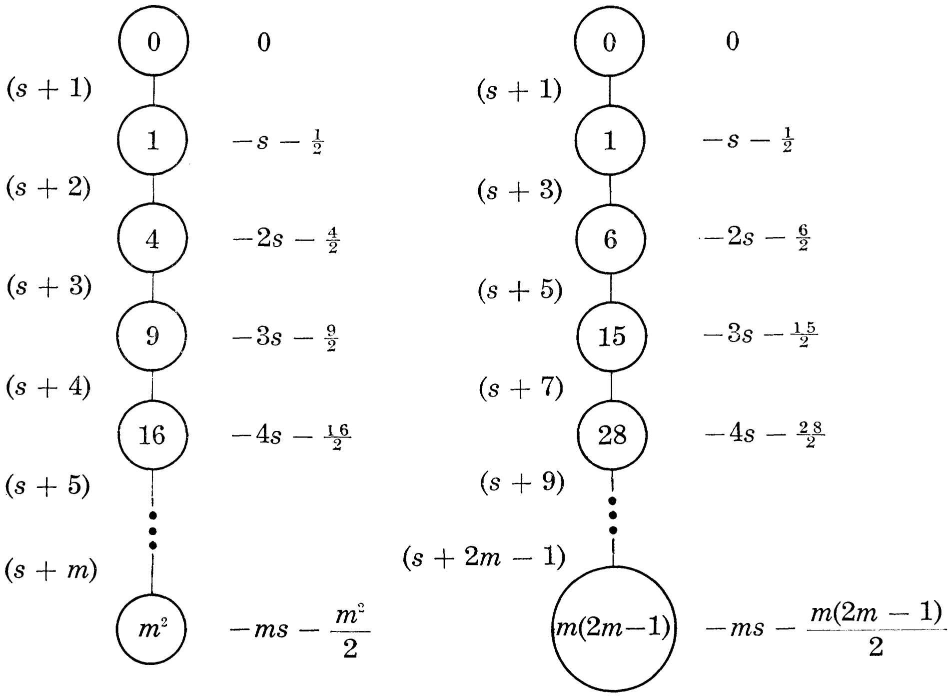
§ 3. $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ with $n \geq 2m$

The representation space V can be identified with the totality of $2n \times 2m$ matrices. Then the action $\rho = \Lambda_1 \otimes \Lambda_1$ is given by $\rho(g)X = g_1 X {}^t g_2$ for $g = (g_1, g_2) \in G = Sp(n) \times GL(2m)$, $X \in V$. Let X be an element of V such that $\operatorname{rank} X = \nu$ and $\operatorname{rank} {}^t XJX = 2\mu (2m \geqq \nu \geqq 2\mu \geqq 0)$ where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Then by the action of $GL(2m)$, we may assume that $X = (X', 0)$ with $X' \in M(2n, \nu)$ satisfying ${}^t X' J X' = \begin{pmatrix} 0 & 0 & I_\mu \\ 0 & 0 & 0 \\ -I_\mu & 0 & 0 \end{pmatrix}$. Put $X_{\nu, 2\mu}$ as follows.



$(m+1)$-orbits $\deg f=m$ $\dim V=\binom{m}{2}\ell+m$

(1) $\ell=1$



(2) $\ell=2$

(3) $\ell=4$

Figure 2-4

(3.1)
$$X_{\nu,2\mu} = \begin{array}{r} \\ \\ \\ \\ \nu-2\mu\,\{ \\ \\ \end{array} \left(\begin{array}{c|c|c|c} I_\mu & & & \\ \hline & I_{\nu-2\mu} & & \\ \hline & & & \\ \hline & & I_\mu & \\ \hline & & & \\ \hline & & & \\ & & & \underbrace{\qquad}_{2m-\nu} \end{array}\right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} n \end{array}$$

Then $X_{\nu,2\mu}$ satisfies the same condition as X and hence there exists an element g_1 of $Sp(n)$ satisfying $g_1X = X_{\nu,2\mu}$. This implies that $S_{\nu,2\mu} = \{X \in V;$ rank $X = \nu$, rank ${}^tXJX = 2\mu\}$ $(2m \geqq \nu \geqq 2\mu \geqq 0)$ consists of a single G-orbit, and we complete the orbital decomposition of this space. Put $A \in \mathfrak{sp}(n)$ and $D \in \mathfrak{gl}(2m)$ as follows:

(3.2)
$$A = \left(\begin{array}{ccc|ccc} A_1 & A_{12} & A_{13} & B_1 & B_{12} & B_{13} \\ A_{21} & A_2 & A_{23} & {}^tB_{12} & B_2 & B_{23} \\ A_{31} & A_{32} & A_3 & {}^tB_{13} & {}^tB_{23} & B_3 \\ \hline C_1 & C_{12} & C_{13} & -{}^tA_1 & -{}^tA_{21} & -{}^tA_{31} \\ {}^tC_{12} & C_2 & C_{23} & -{}^tA_{12} & -{}^tA_2 & -{}^tA_{32} \\ \underbrace{{}^tC_{13}}_{\mu} & \underbrace{{}^tC_{23}}_{\nu-2\mu} & \underbrace{C_3}_{n-\nu+\mu} & \underbrace{-{}^tA_{13}}_{\mu} & \underbrace{-{}^tA_{23}}_{\nu-2\mu} & \underbrace{-{}^tA_3}_{n-\nu+\mu} \end{array}\right),$$

$$D = \left(\begin{array}{cccc} D_1 & D_{12} & D_{13} & D_{14} \\ D_{21} & D_2 & D_{23} & D_{24} \\ D_{31} & D_{32} & D_3 & D_{34} \\ \underbrace{D_{41}}_{\mu} & \underbrace{D_{42}}_{\nu-2\mu} & \underbrace{D_{43}}_{\mu} & \underbrace{D_4}_{2m-\nu} \end{array}\right)$$

where ${}^tB_i = B_i$, ${}^tC_i = C_i$ $(i = 1, 2, 3)$.

Then, for $\tilde{A} = A \oplus D \in \mathfrak{g}$, we have

(3.3)
$$d\rho(\tilde{A})X_{\nu,2\mu} = AX_{\nu,2\mu} + X_{\nu,2\mu}{}^tD$$
$$= \left(\begin{array}{cc|cc} A_1 + {}^tD_1 & A_{12} + {}^tD_{21} & B_1 + {}^tD_{31} & {}^tD_{41} \\ A_{21} + {}^tD_{12} & A_2 + {}^tD_2 & {}^tB_{12} + {}^tD_{32} & {}^tD_{42} \\ A_{31} & A_{32} & {}^tB_{13} & 0 \\ \hline C_1 + {}^tD_{13} & C_{12} + {}^tD_{23} & -{}^tA_1 + {}^tD_3 & {}^tD_{43} \\ {}^tC_{12} & C_2 & -{}^tA_{12} & 0 \\ {}^tC_{13} & {}^tC_{23} & -{}^tA_{13} & 0 \end{array}\right)$$

and hence the isotropy subalgebra $\mathfrak{g}_{X_{\nu,2\mu}}$ is given as follows:

$$\mathfrak{g}_{X_{\nu,2\mu}} = \left\{ \tilde{A} = \left[\begin{array}{ccc|ccc} A_1 & 0 & 0 & B_1 & B_{12} & 0 \\ A_{21} & A_2 & A_{23} & {}^tB_{12} & B_2 & B_{23} \\ 0 & 0 & A_3 & 0 & {}^tB_{23} & B_3 \\ \hline C_1 & 0 & 0 & -{}^tA_1 & -{}^tA_{21} & 0 \\ 0 & 0 & 0 & 0 & -{}^tA_2 & 0 \\ 0 & 0 & C_3 & 0 & -{}^tA_{23} & -{}^tA_3 \end{array}\right] \right.$$

$$\left. \oplus \left[\begin{array}{cc|cc} -{}^tA_1 & -{}^tA_{21} & -C_1 & D_{14} \\ 0 & -{}^tA_2 & 0 & D_{24} \\ \hline -B_1 & -B_{12} & A_1 & D_{34} \\ 0 & 0 & 0 & D_4 \end{array}\right] \right\}$$

$$\cong \left\{ \left[\begin{array}{c|cc|cc|c} A_2 & A_{21} & {}^tB_{12} & A_{23} & B_{23} & B_2 \\ \hline 0 & A_1 & B_1 & & 0 & B_{12} \\ & C_1 & -{}^tA_1 & & & -{}^tA_{21} \\ \hline 0 & & 0 & A_3 & B_3 & {}^tB_{23} \\ & & & C_3 & -{}^tA_3 & -{}^tA_{23} \\ \hline 0 & & 0 & & 0 & -{}^tA_2 \end{array}\right] \right. \tag{3.4}$$

$$\left. \oplus \left[\begin{array}{cc|c|c} A_1 & B_1 & -B_{12} & D_{34} \\ C_1 & -{}^tA_1 & {}^tA_{21} & -D_{14} \\ \hline & & -{}^tA_2 & D_{24} \\ \cline{3-4} & 0 & 0 & D_4 \end{array}\right] \right\}$$

$$\cong (\mathfrak{gl}(\nu - 2\mu) \oplus \mathfrak{gl}(2m - \nu) \oplus \mathfrak{sp}(\mu) \oplus \mathfrak{sp}(n - \nu + \mu)) \oplus \mathfrak{u}(k)$$

where $\mathfrak{u}(k)$ denotes the Lie algebra of a k-dimensional unipotent group with $k = \frac{1}{2}(4n + 1)(\nu - 2\mu) - \frac{3}{2}(\nu - 2\mu)^2 + \nu(2m - \nu)$. In this paper, we make a convention that the first (resp. second) $\oplus$ implies the direct sum as Lie algebras (resp. vector spaces) for $(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \oplus \mathfrak{g}_3$.

We identify the dual space V^* of V with V by $\langle X, Y \rangle = \mathrm{tr}\, X {}^tY$ for $X, Y \in V = M(2n, 2m)$, and hence we have $\rho^*(g)Y = {}^tg_1^{-1} Y g_2^{-1}$ for $g = (g_1, g_2) \in G$, $Y \in V$ and $d\rho^*(\tilde{A})Y = -{}^tAY - YD$ for $\tilde{A} = (A, D) \in \mathfrak{g}$. From (3.3), the conormal vector space $V^*_{X_{\nu,2\mu}}$ is given by

$$V^*_{X_{\nu,2\mu}} = \left\{ \tilde{Y} = \begin{array}{c} \left[\begin{array}{c|c|c|c} & & & 0 \\ 0 & 0 & 0 & 0 \\ & & & Y \\ \hline & 0 & & 0 \\ 0 & X & 0 & Z \\ & 0 & & W \end{array}\right] \begin{array}{l} \\ \\ \\ \}\mu \\ \}\nu-2\mu \\ \}n-\nu+\mu \end{array} \\ \begin{array}{cccc} \underbrace{\quad}_{\mu} & \underbrace{\quad}_{\nu-2\mu} & \underbrace{\quad}_{\mu} & \underbrace{\quad}_{2m-\nu} \end{array} \end{array} ;\ {}^tX = -X \right\} \tag{3.5}$$

$$\cong \left\{ \tilde{Y}' = \underbrace{\left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline 0 & Y_3 \end{array}\right)}_{\nu-2\mu\ 2m-\nu} \begin{array}{l} \}\nu-2\mu \\ \}2(n-\nu+\mu) \end{array} ;\ {}^tY_1 = -\ Y_1 \right\} .$$

Here the isomorphism is obtained by putting $Y_1 = X$, $Y_2 = Z$ and $Y_3 = \begin{bmatrix} Y \\ W \end{bmatrix}$. Then the action $d\rho_{X_{\nu,2\mu}}$ of $\mathfrak{g}_{X_{\nu,2\mu}}$ on $V^*_{X_{\nu,2\mu}}$ is given as follows.

$$d\rho_{X_{\nu,2\mu}}(\tilde{A})\tilde{Y}' = \left(\begin{array}{c|cc} A_2 & -B_{23} & A_{23} \\ \hline 0 & -{}^tA_3 & -C_3 \\ & -B_3 & A_3 \end{array}\right)\left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline 0 & Y_3 \end{array}\right) + \left(\begin{array}{c|c} Y_1 & Y_2 \\ \hline 0 & Y_3 \end{array}\right)\left(\begin{array}{c|c} {}^tA_2 & -D_{24} \\ \hline 0 & -D_4 \end{array}\right) .$$

Thus the action on Y_1-space is isomorphic to $(GL(\nu-2\mu), \Lambda_2, V(\frac{1}{2}(\nu-2\mu) \times (\nu-2\mu-1)))$ and the action on Y_3-space is isomorphic to $(Sp(n-\nu+\mu) \times GL(2m-\nu), \Lambda_1 \otimes \Lambda_1, V(2n-2\nu+2\mu) \otimes V(2m-\nu))$. First we shall consider the case when ν is even, i.e., $\nu = 2\nu'$. Let $\tilde{Y}_0$ be an element of $V^*_{X_{\nu,2\mu}}$ with $X = \begin{pmatrix} 0 & I_{\nu'-\mu} \\ -I_{\nu'-\mu} & 0 \end{pmatrix}$, $Y = \begin{pmatrix} I_{m-\nu'} & 0 \\ 0 & \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu} \\ & 0 \end{pmatrix}$ and $Z = 0$ in (3.5). Then $\tilde{Y}_0$ is a generic point and $\tilde{Y}_0 \in S^*_{2m-2\mu,2m-2\nu'}$, i.e., $\Lambda_{2\nu',2\mu} = \Lambda^*_{2m-2\mu,2m-2\nu'}$ where $\Lambda_{\nu,2\mu}$ (resp. $\Lambda^*_{\nu,2\mu}$) denotes the conormal bundle of $S_{\nu,2\mu}$ (resp. $S^*_{\nu,2\mu}$). We shall calculate the order $\mathrm{ord}_{\Lambda_{2\nu',2\mu}} f^s$ where $f(X) = Pf\,{}^tXJX$. Let $\tilde{A}_0$ be the element of $\mathfrak{g}_{X_{\nu,2\mu}}$ with $A_2 = \frac{1}{2}I_{2(\nu'-\mu)}$, $D_4 = -\ I_{2(m-\nu')}$, all remaining parts zero in (3.4). Then we have $d\rho(\tilde{A}_0)X_{\nu,2\mu} = 0$ and $d\rho^*(\tilde{A}_0)\tilde{Y}_0 = \tilde{Y}_0$ Since $\delta\chi(\tilde{A}_0) = -\ (2m-\nu'-\mu)$, $\mathrm{tr}_{V^*_{X_{\nu,2\mu}}}\tilde{A}_0 = (\nu'-\mu)(2\nu'-2\mu-1) + 4(m-\nu')(n-2\nu'+\mu) + 6(m-\nu')(\nu'-\mu)$ and $\dim V^*_{X_{\nu,2\mu}} = (\nu'-\mu)(2\nu'-2\mu-1) + 4(m-\nu')(n-2\nu'+\mu) + 4(m-\nu')(\nu'-\mu)$, we have

$$\text{(3.6)} \qquad \mathrm{ord}_{\Lambda_{2\nu',2\mu}} f^s = -\ (2m-\nu'-\mu)s - \tfrac{1}{2}(\nu'-\mu)(2\nu'-2\mu-1) \\ -\ 2(m-\nu')(n-2\nu'+\mu) - 4(m-\nu')(\nu'-\mu) .$$

Let $\tilde{Y}_1$ be the element of $V^*_{X_{\nu,2\mu}}$ with $X = \begin{pmatrix} 0 & I_{\nu'-\mu} \\ -I_{\nu'-\mu} & 0 \end{pmatrix}$, $Y = \begin{pmatrix} I_{m-\nu'+1} & 0 \\ 0 & \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu'-1} \\ & 0 \end{pmatrix}$ and $Z = 0$. Since $\tilde{Y}_1$ is a point of a one-codimensional orbit and $\tilde{Y}_1 \in S^*_{2m-2\mu,2(m-\nu'-1)}$, we have $\Lambda_{2\nu',2\mu} \cap \Lambda_{2(\nu'+1),2\mu} = \dim V - 1$. They intersect regularly. By Corollary 1–2, we have

$$\text{(3.7)} \qquad b_{\Lambda_{2\nu',2\mu}}(s)/b_{\Lambda_{2(\nu'+1),2\mu}}(s) = s + 2n - 2\nu' \qquad (m-1 \geqq \nu' \geqq 0) .$$

Now let $\tilde{Y}_2$ be the element of $V^*_{X_{\nu,2\mu}}$ with $X = \left(\begin{array}{cc|c} 0 & I_{\nu'-\mu-1} & 0 \\ -I_{\nu'-\mu-1} & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$, $Y = \begin{pmatrix} I_{m-\nu'} & 0 \\ 0 & \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu'} \\ & 0 \end{pmatrix}$ and $Z = 0$. Since $\tilde{Y}_2$ is a point of the other one-codimensional orbit and $\tilde{Y}_2 \in S^*_{2(m-\mu-1),2(m-\nu')}$, we have $\dim \Lambda_{2\nu',2\mu} \cap \Lambda_{2\nu',2(\mu+1)} = \dim V - 1$. They intersect regularly. By Corollary 1–2, we have

$$(3.8) \qquad b_{\Lambda_{2\nu',2\mu}}(s)/b_{\Lambda_{2\nu',2(\mu+1)}} = s + 2m - 2\mu - 1 \qquad (m - 1 \geq \mu \geq 0).$$

Now we shall show that $\Lambda_{\nu,2m}$ is not a good holonomic variety when ν is odd, i.e., $\nu = 2\nu' + 1$. Let $\tilde{Y}_0$ be the element of $V^*_{X_{\nu,2\mu}}$ with $X = \left(\begin{array}{cc|c} 0 & I_{\nu'-\mu} & 0 \\ -I_{\nu'-\mu} & & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right)$, $Y = \begin{pmatrix} I_{m-\nu'-1} & 0 \\ & 0 \end{pmatrix}$, $W = \begin{pmatrix} 0 & I_{m-\nu'} \\ & 0 \end{pmatrix}$ and $Z = 0$ in (3.5). Then it is a generic point of the conormal vector space. Let $\tilde{A}_0$ be the element of $\mathfrak{g}_{X_{\nu,2\mu}}$ with $A_2 = \begin{pmatrix} \frac{1}{2}I_{2(\nu'-\mu)} & 0 \\ 0 & \beta \end{pmatrix}$, $D_4 = -I_{2m-\nu}$, all remaining parts zero. Then we have $d\rho(\tilde{A}_0)X_{\nu,2\mu} = 0$ and $d\rho^*(\tilde{A}_0)\tilde{Y}_0 = \tilde{Y}_0$. Therefore, if $\Lambda_{\nu,2\mu}$ is a good holonomic variety, $m_{\Lambda_{\nu,2\mu}} = -\delta\chi(\tilde{A}_0) = 2m - \nu' - \mu - 1 + \beta$ is a non-negative integer which is a contradiction. Thus we obtain the following proposition.

PROPOSITION 3–1. *The irreducible regular P.V.* $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ $(n \geqq 2m)$ *has finitely many orbits* $S_{\nu,2\mu} = \{X \in M(2n, 2m);$ *rank* $X = \nu$, $\mathrm{rank}\, {}^tXJX = 2\mu\}$ $(2m \geqq \nu \geqq 2\mu \geqq 0)$. *When* ν *is odd, the conormal bundle* $\Lambda_{\nu,2\mu}$ *of* $S_{\nu,2\mu}$ *is outside* W, *i.e.,* $\Lambda_{\nu,2\mu}$ *is not a good holonomic variety. When* ν *is even* $(\nu = 2\nu')$, $\Lambda_{\nu,2\mu}$ *is a good holonomic variety and* $\mathrm{ord}_{\Lambda_{\nu,2\mu}} f^s = -(2m - \nu' - \mu)s - \frac{1}{2}(\nu' - \mu)(2\nu' - 2\mu - 1) - 2(m - \nu')(n - 2\nu' + \mu) - 4(m - \nu')(\nu' - \mu)$. *We have* $\dim \Lambda_{\nu,2\mu} \cap \Lambda_{\nu,2(\mu+1)} = \dim \Lambda_{\nu,2\mu} \cap \Lambda_{\nu+2,2\mu} = \dim V - 1$. *The b-function* $b(s)$ *is given by* $b(s) = \prod_{k=1}^{m}(s + 2k - 1) \cdot \prod_{\ell=0}^{m-1} (s + 2n - 2\ell)$.

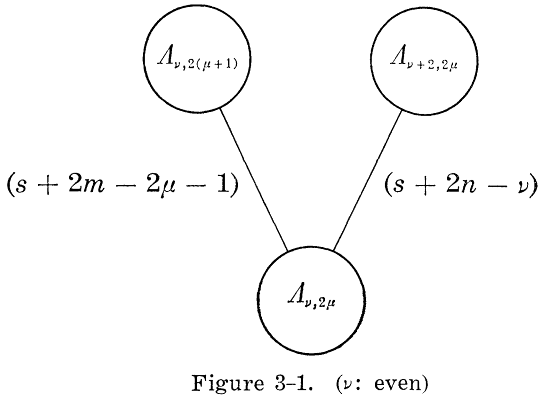


Figure 3-1. (ν: even)

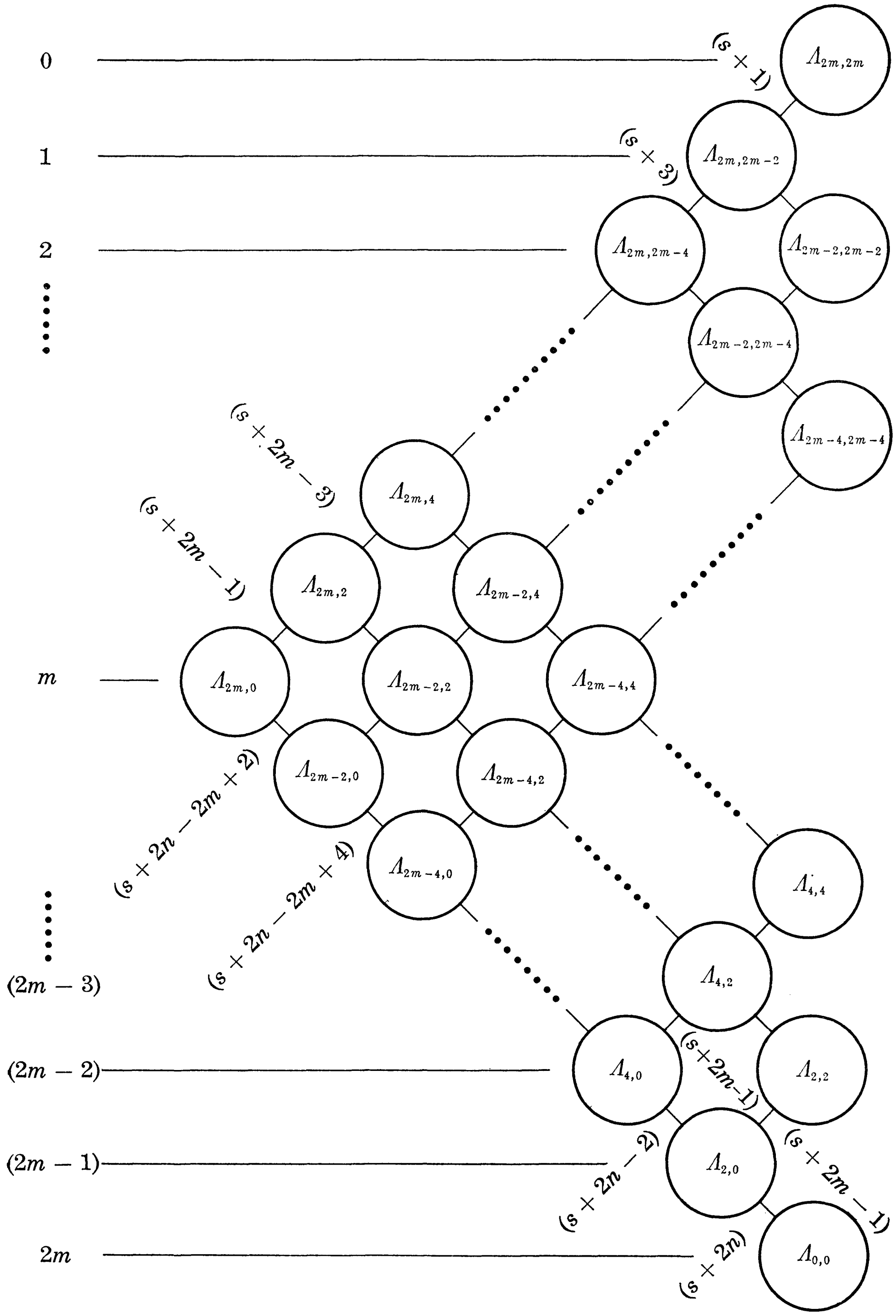


Figure 3-2. Holonomy diagram of $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ with $n \geqq 2m$.

§ 4. (Spin(10) × $GL(2)$, half-spin rep. $\otimes \Lambda_1$, $V(16) \otimes V(2)$)

The representation space $V(16) \otimes V(2)$ is identified with $V = V(16) \oplus V(16)$ where $V(16)$ is spanned by 1, e_ie_j, $e_ke_\ell e_me_n$ $(1 \leq i < j \leq 5, 1 \leq k < \ell < m < n \leq 5)$ (See p. 110–112 in [1]). The action $\rho = \rho_1 \otimes \Lambda_1$ is given by $\rho(g)x = (\rho_1(g_1)X, \rho_1(g_1)Y)^t g_2$ for $g = (g_1, g_2) \in \text{Spin}(10) \times GL(2)$, $x = (X, Y) \in V = V(16) \oplus V(16)$ where ρ_1 denotes the even half-spin representation of Spin(10) on $V(16)$. First of all, we shall complete the orbital decomposition of this space. J-I. Igusa completed the orbital decomposition of (Spin(10), ρ_1, $V(16)$) (See [3]). There exist three orbits $S'_m = \rho_1(\text{Spin}(10)) \cdot x'_m$ $(m = 0, 5, 16)$ where S'_m denotes the m-codimensional Spin(10)-orbit and $x'_0 = 1 + e_1e_2e_3e_4$, $x'_5 = 1$, $x'_{16} = 0$. If $\lambda \in C^\times$, for any index i satisfying $1 \leq i \leq 5$, we put $S_i(\lambda) = \lambda^{-1} + (\lambda - \lambda^{-1})e_if_i$. Then $S_i(\lambda)$ is an element of Spin(10). For any two distinct indices i, j satisfying $1 \leq i, j \leq 10$, $j \neq i + 5$, $i \neq j + 5$, we put $S_{ij}(\lambda) = 1 + \lambda e_ie_j = \exp(\lambda e_ie_j)$ where $e_k = f_{k-5}$ for $6 \leq k \leq 10$ (See [1], [3]). Then $S_{ij}(\lambda)$ is an element of Spin(10) satisfying $S_{ij}(\lambda)S_{ji}(\lambda) = 1$.

PROPOSITION 4–1. *The triplet* (Spin(10) × $GL(2)$, half-spin rep. $\otimes \Lambda_1$, $V(16) \otimes V(2)$) *has nine orbits* $S_m = \rho(G)x_m$ $(m = 0, 1, 4, 8, 9, 13, 15, 20, 32)$ *where* S_m *denotes the m-codimensional orbit.*

(1) $x_0 = (1 + e_1e_2e_3e_4, e_1e_5 + e_2e_3e_4e_5)$
(2) $x_1 = (1 + e_1e_2e_3e_4, e_1e_2 + e_2e_3e_4e_5)$
(3) $x_4 = (1, e_1e_5 + e_2e_3e_4e_5)$
(4) $x_8 = (1, e_1e_2e_3e_4)$
(5) $x_9 = (1, e_1e_2 + e_3e_4)$
(6) $x_{13} = (1, e_1e_2)$
(7) $x_{15} = (1 + e_1e_2e_3e_4, 0)$
(8) $x_{20} = (1, 0)$
(9) $x_{32} = (0, 0)$

Proof. Let $\tilde{x} = (x, y)$ be a representative of one of the orbits of $V = V(16) \oplus V(16)$. Then we may assume that $x = 0$, 1, or $1 + e_1e_2e_3e_4$ by the action of Spin(10). If $x = 0$, then we have also $y = 1 + e_1e_2e_3e_4$, 1, 0, i.e., (7), (8), (9) respectively. Note that we can exchange x and y in $\tilde{x} = (x, y)$ by the action of $GL(2)$. Assume that $x = 1$. We may put $y = y_0 + y_2 + y_4 \neq 0$ where $y_0 = y_0 \cdot 1$, $y_2 = \sum y_{ij}e_ie_j$ and $y_4 = \sum y_{rstu}e_re_se_te_u$. We may assume that $y_0 = 0$ by the action of $\begin{pmatrix} 1 & 0 \\ -y_0 & 1 \end{pmatrix}$. If $y = y_2 \neq 0$, we may assume that $y_{12} = 1$ by the action of some $S_{ij}(\lambda)$ $(i = 1, 2; j \geq 6)$ and $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ if necessary.

In this case, we have $y = e_1e_2 + y_{34}e_3e_4 + y_{35}e_3e_5 + y_{45}e_4e_5$ by $S_{j7}(-y_{1j})$ and $S_{j6}(y_{2j})$ for $j = 3, 4, 5$. If $y_{34} = y_{35} = y_{45} = 0$, we have (6), and otherwise we may assume that $y_{34} = 1$, $y_{35} = y_{45} = 0$ by the action of suitable elements of $\{S_{3,10}(\lambda), S_{4,10}(\lambda), S_{58}(\lambda), S_{59}(\lambda); \lambda \in \boldsymbol{C}\}$, i.e., (5). If $y_4 \neq 0$, we may assume that $y_4 = e_1e_2e_3e_4$. By the action of $S_{89}(y_{12})$ and $\begin{pmatrix} 1 & 0 \\ y_{12}y_{34} & 1 \end{pmatrix}$, we have $y_{12} = 0$. Similarly $y_{ij} = 0$ for $1 \leq i < j \leq 4$, and hence $y = \sum_{j=1}^{4} y_{j5}e_je_5 + e_1e_2e_3e_4$. If $y_{j5} = 0$ for all $j = 1, \cdots, 4$, we have (4). In the other case, we may assume that $y_{15} = 1$ and $y_{j5} = 0$ $(2 \leq j \leq 4)$. By the action of $S_{56}(-1)$ and $S_{1,10}(1)$, we have (3).

Finally assume that $x = 1 + e_1e_2e_3e_4$. We may put $y = y_2 + y_4$. If $y_4 \neq 0$, we may assume that $y_4 = e_2e_3e_4e_5$ or $y_4 = e_1e_2e_3e_4$. In the former case, if $y_{15} \neq 0$, we may assume that $y_{15} = 1$ by the action of $S_1(\lambda)S_5(\lambda)S_2(\lambda^{-1})$ and λI_2 where $\lambda^4 \cdot y_{15} = 1$. Then by the action of $S_{j10}(-y_{1j})$, $S_{j6}(-y_{j5})$ $(j = 2, 3, 4)$, $S_{9,10}(y_{23})$, $S_{8,10}(-y_{24})$ and $S_{7,10}(y_{34})$, we have (1). If $y_{15} = 0$, we may assume that $y_{35} = y_{45} = 0$ by $\{S_{28}(\lambda), S_{29}(\lambda), S_{37}(\lambda), S_{47}(\lambda); \lambda \in \boldsymbol{C}\}$. Then by $S_{8,10}(-y_{24})$ and $S_{9,10}(y_{23})$, we may assume that $y_{24} = y_{23} = 0$. By some $S_{39}(\lambda)$ and $S_{28}(\lambda)$, we may also assume that $y_{14} = 0$, i.e., $y = y_{12}e_1e_2 + y_{13}e_1e_3 + y_{34}e_3e_4$ $y_{25}e_2e_5 + e_2e_3e_4e_5$. By the action of $S_{7,10}(y_{34})$, $\begin{pmatrix} 1 & 0 \\ y_{25}y_{34} & 1 \end{pmatrix}$ and $S_{1,10}(y_{25}y_{34})$, we have $y_{34} = 0$. By $S_{89}(y_{25})$ and $S_{12}(y_{25})$, we also have $y_{25} = 0$, i.e., $y = y_{12}e_1e_2 + y_{13}e_1e_3 + e_2e_3e_4e_5$, where we may assume that $y_{13} = 0$. If $y_{12} \neq 0$, we have (2). If $y_{12} = 0$, it is transferred to x_4 by $S_{12}(-1)$, $S_{89}(-1)$, $S_{34}(-1)$, $S_{67}(-1)$, $S_{17}(1)$, $S_{26}(-1)$, $S_{56}(-1)$, $S_{1,10}(1)$, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S_{6,10}(1)$, $S_{15}(1)$ and $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Now consider the latter case, i.e., $y_4 = e_1e_2e_3e_4$. If some of y_{j5} $(1 \leq j \leq 4)$ is not zero, we may assume that $y = e_1e_5 + y_{23}e_2e_3 + e_1e_2e_3e_4$. If $y_{23} = 0$, it is transferred to x_4 by $S_{15}(1)$, $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $S_{56}(-1)$ and $S_{1,10}(1)$. If $y_{23} \neq 0$, we may assume that $y_{23} = 1$. In this case, it is transferred to x_1 by $S_{69}(1)$, $S_{23}(1)$, $S_{4,10}(-1)$, $S_{78}(1)$, $S_{14}(1)$, $S_{46}(1)$, $S_{19}(-1)$, $S_{29}(-1)$, $S_{47}(1)$. When all $y_{j5} = 0$ for $1 \leq j \leq 4$, $y = \sum_{1 \leq i < j \leq 4} y_{ij}e_ie_j + e_1e_2e_3e_4$. If all $y_{ij} = 0$ for $1 \leq i < j \leq 4$, it is transferred to x_8 by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. In the other case, we may assume that $y = e_1e_2 + y_{34}e_3e_4 + e_1e_2e_3e_4$. By the action of $S_{67}(\lambda)$, $S_{34}(\lambda)$ and $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$ with $\lambda^2 - \lambda - y_{34} = 0$, we have $y = e_1e_2 + (1 + 2\lambda)e_1e_2e_3e_4$. If $(1 + 2\lambda) \neq 0$, it is transferred to x_8 by $S_{89}(\mu)$, $S_{12}(\mu)$ and $\begin{pmatrix} 1 & -\mu \\ 0 & \mu \end{pmatrix}$ with $\mu = \dfrac{1}{1 + 2\lambda}$. If $(1 + 2\lambda) = 0$, it is equivalent to x_9 by $S_{67}(-1)$, $S_{12}(-1)$, $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $S_1(\sqrt{-1})$, and $\sqrt{-1}\ I_2$.

Finally consider the case $y = y_2$, i.e., $y_4 = 0$. Since $y \neq 0$, we may assume that $y = e_1e_2 + y_{34}e_3e_4 + \sum_{j=1}^{4} y_{j5}e_je_5$. If $y_{34} = y_{j5} = 0$ for $1 \leq j \leq 4$, it is equivalent to x_9 as we have already seen. If $y_{34} \neq 0$ and $y_{j5} = 0$ for $1 \leq j \leq 4$, it is transferred to x_8 by $S_{34}(\lambda)$, $\begin{pmatrix} 1 & -1/\lambda \\ 1 & 0 \end{pmatrix}$, $S_{12}(1/\lambda)$, $\begin{pmatrix} 1 & 0 \\ 0 & 1/2\lambda \end{pmatrix}$, $S_{67}(\lambda/2)$, $S_{89}(1/2\lambda)$, $\begin{pmatrix} 1 & 0 \\ 1/4 & 1 \end{pmatrix}$ with $\lambda^2 = y_{34}$. If some of y_{j5} $(1 \leq j \leq 4)$ is not zero, y is equivalent to an element of the form $e_1e_5 + y_{23}e_2e_3 + y_{24}e_2e_4 + y_{34}e_3e_4$. If $y_{ij} = 0\,(2 \leq i < j \leq 4)$, it is equivalent to x_4 as we have already seen. In the other case, we have $y = e_1e_5 + e_3e_4$. By the action of $S_{26}(-1)$, $S_{17}(1)$, $S_{34}(-1)$, $S_{67}(-1)$, $S_{7,10}(1)$, $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $S_{89}(-1)$, $S_{12}(-1)$, $S_{1,10}(-1)$, it is equivalent to x_1. About the codimension of these orbits, we will see later. Q.E.D.

By the degree formula (See Proposition 15, § 4 in [1]), we know that there exists a relatively invariant irreducible polynomial $f(x, y)$ of degree four which is unique up to a constant multiple. We shall give an explicit form of $f(x, y)$ after H. Kawahara's work (Master Thesis in Japanese, University of Tokyo, 1974).

For an element $x = x_0 + \sum_{i<j} x_{ij}e_ie_j + \sum_k x_k^* e_k^*$ of $V(16)$ where $e_ke_k^* = e_1e_2e_3e_4e_5$ for $1 \leq k \leq 5$, let $X = (x_{ij})$ be the skew-symmetric matrix of degree five determined by x_{ij}, and X_i the skew-symmetric matrix of degree four obtained from $(-1)^i X$ by crossing out its i-th line and column $(1 \leq i \leq 5)$. We denote by $\mathrm{Pf}(Y)$ the Pfaffian of the skew-symmetric matrix $Y = (y_{ij})$ of degree four, i.e., $\mathrm{Pf}(Y) = y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23}$. We define ten quadratic forms $Q_i(x)$ on $V(16)$ by $Q_i(x) = \sum_{j=1}^{5} x_{ij}x_j^*$ and $Q_{i+5}(x) = x_0x_i^* + \mathrm{Pf}(X_i)$ for $1 \leq i \leq 5$.

PROPOSITION 4–2 (H. Kawahara).

(1) *$\rho_1(\mathrm{Spin}(10)) \cdot 1 = \{x \in V(16)\,;\, Q_i(x) = 0 \ (1 \leq i \leq 10)\} - \{0\}$, where ρ_1 denotes the even half-spin representation. Moreover, this is the totality of pure spinors.*

(2) *The relative invariant $f(x, y)$ of $(\mathrm{Spin}(10) \times GL(2), \rho_1 \otimes \Lambda_1, V(16) \oplus V(16))$ is given by $f(x, y) = \sum_{i=1}^{5} B_i(x, y)B_{i+5}(x, y)$ for $(x, y) \in V(16) \oplus V(16)$ where $B_i(x, y) = Q_i(x + y) - Q_i(x) - Q_i(y)$ is the associated bilinear form of $Q_i(x)$ for $1 \leq i \leq 10$.*

Proof. We shall use the same notation as in [4]. By simple calculation, we have $\beta_1(x, x) = (1/8) \sum_{i=1}^{10} Q_i(x)e_i$. Since $\beta_1(\rho_1(s)x, \rho_1(s)x) = \lambda(s) \cdot \zeta_1(\chi(s)) \cdot \beta_1(x, x)$ for $s \in \mathrm{Spin}(10)$ where ζ_1 is the representation Λ_1 of $SO(10) = \chi(\mathrm{Spin}(10))$ (See p. 90 in [4]), we have

$$\sum_{i=1}^{10} Q_i(\rho_1(s)x)e_i = \lambda(s)\cdot\zeta_1(\chi(s))\cdot\sum_{i=1}^{10} Q_i(x)e_i. \tag{4.1}$$

This implies that $W = \{x \in V(16)\,;\, Q_i(x) = 0, 1 \leq i \leq 10\}$ is a Spin(10)-invariant subspace. From the orbital decomposition, it is clear that $W = S_5' \cup S_{16}'$, i.e., $S_5' = W - \{0\}$. Since the totality of pure spinors in $V(16)$ is a single Γ^+-orbit where Γ^+ denotes the even Clifford group, and $\beta_1(x, x) = 0$ for a pure spinor x (See [4]), we have (1). From (4.1), $F(x) = \sum_{i=1}^{5} Q_i(x) Q_{i+5}(x)$ is invariant under the action ρ_1 of Spin(10) since $\tilde{f}(y) = \sum_{i=1}^{5} y_i y_{i+5}$ for $y = \sum_{i=1}^{10} y_i e_i$ is invariant under the action ζ_1 of $SO(10) = \chi(\text{Spin}(10))$. The triplet (Spin(10), ρ_1, $V(10)$) has no relative invariant (See [1]) and hence we have $F(x) \equiv 0$. By using (4.1), it is clear that $f(x, y)$ is invariant under the action of Spin(10). We shall show that $f(x, y)$ is relatively invariant under $GL(2)$. Assume that $Q_i(x)$ (resp. $Q_{i+5}(x)$) has a term $x_{i_1}x_{i_2}$ (resp. $x_{i_3}x_{i_4}$) $(1 \leq i \leq 5)$. Since $F(x) \equiv 0$, we may assume that $Q_j(x)$ (resp. $Q_{j+5}(x)$) has a term $x_{i_1}x_{i_3}$ (resp. $x_{i_2}x_{i_4}$) for some j satisfying $1 \leq j \leq 5$. This implies that $f(x, y) = \sum_{i=1}^{5} B_i(x, y)B_{i+5}(x, y)$ is a linear combination of terms of the following form:

$$\begin{aligned}(x_{i_1}y_{i_2} + y_{i_1}x_{i_2})(x_{i_3}y_{i_4} + y_{i_3}x_{i_4}) - (x_{i_1}y_{i_3} + y_{i_1}x_{i_3})(x_{i_2}y_{i_4} + y_{i_2}x_{i_4})\\ = \det\begin{pmatrix} x_{i_2} & y_{i_2} \\ x_{i_3} & y_{i_3} \end{pmatrix}\cdot\det\begin{pmatrix} x_{i_4} & y_{i_4} \\ x_{i_1} & y_{i_1} \end{pmatrix}.\end{aligned} \tag{4.2}$$

Hence it is clear that $f(x, y)$ is relatively invariant under $GL(2)$. Since $f(1 + e_1e_2e_3e_4,\ e_1e_5 + e_2e_3e_4e_5) = 1$, it is not identically zero. Q.E.D.

Now we shall consider the micro-differential equation $\mathfrak{M} = \mathscr{E}f(x, y)^s$ and by constructing its holonomy diagram, we shall calculate the b-function of this space.

Since $G = \text{Spin}(10) \times GL(2)$ is reductive, we have $(G, \rho^*, V^*) \cong (G, \rho, V)$ and hence the dual space V^* has also nine G-orbits $S_m^*(m = 0, 1, 4, 8, 9, 13, 15, 20, 32)$. We identify V and V^* by taking $(e_{i_1}\cdots e_{i_k}, e_{j_1}\cdots e_{j_\ell})$ $(k, \ell = 0, 2, 4)$ as a dual basis, where $e_{i_1}\cdots e_{i_k} = 1$ for $k = 0$. We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m (resp. S_m^*).

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at $x_0 = (1 + e_1e_2e_3e_4, e_1e_5 + e_2e_3e_4e_5)$ is isomorphic to $(\mathfrak{g}_2) \oplus \mathfrak{sl}(2)$ (See (5.40) and (5.42) in [1]). Since $\Lambda_0 = V \times \{0\} = \Lambda_{32}^*$, Λ_0 is a good holonomic variety and we have $\text{ord}_{\Lambda_0} f^s = 0$.

(2) The isotropy subalgebra $\mathfrak{g}_{x_1}$ at $x_1 = (1 + e_1e_2e_3e_4, e_1e_2 + e_2e_3e_4e_5)$ is isomorphic to $(\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(11)$ (See (5.43) in [1]). The conormal vector space $V_{x_1}^*$ is spanned by $(e_1e_3e_4e_5, -e_1e_5) = y_1 \in S_{13}^*$. Hence $\Lambda_1 =$

$\overline{G(x_1, y_1)} = \Lambda_{13}^*$ and $\Lambda_{13} = \Lambda_1^*$. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $d_{11} = d_{22} = -1/4$, all remaining parts zero in (5.43) of [1]. Then we have $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$. Since $\delta\chi(A_0) = 2(d_{11} + d_{22}) = -1$, $\mathrm{tr}_{V_{x_1}^*} A_0 = \dim V_{x_1}^* = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - 1/2$. It is clear that Λ_0 and Λ_1 intersect regularly and G_0-prehomogeneously with codimension one. Hence we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$. Note that $G_0 = \{g \in G\,;\, \chi(g) = 1\}$.

(3) The isotropy subalgebra $\mathfrak{g}_{x_4}$ at $x_4 = (1, e_1e_5 + e_2e_3e_4e_5)$ is, by simple calculation using (5.38) in [1], given as follows:

$$(4.3)\qquad \mathfrak{g}_{x_4} = \left\{ \tilde{A} = \left(\begin{array}{c|c} A & 0 \\ \hline C & -{}^tA \end{array}\right) \oplus \begin{pmatrix} 3\varepsilon - \eta & 0 \\ c & \eta \end{pmatrix};\ A = \left(\begin{array}{c|c|c} 3\varepsilon & 0 & 0 \\ \hline * & \varepsilon I_3 + X & 0 \\ \hline * & * & -2n \end{array}\right), \right.$$

$$\left. X \in \mathfrak{sl}(3),\ {}^tC = -C \quad \text{with} \quad c_{i5} = 0,\ i = 1, \cdots, 4 \right\}.$$

Put $\omega_1 = (e_1e_3e_4e_5, 0)$, $\omega_2 = (-e_1e_2e_4e_5, 0)$, $\omega_3 = (e_1e_2e_3e_5, 0)$, and $\omega_4 = (e_1e_2e_3e_4, 0)$. Then the conormal vector space $V_{x_4}^*$ is spanned by $\omega_1, \cdots, \omega_4$. The action $d\rho_{x_4}$ of $\mathfrak{g}_{x_4}$ on $V_{x_4}^*$ is given as follows:

$$(4.4)\qquad d\rho_{x_4}(\tilde{A})(\omega_1, \cdots, \omega_4) = (\omega_1, \cdots, \omega_4)\left(\begin{array}{ccc|c} \multicolumn{3}{c|}{(2\eta - 5\varepsilon)I_3 + X} & 0 \\ \hline * & * & * & -6\varepsilon \end{array}\right).$$

Since ω_1 is a generic point, we have $\Lambda_4 = \Lambda_{20}^*$ and $\Lambda_{20} = \Lambda_4^*$. Let A_0 be an element of $\mathfrak{g}_{x_4}$ with $2\eta - 5\varepsilon = 1$, all remaining parts zero except ε and η in (4.3). Then $d\rho(A_0)x_4 = 0$ and $d\rho^*(A_0)\omega_1 = \omega_1$. However we have $\delta\chi(A_0) = 6\varepsilon$ which is not definite. If Λ_4 is a good holonomic variety, this must be definite by Proposition 1–3, and hence Λ_4 is not a good holonomic variety, i.e., $\Lambda_4 \not\subset W$. Note that the P.V. $(G_{x_4}, \rho_{x_4}, V_{x_4}^*)$ has no relative invariant.

(4) The isotropy subalgebra $\mathfrak{g}_{x_8}$ at $x_8 = (1, e_1e_2e_3e_4)$ is given as follows:

$$(4.5)\qquad \mathfrak{g}_{x_8} = \left\{ \tilde{X} = \left[\begin{array}{c|c|c|c} \varepsilon I_4 + X & \gamma & 0 & 0 \\ \hline 0 & 2\eta & 0 & 0 \\ \hline 0 & \delta & -\varepsilon I_4 - {}^tX & 0 \\ \hline -{}^t\delta & 0 & -{}^t\gamma & -2\eta \end{array}\right] \oplus \begin{pmatrix} \eta + 2\varepsilon & 0 \\ 0 & \eta - 2\varepsilon \end{pmatrix}; \right.$$

$$\left. X \in \mathfrak{sl}(4),\ \gamma, \delta \in \boldsymbol{C}^4 \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(4)) \oplus \mathfrak{u}(8).$$

Put $\omega_1 = (e_2e_3e_4e_5, 0)$, $\omega_2 = -(e_1e_3e_4e_5, 0)$, $\omega_3 = (e_1e_2e_4e_5, 0)$, $\omega_4 = -(e_1e_2e_3e_5, 0)$, $\omega_5 = (0, e_1e_5)$, $\omega_6 = (0, e_2e_5)$, $\omega_7 = (0, e_3e_5)$, $\omega_8 = (0, e_4e_5)$. Then the conormal vector space $V_{x_8}^*$ is spanned by $\omega_1, \cdots, \omega_8$, and the action $d\rho_{x_8}$ of $\mathfrak{g}_{x_8}$ on $V_{x_8}^*$ is given as follows.

$$(4.6) \qquad d\rho_{x_8}(\tilde{X})(\omega_1, \cdots, \omega_8) = (\omega_1, \cdots, \omega_8)\left(\begin{array}{c|c} (3\varepsilon - 2\eta)I_4 + X & 0 \\ \hline 0 & -(3\varepsilon + 2\eta)I_4 - {}^tX \end{array}\right)$$

where $\tilde{X} \in \mathfrak{g}_{x_8}$ in (4.5).

Any relative invariant of $(G_{x_8}, \rho_{x_8}, V_{x_8}^*)$ is of the form $c \cdot g(x)^m$ $(c \in \boldsymbol{C}, m \in \boldsymbol{Z})$ where $g(x) = \sum_{i=1}^4 x_i x_{i+4}$ for $x = \sum_{i=1}^8 x_i\omega_i$. Clearly $y_8 = \omega_1 + \omega_5 = (e_2e_3e_4e_5, e_1e_5)$ is a generic point, and $y_8' = \omega_1 + \omega_6 = (e_2e_3e_4e_5, e_2e_5)$ is a point of the one-codimensional orbit. Hence we have $\Lambda_8 = \Lambda_8^*$ and $\dim \Lambda_1 \cap \Lambda_8 = \dim V - 1$. Since $\Lambda_{13} = \Lambda_1^*$, we have also $\dim \Lambda_8 \cap \Lambda_{13} = \dim V - 1$. Note that $(G_{x_8}, \rho_{x_8}, V_{x_8}^*)$ is a regular P.V. since $\rho_{x_8}(G_{x_8})$ and its generic isotropy subgroup are reductive (See [1]). By Corollary 1–7, Λ_8 is a good holonomic variety. Let $\tilde{X}_0$ be an element of $\mathfrak{g}_{x_8}$ with $\eta = -\frac{1}{2}$, all remaining parts zero in (4.5). Then $d\rho(\tilde{X}_0)x_8 = 0$ and $d\rho^*(\tilde{X}_0)y_8 = y_8$. Since $\delta\chi(\tilde{X}_0) = 4\eta = -2$, $\mathrm{tr}_{V_{x_8}^*}\tilde{X}_0 = -16\eta = 8$ and $\dim V_{x_8}^* = 8$, we have $\mathrm{ord}_{\Lambda_8} f^s = -2s - \frac{8}{2}$. Since $m_{\Lambda_8} - m_{\Lambda_1} = 1$, they intersect regularly. By Corollary 1–2, we have $b_{\Lambda_8}(s)/b_{\Lambda_1}(s) = (s + 4)$.

(5) We shall calculate the isotropy subalgebra at $x_9' = (1, e_1e_3 + e_2e_4)$ instead of $x_9 = (1, e_1e_2 + e_3e_4)$. It is given as follows.

$$(4.7) \qquad \mathfrak{g}_{x_9'} = \left\{ \tilde{A} = \left[\begin{array}{c|c} \begin{array}{c|c} \varepsilon I_4 + A & B \\ \hline 0 & 2\eta \end{array} & 0 \\ \hline C & \begin{array}{c|c} -\varepsilon I_4 - {}^tA & 0 \\ \hline -{}^tB & -2\eta \end{array} \end{array}\right] + \left(\begin{array}{c|c} 2\varepsilon + \eta & 0 \\ \hline c_{13} + c_{24} & \eta \end{array}\right); \right.$$

$$\left. A \in \mathfrak{sp}(2),\ B \in \boldsymbol{C}^4,\ C = -{}^tC = (c_{ij}) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sp}(2)) \oplus \mathfrak{u}(4).$$

Put $\omega_1 = (e_2e_3e_4e_5, 0)$, $\omega_2 = -(e_1e_3e_4e_5, 0)$, $\omega_3 = (e_1e_2e_4e_5, 0)$, $\omega_4 = -(e_1e_2e_3e_5, 0)$, $\omega_5 = (e_1e_2e_3e_4, 0)$, $\omega_6 = -(e_3e_5, e_2e_3e_4e_5)$, $\omega_7 = (-e_4e_5, e_1e_3e_4e_5)$, $\omega_8 = (e_1e_5, -e_1e_2e_4e_5)$, $\omega_9 = (e_2e_5, e_1e_2e_3e_5)$. Then the conormal vector space $V_{x_9'}^*$ is spanned by these $\omega_1, \cdots, \omega_9$ and the action $d\rho_{x_9'}$ of $\mathfrak{g}_{x_9'}$ on $V_{x_9'}^*$ is given as follows:

$$(4.8)\qquad d\rho_{x_9'}(\tilde{A})(\omega_1, \cdots, \omega_9) = (\omega_1, \cdots, \omega_9)\left(\begin{array}{c|c|c} A-(3\varepsilon+2\eta)I_4 & B & C' \\ \hline 0 & -4\varepsilon & 0 \\ \hline 0 & 0 & A-(\varepsilon+2\eta)I_4 \end{array}\right)$$

$$\text{with}\quad C' = \left[\begin{array}{c|c|c|c} c_{13}+2c_{24} & -c_{23} & 0 & c_{34} \\ \hline -c_{14} & 2c_{13}+c_{24} & -c_{34} & 0 \\ \hline 0 & -c_{12} & c_{13}+2c_{24} & -c_{14} \\ \hline c_{12} & 0 & -c_{23} & 2c_{13}+c_{24} \end{array}\right].$$

Clearly, $y_9 = \omega_5 + \omega_6$ is its generic point and $y_9' = \omega_1 + \omega_8$ is a point of the one-codimensional orbit. Note that $(G_{x_9'}, \rho_{x_9'}, V^*_{x_9'})$ has only one orbit of codimension one. Since $y_9, y_9' \in S_9^*$, we have $\Lambda_9 = \Lambda_9^*$, and Λ_9 has no one-codimensional intersection with other conormal bundles. Let $\tilde{A}_0$ be an element of $\mathfrak{g}_{x_9'}$ with $\varepsilon = -\frac{1}{4}$, $\eta = -\frac{3}{8}$, all remaining parts zero in (4.7). Then $d\rho(\tilde{A}_0)x_9' = 0$ and $d\rho^*(\tilde{A}_0)y_9 = y_9$. We have $\delta\chi(\tilde{A}_0) = 2\{(2\varepsilon + \eta) + \eta\} = -\frac{5}{2}$, we have $m_{\Lambda_9} = \frac{5}{2}$. This implies that the conormal bundle Λ_9 is not a good holonomic variety, i.e., $\Lambda_9 \not\subset W$ since otherwise m_{Λ_9} must be a non-negative integer (See § 1 or [1]).

(6) The isotropy subalgebra $\mathfrak{g}_{x_{13}}$ at $x_{13} = (1, e_1e_2)$ is given as follows.

$$(4.9)\qquad \mathfrak{g}_{x_{13}} = \left\{\tilde{X} = \left[\begin{array}{c|c|c|c} \varepsilon_1 I_2 + X & Z & \begin{matrix} 0 & -b \\ b & 0 \end{matrix} & 0 \\ \hline 0 & 2\varepsilon I_3 + Y & 0 & 0 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{C}} & -\varepsilon_1 I_2 - {}^tX & 0 \\ \cline{3-4} \multicolumn{2}{c|}{} & -{}^tZ & -2\varepsilon I_3 - {}^tY \end{array}\right] \right.$$

$$\left. \oplus \left(\begin{array}{c|c} 3\varepsilon+\varepsilon_1 & b \\ \hline c_{12} & 3\varepsilon-\varepsilon_1 \end{array}\right);\ X \in \mathfrak{sl}(2),\ Y \in \mathfrak{sl}(3),\ {}^tC = -C \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(3)) \oplus \mathfrak{u}(15).$$

Since $\Lambda_{13} = \Lambda_1^*$, $\Lambda_8 = \Lambda_8^*$, and $\dim \Lambda_1^* \cap \Lambda_8^* = \dim V - 1$, the conormal bundle Λ_{13} is a good holonomic variety and $\dim \Lambda_8 \cap \Lambda_{13} = \dim V - 1$. They intersect regularly. Put $\omega_1 = (0, e_2e_3e_4e_5)$, $\omega_2 = (e_2e_3e_4e_5, 0)$, $\omega_3 = (0, e_1e_3e_4e_5)$, $\omega_4 = (e_1e_3e_4e_5, 0)$, $\omega_5 = (e_4e_5, -e_1e_2e_4e_5)$, $\omega_6 = (e_3e_5, -e_1e_2e_3e_5)$, $\omega_7 = (e_3e_4, -e_1e_2e_3e_4)$, $\omega_8 = (0, e_4e_5)$, $\omega_9 = (0, e_3e_5)$, $\omega_{10} = (0, e_3e_4)$, $\omega_{11} = (e_1e_2e_4e_5, 0)$, $\omega_{12} = (e_1e_2e_3e_5, 0)$, $\omega_{13} = (e_1e_2e_3e_4, 0)$. Then the conormal vector space $V^*_{x_{13}}$ is spanned by these $\omega_1, \cdots, \omega_{13}$ and the action $d\rho_{x_{13}}$ of $\mathfrak{g}_{x_{13}}$ on $V^*_{x_{13}}$ is given as follows:

$$
\begin{aligned}
&d\rho_{x_{13}}(\tilde{X})(\omega_1, \cdots, \omega_{13}) \\
&= (\omega_1, \cdots, \omega_{13})\left(\begin{array}{c|c} -6\varepsilon I_4 + d\rho_1(X \oplus W) & * \\ \hline 0 & -4\varepsilon I_9 + d\rho_2(W \oplus Y) \end{array}\right)
\end{aligned} \tag{4.10}
$$

where $\rho_1 = \Lambda_1 \otimes \Lambda_1$ for $SL(2) \times SL(2)$, $\rho_2 = (2\Lambda_1) \otimes \Lambda_1$ for $SL(2) \times SL(3)$ and $W = \begin{pmatrix} \varepsilon_1 & b \\ c_{12} & -\varepsilon_1 \end{pmatrix} \in \mathfrak{sl}(2)$.

As a generic point, we may take $y_{13} = \omega_5 + \omega_9 + \omega_{13} = (e_4e_5 + e_1e_2e_3e_4, e_3e_5 - e_1e_2e_4e_5)$. Let $\tilde{X}_0$ be an element of $\mathfrak{g}_{x_{13}}$ with $\varepsilon = -\frac{1}{4}$, all remaining parts zero. Then $d\rho(\tilde{X}_0)x_{13} = 0$ and $d\rho^*(\tilde{X}_0)y_{13} = y_{13}$. Since $\delta\chi(\tilde{X}_0) = 12\varepsilon = -3$, $\mathrm{tr}_{V^*_{x_{13}}} \tilde{X}_0 = -60\varepsilon = 15$ and $\dim V^*_{x_{13}} = 13$, we have $\mathrm{ord}_{\Lambda_{13}} f^s = -3s - \frac{17}{2}$. By Corollary 1–2, we have $b_{\Lambda_{13}}(s)/b_{\Lambda_8}(s) = (s+5)$. By (4.10), we can see that $(G_{x_{13}}, \rho_{x_{13}}, V^*_{x_{13}})$ has the unique relative invariant (See Lemma 4 and Proposition 5 in § 4 in [1]), i.e., it has the unique one-codimensional orbit.

(7) The isotropy subalgebra $\mathfrak{g}_{x_{15}}$ at $x_{15} = (1 + e_1e_2e_3e_4, 0)$ is given as follows.

$$
\begin{aligned}
\mathfrak{g}_{x_{15}} = &\left\{ \tilde{X} = \left[\begin{array}{c|c|c|c} X & Y & C'' & 0 \\ \hline 0 & 2\varepsilon & 0 & 0 \\ \hline C & C' & -{}^tX & 0 \\ \hline -{}^tC' & 0 & -{}^tY & -2\varepsilon \end{array}\right] \oplus \begin{pmatrix} \varepsilon & b \\ 0 & \eta \end{pmatrix}; \right. \\
&\qquad \left. X \in \mathfrak{sl}(4),\ CC'' = -\operatorname{Pf} C \cdot I_4,\ {}^tC = -C \in M(4) \right\} \\
\cong\ &(\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{o}(7)) \oplus \mathfrak{u}(9)\ .
\end{aligned} \tag{4.11}
$$

Note that, in (4.11), $\tilde{X}_0 = \left(\begin{array}{c|c} X & C'' \\ \hline C & -{}^tX \end{array}\right)$ is the spin representation of X_0 in $\mathfrak{o}(7)$. Put $\omega_1 = (0, e_1e_5)$, $\omega_2 = (0, e_2e_5)$, $\omega_3 = (0, e_3e_5)$, $\omega_4 = (0, e_4e_5)$, $\omega_5 = (0, e_2e_3e_4e_5)$, $\omega_6 = (0, -e_1e_3e_4e_5)$, $\omega_7 = (0, e_1e_2e_4e_5)$, $\omega_8 = (0, -e_1e_2e_3e_5)$, $\omega_9 = (0, \frac{1}{2}(1 - e_1e_2e_3e_4))$, $\omega_{10} = (0, e_2e_3)$, $\omega_{11} = (0, -e_2e_4)$, $\omega_{12} = (0, e_3e_4)$, $\omega_{13} = (0, e_1e_4)$, $\omega_{14} = (0, e_1e_3)$, $\omega_{15} = (0, e_1e_2)$. The conormal vector space $V^*_{x_{15}}$ is spanned by $\omega_1, \cdots, \omega_{15}$. Then $y_{15} = \omega_9$ is its generic point and $y'_{15} = \omega_{10} + \omega_{14}$ is a point of the unique one-codimensional orbit. Since $y_{15}, y'_{15} \in S^*_{15}$, we have $\Lambda_{15} = \Lambda^*_{15}$, and Λ_{15} has no one-codimensional intersection with any other conormal bundle. Let $\tilde{X}_1$ be an element of $\mathfrak{g}_{x_{15}}$ with $\varepsilon = \beta + 1$, $\eta = \beta$, all remaining parts zero in (4.11). Then $d\rho(\tilde{X}_1)x_{15} = 0$ and $d\rho^*(\tilde{X}_1)y_{15} = y_{15}$. Since $\delta\chi(\tilde{X}_1) = 2(\varepsilon + \eta) = 2(2\beta + 1)$ is not definite, the conormal bundle Λ_{15} is not a good holonomic variety, i.e., $\Lambda_{15} \not\subset W$.

(8) Since $\Lambda_{20} = \Lambda_4^*$ and $\Lambda_4 \not\subset W$, the conormal bundle Λ_{20} is not a good holonomic variety. Note that $W \subset V \times V^*$ is symmetric with respect to V and V^*.

(9) Since $\Lambda_{32} = \{0\} \times V^*$, the conormal bundle Λ_{32} is a good holonomic variety. Put $A_0 = (0) \oplus (-I_2)$. Then $d\rho(A_0)x_{32} = 0$ and $d\rho^*(A_0)y_{32} = y_{32}$ where y_{32} is a generic point of (G, ρ^*, V^*). Since $\delta\chi(A_0) = -4$, $\mathrm{tr}_{V^*_{x_{32}}} A_0 = 32$ and $\dim V^*_{x_{32}} = 32$, we have $\mathrm{ord}_{\Lambda_{32}} f^s = -4s - \frac{32}{2}$ and hence by Corollary 1–2, we have $b_{\Lambda_{32}}(s)/b_{\Lambda_{13}}(s) = s + 8$. Note that $\Lambda_{32} = \Lambda_0^*$ and $\Lambda_{13} = \Lambda_1^*$. Since $b_{\Lambda_0}(s) = 1$ and $b_{\Lambda_{32}}(s) = b(s)$, we have the b-function $b(s) = (s+1)(s+4)(s+5)(s+8)$, and the holonomy diagram (Figure 4–1). We denote Λ_m by ⓜ.

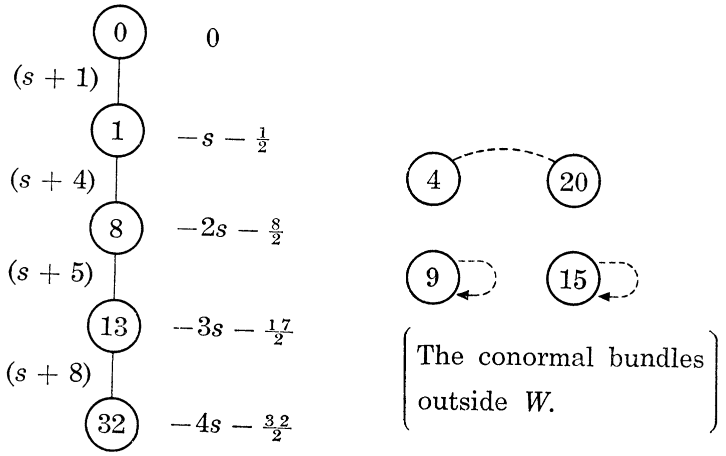


Figure 4–1. Holonomy diagram of (Spin(10) × $GL(2)$, half-spin rep. ⊗ Λ_1, $V(16) \otimes V(2)$)

§ 5. ($GL(1)$ × Spin(12), □ ⊗ half-spin rep., $V(1) \otimes V(32)$)

The representation space $V = V(1) \otimes V(32)$ is spanned by 1, e_ie_j, $e_re_se_te_u$, $e_1e_2e_3e_4e_5e_6$ $(1 \leq i < j \leq 6,\ 1 \leq r < s < t < u \leq 6)$ (See [1], [4]). J-I. Igusa has completed the orbital decomposition of this space (See [3]). There exist five G-orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 7, 16, 32)$ where S_m denotes the m-codimensional orbit and $x_0 = 1 + e_1e_2e_3e_4e_5e_6$, $x_1 = 1 + e_2e_3e_5e_6 + e_1e_3e_4e_6$, $x_7 = 1 + e_2e_3e_5e_6$, $x_{16} = 1$, $x_{32} = 0$. We identify V^* with V by taking $\{1, e_ie_j, e_re_se_te_u, e_1e_2e_3e_4e_5e_6\}$ as a dual basis. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also five orbits $S_m^* (m = 0, 1, 7, 16, 32)$ in V^*. We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m (resp. S_m^*). Clearly, we have $\Lambda_0 = V \times \{0\} = \Lambda_{32}^*$ and $\Lambda_{32} = \{0\} \times V^* = \Lambda_0^*$. The Lie algebra $\mathfrak{g}$ of $GL(1)$ × Spin(12) is given as follows:

$$\mathfrak{g}=\left\{(d)\oplus\left(\begin{array}{c|c} A & B \\ \hline C & -{}^tA\end{array}\right);\ A,B,C\in M(6),\ {}^tB=-B,\ {}^tC=-C\right\}. \tag{5.1}$$

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is given as follows (See [1]).

$$\mathfrak{g}_{x_0}=\left\{(0)\oplus\left(\begin{array}{c|c} A & 0 \\ \hline 0 & -{}^tA\end{array}\right);\ A\in\mathfrak{sl}(6)\right\}\cong\mathfrak{sl}(6). \tag{5.2}$$

Since $\Lambda_0=V\times\{0\}$, we have $\mathrm{ord}_{\Lambda_0}f^s=0$, where f denotes the relative invariant of degree four (See [1], [3]).

(2) By using (5.29) in [1], we can calculate the isotropy subalgebra $\mathfrak{g}_{x_1}$.

$$\begin{aligned}\mathfrak{g}_{x_1}&=\left\{\tilde{A}=(d)\oplus\left\{\begin{array}{c|c} A & B \\ \hline C & -{}^tA\end{array}\right\};\ a_1+a_4=a_2+a_5=-a_3-a_6=2d,\ c_{36}=0\right\}\\ &\cong\left\{(d)\oplus\left(\begin{array}{c|c} dI_6+A_0 & B_0 \\ \hline 0 & -dI_6-{}^tA_0\end{array}\right);\ A_0\in\mathfrak{sp}(3),\ {}^tB_0=-B_0,\ \mathrm{tr}\,B_0J=0\right\}\\ &\qquad\text{with}\quad J=\left(\begin{array}{c|c} 0 & I_3 \\ \hline -I_3 & 0\end{array}\right),\\ &\cong(\mathfrak{gl}(1)\oplus\mathfrak{sp}(3))\oplus V(14)\quad\text{where}\end{aligned} \tag{5.3}$$

$$A=\begin{bmatrix} a_1 & a_{12} & 0 & a_{14} & a_{15} & 0\\ a_{21} & a_2 & 0 & a_{15} & a_{25} & 0\\ a_{31} & a_{32} & a_3 & a_{34} & a_{35} & a_{36}\\ a_{41} & a_{42} & 0 & a_4 & -a_{21} & 0\\ a_{42} & a_{52} & 0 & -a_{12} & a_5 & 0\\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_6\end{bmatrix},\quad B=\begin{bmatrix} 0 & 0 & c_{46} & 0 & 0 & c_{34}\\ 0 & 0 & c_{56} & 0 & 0 & c_{35}\\ -c_{46} & -c_{56} & 0 & c_{16} & c_{26} & b_{36}\\ 0 & 0 & -c_{16} & 0 & 0 & c_{13}\\ 0 & 0 & -c_{26} & 0 & 0 & c_{23}\\ -c_{34} & -c_{35} & -b_{36} & -c_{13} & -c_{23} & 0\end{bmatrix}$$

with $b_{36}+c_{14}+c_{25}=0$.

The conormal vector space $V^*_{x_1}$ is spanned by $e_1e_2e_4e_5$ on which $\mathfrak{g}_{x_1}$ acts as $d\rho_{x_1}(\tilde{A})e_1e_2e_4e_5=-4de_1e_2e_4e_5$ for $\tilde{A}\in\mathfrak{g}_{x_1}$. This implies that $\Lambda_1=\overline{G(x_1,e_1e_2e_4e_5)}=\Lambda^*_{16}$. Since 0 is the point of the one-codimensional orbit, we have $\dim\Lambda_0\cap\Lambda_1=\dim V-1$ and $\Lambda_0\cap\Lambda_1$ is G_0-prehomogeneous, i.e., Λ_1 is a good holonomic variety by Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $-4d=1$. Then $d\rho(A_0)x_1=0$ and $d\rho^*(A_0)y_1=y_1$ where $y_1=e_1e_2e_4e_5$. Since $\delta\chi(A_0)=4d=-1$, $\mathrm{tr}_{V^*_{x_1}}A_0=\dim V^*_{x_1}=1$, we have $\mathrm{ord}_{\Lambda_1}f^s=-s-\frac{1}{2}$. By Proposition 1–4, Λ_0 and Λ_1 intersect regularly and hence $b_{\Lambda_1}(s)/b_{\Lambda_0}(s)=(s+1)$ by Corollary 1–2.

(3) By using (5.29) in [1], we can calculate the isotropy subalgebra $\mathfrak{g}_{x_7}$.

$$\mathfrak{g}_{x_7}=\left\{\tilde{A}=(d)\oplus\left(\begin{array}{c|c} A & C' \\ \hline C & -{}^tA\end{array}\right);\ d=\frac{a_1+a_4}{2},\ a_2+a_3+a_5+a_6=0,\right.$$

$$\left.{}^tC = C,\ A \text{ and } C' \text{ are given as follows}\right\}$$

$$(5.4)\qquad \cong \left\{(d) \oplus \left(\begin{array}{c|c|c} dI_2 + A_1 & 0 & 0 \\ \hline U & d\rho_1(V) & 0 \\ \hline W & -{}^tUS & -dI_2 - {}^tA_1 \end{array}\right);\ \begin{array}{l} A_1 \in \mathfrak{sl}(2),\ W \in \mathfrak{o}(2), \\ V \in \mathfrak{o}(7),\ U \in M(8.2) \end{array}\right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{o}(7) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(17)$$

where ρ_1 is the spin-representation of Spin(7), $S = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix}$, and

$$A = \begin{bmatrix} a_1 & 0 & 0 & a_{14} & 0 & 0 \\ a_{21} & a_2 & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_3 & a_{34} & a_{35} & a_{36} \\ a_{41} & 0 & 0 & a_4 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_5 & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_6 \end{bmatrix},\quad C' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{56} & 0 & -c_{36} & c_{35} \\ 0 & -c_{56} & 0 & 0 & c_{26} & -c_{25} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{36} & -c_{26} & 0 & 0 & c_{23} \\ 0 & -c_{35} & c_{25} & 0 & -c_{23} & 0 \end{bmatrix}.$$

Put $\omega_1 = e_1e_4 - e_1e_2e_3e_4e_5e_6$, $\omega_2 = e_1e_2e_4e_5$, $\omega_3 = e_1e_2e_3e_4$, $\omega_4 = e_1e_2e_4e_6$, $\omega_5 = e_1e_3e_4e_5$, $\omega_6 = e_1e_3e_4e_6$, $\omega_7 = e_1e_4e_5e_6$. The conormal vector space $V_{x_7}^*$ is spanned by these $\omega_1, \cdots, \omega_7$, and $(G_{x_7}, \rho_{x_7}, V_{x_7}^*) \cong (GL(1) \times SO(7), \square \otimes \Lambda_1, V(1) \otimes V(7))$. Then ω_1 is its generic point and $\omega_2 = e_1e_2e_4e_5$ is a point of the one-codimensional orbit. Since $\Lambda_1 = \overline{G(x_1, \omega_2)}$, we have codim $\Lambda_1 \cap \Lambda_7 = 1$. Since $\Lambda_1 \cap \Lambda_7$ is G_0-prehomogeneous, Λ_7 is a good holonomic variety. We have $\Lambda_7 = \Lambda_7^*$.

Let A'_β be an element of $\mathfrak{g}_{x_7}$ with $4d = 2(a_1 + a_4) = -\beta - 1$, $2(a_3 + a_6) = -2(a_2 + a_5) = 1 - \beta$, all remaining parts zero in (5.4). Then we have $d\rho(A'_\beta)x_7 = 0$, $d\rho^*(A'_\beta)\ \omega_2 = \omega_2$ and $\mathrm{tr}_{\tilde{V}} A'_\beta = \beta$ where $\tilde{V} = V_{x_7}^* \bmod d\rho_{x_7}(\mathfrak{g}_{x_7})\omega_2$. This implies that Λ_1 and Λ_7 intersect regularly by Proposition 1–4. Let A_0 be an element of $\mathfrak{g}_{x_7}$ with $d = -\frac{1}{2}$, all remaining parts zero in (5.4). Then $d\rho(A_0)x_7 = 0$, $d\rho^*(A_0)\omega_1 = \omega_1$ Since $\delta\chi(A_0) = 4d = -2$, $\mathrm{tr}_{V_{x_7}^*} A_0 = -14\mathrm{d} = 7$, $\dim V_{x_7}^* = 7$, we have $\mathrm{ord}_{\Lambda_7} f^s = -2s - \frac{7}{2}$. By Corollary 1–2, we have $b_{\Lambda_7}(s)/b_{\Lambda_1}(s) = (s + \frac{7}{2})$.

(4) Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, $\Lambda_7^* = \Lambda_7$ and $\Lambda_1^* = \Lambda_{16}$, we have $\dim \Lambda_7 \cap \Lambda_{16} = \dim V - 1$ and they intersect regularly. Since $\Lambda_7 \cap \Lambda_{16} = \Lambda_1^* \cap \Lambda_7^*$ is G_0-prehomogeneous, Λ_{16} is a good holonomic variety. Since $d\rho(\tilde{A}) \cdot 1 = d - (\mathrm{tr}\, A/2) + \sum_{i<j} b_{ij}e_ie_j$, the isotropy subalgebra $\mathfrak{g}_{x_{16}}$ at $x_{16} = 1$ is

$$(5.5)\qquad \mathfrak{g}_{x_{16}} = \left\{\tilde{A} = \left(\frac{\mathrm{tr}\, A}{2}\right) \oplus \begin{pmatrix} A & 0 \\ C & -{}^tA \end{pmatrix};\ {}^tC = -C,\ A,\ C \in M(3)\right\}$$

$$\cong \mathfrak{gl}(6) \oplus V(15).$$

The conormal vector space $V^*_{x_{16}}$ is spanned by $e_1e_2e_3e_4e_5e_6$ and $e_ie_je_ke_\ell$ $(1 \leq i < j < k < \ell \leq 6)$. Then the action $d\rho_{x_{16}}$ of $\mathfrak{g}_{x_{16}}$ on $V^*_{x_{16}}$ is given by

$$d\rho_{x_{16}}(\tilde{A})(\omega_1, \cdots, \omega_{16}) = (\omega_1, \cdots, \omega_{16})\begin{pmatrix} -\operatorname{tr} A & c' \\ 0 & d\rho_1^*(A) \end{pmatrix} \tag{5.6}$$

where $\omega_1 = e_1e_2e_3e_4e_5e_6$, $\{\omega_2, \cdots, \omega_{16}\} = \{e_ie_je_ke_\ell, 1 \leq i < j < k < \ell \leq 6\}$, ${}^tc' \in \boldsymbol{C}^{15}$, $\rho_1 = \Lambda_2$ for $GL(6)$.

Then $y_{16} = e_1e_2e_4e_5 + e_1e_3e_4e_6 + e_2e_3e_5e_6$ is its generic point. Let A_0 be an element of $\mathfrak{g}_{x_{16}}$ with $A = -\frac{1}{4}I_6$, $C = 0$ in (5.5). Then $d\rho(A_0)x_{16} = 0$ and $d\rho^*(A_0)y_{16} = y_{16}$. Since $\delta\chi(A_0) = 4d = 2\operatorname{tr}(-\frac{1}{4}I_6) = -3$, $\operatorname{tr}_{V^*_{x_{16}}} A_0 = -11\operatorname{tr}(-\frac{1}{4}I_6) = \frac{33}{2}$ and $\dim V_{x^*_{16}} = 16$, we have $\operatorname{ord}_{\Lambda_{16}} f^s = s\delta\chi(A_0) - \operatorname{tr}_{V^*_{x_{16}}} A_0 + \frac{1}{2}\dim V_{x^*_{16}} = -3s - \frac{17}{2}$. By Corollary 1–2, we have $b_{\Lambda_{16}}(s)/b_{\Lambda_7}(s) = (s + \frac{11}{2})$. By (5.6), the character group of $\rho_{x_{16}}(G_{x_{16}})$ is one-dimensional and hence $(G_{x_{16}}, \rho_{x_{16}}, V^*_{x_{16}})$ has (at most) the unique one-codimensional orbit.

(5) Since $\Lambda_{32} = \Lambda_0^*$ and $\Lambda_{16} = \Lambda_1^*$, they intersect regularly with codimension one. We shall calculate the order $\operatorname{ord}_{\Lambda_{32}} f^s$. Since $(G_{x_{32}}, \rho_{x_{32}}, V_{x_{32}}) \cong (G, \rho^*, V^*)$, $y_{32} = 1 + e_1e_2e_3e_4e_5e_6$ is its generic point. Let A_0 be an element of $\mathfrak{g}$ with $d = -1$, all remaining parts zero in (5.1). Then $d\rho(A_0)x_{32} = 0$, $d\rho^*(A_0)y_{32} = y_{32}$. Since $\delta\chi(A_0) = -4$, $\operatorname{tr}_{V^*_{32}} A_0 = -32d = 32$, $\dim V_{x^*_{32}} = 32$, we have $\operatorname{ord}_{\Lambda_{32}} f^s = -4s - \frac{32}{2}$. By Corollary 1–2, we have $b_{\Lambda_{32}}(s)/b_{\Lambda_{16}}(s) = s + 8$. Since $b_{\Lambda_0}(s) = 1$ and $b_{\Lambda_{32}}(s) = b(s)$, we obtain the b-function $b(s) = (s + 1)(s + \frac{7}{2})(s + \frac{11}{2})(s + 8)$ and the holonomy diagram (Figure 5–1). We denote $\bigcirc\!\!\!\!\!\Lambda_m$ by $\bigcirc\!\!\!\!\!m$.

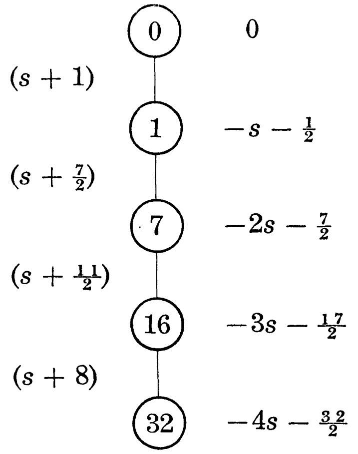


Figure 5–1. Holonomy diagram of $(GL(1) \times \mathrm{Spin}(12)$, $\square \otimes$ half-spin rep., $V(1) \otimes V(32))$.

Remark. (1) $(GL(1) \times \mathrm{Spin}(7),\ \square \otimes \text{spin rep.},\ V(1) \otimes V(8))$

(2) $(\mathrm{Spin}(7) \times GL(2),\ \text{spin rep.} \otimes \Lambda_1,\ V(8) \otimes V(2))$

(3) $(\mathrm{Spin}(7) \times GL(3),\ \text{spin rep.} \otimes \Lambda_1,\ V(8) \otimes V(3))$

(4) $(GL(1) \times \mathrm{Spin}(9),\ \square \otimes \text{spin rep.},\ V(1) \otimes V(16))$

(5) $(GL(1) \times (G_2),\ \square \otimes \Lambda_2,\ V(1) \otimes V(7))$

(6) $((G_2) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(2))$

(7) $(GL(1) \times \mathrm{Spin}(11),\ \square \otimes \text{spin rep.},\ V(1) \otimes V(32))$

Since $\mathrm{Spin}(7) \hookrightarrow SO(8)$ by the spin representation, the first three triplets (1), (2), (3) are reduced to the triplet $(SO(8) \times GL(n),\ \Lambda_1 \otimes \Lambda_1,\ V(8) \otimes V(n))$ $(n = 1, 2, 3)$ (See [1]). Since $\mathrm{Spin}(9) \hookrightarrow SO(16)$ by the spin representation, (4) is reduced to $(SO(16) \times GL(1),\ \Lambda_1 \otimes \Lambda_1,\ V(16) \otimes V(1))$ (See [1]). Since $(G_2) \hookrightarrow SO(7)$ by Λ_2, (5) and (6) are reduced to $(SO(7) \times GL(n),\ \Lambda_1 \otimes \Lambda_1,\ V(7) \otimes V(n))$ $(n = 1, 2)$ (See [1]).

Since the spin representation of Spin(11) is obtained by the restriction of the half-spin representation of Spin(12) to Spin(11), (7) is reduced to Spin (12) in § 5. Note that the b-function depends essentially on the relative invariant itself, not on the group.

§ 6. $(GL(1) \times E_6,\ \square \otimes \Lambda_1,\ V(1) \otimes V(27))$

The Lie algebra $\mathfrak{g}$ of $G = GL(1) \times E_6$ can be written as $\mathfrak{g} = \mathscr{D}_0 \oplus \mathscr{T}_1 \oplus \mathscr{T}_2 \oplus \mathscr{T}_3 \oplus \{R_y\}_{y \in \mathscr{J}}$ (See Proposition 37 and Example 39 of § 1 in [1]). The representation space is identified with the simple Jordan algebra $\mathscr{J}$.

$$\mathscr{J} = \left\{ X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix};\ \xi_1, \xi_2, \xi_3 \in \boldsymbol{C};\ x_1, x_2, x_3 \in \mathscr{L} \right\} \tag{6.1}$$

where $\mathscr{L}$ denotes the complex Cayley algebra.

Definition 6–1. For $a \in \mathscr{L}$, we define elements $T_i(a)$ and $T_i'(a)$ $(i = 1, 2, 3)$ of $\mathfrak{g}$ as follows:

$$T_1(a) \cdot X = [R_{A_1(a)} + \mathscr{T}_1(2a)]X = \begin{pmatrix} 0 & 0 & x_3 a \\ 0 & 0 & a\xi_2 \\ \overline{x_3 a} & \overline{a\xi_2} & \mathrm{tr}(\bar{x}_1 a) \end{pmatrix}$$

$$T_1'(a) \cdot X = [R_{A_1(a)} - \mathscr{T}_1(2a)]X = \begin{pmatrix} 0 & \overline{ax_2} & 0 \\ ax_2 & \mathrm{tr}(a\bar{x}_1) & a\xi_3 \\ 0 & \overline{a\xi_3} & 0 \end{pmatrix}$$

$$T_2(a) \cdot X = [R_{A_2(\bar{a})} + \mathscr{T}_2(2\bar{a})]X = \begin{pmatrix} 0 & 0 & a\xi_1 \\ 0 & 0 & \bar{x}_3 a \\ \overline{a\xi_1} & \bar{a}x_3 & \mathrm{tr}(x_2 a) \end{pmatrix}$$

$$T_2'(a)\cdot X = [R_{A_2(\bar{a})} - \mathscr{T}_2(2\bar{a})]X = \begin{pmatrix} \mathrm{tr}(ax_2) & a\bar{x}_1 & a\xi_3 \\ x_1\bar{a} & 0 & 0 \\ \overline{a\xi_3} & 0 & 0 \end{pmatrix}$$

$$T_3(a)\cdot X = [R_{A_3(a)} + \mathscr{T}_3(2a)]X = \begin{pmatrix} 0 & a\xi_1 & 0 \\ \overline{a\xi_1} & \mathrm{tr}(\bar{a}x_3) & \overline{x_2a} \\ 0 & x_2a & 0 \end{pmatrix}$$

$$T_3'(a)\cdot X = [R_{A_3(a)} - \mathscr{T}_3(2a)]X = \begin{pmatrix} \mathrm{tr}(a\bar{x}_3) & a\xi_2 & ax_1 \\ \overline{a\xi_2} & 0 & 0 \\ \overline{ax_1} & 0 & 0 \end{pmatrix}$$

where $A_i(a)$ denotes the element of $\mathscr{J}$ with $x_i = a$, all remaining terms zero in (6.1) for $i = 1, 2, 3$, and $\mathrm{tr}\, b = b + \bar{b}$ for $b \in \mathscr{L}$. Thus we have $\mathfrak{g} = \mathscr{D}_0 \oplus T_1 \oplus T_2 \oplus T_3 \oplus T_1' \oplus T_2' \oplus T_3' \oplus \{R_{\left(\begin{smallmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & \eta_3 \end{smallmatrix}\right)}\}$. For $a \in \mathscr{L}$, we put $t_i(a) = \exp T_i(a)$ and $t_i'(a) = \exp T_i'(a)$ for $i = 1, 2, 3$. They are elements of G. For $\xi \in \boldsymbol{C}$, let $B_i(\xi)$ be the element of $\mathscr{J}$ with $\xi_i = \xi$, all remaining terms zero in (6.1) for $i = 1, 2, 3$ and put $c = \exp \xi$. We define an element $S_i(c)$ of G by $S_i(c) = \exp R_{B_i(\xi)}$ for $i = 1, 2, 3$. The following proposition is well-known.

PROPOSITION 6–2. *There exist four orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 10, 27)$ where S_m denotes the m-codimensional orbit, and x_m is given as follows*:

$$x_0 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \ x_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \ x_{10} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \ x_{27} = (0).$$

Proof. Let X be a non-zero element of $\mathscr{J}$. Then we may assume that $\xi_1 = 1$ by t_i, t_i' and S_1. By $t_2(-\bar{x}_2)$ and $t_3(-x_3)$, we have $x_2 = x_3 = 0$. Unless $\xi_2 = \xi_3 = x_1 = 0$, we have $\xi_2 = 1$ by t_1' and S_2. Then by $t_1(-x_1)$, we have $x_1 = 0$. If $\xi_3 \neq 0$, we have $\xi_3 = 1$ by S_3. Thus we obtain four orbits. We shall calculate their codimension later. Q.E.D.

DEFINITION 6–3. We identify the dual vector space V^* of $V = \mathscr{J}$ with V by $\langle X, Y \rangle = \mathrm{tr}\, X \circ Y$. Then the dual actions are given as follows: (i) $D^*Y = DY$ for $D \in \mathscr{D}_0$. (ii) $T_i^*(a)Y = -T_i'(a)Y$ for $a \in \mathscr{L}$ and $i = 1, 2, 3$. (iii) $T_i'^*(a)Y = -T_i(a)Y$ for $i = 1, 2, 3$ and $a \in \mathscr{L}$. (iv) $R_z^*Y = -R_zY$ for $z \in \mathscr{J}$.

DEFINITION 6–4. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, the dual space has also four orbits S_m^* $(m = 0, 1, 10, 27)$. We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m(resp. S_m^*). Clearly we have $\Lambda_0 = V \times \{0\} = \Lambda_{27}^*$ and $\Lambda_{27} = \{0\} \times V^* = \Lambda_0^*$.

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is $\mathscr{D}_0$ which is the Lie algebra of F_4 (See [1]). Since $\Lambda_0 = V \times \{0\}$, we have $\mathrm{ord}_{\Lambda_0} f^s = 0$.

(2) For $\tilde{A} = D \oplus \sum_{i=1}^3 (T_i(a_i) \oplus T_i'(a_i')) \oplus R_{\begin{pmatrix}\alpha_1 & & 0\\ & \alpha_2 & \\ 0 & & \alpha_3\end{pmatrix}}$, we have $d\rho(\tilde{A})x_1 = \begin{pmatrix}\alpha_1 & a_3+a_3' & a_2\\ \overline{a_3+a_3'} & \alpha_2 & a_1\\ \bar{a}_2 & \bar{a}_1 & 0\end{pmatrix}$, and hence $\tilde{A}$ is an element of the isotropy subalgebra $\mathfrak{g}_{x_1}$ at x_1 if and only if $\tilde{A} = D \oplus T_1'(a_1') \oplus T_2'(a_2') \oplus [T_3(a_3) \oplus T_3'(-a_3)] \oplus R_{\begin{pmatrix}0 & & \\ & 0 & \\ & & \alpha_3\end{pmatrix}}$. The conormal vector space $V_{x_1}^*$ is given by $V_{x_1}^* = \left\{\begin{pmatrix}0 & & 0\\ & 0 & \\ 0 & & \eta\end{pmatrix}; \eta \in \boldsymbol{C}\right\} \cong \{\eta\}$ and $d\rho_{x_{10}}(\tilde{A})\eta = -\alpha_3\eta$. For $\tilde{A}_0 = R_{\begin{pmatrix}0 & & \\ & 0 & \\ & & -1\end{pmatrix}}$, we have $d\rho(\tilde{A}_0)x_1 = 0$ and $d\rho^*(\tilde{A}_0)y_1 = y_1$ where $y_1 = \begin{pmatrix}0 & & \\ & 0 & \\ & & 1\end{pmatrix}$. Since $\delta\chi(\tilde{A}_0) = -1$, $\mathrm{tr}_{V_{x_1}^*} \tilde{A}_0 = \dim V_{x_1}^* = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$. By Corollary 1–2, we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$.

(3) For $\tilde{A} = D \oplus \sum_{i=1}^3 (T_i(a_i) \oplus T_i'(a_i')) \oplus R_{\begin{pmatrix}\alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3\end{pmatrix}}$, we have $d\rho(\tilde{A})x_{10} = \begin{pmatrix}\alpha_1 & a_3 & a_2\\ \bar{a}_3 & 0 & 0\\ \bar{a}_2 & 0 & 0\end{pmatrix}$ and hence $\tilde{A} \in \mathfrak{g}_{x_{10}}$ if and only if $\tilde{A} = D \oplus T_1(a_1) \oplus T_1'(a_1') \oplus T_2'(a_2') \oplus T_3'(a_3') \oplus R_{\begin{pmatrix}0 & & \\ & \alpha_2 & \\ & & \alpha_3\end{pmatrix}}$. In this case, $\tilde{A}$ acts on the conormal vector space as follows:

$$d\rho_{x_{10}}(\tilde{A}) \cdot \begin{pmatrix}0 & 0 & 0\\ 0 & \eta_2 & y_1\\ 0 & \bar{y}_1 & \eta_3\end{pmatrix} = \left\{\begin{array}{c|c|c} 0 & & \\ \hline & \begin{matrix}-\alpha_2\eta_2\\ -\mathrm{tr}(y_1\bar{a}_1)\end{matrix} & \begin{matrix}Uy_1 - a_1\eta_3 - a_1'\eta_2\\ -\frac{1}{2}(\alpha_2+\alpha_3)y_1\end{matrix} \\ \hline & \begin{matrix}\overline{Uy_1 - a_1\eta_3 - a_1'\eta_2}\\ -\frac{1}{2}(\alpha_2+\alpha_3)\bar{y}_1\end{matrix} & -\alpha_3\eta_3 - \mathrm{tr}(\bar{y}_1 a_1')\end{array}\right] .$$

Let y_{10} (resp. y_{10}') be the element of $\mathscr{J}$ with $x_1 = 1$ (resp. $\xi_3 = 1$), all remaining parts zero in (6.1). Then y_{10} is a generic point of $(G_{x_{10}}, \rho_{x_{10}}, V_{x_{10}}^*)$ and y_{10}' is a point of the one-codimensional orbit. Thus we have $\Lambda_{10} = \Lambda_1^*$, $\Lambda_1 = \Lambda_{10}^*$ and $\dim \Lambda_1 \cap \Lambda_{10} = \dim V - 1$. Put $\tilde{A}_0 = R_{\begin{pmatrix}0 & & \\ & -1 & \\ & & -1\end{pmatrix}}$. Then we have $d\rho(\tilde{A}_0)x_{10} = 0$ and $d\rho^*(\tilde{A}_0)y_{10} = y_{10}$. Since $\delta\chi(\tilde{A}_0) = -2$, $\mathrm{tr}_{V_{x_{10}}^*} \tilde{A}_0 = 10 = \dim V_{x_{10}}^*$, we have $\mathrm{ord}_{\Lambda_{10}} f^s = -2s - \frac{10}{2}$. By Corollary 1–2, we have $b_{\Lambda_{10}}(s)/b_{\Lambda_1}(s) = (s+5)$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_{27}}$ is $\mathfrak{g}$. Put $y_{27} = x_0$ and $y_{27}' = x_1'$. Then $d\rho(\tilde{A}_0)x_{27} = 0$ and $d\rho^*(\tilde{A}_0)y_{27} = y_{27}$ for $\tilde{A}_0 = R_{-I_3}$. Since $\delta\chi(\tilde{A}_0) = -3$, $\mathrm{tr}_{V_{x_{27}}^*} \tilde{A}_0$

$= \dim V^*_{x_{27}} = 27$, we have $\mathrm{ord}_{\Lambda_{27}} f^s = -3s - \frac{27}{2}$. Since $\Lambda_{10} = \Lambda_1^*$, $\Lambda_{27} = \Lambda_0^*$, we have codim $\Lambda_{10} \cap \Lambda_{27} = 1$ and $b_{\Lambda_{27}}(s)/b_{\Lambda_{10}}(s) = (s+9)$. Thus we obtain the b-function $b(s) = (s+1)(s+5)(s+9)$ and the holonomy diagram (Figure 6-1). Note that the relative invariant $f(X)$ is given by the determinant $\det X$ of X in $\mathscr{J}$.

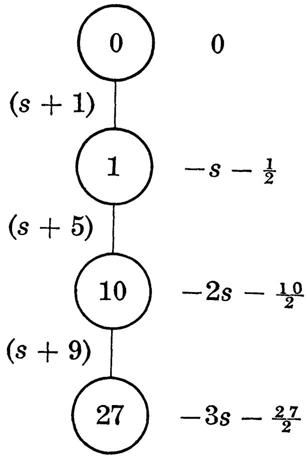


Figure 6-1. Holonomy diagram of $(GL(1) \times E_6,$
$\square \otimes \Lambda_1,\ V(1) \otimes V(27))$

§ 7. $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$

The representation space $V(1) \otimes V(56)$ is identified with

$$V = \{X = (x, x');\ x, x' \in M(8),\ {}^t x = -x,\ {}^t x' = -x'\}\,. \tag{7.1}$$

Then the infinitesimal action $d\rho$ of $\mathfrak{g} = \mathfrak{gl}(1) \oplus E_7$ is given by

$$\begin{aligned} &(1)\quad (x, x') \overset{p}{\mapsto} (px + x^t p,\ -{}^t p y - yp) \quad \text{for} \quad p \in SL(8, C) \\ &(2)\quad (x, x') \overset{c}{\mapsto} (cx, cx') \quad \text{for} \quad c \in \mathfrak{gl}(1) \\ &(3)\quad ((x_{ij}), (x'_{ij})) \overset{\theta}{\mapsto} \left(\left(\sum_{m,n=1}^{8} \vartheta^{ijmn} y_{mn}\right),\ \left(-\sum_{m,n=1}^{8} \vartheta_{ijmn} x_{mn}\right)\right) \end{aligned} \tag{7.2}$$

where ϑ denotes a tensor, antisymmetric in its indices, and upper, lower indices satisfy the relation

$$\vartheta_{i_1,\cdots,i_4} = \frac{1}{4!} \sum_{j_1,\cdots,j_4} I^{1,\cdots,8}_{i_1,\cdots,i_4,j_1,\cdots,j_4} \vartheta^{j_1,\cdots,j_4}\,.$$

Here $I^{1,\cdots,8}_{i_1,\cdots,i_4,j_1,\cdots,j_4}$ denotes the signature of the permutation $\begin{pmatrix} 1, & \cdots & \cdots, & 8 \\ i_1, & \cdots, & i_4, j_1, & \cdots, j_4 \end{pmatrix}$ when $\{i_1, \cdots, i_4, j_1, \cdots, j_4\} = \{1, \cdots, 8\}$, and 0 otherwise. The product as a Lie algebra is given as follows:

$$
\begin{aligned}
&(1)\quad [p, p'] = pp' - p'p \text{ where } pp' \text{ denotes the matrix multiplication} \\
(7.3)\quad &(2)\quad [p, \vartheta] = \vartheta' \text{ where } (\vartheta')^{ijk\ell} = \sum_m (\vartheta^{mjkl} p_{im} + \vartheta^{imk\ell} p_{jm} + \vartheta^{ijm\ell} p_{km} + \\
&\qquad \vartheta^{ijkm} p_{\ell m}) \\
&(3)\quad [\vartheta, \vartheta'] = p \text{ where } p_{ij} = \tfrac{2}{3}\left(\sum_{\ell,m,n} \left(\vartheta^{\ell m n i} (\vartheta')_{\ell m n j} - \tfrac{1}{8}\left(\sum_r \vartheta^{\ell m n r} (\vartheta')_{\ell m n r} \right) \delta_{ij} \right)\right)
\end{aligned}
$$

PROPOSITION 7–1 (Stephen J. Haris). *There exist five orbits* $S_m = \rho(G)x_m$ ($m = 0, 1, 11, 28, 56$) *where* S_m *denotes the m-codimensional G-orbit and* x_m *is given as follows.*

$$
x_0 = \left(\left[\begin{array}{cccc|cccc}
 & 1 & & & & & & \\
-1 & & & & & & & \\
 & & & 1 & & & & \\
 & & -1 & & & & & \\
\hline
 & & & & & 1 & & \\
 & & & & -1 & & & \\
 & & & & & & & 1 \\
 & & & & & & -1 &
\end{array} \right], 0 \right), \quad
x_1 = \left(\left[\begin{array}{cccc|cccc}
 & 1 & & & & & & \\
-1 & & & & & & & \\
 & & & 1 & & & & \\
 & & -1 & & & & & \\
\hline
 & & & & & 1 & & \\
 & & & & -1 & & & \\
 & & & & & & & \\
 & & & & & & 0 &
\end{array} \right], 0 \right)
$$

$$
x_{11} = \left(\left[\begin{array}{cccc|cccc}
 & 1 & & & & & & \\
-1 & & & & & & & \\
 & & & 1 & & & & \\
 & & -1 & & & & & \\
\hline
 & & & & & & & \\
 & & & & & & & \\
 & & & & & 0 & & \\
 & & & & & & &
\end{array} \right], 0 \right), \quad
x_{28} = \left(\left[\begin{array}{cccc|cccc}
 & 1 & & & & & & \\
-1 & & & & & & & \\
 & & 0 & & & & & \\
 & & & & & & & \\
\hline
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & &
\end{array} \right], 0 \right)
$$

and $x_{56} = (0, 0)$.

Proof. See [5]. Q.E.D.

We identify the dual vector space V^* with V by $\langle X, Y \rangle = \operatorname{tr} xx' + \operatorname{tr} yy'$ for $X = (x, x')$, $Y = (y, y') \in V$.

Then the dual action $d\rho^*$ is given as follows:

$$
\begin{aligned}
&(1)\quad (y, y') \overset{p^*}{\mapsto} (-{}^t py - yp, py' + y'{}^t p) \\
(7.4)\quad &(2)\quad (y, y') \overset{c^*}{\mapsto} (-\,cy, -\,cy') \\
&(3)\quad ((y_{ij}), (y'_{ij})) \overset{\theta^*}{\mapsto} \left(\left(\sum_{m,n=1}^{8} \vartheta_{ijmn} y'_{mn} \right), \left(-\sum_{m,n=1}^{8} \vartheta^{ijmn} y_{mn} \right) \right)
\end{aligned}
$$

Since $G = GL(1) \times E_7$ is reductive, the dual triplet (G, ρ^*, V^*) has also five orbits S^*_m ($m = 0, 1, 11, 28, 56$). We denote by Λ_m (resp. Λ^*_m) the conormal bundle of S_m (resp. S^*_m).

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is the Lie algebra of E_6 (See [5], [1]). Since $\Lambda_0 = V \times \{0\} = \Lambda^*_{56}$, we have $\mathrm{ord}_{\Lambda_0} f^s = 0$ where $f(X) = \mathrm{Pf}(x) + \mathrm{Pf}(x') - \frac{1}{4}\mathrm{tr}(xx'xx') + \frac{1}{16}\mathrm{tr}(xx')^2$ for $X = (x, x') \in V$.

(2) The isotropy subalgebra $\mathfrak{g}_{x_1}$ at x_1 is the set $\{c \oplus p \oplus \theta\}$ satisfying the following conditions:

$$(7.4) \quad p = \begin{bmatrix} -\frac{c}{2} I_6 & \\ & \frac{3}{2} c I_2 \end{bmatrix} + \begin{bmatrix} p_1 & p_{12} & p_{13} & p_{14} \\ p'_{12} & p_2 & p_{23} & p_{24} \\ p'_{13} & p'_{23} & p_3 & p_{34} \\ 0 & 0 & 0 & p_4 \end{bmatrix} \quad \text{where } \mathrm{tr}\, p_i = 0$$

for $i = 1, \cdots, 4$, and $p_{ij} p'_{ij} = -\det p_{ij} \cdot I_2$ for $1 \leq i < j \leq 3$. $\vartheta = (\vartheta_{ijkl})$ satisfies $\vartheta_{12ij} + \vartheta_{34ij} + \vartheta_{56ij} = 0$.

In fact, the isotropy subgroup is connected, and is isomorphic to $(GL(1) \times F_4) \cdot U$ where U is unipotent of dimension 26 (See [5]). The conormal vector space $V^*_{x_1}$ is given by

$$(7.5) \qquad V^*_{x_1} = \left\{ x = \left(\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{array}{cc} 0 & x \\ -x & 0 \end{array} \end{array} \right), 0 \right) ;\ x \in C \right\}.$$

Let y_1 be the element of $V^*_{x_1}$ with $x = 1$ in (7.5). Then it is a generic point, and $y'_1 = 0$ is the point of the one-codimensional orbit. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $c = -\frac{1}{4}$, all remaining parts zero in (7.4). Then $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$. Since $\delta\chi(A_0) = -1$, $\mathrm{tr}_{V^*_{x_1}} A_0 = \dim V^*_{x_1} = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ and $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$.

(3) The isotropy subalgebra $\mathfrak{g}_{x_{11}}$ at x_{11} is the set $\{c \oplus p \oplus \theta\}$ satisfying the following conditions

$$(7.6) \quad \begin{aligned} p &= \left[\begin{array}{c|c} -\frac{c}{2} I_4 + p_1 & p_2 \\ \hline 0 & \frac{c}{2} I_4 + p_4 \end{array} \right] \quad \text{where } K^{-1} p_1 K \in \mathfrak{sp}(2) \text{ with} \\ K &= \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix}, \ \mathrm{tr}\, p_4 = 0,\ (\vartheta) \text{ with } \vartheta_{12ij} + \vartheta_{34ij} = 0 \text{ for all } i, j. \end{aligned}$$

The conormal vector space $V^*_{x_{11}}$ is given by

$$V^*_{x_{11}} = \left\{ \tilde{Y} = \left(\begin{pmatrix} 0 & 0 \\ 0 & Y_4 \end{pmatrix}, \begin{pmatrix} Y'_1 & 0 \\ 0 & 0 \end{pmatrix} \right); y'_{12} + y'_{34} = 0 \right\} . \tag{7.7}$$

Then, for $A = c \oplus p \oplus \theta$ in $\mathfrak{g}_{x_{11}}$, we have $d\rho^*(A)\tilde{Y} = \left(\begin{pmatrix} 0 & 0 \\ 0 & Z_4 \end{pmatrix}, \begin{pmatrix} Z'_1 & 0 \\ 0 & 0 \end{pmatrix} \right)$ where

$$Z_4 = -2cY_4 - {}^tP_4Y_4 - Y_4P_4 + \left(\sum_{m,n=1}^{4} \vartheta_{ijmn} y'_{mn} \right)$$

and

$$Z'_1 = -2cY'_1 + P_1Y'_1 + Y'_1{}^tP_1 - \left(\sum_{m,n=5}^{8} \vartheta^{ijmn} y_{mn} \right) .$$

Put $y_{11} = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{matrix} \end{array} \right], 0 \right\}$ and $y'_{11} = \left\{ \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} & & & \\ & & & \\ & & & 1 \\ & & -1 & \end{matrix} \end{array} \right], 0 \right\}$.

Then y_{11} is a generic point and y'_{11} is a point of the unique one-codimensional orbit. Thus we have $\Lambda_{11} = \Lambda^*_{11}$ and $\dim \Lambda_1 \cap \Lambda_{11} = \dim V - 1$. Let A_0 be an element of $\mathfrak{g}_{x_{11}}$ with $c = -\frac{1}{2}$, all remaining parts zero in (7.7). Then $d\rho(A_0)x_{11} = 0$ and $d\rho^*(A_0)y_{11} = y_{11}$. Since $\delta\chi(A_0) = 4c = -2$, $\operatorname{tr}_{V^*_{x_{11}}} A_0 = -22c = 11$ and $\dim V^*_{x_{11}} = 11$, we have $\operatorname{ord}_{\Lambda_{11}} f^s = -2s - \frac{11}{2}$ and hence $b_{\Lambda_{11}}(s)/b_{\Lambda_1}(s) = (s + \frac{11}{2})$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_{28}}$ at x_{28} is the set $\{c \oplus p \oplus \theta\}$ satisfying the following conditions:

$$p = \left[\begin{array}{c|c} -\frac{c}{2} I_2 + p_1 & p_2 \\ \hline 0 & \frac{c}{6} I_2 + p_4 \end{array} \right] \quad \text{with} \quad \operatorname{tr} p_1 = \operatorname{tr} p_4 = 0 \tag{7.8}$$

$$\vartheta = (\vartheta^{ijk\ell}) \quad \text{with} \quad \vartheta_{12ij} = 0 \quad \text{for all } i, j.$$

The conormal vector space $V^*_{x_{28}}$ is given by

$$V^*_{x_{28}} = \left\{ \tilde{Y} = \left(\begin{pmatrix} \underbrace{\begin{matrix} 0 \\ 0 \end{matrix}}_{2} & \underbrace{\begin{matrix} 0 \\ Y_4 \end{matrix}}_{6} \end{pmatrix}, \begin{pmatrix} \underbrace{\begin{matrix} Y'_1 \\ -{}^tY'_3 \end{matrix}}_{2} & \underbrace{\begin{matrix} Y'_3 \\ 0 \end{matrix}}_{6} \end{pmatrix} \right) \in V^* \right\} . \tag{7.9}$$

Then for $A = c \oplus p \oplus \theta$ in (7.8), we have

$$d\rho^*(A)\tilde{Y} = \left(\begin{pmatrix} 0 & 0 \\ 0 & Z_4 \end{pmatrix}, \begin{pmatrix} Z'_1 & Z'_3 \\ -{}^tZ'_3 & 0 \end{pmatrix} \right)$$

where

$$Z_4 = -\frac{4}{3}cY_4 - {}^tp_4Y_4 - Y_4p_4 + \left(\sum_{m,n}\vartheta_{ijmn}y'_{mn}\right)$$

$$Z'_1 = -2cY'_1 + p_1Y'_1 + Y'_1{}^tp_1 - p_2{}^tY'_3 + Y'_3{}^tp_2 - \left(\sum_{m,n=3}^{8}\vartheta^{ijmn}y_{mn}\right)$$

$$Z'_3 = -\frac{4}{3}cY'_3 + p_1Y'_3 + Y'_3{}^tp_4 - \left(\sum_{m,n=3}^{8}\vartheta^{ijmn}y_{mn}\right).$$

Therefore, one can see that the colocalization at x_{28} has at most unique one-codimensional orbit.

Since $\Lambda_{28} = \Lambda_1^*$ and $\Lambda_{11} = \Lambda_{11}^*$, Λ_{28} is a good holonomic variety and $\dim \Lambda_{28} \cap \Lambda_{11} = \dim V - 1$.

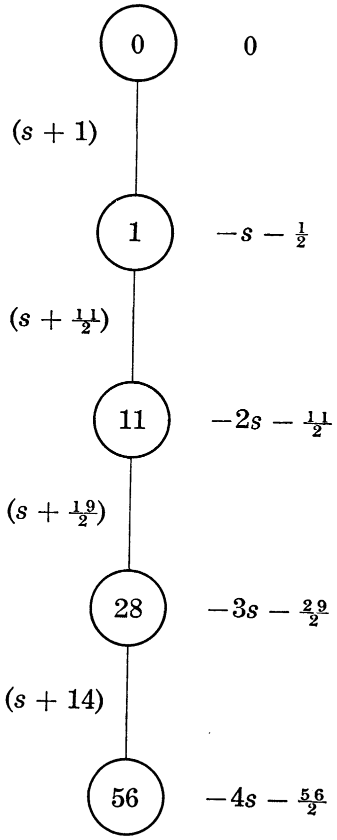


Figure 7-1. Holonomy diagram of $(GL(1)\times E_7,$
$\square\otimes\Lambda_1,\ V(1)\otimes V(56))$

Put $$y_{28} = \left\{\left[\begin{array}{c|cccccc} 0 & & & & & & \\ \hline & & 1 & & & & \\ & -1 & & & & & \\ & & & & 1 & & \\ & & & -1 & & & \\ & & & & & & 1 \\ & & & & & -1 & \end{array}\right], 0\right\}.$$ Then y_{28} is a generic point.

Let A_0 be an element of $\mathfrak{g}_{x_{28}}$ with $c = -\frac{3}{4}$, all remaining parts zero in (7.8). Then we have $d\rho(A_0)x_{28} = 0$ and $d\rho^*(A_0)y_{28} = y_{28}$. Since $\delta\chi(A_0) = 4c = -3$, $\operatorname{tr}_{V^*_{x_{28}}} A_0 = -38c = \frac{57}{2}$ and $\dim V^*_{x_{28}} = 28$, we have $\operatorname{ord}_{\Lambda_{28}} f^s = -3s - \frac{29}{2}$ and hence $b_{\Lambda_{28}}(s)/b_{\Lambda_{11}}(s) = (s + \frac{19}{2})$ by Corollary 1–2.

(5) Since $x_{56} = 0$, we have $(G_{x_{56}}, \rho_{x_{56}}, V^*_{x_{56}}) = (G, \rho^*, V^*)$. Since $\Lambda_{56} = \{0\} \times V^* = \Lambda_0^*$ and $\Lambda_{28} = \Lambda_1^*$, we have $\dim \Lambda_{56} \cap \Lambda_{28} = \dim V - 1$ and they intersect regularly. Let A_0 be an element of $\mathfrak{g}_{x_{28}}$ with $c = -1$, all remaining parts zero in (7.2). Then $d\rho(A_0)x_{56} = 0$ and $d\rho^*(A_0)y_{56} = y_{56}$ where $y_{56} = x_0$. Since $\delta\chi(A_0) = -4$, $\operatorname{tr}_{V^*_{x_{56}}} A_0 = \dim V^m_{x_{56}} = 56$, we have $\operatorname{ord}_{\Lambda_{56}} f^s = -4s - \frac{56}{2}$, and hence $b_{\Lambda_{56}}(s)/b_{\Lambda_{28}}(s) = s + 14$. Thus we obtain the b-function $b(s) = (s+1)(s + \frac{11}{2})(s + \frac{19}{2})(s + 14)$ and the holonomy diagram (Figure 7–1).

§ 8. $(GL(6), \Lambda_3, V(20))$

Let V_1 be a 6-dimensional vector space spanned by $u_1, \cdots, u_6$. Then $G = GL(6)$ acts on V_1 by $\rho_1(g)(u_1, \cdots, u_6) = (u_1, \cdots, u_6)g$ for $g \in G$. The representation space $V = V(20)$ is spanned by skew-tensors $u_i \wedge u_j \wedge u_k$ $(1 \leq i < j < k \leq 6)$, and $\rho = \Lambda_3$ is given by $\rho(g)(u_i \wedge u_j \wedge u_k) = \rho_1(g)u_i \wedge \rho_1(g)u_j \wedge \rho_1(g)u_k$ for $1 \leq i < j < k \leq 6$, and $g \in G$. Then it is well-known (and also one can easily check) that there exist five G-orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 5, 10, 20)$ where S_m denotes the m-codimensional orbit, and $x_0 = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$, $x_1 = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5 + u_2 \wedge u_4 \wedge u_6$, $x_5 = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5$, $x_{10} = u_1 \wedge u_2 \wedge u_3$, and $x_{20} = 0$. We identify the dual space V^* with V by $\langle \sum a_{ijk} u_i \wedge u_j \wedge u_k, \sum b_{rst} u_r \wedge u_s \wedge u_t \rangle = \sum_{1 \leq i < j < k \leq 6} a_{ijk} b_{ijk}$. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also five orbits S_m^* $(m = 0, 1, 5, 10, 20)$ in V^*. We denote by Λ_m the conormal bundle of S_m. The isotropy subalgebra $\mathfrak{g}_x$ at $x \in V(20)$ is, by definition, $\mathfrak{g}_x = \{A \in \mathfrak{gl}(6);\ d\rho(A)x = 0\}$ where $d\rho(A)(u_i \wedge u_j \wedge u_k) = d\rho_1(A)u_i \wedge u_j \wedge u_k + u_i \wedge d\rho_1(A)u_j \wedge u_k + u_i \wedge u_j \wedge d\rho_1(A)u_k$.

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ is, by simple calculation, given as follows:

$$\mathfrak{g}_{x_0} = \left\{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{gl}(6);\, A, B \in \mathfrak{sl}(3)\right\}. \tag{8.1}$$

We have $\Lambda_0 = V \times \{0\}$, and hence $\mathrm{ord}_{\Lambda_0} f^s = 0$ where f denotes the relatively invariant irreducible polynomial of degree four (See [1], [14]).

(2) Put $x_1' = u_1 \wedge u_2 \wedge u_6 + u_2 \wedge u_3 \wedge u_4 - u_1 \wedge u_3 \wedge u_5$. Then $x_1' \in S_1$. By simple calculation, the isotropy subalgebra $\mathfrak{g}_{x_1'}$ at x_1' is given by

$$\mathfrak{g}_{x_1'} = \left\{\tilde{A} = \left(\begin{array}{c|c} A & B \\ \hline 0 & A - (\mathrm{tr}\, A)\cdot I_3 \end{array}\right) \in \mathfrak{gl}(6);\, A, B \in M(3), \mathrm{tr}\, B = 0\right\}. \tag{8.2}$$

Therefore we have $G_{x_1'} \sim GL(3)\cdot(G_a)^8$ where $\cdot$ denotes a semi-direct product and $G_a \cong \boldsymbol{C}$. The conormal vector space $V_{x_1'}^*$ is of one-dimension with a basis $u_4 \wedge u_5 \wedge u_6$. The action $d\rho_{x_1'}$ of $\mathfrak{g}_{x_1'}$ on $V_{x_1'}^*$ is $d\rho_{x_1'}(\tilde{A})u_4 \wedge u_5 \wedge u_6 = -2\mathrm{tr}\,\tilde{A}\cdot u_4 \wedge u_5 \wedge u_6$. Take $A_0 \in \mathfrak{g}_{x_1'}$ with $\mathrm{tr}\, A_0 = -\frac{1}{2}$. Then we have $d\rho(A_0)x_1' = 0$ and $d\rho^*(A_0)y_1 = y_1$ where $y_1 = u_4 \wedge u_5 \wedge u_6$. Since $\delta\chi(A_0) = (\deg f/\dim V)\,\mathrm{tr}\, d\rho(A_0) = \frac{4}{20} \times (10\mathrm{tr}\, A_0) = -1$ and $\mathrm{tr}_{V_{x_1'}^*} A_0 = \dim V_{x_1'}^* = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ by Proposition 1–3. Since 0 is the point of the one-codimensional orbit, we have $\dim \Lambda_0 \cap \Lambda_1 = \dim V - 1$ and $\Lambda_0 \cap \Lambda_1$ is G_0-prehomogeneous, i.e., Λ_1 is a good holonomic variety by Proposition 1–5. Also we have $\mu = 1$ and $\nu = 0$ by Proposition 1–4, i.e., Λ_0 and Λ_1 intersect regularly. By Corollary 1–2, we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$.

(3) Put $x_5' = u_1 \wedge (u_2 \wedge u_4 + u_3 \wedge u_5) \in S_5$. Then the isotropy subalgebra $\mathfrak{g}_{x_5'}$ is given as follows:

$$\begin{aligned}\mathfrak{g}_{x_5'} &= \left\{\left(\begin{array}{c|c|c} -2\varepsilon & B & C \\ \hline 0 & A+\varepsilon I_4 & D \\ \hline 0 & 0 & \eta \end{array}\right) \in \mathfrak{gl}(6);\, A \in \mathfrak{sp}(2), {}^tB, D \in \boldsymbol{C}^4, C \in \boldsymbol{C}\right\} \\ &= (\mathfrak{sp}(2) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(9)\end{aligned} \tag{8.3}$$

where $\mathfrak{u}(9)$ denotes the Lie algebra of 9-dimensional unipotent group. Put $\omega_1 = (u_2 \wedge u_4 - u_3 \wedge u_5) \wedge u_6$, $\omega_2 = u_4 \wedge u_5 \wedge u_6$, $\omega_3 = u_3 \wedge u_4 \wedge u_6$, $\omega_4 = u_2 \wedge u_5 \wedge u_6$ and $\omega_5 = u_2 \wedge u_3 \wedge u_6$. Then the conormal vector space $V_{x_5'}^*$ is spanned by $\omega_1, \cdots, \omega_5$ and $(G_{x_5'}, \rho_{x_5'}, V_{x_5'}^*) \cong (GL(1) \times Sp(2), \Lambda_1 \otimes \Lambda_2, V(1) \otimes V(5)) \cong (GL(1) \times SO(5), \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(5))$, where ω_1 is a generic point and $\omega_2 = u_4 \wedge u_5 \wedge u_6$ is a point of the one-codimensional orbit. Therefore we have $\dim \Lambda_1 \cap \Lambda_5 = \dim V - 1$. Since the $(G_{x_5'} \cap G_0)$-orbit of ω_2 is one-codimensional in $V_{x_5'}^*$, i.e., $\Lambda_1 \cap \Lambda_5$ is G_0-prehomogeneous, Λ_5 is a good holonomic variety by (2) and Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_5'}$ with $\eta = -1$ and all remaining parts zero in (8.3). Then we have

$d\rho(A_0)x_5' = 0$ and $d\rho^*(A_0)\omega_1 = \omega_1$. Since $\delta\chi(A_0) = 2\mathrm{tr}\, A_0 = -2$, $\mathrm{tr}_{V_{x_5'}}A_0 = -5(2\varepsilon + \eta) = 5$, and $\dim V_{x_5'}^* = 5$, we have $\mathrm{ord}_{\Lambda_5} f^s = -2s - \frac{5}{2}$. Put $A_\beta = \beta(E_{22} - E_{44}) + (\beta + 1)E_{66}$ for $\beta \in \boldsymbol{C}$ where E_{ij} denotes the matrix unit. Then $d\rho(A_\beta)x_5' = 0$ and $d\rho^*(A_\beta)\omega_2 = \omega_2$. Since $\tilde{V} = V_{x_5'}^* \bmod d\rho_{x_5'}(\mathfrak{g}_{x_5'})\omega_2$ is spanned by $u_2 \wedge u_3 \wedge u_6$, we have $\mathrm{tr}_{\tilde{V}} A_\beta = 2\beta + 1$. Hence we have $\mu = 1$ and $\nu = 0$ by Proposition 1–4, i.e., Λ_1 and Λ_5 intersect regularly. One can also get this from the fact $m_{\Lambda_5} - m_{\Lambda_1} = 1$. By Corollary 1–2, we have $b_{\Lambda_5}(s)/b_{\Lambda_1}(s) = s + \frac{5}{2}$.

(4) Put $x_{10} = u_1 \wedge u_2 \wedge u_3 \in S_{10}$. Then the isotropy subalgebra $\mathfrak{g}_{x_{10}}$ is given as follows:

$$(8.4) \qquad \mathfrak{g}_{x_{10}} = \left\{ \tilde{A} = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array} \right) \in \mathfrak{gl}(6);\, A, B, C \in M(3), \mathrm{tr}\, A = 0 \right\}$$
$$\cong (\mathfrak{sl}(3) \oplus \mathfrak{gl}(3)) \oplus V(9), \text{ i.e., } G_{x_{10}} \sim (SL(3) \times GL(3)) \cdot (G_a)^9.$$

In general, we write $G_1 \sim G_2$ when two groups G_1 and G_2 are locally isomorphic to each other. Put $\omega_1 = u_1 \wedge u_4 \wedge u_5$, $\omega_2 = u_1 \wedge u_4 \wedge u_6$, $\omega_3 = u_1 \wedge u_5 \wedge u_6$, $\omega_4 = u_2 \wedge u_4 \wedge u_5$, $\omega_5 = u_2 \wedge u_4 \wedge u_6$, $\omega_6 = u_2 \wedge u_5 \wedge u_6$, $\omega_7 = u_3 \wedge u_4 \wedge u_5$, $\omega_8 = u_3 \wedge u_4 \wedge u_6$, $\omega_9 = u_3 \wedge u_5 \wedge u_6$ and $\omega_{10} = u_4 \wedge u_5 \wedge u_6$. Then the conormal vector space $V_{x_{10}}^*$ is spanned by $\omega_1, \cdots, \omega_{10}$. The action $d\rho_{x_{10}}$ of $\mathfrak{g}_{x_{10}}$ on $V_{x_{10}}^*$ is given by

$$(8.5) \qquad d\rho_{x_{10}}(\tilde{A})(\omega_1, \cdots, \omega_{10}) = (\omega_1, \cdots, \omega_{10}) \left(\begin{array}{c|c} d\tilde{\rho}(A \oplus C) & 0 \\ \hline B' & -\mathrm{tr}\, A \end{array} \right) \quad ({}^tB' \in \boldsymbol{C}^9)$$

where $\tilde{\rho} = \Lambda_2 \otimes \Lambda_2^*$, i.e., the action of G_{x_0} induced on the subspace spanned by $\omega_1, \cdots, \omega_9$ is isomorphic to a triplet $(SL(3) \times GL(3), \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3))$ as a triplet (See [1]). Then $\omega_1 + \omega_5 + \omega_9 \in S_1^*$ is a generic point and $\omega_1 + \omega_5 \in S_5^*$ is a point of the one-codimensional orbit. This implies that $\dim \Lambda_5 \cap \Lambda_{10} = \dim V - 1$. Since $\Lambda_5 \cap \Lambda_{10}$ is G_0-prehomogeneous, Λ_{10} is a good holonomic variety by Proposition 1–5. Let $\tilde{A}$ be an element of $\mathfrak{g}_{x_{10}}$ with $A = B = 0$ and $C = -\frac{1}{2}I_3$ in (8.4). Then $d\rho(\tilde{A})x_{10} = 0$ and $d\rho^*(\tilde{A})(\omega_1 + \omega_5 + \omega_9) = (\omega_1 + \omega_5 + \omega_9)$. Since $\delta\chi(\tilde{A}) = 2 \cdot \mathrm{tr}\, \tilde{A} = -3$, $\mathrm{tr}_{V_{x_{10}}^*} \tilde{A} = -7\mathrm{tr}\, C = \frac{21}{2}$ and $\dim V_{x_{10}}^* = 10$, we have $\mathrm{ord}_{\Lambda_{10}} f^s = -3s - \frac{11}{2}$ by Proposition 1–3. Let A_β be an element of $\mathfrak{g}_{x_{10}}$ with $A = B = 0$ and $C = \begin{pmatrix} 1-(\beta/2) & 0 \\ 0 & (\beta/2)I_2 \end{pmatrix}$ in (8.4). Then $d\rho(A_\beta)x_{10} = 0$ and $d\rho(A_\beta)(\omega_1 + \omega_5) = (\omega_1 + \omega_5)$. Since $\tilde{V} = V_{x_{10}}^* \bmod d\rho_{x_{10}}(\mathfrak{g}_{x_{10}})(\omega_1 + \omega_5)$ is spanned by $u_3 \wedge u_5 \wedge u_6$, we have $\mathrm{tr}_{\tilde{V}} A_\beta = \beta$. This implies that $\mu = 1$ and $\nu = 0$, i.e., Λ_5 and Λ_{10} intersect regularly by Proposition 1–4. One can also get this from the fact $m_{\Lambda_{10}} - m_{\Lambda_5} = 1$. By

Corollary 1–2, we have $b_{\Lambda_{10}}(s)/b_{\Lambda_5}(s) = s + \frac{7}{2}$.

(5) Put $x_{20} = 0 \in S_{20}$. In this case, $(G_{x_{20}}, \rho_{x_{20}}, V^*_{x_{20}}) \cong (GL(6), \Lambda_3, V(20))$. $\Lambda_{x_{20}} = \{0\} \times V^*$ is a good holonomic variety. Put $\tilde{A} = -\frac{1}{3}I_6$. Then $d\rho(\tilde{A})x_{20} = 0$ and $d\rho^*(\tilde{A})x_0^* = x_0^*$ where $x_0^* = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 \in S_0^*$. Since $\delta\chi(\tilde{A}) = 2\mathrm{tr}\,\tilde{A} = -4$, $\mathrm{tr}_{V^*_{x_{20}}}\tilde{A} = 20$ and $\dim V^*_{x_{20}} = 20$, we have $\mathrm{ord}_{\Lambda_{20}} f^s = -4s - \frac{20}{2}$ by Proposition 1–3. Put $A_\beta = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_6 \end{pmatrix}$ with $a_1 = a_2 = a_4 = 1/2 - \beta/6$, $a_3 = a_5 = a_6 = \beta/3$. Then $d\rho(A_\beta)x_{20} = 0$ and $d\rho^*(A_\beta)x_1^* = x_1^*$ where $x_1^* = u_1 \wedge u_2 \wedge u_3 + u_1 \wedge u_4 \wedge u_5 + u_2 \wedge u_4 \wedge u_6$. Since $\tilde{V} = V^*_{x_{20}} \bmod d\rho_{x_{20}}(\mathfrak{g}_{x_{20}})x_1^*$ is spanned by $u_3 \wedge u_5 \wedge u_6$, we have $\mathrm{tr}_{\tilde{V}} A_\beta = \beta$. This implies that $\mu = 1$ and $\nu = 0$, i.e., Λ_{20} and Λ_{10} intersect regularly. One can also get this from $m_{\Lambda_{20}} - m_{\Lambda_{10}} = 1$. By Corollary 1–2, we have $b_{\Lambda_{20}}(s)/b_{\Lambda_{10}}(s) = s + 5$. Thus we obtain the b-function $b(s) = (s+1)(s+\frac{5}{2})(s+\frac{7}{2})(s+5)$ and the holonomy diagram (Figure 8–1). We denote $\textcircled{\Lambda_m}$ by $\textcircled{m}$.

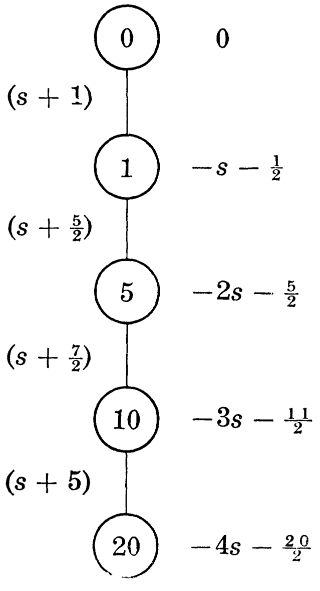


Figure 8–1. Holonomy diagram of $(GL(6), \Lambda_3, V(20))$.

§ 9. $(GL(1) \times Sp(3),\ \square \otimes \Lambda_3,\ V(1) \otimes V(14))$

Put $\omega_1 = u_1 \wedge u_2 \wedge u_3$, $\omega_2 = u_4 \wedge u_5 \wedge u_6$, $\omega_3 = u_2 \wedge u_3 \wedge u_4$, $\omega_4 = u_1 \wedge u_5 \wedge u_6$, $\omega_5 = u_1 \wedge u_3 \wedge u_5$, $\omega_6 = u_2 \wedge u_4 \wedge u_6$, $\omega_7 = u_1 \wedge u_2 \wedge u_6$, $\omega_8 = u_3 \wedge u_4 \wedge u_5$, $\omega_9 = u_1 \wedge u_2 \wedge u_5 - u_1 \wedge u_3 \wedge u_6$, $\omega_{10} = u_2 \wedge u_4 \wedge u_5 - u_3 \wedge u_4 \wedge u_6$,

$\omega_{11} = u_1 \wedge u_2 \wedge u_4 + u_2 \wedge u_3 \wedge u_6$, $\omega_{12} = u_1 \wedge u_4 \wedge u_5 + u_3 \wedge u_5 \wedge u_6$, $\omega_{13} = u_1 \wedge u_3 \wedge u_4 - u_2 \wedge u_3 \wedge u_5$, $\omega_{14} = u_1 \wedge u_4 \wedge u_6 - u_2 \wedge u_5 \wedge u_6$. Then the representation space V is identified with the subspace of $V(20)$ in § 8 generated by $\omega_1, \cdots, \omega_{14}$. Then the representation $\rho = \square \otimes \Lambda_3$ is the restriction of Λ_3 for $GL(6)$ to $G = GL(1) \times Sp(3)$. The orbital decomposition of this space has been completed by J-I. Igusa (See [3]). There exist five G-orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 4, 7, 14)$ where S_m denotes the m-codimensional orbit, and $x_0 = \omega_1 + \omega_2$, $x_1 = \omega_7 + \omega_{13}$, $x_4 = \omega_{13}$, $x_7 = \omega_1$, $x_{14} = 0$. We identify the dual space V^* with V by $(\sum_{i=1}^{14} a_i\omega_i, \sum_{j=1}^{14} b_j\omega_j) = \sum_{k=1}^{14} a_k b_k$. Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also five G-orbits S_m^* $(m = 0, 1, 4, 7, 14)$ in V^*. We denote by Λ_m the conormal bundle of S_m. The Lie algebra $\mathfrak{g}$ of $G = GL(1) \times Sp(3)$ is given as follows:

$$(9.1) \quad \mathfrak{g} = \left\{ \tilde{A} = (d) \oplus \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix};\ A, B, C \in M(3),\ {}^tB = B,\ {}^tC = C \right\}.$$

(1) Since $d\rho(\tilde{A})x_0 = (d + \operatorname{tr} A)\omega_1 + (d - \operatorname{tr} A)\omega_2 + c_1\omega_3 + b_1\omega_4 - c_2\omega_5 - b_2\omega_6 + c_3\omega_7 + b_3\omega_8 + c_{23}\omega_9 + b_{23}\omega_{10} + c_{13}\omega_{11} + b_{13}\omega_{12} - c_{12}\omega_{13} - b_{12}\omega_{14}$ where $c_i = c_{ii}$ and $b_i = b_{ii}$ for $i = 1, 2, 3$, we have

$$(9.2) \qquad \mathfrak{g}_{x_0} = \left\{ \tilde{A} = (0) \oplus \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix};\ A \in \mathfrak{sl}(3) \right\} \cong \mathfrak{sl}(3).$$

We have $\Lambda_0 = V \times \{0\}$, and hence $\operatorname{ord}_{\Lambda_0} f^s = 0$ where f denotes the relatively invariant irreducible polynomial of degree four (See [1], [3]).

(2) Since $d\rho(\tilde{A})x_1 = (b_3 - 2b_{12})\omega_1 + 2a_{21}\omega_3 + c_2\omega_4 - 2a_{12}\omega_5 - c_1\omega_6 + (d + a_1 + a_2 - a_3)\omega_7 + 2c_{12}\omega_8 + (a_{13} - a_{32})\omega_9 - c_{13}\omega_{10} + (a_{23} - a_{31})\omega_{11} - c_{23}\omega_{12} + (d + a_3)\omega_{13} + (c_{12} - c_3)\omega_{14}$ where $a_i = a_{ii}$ for $i = 1, 2, 3$, we have

$$(9.3) \qquad \mathfrak{g}_{x_1} = \left\{ (d) \oplus \left[\begin{array}{ccc|ccc} -d+\alpha & 0 & \beta & b_1 & b_{12} & b_{13} \\ 0 & -d-\alpha & \gamma & b_{12} & b_2 & b_{23} \\ \gamma & \beta & -d & b_{13} & b_{23} & 2b_{12} \\ \hline & & & d-\alpha & 0 & -\gamma \\ & 0 & & 0 & d+\alpha & -\beta \\ & & & -\beta & -\gamma & d \end{array} \right] \right\}$$
$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{o}(3)) \oplus V(5)$$

where $V(5)$ denotes the Lie algebra of $(G_a)^5$.

The conormal vector space $V_{x_1}^*$ is one-dimensional with a basis ω_2. The action $d\rho_{x_1}$ of $\mathfrak{g}_{x_1}$ on $V_{x_1}^*$ is given by $d\rho_{x_1}(\tilde{A})\omega_2 = (-d + a_1 + a_2 + a_3)\omega_2 = -4d\omega_2$. Therefore we have $\Lambda_1 = \overline{G(x_1, y_1)}$ where $y_1 = \omega_2$. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $d = -\frac{1}{4}$, all remaining parts zero in (9.3). Then we have $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$. Since $\delta\chi(A_0) = 4d = -1$, $\mathrm{tr}_{V^*_{x_1}}\tilde{A} = \dim V^*_{x_1} = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ by Proposition 1–3. Since 0 is the point of the one-codimensional orbit, we have $\dim \Lambda_0 \cap \Lambda_1 = \dim V - 1$ and $\Lambda_0 \cap \Lambda_1$ is G_0-prehomogeneous, i.e., Λ_1 is a good holonomic variety by Proposition 1–5. Also we have $\mu = 1$ and $\nu = 0$ by Proposition 1–4, i.e., Λ_0 and Λ_1 intersect regularly. By Corollary 1–2, we have $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$.

(3) Since $d\rho(\tilde{A})x_4 = -2b_{12}\omega_1 + 2a_{21}\omega_3 - 2a_{12}\omega_5 + 2c_{12}\omega_8 + a_{13}\omega_9 - c_{13}\omega_{10} + a_{23}\omega_{11} - c_{23}\omega_{12} + (d + a_3)\omega_{13} - c_3\omega_{14}$, we have

$$\mathfrak{g}_{x_4} = \left\{(d) \oplus \left[\begin{array}{ccc|ccc} a_1 & 0 & 0 & b_1 & 0 & \beta \\ 0 & a_2 & 0 & 0 & b_2 & \delta \\ \alpha & \gamma & -d & \beta & \delta & \varepsilon \\ \hline c_1 & 0 & 0 & -a_1 & 0 & -\alpha \\ 0 & c_2 & 0 & 0 & -a_2 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & d \end{array}\right]\right\}$$

$$(9.4) \qquad \cong \left\{(d) \oplus \left[\begin{array}{c|cc|cc|c} -d & \alpha & \beta & \gamma & \delta & \varepsilon \\ \hline 0 & a_1 & b_1 & 0 & & \beta \\ & c_1 & -a_1 & & & -\alpha \\ \hline 0 & 0 & & a_2 & b_2 & \delta \\ & & & c_2 & -a_2 & -\gamma \\ \hline 0 & 0 & & 0 & & d \end{array}\right]\right\}$$

$$\simeq (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(5)\,.$$

The conormal vector space $V^*_{x_4}$ is spanned by $\omega_2, \omega_4, \omega_6, \omega_7$ on which $\mathfrak{g}_{x_4}$ acts as follows:

$$(\omega_2, \omega_4, \omega_6, \omega_7) \mapsto (\omega_2, \omega_4, \omega_6, \omega_7)\begin{pmatrix} A_1 & -b_1 & b_2 & 0 \\ -c_1 & A_2 & 0 & -b_2 \\ c_2 & 0 & A_3 & b_1 \\ 0 & -c_2 & c_1 & A_4 \end{pmatrix}$$

where $A_1 = a_1 + a_2 - 2d$, $A_2 = -a_1 + a_2 - 2d$, $A_3 = a_1 - a_2 - 2d$, $A_4 = -a_1 - a_2 - 2d$.

Hence we have $(G_{x_4}, \rho_{x_4}, V^*_{x_4}) \cong (SL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2)) \cong (GL(1) \times SO(4), \square \otimes \Lambda_1, V(1) \otimes V(4))$.

Clearly, $y_4 = \omega_4 + \omega_6$ is its generic point, and ω_2 is a point of the one-codimensional orbit. Since $\Lambda_1 = \overline{G(x_1, \omega_2)}$, we have $\dim \Lambda_1 \cap \Lambda_4 = \dim V - 1$. Since $\Lambda_1 \cap \Lambda_4$ is G_0-prehomogeneous, Λ_4 is a good holonomic variety by (2) and Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_4}$ with d

$= -\frac{1}{2}$ and all remaining parts zero in (9.4). Then $d\rho(A_0)x_4 = 0$ and $d\rho^*(A_0)y_4 = y_4$. Since $\delta\chi(A_0) = 4d = -2$, $\mathrm{tr}_{V^*_{x_4}} A_0 = -8d = 4$ and $\dim V^*_{x_4} = 4$, we have $\mathrm{ord}_{\Lambda_1} f^s = -2s - \frac{4}{2}$. Let A_β be an element of $\mathfrak{g}_{x_4}$ with $a_1 = \frac{1}{2}(1-\beta)$, $d = -\frac{1}{4}(\beta+1)$, all remaining parts zero in (9.4). Then we have $d\rho(A_\beta)x_4 = 0$, $d\rho^*(A_\beta)\omega_2 = \omega_2$ and $\mathrm{tr}_{\tilde{V}} A_\beta = \beta$ where $\tilde{V} = V^*_{x_4} \bmod d\rho_{x_4}(\mathfrak{g}_{x_4})\omega_2$. This implies that Λ_4 and Λ_1 intersect regularly by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_4}(s)/b_{\Lambda_1}(s) = (s+2)$.

(4) Since $d\rho(\tilde{A})x_7 = (d + a_1 + a_2 + a_3)\omega_1 + c_1\omega_3 - c_2\omega_5 + c_3\omega_7 + c_{23}\omega_9 + c_{13}\omega_{11} - c_{12}\omega_{13}$, we have

$$(9.5)\qquad \mathfrak{g}_{x_7} = \left\{\tilde{A} = (-\mathrm{tr}\,A) \oplus \left(\begin{array}{c|c} A & B \\ \hline 0 & -{}^tA \end{array}\right);\ {}^tB = B\right\} \simeq \mathfrak{gl}(3) \oplus V(6).$$

The conormal vector space $V^*_{x_7}$ is spanned by $\omega_2, \omega_4, \omega_6, \omega_8, \omega_{10}, \omega_{12}, \omega_{14}$, and $\mathfrak{g}_{x_7}$ acts on $V^*_{x_7}$ as follows:

$$(9.6)\quad d\rho_{x_7}(\tilde{A})(\omega_2, \omega_4, \cdots, \omega_{14}) = (\omega_2, \omega_4, \cdots, \omega_{14})\left(\begin{array}{c|c} 2\,\mathrm{tr}\,A & B \\ \hline 0 & 2\,\mathrm{tr}\,A\cdot I_6 \oplus d\rho_1^*(A) \end{array}\right)$$

where ${}^tB \in \boldsymbol{C}^6$ and $\rho_1 = 2\Lambda_1$.

Then $y_7 = \omega_4 + \omega_{10}$ is its generic point, and $\omega_4 + \omega_6$ is a point of the one-codimensional orbit. Since $\Lambda_4 = \overline{G(x_4, \omega_4 + \omega_6)}$, we have $\dim \Lambda_4 \cap \Lambda_7 = \dim V - 1$. Since $\Lambda_4 \cap \Lambda_7$ is G_0-prehomogeneous, Λ_7 is a good holonomic variety by (3) and Proposition 1–5. Let A_0 be an element of $\mathfrak{g}_{x_7}$ with $A = \frac{1}{4}I_3$ and $B = 0$ in (9.5). Then $d\rho(A_0)x_7 = 0$ and $d\rho^*(A_0)y_7 = y_7$. Since $\delta\chi(A_0) = -4\,\mathrm{tr}\,A = -3$, $\mathrm{tr}_{V^*_{x_7}} A_0 = 10\,\mathrm{tr}\,A = \frac{15}{2}$ and $\dim V^*_{x_7} = 7$, we have $\mathrm{ord}_{\Lambda_7} f^s = -3s - \frac{8}{2}$. Let A_β be an element of $\mathfrak{g}_{x_7}$ with $a_1 = a_2 = \frac{\beta}{4}$, $a_3 = \frac{1}{2} - \frac{\beta}{4}$, all remaining parts zero in (9.5). Then $d\rho(A_\beta)x_7 = 0$, $d\rho(A_\beta)(\omega_4 + \omega_6) = (\omega_4 + \omega_6)$ and $\mathrm{tr}_{\tilde{V}} A_\beta = \beta$ where $\tilde{V} = V^*_{x_7} \bmod d\rho_{x_7}(\mathfrak{g}_{x_7})(\omega_4 + \omega_6) = \boldsymbol{C}\omega_8$. This implies that Λ_4 and Λ_7 intersect regularly by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_7}(s)/b_{\Lambda_4}(s) = (s + \frac{5}{2})$.

(5) Since $x_{14} = 0$, we have $(G_{x_{14}}, \rho_{x_{14}}, V^*_{x_{14}}) \cong (GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$ and $\Lambda_{14} = \{0\} \times V^*$ is a good holonomic variety. Take $\tilde{A} = (-1) \oplus (0) \in \mathfrak{g} = \mathfrak{gl}(1) \oplus \mathfrak{sp}(3)$. Then $d\rho(\tilde{A})x_{14} = 0$, $d\rho^*(\tilde{A})(\omega_1 + \omega_2) = (\omega_1 + \omega_2)$. Since $\delta\chi(\tilde{A}) = -4$, $\mathrm{tr}_{V^*_{x_{14}}} \tilde{A} = 14$ and $\dim V^*_{x_{14}} = 14$, we have $\mathrm{ord}_{\Lambda_{14}} f^s = -4s - \frac{14}{2}$. Since $\Lambda_{14} = \Lambda_0^*$, $\Lambda_7 = \Lambda_1^*$ where Λ_m^* denotes the conormal bundle of $S_m^*(\subset V^*)$, they intersect regularly by (2). Note that $(G, \rho, V) \cong (G, \rho^*, V^*)$ since $G = GL(1) \times Sp(3)$ is reductive. By Corollary 1–2, we have $b_{\Lambda_{14}}(s)/b_{\Lambda_7}(s) = s + \frac{7}{2}$. Since $b_{\Lambda_{14}}(s) = b(s)$ and $b_{\Lambda_0}(s) = 1$, we obtain the b-function

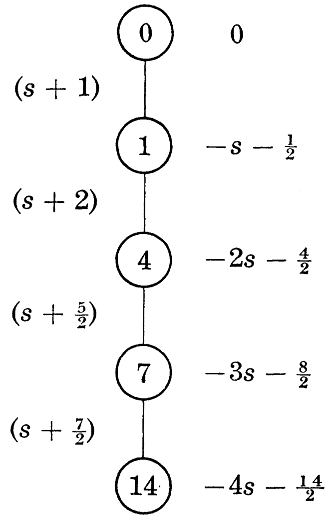


Figure 9-1. Holonomy diagram of $(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$.

$b(s) = (s+1)(s+2)(s+\frac{5}{2})(s+\frac{7}{2})$ and the holonomy diagram (Figure 9-1). We denote Λ_m by ⓜ.

§ 10. $(GL(7), \Lambda_3, V(35))$

The representation space $V = V(35)$ is spanned by the skew-tensors $u_i \wedge u_j \wedge u_k$ $(1 \leq i < j < k \leq 7)$ of degree three, on which $G = GL(7)$ acts as in § 8. Then it is known (See [6], [7]) that there exist ten orbits $S_m = \rho(G)x_m$ $(m = 0, 1, 4, 7, 9, 10, 14, 15, 22, 35)$, where S_m denotes the m-codimensional orbit, and $x_0 = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$, $x_1 = u_2 \wedge u_3 \wedge u_5 + u_3 \wedge u_4 \wedge u_6 + u_1 \wedge (u_2 \wedge u_7 - u_4 \wedge u_5)$, $x_4 = u_1 \wedge u_3 \wedge u_4 + u_2 \wedge u_5 \wedge u_6 + u_1 \wedge u_2 \wedge u_7$, $x_7 = u_2 \wedge u_3 \wedge u_4 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$, $x_9 = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$, $x_{10} = u_1 \wedge u_2 \wedge u_6 - u_1 \wedge u_3 \wedge u_5 + u_2 \wedge u_3 \wedge u_4$, $x_{14} = u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$, $x_{15} = u_1 \wedge (u_2 \wedge u_4 + u_3 \wedge u_5)$, $x_{22} = u_1 \wedge u_2 \wedge u_3$ and $x_{35} = 0$. Note that we chose these representative points x_m so that the isotropy subalgebra $\mathfrak{g}_{x_m}$ at x_m will be the standard form. The relative invariant $f(x)$ of this space is of degree seven (See [1], [14]). Since $(G, \rho, V) \cong (G, \rho^*, V^*)$, there exist also ten G-orbits S_m^* $(m = 0, 1, 4, 7, 9, 10, 14, 15, 22, 35)$ in V^*. We denote by Λ_m (resp. Λ_m^*) the conormal bundle of S_m (resp. S_m^*). Clearly we have $\Lambda_0 = V \times \{0\} = \Lambda_{35}^*$ and $\Lambda_{35} = \{0\} \times V^* = \Lambda_0^*$.

(1) The isotropy subalgebra $\mathfrak{g}_{x_0}$ at x_0 is given as follows (See [1]).

$$\text{(10.1)}\quad \mathfrak{g}_{x0} = \left\{ \left(\begin{array}{c|ccc|ccc} & 2d & 2e & 2f & 2a & 2b & 2c \\ \hline a & & & & 0 & f & -e \\ b & & X & & -f & 0 & d \\ c & & & & e & -d & 0 \\ \hline d & 0 & -c & b & & & \\ e & c & 0 & -a & & -{}^tX & \\ f & -b & a & 0 & & & \end{array} \right) ; X \in \mathfrak{sl}(3) \right\} \cong \mathfrak{g}_2 .$$

Since $\Lambda_0 = V \times \{0\}$, we have $\mathrm{ord}_{\Lambda_0} f^s = 0$.

(2) The isotropy subalgebra $\mathfrak{g}_{x_1}$ at x_1 is given as follows.

$$\text{(10.2)}\quad \mathfrak{g}_{x_1} = \left\{ \left(\begin{array}{cc|cc|ccc} \frac{d}{2}+\alpha+\beta & a_{12} & & & \gamma_1 & \gamma_2 & \gamma_3 \\ a_{21} & \frac{d}{2}-\alpha+\beta & b_{12}I_2 & & \gamma_4 & \gamma_5 & \gamma_6 \\ \hline & & \frac{d}{2}+\alpha-\beta & a_{12} & -\gamma_2 & \gamma_7 & -\gamma_1 \\ b_{21}I_2 & & a_{21} & \frac{d}{2}-\alpha-\beta & -\gamma_5 & \gamma_8 & -\gamma_4 \\ \hline & & & & -d & 2b_{21} & 2b_{12} \\ & & 0 & & b_{12} & -d+2\beta & 0 \\ & & & & b_{21} & 0 & -d-2\beta \end{array} \right) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(8).$$

The conormal vector space $V_{x_1}^*$ is spanned by $u_5 \wedge u_6 \wedge u_7$. Then $d\rho_{x_1}(A)u_5 \wedge u_6 \wedge u_7 = 3d\, u_5 \wedge u_6 \wedge u_7$ for $A \in \mathfrak{g}_{x_1}$. Since 0 is the point of the one-codimensional G-orbit, Λ_0 and Λ_1 intersect regularly with codimension one. Let A_0 be an element of $\mathfrak{g}_{x_1}$ with $d = \frac{1}{3}$, all remaining parts zero in (10.2). Then $d\rho(A_0)x_1 = 0$ and $d\rho^*(A_0)y_1 = y_1$ where $y_1 = u_5 \wedge u_6 \wedge u_7$. Since $\delta\chi(A_0) = (\deg f/\dim V) \cdot \mathrm{tr}_V A_0 = 3\,\mathrm{tr}\, A_0 = -3d = -1$ (See Proposition 1–9), $\mathrm{tr}_{V_{x_1}^*} A_0 = \dim V_{x_1}^* = 1$, we have $\mathrm{ord}_{\Lambda_1} f^s = -s - \frac{1}{2}$ and hence $b_{\Lambda_1}(s)/b_{\Lambda_0}(s) = (s+1)$. We have also $\Lambda_1 = \Lambda_{22}^*$, and hence $\Lambda_{22} = \Lambda_1^*$.

(3) The isotropy subalgebra $\mathfrak{g}_{x_4}$ at x_4 is given as follows.

$$\text{(10.3)}\quad \mathfrak{g}_{x4} = \left\{ \left(\begin{array}{cc|cc|cc|c} -\mathrm{tr}\, X & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ 0 & -\mathrm{tr}\, Y & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \hline & & & & & & \beta_2 \\ & 0 & & X & & 0 & -\beta_1 \\ \hline & & & & & & -\alpha_4 \\ & 0 & & 0 & & Y & \alpha_3 \\ \hline & 0 & & 0 & & 0 & \mathrm{tr}(X+Y) \end{array} \right) \right\}$$

$$\cong (\mathfrak{gl}(2) \oplus \mathfrak{gl}(2)) \cdot \mathfrak{u}(10).$$

The conormal vector space $V_{x_4}^*$ is spanned by $\omega_1 = u_3 \wedge u_5 \wedge u_7$, $\omega_2 = u_3$

$\wedge\ u_6 \wedge u_7$, $\omega_3 = u_4 \wedge u_5 \wedge u_7$, $\omega_4 = u_4 \wedge u_6 \wedge u_7$. Then we have $(G_{x_4}, \rho_{x_4}, V_{x_4}^*) \cong (GL(2) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(2) \otimes V(2))$, and $y_4 = \omega_1 + \omega_4$ is its generic point, $y_4' = \omega_1$ is a point of the one-codimensional orbit. Since the colocalization $(G_{x_4}, \rho_{x_4}, V_{x_4}^*)$ is an irreducible regular P.V., Λ_4 is a good holonomic variety by Corollary 1–8. Let A_0 be an element of $\mathfrak{g}_{x_4}$ with $X = -\frac{1}{3}I_2$, all remaining parts zero in (10.3). Then $d\rho(A_0)x_4 = 0$ and $d\rho^*(A_0)y_4 = y_4$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = -2$, $\mathrm{tr}_{V_{x_4}^*}A_0 = 4\dim V_{x_4}^*$, we have $\mathrm{ord}_{\Lambda_4}f^s = -2s - \frac{4}{2}$ by Proposition 1–3. We have also $d\rho^*(A_0)y_4' = y_4'$ and $\mathrm{tr}_{\tilde{V}}A_0 = 1$ where $\tilde{V} = V_{x_4}^* \bmod d\rho_{x_4}(\mathfrak{g}_{x_4})y_4' = C\omega_4$. This implies that Λ_1 and Λ_4 intersect regularly with codimension one by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_4}(s)/b_{\Lambda_1}(s) = (s + 2)$. We have also $\Lambda_4 = \Lambda_{15}^*$ and hence $\Lambda_{15} = \Lambda_4^*$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_7}$ at x_7 is given as follows.

$$(10.4)\qquad \mathfrak{g}_{x_7} = \left\{ \left(\begin{array}{c|ccc|ccc} \varepsilon & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \hline & & & & \delta_4 & (\gamma_3+\delta_3) & \delta_2 \\ 0 & & X & & \delta_3 & \delta_5 & (\gamma_1+\delta_1) \\ & & & & (\gamma_2+\delta_2) & \delta_1 & \delta_6 \\ \hline 0 & & 0 & & & -\varepsilon I_3 - {}^tX & \end{array} \right) ; X \in \mathfrak{sl}(3) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(3)) \oplus \mathfrak{u}(12).$$

Put $\omega_1 = u_5 \wedge u_6 \wedge u_7$, $\omega_2 = u_2 \wedge u_6 \wedge u_7$, $\omega_3 = u_4 \wedge u_5 \wedge u_6$, $\omega_4 = u_3 \wedge u_5 \wedge u_7$, $\omega_5 = u_2 \wedge u_5 \wedge u_7 - u_3 \wedge u_6 \wedge u_7$, $\omega_6 = u_4 \wedge u_5 \wedge u_7 - u_3 \wedge u_5 \wedge u_6$, $\omega_7 = u_2 \wedge u_5 \wedge u_6 + u_4 \wedge u_6 \wedge u_7$. Then the conormal vector space $V_{x_7}^*$ is spanned by these $\omega_1, \cdots, \omega_7$. The action $d\rho_{x_7}$ of $\mathfrak{g}_{x_7}$ on $V_{x_7}^*$ is given by

$$(10.5)\qquad d\rho_{x_7}(A)(\omega_1, \cdots, \omega_7) = (\omega_1, \cdots, \omega_7)\left(\begin{array}{c|c} 3\varepsilon & * * * * * * \\ \hline 0 & 2\varepsilon I_6 \oplus d\rho_1^*(X) \end{array} \right)$$

where $\rho_1 = 2\Lambda_1$ for $SL(3)$.

Here $y_7 = \omega_2 + \omega_3 + \omega_4$ is its generic point, and $y_7' = \omega_2 + \omega_3$ is a point of the one-codimensional orbit. This implies that $\Lambda_7 = \Lambda_{10}^*$, $\Lambda_{10} = \Lambda_7^*$ and $\dim \Lambda_4 \cap \Lambda_7 = \dim V - 1$. Since $\Lambda_4 \cap \Lambda_7$ is G_0-prehomogeneous, Λ_7 is a good holonomic variety. Let A_0 be an element of $\mathfrak{g}_{x_7}$ with $\varepsilon = \frac{1}{2}$, all remaining parts zero in (10.4). Then $d\rho(A_0)x_7 = 0$ and $d\rho^*(A_0)y_7 = y_7$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = -6\varepsilon = -3$, $\mathrm{tr}_{V_{x_7}^*}A_0 = \frac{15}{2}$ and $\dim V_{x_7}^* = 7$, we have $\mathrm{ord}_{\Lambda_7}f^s = -3s - \frac{8}{2}$ by Proposition 1–3. Let A_β be an element of $\mathfrak{g}_{x_7}$ with $\varepsilon = \frac{\beta}{6} + \frac{1}{3}$, $X = \begin{pmatrix} \eta & & \\ & \eta & \\ & & -2\eta \end{pmatrix}$ with $\eta = \frac{\beta}{6} - \frac{1}{6}$, all remaining parts zero in (10.4). Then we have $d\rho(A_\beta)x_7 = 0$, $d\rho(A_\beta)y_7' = y_7'$ and $\mathrm{tr}_{\tilde{V}}A_\beta = \beta$ where $\tilde{V} = V_{x_7}^* \bmod d\rho_{x_7}(\mathfrak{g}_{x_7})y_7' = C\omega_4$. This implies that Λ_4 and Λ_7 intersect regularly by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_7}(s)/b_{\Lambda_4}(s) = (s + \frac{5}{2})$.

(5) The isotropy subalgebra $\mathfrak{g}_{x_9}$ at x_9 is given as follows.

$$\text{(10.6)} \qquad \mathfrak{g}_{x_9} = \left\{ \begin{pmatrix} X & 0 & Z \\ 0 & Y & W \\ 0 & 0 & \varepsilon \end{pmatrix} ; X, Y \in \mathfrak{sl}(3), Z, W \in \boldsymbol{C}^3 \right\}$$
$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(3)) \oplus V(6).$$

The conormal vector space $V^*_{x_9}$ is spanned by $u_i \wedge u_j \wedge u_7 (1 \leq i \leq 3; 4 \leq j \leq 6)$. By seeing the weights, we have $(G_{x_9}, \rho_{x_9}, V^*_{x_9}) \cong (SL(3) \times GL(3), \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3))$. Since this is an irreducible regular P.V., Λ_9 is a good holonomic variety by Corollary 1–8. As a generic point, we may take $y_9 = (u_1 \wedge u_4 + u_2 \wedge u_5 + u_3 \wedge u_6) \wedge u_7$, and $y'_9 = (u_1 \wedge u_4 + u_2 \wedge u_5) \wedge u_7$ is a point of the one-codimensional orbit. This implies that $\Lambda_9 = \Lambda^*_{14}$, $\Lambda_{14} = \Lambda^*_9$ and $\dim \Lambda_4 \cap \Lambda_9 = \dim V - 1$. Let A_0 be an element of $\mathfrak{g}_{x_9}$ with $\varepsilon = -1$, all remaining parts zero in (10.6). Then $d\rho(A_0)x_9 = 0$, $d\rho^*(A_0)y_9 = y_9$. Since $\delta\chi(A_0) = 3 \operatorname{tr} A_0 = -3$, $\operatorname{tr}_{V^*_{x_9}} A_0 = 9\varepsilon = -9$, $\dim V^*_{x_9} = 9$, we have $\operatorname{ord}_{\Lambda_9} f^s = -3s - \frac{9}{2}$. Let A_β be an element of $\mathfrak{g}_{x_9}$ with $\varepsilon = ((\beta + 2)/3)$, $X = \begin{pmatrix} \eta & & \\ & \eta & \\ & & -2\eta \end{pmatrix}$ with $\eta = ((1 - \beta)/3)$, all remaining parts zero in (10.6). Then we have $d\rho(A_0)x_9 = 0$, $d\rho^*(A_0)y'_9 = y'_9$ and $\operatorname{tr}_{\tilde{V}} A_0 = \beta$ where $\tilde{V} = V^*_{x_9} \operatorname{mod} d\rho_{x_9}(\mathfrak{g}_{x_9})y'_9 = \boldsymbol{C} u_3 \wedge u_6 \wedge u_7$. This implies that Λ_4 and Λ_9 intersect regularly. By Corollary 1–2, we have $b_{\Lambda_9}(s)/b_{\Lambda_4}(s) = (s + 3)$.

(6) The isotropy subalgebra $\mathfrak{g}_{x_{10}}$ at x_{10} is given as follows.

$$\text{(10.7)} \quad \mathfrak{g}_{x_{10}} = \left\{ A = \left(\begin{array}{c|c|c} \varepsilon I_3 + X & B & C \\ \hline 0 & -2\varepsilon I_3 + X & F \\ \hline 0 & 0 & \eta \end{array} \right) ; X \in \mathfrak{sl}(3), \operatorname{tr} B = 0, C, D \in \boldsymbol{C}^3 \right\}$$
$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(3)) \oplus \mathfrak{u}(14) .$$

Put $\omega_1 = u_5 \wedge u_6 \wedge u_7$, $\omega_2 = u_4 \wedge u_6 \wedge u_7$, $\omega_3 = u_4 \wedge u_5 \wedge u_7$, $\omega_4 = u_1 \wedge u_4 \wedge u_7$, $\omega_5 = u_2 \wedge u_5 \wedge u_7$, $\omega_6 = u_3 \wedge u_6 \wedge u_7$, $\omega_7 = (u_1 \wedge u_5 + u_2 \wedge u_4) \wedge u_7$, $\omega_8 = (u_2 \wedge u_6 + u_3 \wedge u_5) \wedge u_7$, $\omega_9 = (u_1 \wedge u_6 + u_3 \wedge u_4) \wedge u_7$, $\omega_{10} = u_4 \wedge u_5 \wedge u_6$. Then the conormal vector space $V^*_{x_{10}}$ is spanned by $\omega_1, \cdots, \omega_{10}$, and the action $d\rho_{x_{10}}$ of $\mathfrak{g}_{x_{10}}$ on $V^*_{x_{10}}$ is given as follows.

$$\text{(10.8)} \quad d\rho_{x_{10}}(A)(\omega_1, \cdots, \omega_{10}) = (\omega_1, \cdots, \omega_{10}) \left(\begin{array}{c|c|c} (4\varepsilon - \eta)I_3 + X & B' & F \\ \hline 0 & (\varepsilon - \eta)I_6 + d\rho_1^*(X) & 0 \\ \hline 0 & 0 & 6\varepsilon \end{array} \right)$$

where $\rho_1 = 2\Lambda_1$ for $SL(3)$.

Then $y_{10} = \omega_4 + \omega_8 + \omega_{10}$ is a generic point. There exist two one-codimensional orbits. As a representative point, we may take $y'_{10} = \omega_8 + \omega_{10}$ and $y''_{10} = \omega_4 + \omega_5 + \omega_6$ respectively. This implies that $\dim \Lambda_7 \cap \Lambda_{10} = \dim \Lambda_9 \cap \Lambda_{10} = \dim V - 1$. Since $\Lambda_{10} = \Lambda_7^*$, Λ_{10} is a good holonomic variety. Let A_0 be an element of $\mathfrak{g}_{x_{10}}$ with $\varepsilon = \frac{1}{6}$, $\eta = -\frac{5}{6}$, all remaining parts zero in (10.7). Then $d\rho(A_0)x_{10} = 0$ and $d\rho^*(A_0)y_{10} = y_{10}$. Since $\delta\chi(A_0) = -9\varepsilon + 3\eta = -4$, $\mathrm{tr}_{V^*_{x_{10}}} A_0 = 24\varepsilon - 9\eta = \frac{23}{2}$, and $\dim V^*_{x_{10}} = 10$, we have $\mathrm{ord}_{\Lambda_{10}} f^s = -4s - \frac{13}{2}$ by Proposition 1–3.

Since $d\rho^*(A_0)y'_{10} = y'_{10}$ and $\mathrm{tr}_{\tilde{V}} A_0 = 1$ where $\tilde{V} = V^*_{x_{10}} \bmod d\rho_{x_{10}}(\mathfrak{g}_{x_{10}})y'_{10} = \boldsymbol{C}\omega_4$, Λ_7 and Λ_{10} intersect regularly by Proposition 1–4. Let A_β be an element of $\mathfrak{g}_{x_{10}}$ with $\varepsilon = \frac{\beta}{6}$, $\eta = \frac{\beta}{6} - 1$, all remaining parts zero in (10.7). Then we have $d\rho(A_\beta)x_{10} = 0$, $d\rho^*(A_\beta)y''_{10} = y''_{10}$, and $\mathrm{tr}_{\tilde{V}} A_\beta = \beta$ where $\tilde{V} = V^*_{x_{10}} \bmod d\rho_{x_{10}}(\mathfrak{g}_{x_{10}})y''_{10} = \boldsymbol{C}\omega_{10}$. This implies that Λ_9 and Λ_{10} intersect regularly by Proposition 1–4. By Corollary 1–2, we have $b_{\Lambda_{10}}(s)/b_{\Lambda_7}(s) = (s+3)$ and $b_{\Lambda_{10}}(s)/b_{\Lambda_9}(s) = (s + \frac{5}{2})$.

(7) The isotropy subalgebra $\mathfrak{g}_{x_{14}}$ at x_{14} is given as follows.

$$(10.9)\quad \mathfrak{g}_{x_{14}} = \left\{ \left(\begin{array}{c|c} -2\varepsilon & Y \\ \hline 0 & \varepsilon I_6 + X \end{array} \right); X \in \mathfrak{sp}(3),\ {}^tY \in \boldsymbol{C}^6 \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{sp}(3)) \oplus V(6)\,.$$

Put $\omega_1 = u_2 \wedge u_3 \wedge u_4$, $\omega_2 = u_5 \wedge u_6 \wedge u_7$, $\omega_3 = u_3 \wedge u_4 \wedge u_5$, $\omega_4 = u_2 \wedge u_6 \wedge u_7$, $\omega_5 = u_2 \wedge u_4 \wedge u_6$, $\omega_6 = u_3 \wedge u_5 \wedge u_7$, $\omega_7 = u_2 \wedge u_3 \wedge u_7$, $\omega_8 = u_4 \wedge u_5 \wedge u_6$, $\omega_9 = u_2 \wedge u_3 \wedge u_6 - u_2 \wedge u_4 \wedge u_7$, $\omega_{10} = u_3 \wedge u_5 \wedge u_6 - u_4 \wedge u_5 \wedge u_7$, $\omega_{11} = u_2 \wedge u_3 \wedge u_5 + u_3 \wedge u_4 \wedge u_7$, $\omega_{12} = u_2 \wedge u_5 \wedge u_6 + u_4 \wedge u_6 \wedge u_7$, $\omega_{13} = u_2 \wedge u_4 \wedge u_5 - u_3 \wedge u_4 \wedge u_6$, $\omega_{14} = u_2 \wedge u_5 \wedge u_7 - u_3 \wedge u_6 \wedge u_7$. The conormal vector space $V^*_{x_{14}}$ is spanned by these $\omega_1, \cdots, \omega_{14}$. By seeing the weights, we have $(G_{x_{14}}, \rho_{x_{14}}, V^*_{x_{14}}) \cong (GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$. Since this is an irreducible regular P.V., Λ_{14} is a good holonomic variety by Corollary 1–8. As we have seen in § 9, $y_{14} = \omega_1 + \omega_2$ is a generic point. Let A_0 be an element of $\mathfrak{g}_{x_{14}}$ with $\varepsilon = -\frac{1}{3}$, $X = Y = 0$ in (10.9). Then $d\rho(A_0)x_{14} = 0$ and $d\rho^*(A_0)y_{14} = y_{14}$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = -4$, $\mathrm{tr}_{V^*_{x_{14}}} A_0 = 14 \times 3\varepsilon = -14$ and $\dim V^*_{x_{14}} = 14$, we have $\mathrm{ord}_{\Lambda_{14}} f^s = -4s - \frac{14}{2}$. Since $\Lambda_{14} = \Lambda_9^*$, $\Lambda_7 = \Lambda_{10}^*$, and Λ_9 and Λ_{10} intersect regularly with codimension one, so do Λ_{14} and Λ_7. By Corollary 1–2, we have $b_{\Lambda_{14}}(s)/b_{\Lambda_7}(s) = (s + \frac{7}{2})$.

(8) The isotropy subalgebra $\mathfrak{g}_{x_{15}}$ at x_{15} is given as follows.

$$(10.10)\quad \mathfrak{g}_{x_{15}} = \left\{ \left(\begin{array}{c|c|c} -2\varepsilon & W & U \\ \hline 0 & \varepsilon I_4 + X & Z \\ \hline 0 & 0 & \eta I_2 + Y \end{array} \right);\ \begin{array}{l} X \in \mathfrak{sp}(2),\ Y \in \mathfrak{sl}(2),\ Z \in M(4,2) \\ {}^tW \in \boldsymbol{C}^2,\ {}^tU \in \boldsymbol{C}^4,\ \varepsilon, \eta \in \boldsymbol{C} \end{array} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sp}(2) \oplus \mathfrak{gl}(2)) \oplus \mathfrak{u}(14).$$

Put $\omega_i = u_{i+1} \wedge u_6 \wedge u_7$ $(1 \leq i \leq 4)$, $\omega_5 = u_1 \wedge u_6 \wedge u_7$, $\omega_{6+j} = (u_2 \wedge u_4 - u_3 \wedge u_5) \wedge u_{6+j}$, $\omega_{8+j} = u_2 \wedge u_3 \wedge u_{6+j}$, $\omega_{10+j} = u_2 \wedge u_5 \wedge u_{6+j}$, $\omega_{12+j} = u_3 \wedge u_4 \wedge u_{6+j}$, $\omega_{14+j} = u_4 \wedge u_5 \wedge u_{6+j}$ $(j = 0, 1)$. Then the conormal vector space $V^*_{x_{15}}$ is spanned by $\omega_1, \cdots, \omega_{15}$. The action $d\rho_{x_{15}}$ of $\mathfrak{g}_{x_{15}}$ on $V^*_{x_{15}}$ is as follows.

$$\begin{aligned} &d\rho_{x_{15}}(A)(\omega_1, \cdots, \omega_{15}) = (\omega_1, \cdots, \omega_{15}) \\ (10.11) \qquad &\times \left(\begin{array}{c|c|c} -(\varepsilon+2\eta)I_4 + d\rho_1(X) & * & * \\ \hline 0 & 2\varepsilon - 2\eta & * \\ \hline 0 & 0 & -(2\varepsilon+\eta)I_{10} + d\rho_2^*(X \oplus Y) \end{array}\right) \end{aligned}$$

where $\rho_1 = \Lambda_1$ for $Sp(2)$ and $\rho_2 = \Lambda_2 \otimes \Lambda_1$ for $Sp(2) \times SL(2)$. Since $\Lambda_{15} = \Lambda_4^*$ and Λ_4 is a good holonomic variety, Λ_{15} is also a good holonomic variety. $y_{15} = \omega_5 + \omega_{11} + \omega_{12}$ is a generic point. Let A_0 be an element of $\mathfrak{g}_{x_{15}}$ with $\varepsilon = -\frac{1}{6}$, $\eta = -\frac{2}{3}$, all remaining parts zero in (10.10). Then $d\rho(A_0)x_{15} = 0$ and $d\rho^*(A_0)y_{15} = y_{15}$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = 6(\varepsilon + \eta) = -5$, $\mathrm{tr}_{V^*_{x_{15}}} A_0 = -22\varepsilon - 20\eta = \frac{102}{6}$, and $\dim V^*_{x_{15}} = 15$, we have $\mathrm{ord}_{\Lambda_{15}} f^s = -5s - \frac{19}{2}$. Since $\Lambda_{15} = \Lambda_4^*$, $\Lambda_{14} = \Lambda_9^*$, $\Lambda_{10} = \Lambda_7^*$, we have $\dim \Lambda_{15} \cap \Lambda_{14} = \dim \Lambda_{15} \cap \Lambda_{10} = \dim V - 1$ and they intersect regularly. By Corollary 1–2, we have $b_{\Lambda_{15}}(s)/b_{\Lambda_{14}}(s) = (s + 3)$ and $b_{\Lambda_{15}}(s)/b_{\Lambda_{10}}(s) = (s + \frac{7}{2})$.

(9) The isotropy subalgebra $\mathfrak{g}_{x_{22}}$ at x_{22} is given as follows.

$$\begin{aligned} (10.12) \qquad \mathfrak{g}_{x_{22}} &= \left\{\tilde{X} = \left(\begin{array}{c|c} X & Z \\ \hline 0 & \varepsilon I_4 + Y \end{array}\right); X \in \mathfrak{sl}(3),\ Y \in \mathfrak{sl}(4),\ Z \in M(3, 4)\right\} \\ &\cong (\mathfrak{sl}(3) \oplus \mathfrak{gl}(4)) \oplus V(12). \end{aligned}$$

The conormal vector space $V^*_{x_{22}}$ is spanned by $u_i \wedge u_j \wedge u_k$ $(4 \leq i < j < k \leq 7)$ and $u_i \wedge u_j \wedge u_k$ $(1 \leq i \leq 3, 4 \leq j < k \leq 7)$. The action $d\rho_{x_{22}}$ of $\mathfrak{g}_{x_{22}}$ is given by

$$\begin{aligned} &d\rho(\tilde{X})(u_5 \wedge u_6 \wedge u_7, \cdots) \\ (10.13) \qquad &= (u_5 \wedge u_6 \wedge u_7, \cdots)\left(\begin{array}{c|c} Y - 3\varepsilon I_4 & * \\ \hline 0 & -2\varepsilon I_{18} + d\rho_1^*(X \oplus Y) \end{array}\right) \end{aligned}$$

where $\rho_1 = \Lambda_1 \otimes \Lambda_2$ for $SL(3) \times SL(4)$. For example, $y_{22} = u_1 \wedge (u_4 \wedge u_5 + u_6 \wedge u_7) + u_2 \wedge u_4 \wedge u_6 + u_3 \wedge u_5 \wedge u_7$ is a generic point. Since $\Lambda_{22} = \Lambda_1^*$, Λ_{22} is a good holonomic variety. Let A_0 be an element of $\mathfrak{g}_{x_{22}}$ with $\varepsilon = -\frac{1}{2}$, $X = Y = Z = 0$ in (10.12). Then $d\rho(A_0)x_{22} = 0$ and $d\rho^*(A_0)y_{22} = y_{22}$. Since $\delta\chi(A_0) = 12\varepsilon = -6$, $\mathrm{tr}_{V^*_{x_{22}}} A_0 = -48\varepsilon = 24$ and $\dim V^*_{x_{22}} = 22$, we have $\mathrm{ord}_{\Lambda_{22}} f^s = -6s - \frac{26}{2}(= -6s - 13)$. Since $\Lambda_{22} = \Lambda_1^*$ and $\Lambda_{15} = \Lambda_4^*$, we have dim $\Lambda_{22}\cap\Lambda_{15}$ = dim $V - 1$ and they intersect regularly.

By Corollary 1–2, we have $b_{\Lambda_{22}}(s)/b_{\Lambda_{15}}(s) = (s+4)$.

(10) The isotropy subalgebra $\mathfrak{g}_{x_{35}}$ at $x_{35} = 0$ is $\mathfrak{g}$ itself and we have $(G_{x_{35}}, \rho_{x_{35}}, V^*_{x_{35}}) = (G, \rho^*, V^*) \cong (GL(7), \Lambda_3, V(35))$. Then $y_{35} = x_0 = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$ is its generic point. Put $A_0 = -\frac{1}{3}I_7$. Then $d\rho(A_0)x_{35} = 0$ and $d\rho^*(A_0)y_{35} = y_{35}$. Since $\delta\chi(A_0) = 3\,\mathrm{tr}\,A_0 = -7$, $\mathrm{tr}_{V^*_{x_{35}}}A_0 = -35$ and $\dim V^*_{x_{35}} = 35$, we have $\mathrm{ord}_{\Lambda_{35}} f^s = -7s - \frac{35}{2}$. Since $\Lambda_{22} = \Lambda_1^*$ and $\Lambda_{35} = \Lambda_0^*$, they intersect regularly with codimension one. By Corollary 1–2, we have $b_{\Lambda_{35}}(s)/b_{\Lambda_{22}}(s) = (s+5)$. Since $b_{\Lambda_0}(s) = 1$ and $b_{\Lambda_{35}}(s) = b(s)$, we obtain the b-function $b(s) = (s+1)(s+2)(s+\frac{5}{2})(s+\frac{7}{2})(s+3)(s+4)(s+5)$, and the holonomy diagram (Figure 10–1). We denote Λ_m by ⓜ.

Note that the colocalization at x_1, x_4, x_7, x_9, x_{14}, x_{22} and x_{35} has the unique one-codimensional orbit respectively, and the colocalization at x_{10} and x_{15} has the two one-codimensional orbits respectively. Therefore we have obtained all one-codimensional intersections among the conormal bundles.

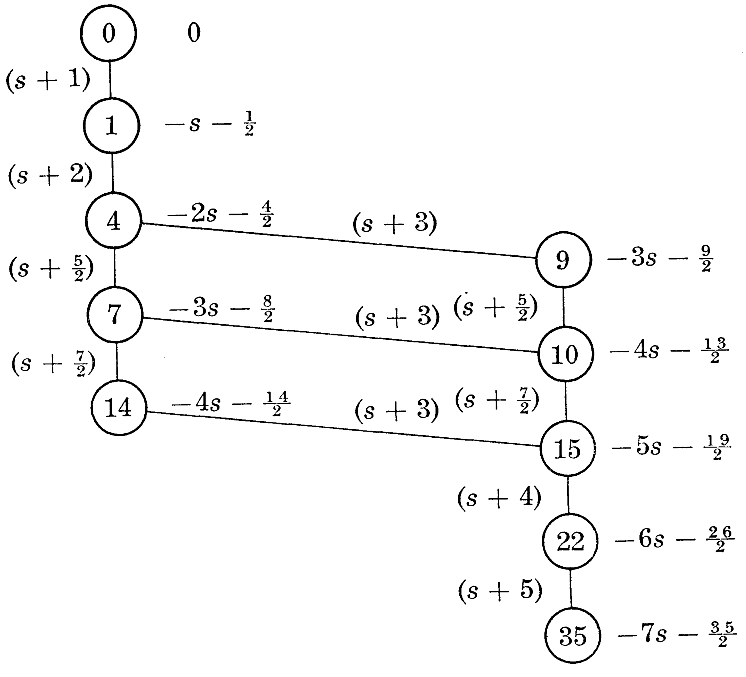


Figure 10–1. Holonomy diagram of $(GL(7), \Lambda_3, V(35))$.

§ 11. $(SL(5) \times GL(3),\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3))$

Let $V(10)$ be a vector space spanned by 2-forms $u_i \wedge u_j$ $(1 \leq i < j \leq 5)$. Then the representation space is identified with $V = V(10) \oplus V(10) \oplus V(10)$ (See [1]). Let Λ be the conormal bundle of an orbit S in V and Λ^* that of an orbit S^* in V^*. When $\Lambda = \Lambda^*$, we say that S and S^* are the dual orbits of each other. We denote by $S^{(k)}_{i,j}$ the i-codimensional orbit whose dual orbit is j-codimensional, where k denotes the dimension of the central torus of the isotropy subgroup of this orbit. When there is no confusion, denote this by S_i or S_{ij}. We denote by $\Lambda^{(k)}_{i,j}$ (resp. $\Lambda_{i,j}$, Λ_i) the conormal bundle of $S^{(k)}_{i,j}$ (resp. $S_{i,j}$, S_i). We identify V and its dual V^* by taking $(u_i \wedge u_{i'}, u_j \wedge u_{j'}, u_k \wedge u_{k'})$ $(i < i', j < j', k < k')$ as a dual basis.

Proposition 11–1. *This space has following twenty five orbits* $S^{(k)}_{i,j}$.

(1) $S^{(0)}_{0,30}$: $(u_1 \wedge u_2 + u_3 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5, u_1 \wedge u_3 + u_2 \wedge u_5)$ $(= x_0)$
(2) $S^{(2)}_{1,21}$: $(u_1 \wedge u_2, u_1 \wedge u_5 + u_3 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5)$ $(= x_1)$
(3) $S^{(2)}_{2,16}$: $(u_1 \wedge u_2, u_2 \wedge u_3 + u_3 \wedge u_4, u_1 \wedge u_3 + u_4 \wedge u_5)$ $(= x_2)$
(4) $S^{(3)}_{3,15}$: $(u_1 \wedge u_2, u_3 \wedge u_4, u_1 \wedge u_5 + u_4 \wedge u_5)$ $(= x_3)$
(5) $S^{(2)}_{3,13}$: $(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5)$ $(= x_3')$
(6) $S^{(3)}_{4,11}$: $(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_4 \wedge u_5)$ $(= x_4)$
(7) $S^{(2)}_{5,8}$: $(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_1 \wedge u_5 + u_3 \wedge u_4)$ $(= x_5)$
(8) $S^{(1)}_{6,12}$: $(u_1 \wedge u_2, u_1 \wedge u_3, u_2 \wedge u_3 + u_4 \wedge u_5)$ $(= x_6)$
(9) $S^{(2)}_{7,9}$: $(u_1 \wedge u_2, u_1 \wedge u_3, u_4 \wedge u_5)$ $(= x_7)$
(10) $S^{(1)}_{7,7}$: $(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_2 \wedge u_3 + u_1 \wedge u_5)$ $(= x_7')$
(11) $S^{(2)}_{7,7}$: $(u_1 \wedge u_2, u_1 \wedge u_3, u_2 \wedge u_4 + u_3 \wedge u_5)$ $(= x_7'')$
(12) $S^{(2)}_{8,18}$: $(u_1 \wedge u_2 + u_3 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5, 0)$ $(= x_8)$
(13) $S^{(1)}_{8,14}$: $(u_1 \wedge u_2, u_3 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_4)$ $(= x_8')$
(14) $S^{(3)}_{8,5}$: $(u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_5 + u_2 \wedge u_4)$ $(= x_8'')$
(15) $S^{(4)}_{9,7}$: $(u_1 \wedge u_2, u_1 \wedge u_3, u_2 \wedge u_4)$ $(= x_9)$
(16) $S^{(3)}_{10,10}$: $(u_1 \wedge u_2, u_3 \wedge u_4 + u_1 \wedge u_5, 0)$ $(= x_{10})$
(17) $S^{(2)}_{11,4}$: $(u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4 + u_2 \wedge u_3)$ $(= x_{11})$
(18) $S^{(3)}_{12,6}$: $(u_1 \wedge u_2, u_3 \wedge u_4, 0)$ $(= x_{12})$
(19) $S^{(3)}_{13,3}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3 + u_2 \wedge u_4,\ 0)$ $(= x_{13})$
(20) $S^{(2)}_{14,8}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_1 \wedge u_4)$ $(= x_{14})$
(21) $S^{(1)}_{15,3}$: $(u_1 \wedge u_2, u_1 \wedge u_3, u_2 \wedge u_3)$ $(= x_{15})$
(22) $S^{(3)}_{16,2}$: $(u_1 \wedge u_2,\ u_1 \wedge u_3,\ 0)$ $(= x_{16})$
(23) $S^{(2)}_{18,8}$: $(u_1 \wedge u_2 + u_3 \wedge u_4, 0, 0)$ $(= x_{18})$
(24) $S^{(2)}_{21,1}$: $(u_1 \wedge u_2, 0, 0)$ $(= x_{21})$

(25) $S^{(1)}_{30,0}$: $(0, 0, 0)$ ($= x_{30}$).

Proof. It is easy to check that the non-regular P.V. $(SL(5) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(2))$ has eight orbits which are represented by the following points; [1] $(0, 0)$, [2] $(u_1 \wedge u_2, 0)$, [3] $(u_1 \wedge u_2 + u_3 \wedge u_4, 0)$, [4] $(u_1 \wedge u_2, u_1 \wedge u_3)$, [5] $(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4)$, [6] $(u_1 \wedge u_2, u_3 \wedge u_4)$, [7] $(u_1 \wedge u_2, u_3 \wedge u_4 + u_1 \wedge u_5)$, [8] $(u_1 \wedge u_2 + u_3 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5)$. Therefore, for a point $x = (x_1, x_2, x_3)$ of V, we may assume that (x_1, x_2) is one of these points. In the first three cases, repeating the same argument, we obtain (12), (16), (18), (19), (22), (23), (24) and (25). For $\lambda \in C$, we define $S_{ij}(\lambda)$ by $S_{ij}(\lambda)u_k = u_k$ for $k \neq i$ and $S_{ij}(\lambda)u_i = u_i + \lambda u_j$. Then $S_{ij}(\lambda)$ is an element of $\rho(G)$. Put $x_3 = \sum_{i<j} a_{ij}u_i \wedge u_j$. First we consider the case [4], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_1 \wedge u_3)$. Assume that $a_{45} \neq 0$. Then we may assume that $x_3 = a_{23}u_2 \wedge u_3 + u_4 \wedge u_5$. In fact, we have $a_{35} = 0$ by $S_{43}(-a_{35}/a_{45})$ and so on. If $a_{23} = 0$, then we have (9). If $a_{23} \neq 0$, then we have (8). Next assume that $a_{45} = 0$. If one of a_{ij} $(i = 2, 3; j = 4, 5)$ is not zero, we may assume that $x_3 = u_2 \wedge u_4 + a_{15}u_1 \wedge u_5 + a_{35}u_3 \wedge u_5$. If $a_{35} \neq 0$ (resp. $a_{35} = 0$ and $a_{15} \neq 0$, $a_{35} = a_{15} = 0$), then we have (11) (resp. (14), (15)). If any $a_{ij} = 0$ $(i = 2, 3; j = 4, 5)$, then we may assume that $x_3 = a_{14}u_1 \wedge u_4 + a_{23}u_2 \wedge u_3$. If $a_{14} \neq 0$ and $a_{23} \neq 0$ (resp. $a_{14} \neq 0$ and $a_{23} = 0$, $a_{14} = 0$ and $a_{23} \neq 0$, $a_{14} = a_{23} = 0$), then we have (17) (resp. (20), (21), (22)). Next we consider the case [5], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4)$. If $a_{35} \neq 0$ or $a_{45} \neq 0$, we may assume that $x_3 = a_{23}u_2 \wedge u_3 + u_4 \wedge u_5$ and hence we have (5) (resp. (6)) for $a_{23} \neq 0$ (resp. $a_{23} = 0$). If $a_{35} = a_{45} = 0$ and one of a_{k5} $(k = 1, 2)$ is not zero, then we may assume that $x_3 = a_{23}u_2 \wedge u_3 + a_{34}u_3 \wedge u_4 + u_1 \wedge u_5$ and hence we have (7) (resp. (10), (14)) for $a_{34} \neq 0$ (resp. $a_{34} = 0$ and $a_{23} \neq 0$, $a_{34} = a_{23} = 0$). If $a_{k5} = 0$ for $1 \leq k \leq 4$, we may assume that $x_3 = a_{13}u_1 \wedge u_3 + a_{14}u_1 \wedge u_4 + a_{23}u_2 \wedge u_3 + a_{34}u_3 \wedge u_4$. Then we have (13) (resp. we have (19); it is reduced to the case [4]) for $a_{34} \neq 0$ (resp. $x_3 = 0$; $a_{34} = 0$ and $x_3 \neq 0$). Now we consider the case [6], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_3 \wedge u_4)$. (i) If $a_{35} \neq 0$ or $a_{45} \neq 0$, we may assume that $x_3 = a_{12}u_1 \wedge u_3 + a_{15}u_1 \wedge u_5 + a_{25}u_2 \wedge u_5 + u_4u_5$. Moreover if $a_{25} \neq 0$, then we have (3) (resp. (4)) for $a_{13} \neq 0$ (resp. $a_{13} = 0$). If $a_{25} = 0$, then we have (4) (resp. it is reduced to the case [4] or [5]) for $a_{15} \neq 0$ (resp. $a_{15} = 0$). (ii) If $a_{35} = a_{45} = 0$, it is reduced to the previous cases. Next we shall consider the case [7], i.e., $(x_1, x_2) = (u_1 \wedge u_2, u_3 \wedge u_4 + u_1 \wedge u_5)$. (i) If $a_{35} \neq 0$ or $a_{45} \neq 0$, then we may assume that $a_{45} = 1$ and $a_{35} = a_{24} = a_{14} = a_{12} = 0$. By $S_{53}(\lambda)$, $S_{41}(\mu)$, $S_{21}(\nu)$ and $GL(3)$, we have $x_3 = (a_{13} + (a_{15} - a_{34})\lambda + (a_{23} + a_{25})\nu + \lambda^2)u_1 \wedge u_3 + (a_{23} + \lambda a_{25})u_2 \wedge u_3$

$+ a_{25}u_2 \wedge u_5 + (a_{15} + \nu a_{25} + \mu + \lambda)u_1 \wedge u_5 + (a_{34} + \mu - \lambda)u_3 \wedge u_4 + u_4 \wedge u_5$. If $a_{25} \neq 0$, we may take λ, μ, ν so that $a_{13} + (a_{15} - a_{34})\lambda + \nu(a_{23} + \lambda a_{25}) + \lambda^2 = a_{15} + \nu a_{25} + \mu + \lambda = a_{34} + \mu - \lambda = 0$ and hence we have $x_3 = \alpha u_2 \wedge u_3 + u_2 \wedge u_5 + u_4 \wedge u_5$. If $\alpha \neq 0$ (resp. $\alpha = 0$), then we have (2) (resp. (3)) by $S_{35}(-1/2\alpha)$, $S_{42}(-1/2)$, $S_{12}(1/4\alpha)$, $\{u_3 \mapsto (1/\sqrt{\alpha})u_3,\ u_4 \mapsto \sqrt{\alpha}\,u_4,\ u_j \mapsto u_j\ (j \neq 3, 4)\}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2\alpha \\ 0 & 0 & 1/\sqrt{\alpha} \end{pmatrix}$ (resp. by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and $\{u_5 \mapsto u_3,\ u_3 \mapsto u_5,\ u_4 \mapsto -u_4,\ u_j \mapsto u_j\ (j = 1, 2)\}$). If $a_{25} = 0$, taking λ and μ satisfying $a_{15} + \mu + \lambda = a_{34} + \mu - \lambda = 0$, we have $x_3 = a'_{13}u_1 \wedge u_3 + a'_{23}u_2 \wedge u_3 + u_4 \wedge u_5$. If $a'_{23} \neq 0$ (resp. $a'_{23} = a'_{13} = 0$), then we have (2) (resp. (6)). If $a'_{23} = 0$ and $a'_{13} \neq 0$, then we have (4) by $S_{41}(\lambda)$, $\begin{pmatrix} 1 & & \\ & 1 & -1/\gamma \\ & 0 & 1 \end{pmatrix}$, $S_{35}\Bigl(-\dfrac{1}{\gamma}\Bigr)$, $S_{53}\Bigl(\dfrac{\gamma}{2}\Bigr)$, $\begin{pmatrix} 1 & & \\ & 1 & \\ & \gamma/2 & 1 \end{pmatrix}$ and $\{u_1 \mapsto (1/2\gamma)u_1,\ u_2 \mapsto 2\gamma u_2,\ u_j \mapsto u_j\ (j \neq 1, 2)\}$ where $\gamma = \sqrt{-a'_{13}}$. (ii) If $a_{35} = a_{45} = 0$, it is reduced to the previous cases. Finally we shall consider the case [8], i.e., $(x_1, x_2) = (u_1 \wedge u_2 + u_3 \wedge u_4,\ u_2 \wedge u_3 + u_4 \wedge u_5)$. The isotropy subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(5) \oplus \mathfrak{gl}(2)$ at this point (x_1, x_2) is given by

$$\mathfrak{h} = \left\{ \begin{bmatrix} a_1 & & \gamma_1 & & \\ \alpha_1 & a_2 & \beta_1 & \gamma_2 & \beta_2 \\ -2\gamma_2 & & a_3 & & 2\gamma_1 \\ \beta_1 & -\gamma_2 & \beta_2 & a_4 & \alpha_2 \\ & & -\gamma_2 & & a_5 \end{bmatrix} \oplus \begin{pmatrix} -(a_1+a_2) & -\gamma_2 \\ \gamma_1 & -(a_4+a_5) \end{pmatrix} \right\}$$

with $a_1 + a_2 = a_3 + a_4$, $a_2 + a_3 = a_4 + a_5$ and $\sum_{i=1}^5 a_i = 0$.

Taking one-parameter subgroups from $\mathfrak{h}$, we obtain the following actions which fix (x_1, x_2). (i) $\alpha_1(\lambda)$: $u_1 \mapsto u_1 + \lambda u_2$, $u_j \mapsto u_j$ $(j \neq 1)$, (ii) $\alpha_2(\lambda)$: $u_5 \mapsto u_5 + \lambda u_4$, $u_j \mapsto u_j$ $(j \neq 5)$, (iii) $\beta_1(\lambda)$: $u_1 \mapsto u_1 + \lambda u_4$, $u_3 \mapsto u_3 + \lambda u_2$, $u_j \mapsto u_j$ $(j \neq 1, 3)$ (iv) $\beta_2(\lambda)$: $u_3 \mapsto u_3 + \lambda u_4$, $u_5 \mapsto u_5 + \lambda u_2$, $u_j \mapsto u_j$ $(j \neq 3, 5)$ (v) $\gamma_1(\lambda)$: $u_2 \mapsto u_2 - \lambda u_4$, $u_3 \mapsto u_3 + \lambda u_1$, $u_5 \mapsto u_5 + 2\lambda u_3 + \lambda^2 u_1$, $u_j \mapsto u_j$ $(j \neq 2, 3, 5)$ and $(x_1, x_2, x_3) \mapsto (x_1, \lambda x_1 + x_2, x_3)$, (vi) $\gamma_2(\lambda)$: $u_1 \mapsto u_1 - 2\lambda u_3 + \lambda^2 u_5$, $u_3 \mapsto u_3 - \lambda u_5$, $u_4 \mapsto u_4 + \lambda u_2$, $u_j \mapsto u_j$ $(j \neq 1, 3, 4)$ and $(x_1, x_2, x_3) \mapsto (x_1 - \lambda x_2, x_2, x_3)$. We have also $\xi_1(\lambda)$ (resp. $\xi_2(\lambda)$): $(x_1, x_2, x_3) \mapsto (x_1, x_2, \lambda x_1 + x_3)$ (resp. $(x_1, x_2, \lambda x_2 + x_3)$) and $\eta(\mu)$: $(x_1, x_2, x_3) \to (x_1, x_2, \mu x_3)$ with $\mu \neq 0$. By using these actions, we shall do the orbital decomposition leaving (x_1, x_2) fixed. If at least one of a_{13}, a_{15} and a_{35} is not zero, then by γ_1, γ_2, ξ_1, ξ_2 and η, we may assume that $a_{13} = 1$, $a_{34} = a_{35} = a_{45} = 0$. (i) If $a_{15} \neq 0$, by α_1, β_1, ξ_1, ξ_2, β_2, α_2, γ_1 and η, we may assume that $x_3 = a_{24}u_2 \wedge u_4 + u_1 \wedge u_5$. If $a_{24} \neq 0$ (resp. $a_{24} = 0$), we have (1) (resp. (2)). (ii) If $a_{15} = 0$, by ξ_1, ξ_2, β_1, β_2 and α_1, we may assume that $x_3 = u_1 \wedge u_3 + a_{24}u_2 \wedge u_4 + a_{25}u_2 \wedge u_5$. If $a_{25} \neq 0$, then we have (1).

If $a_{25}=0$, then it is reduced to previous cases. Finally, if $a_{13}=a_{15}=a_{35}=0$, we may assume that $a_{15}=a_{25}=a_{35}=a_{45}=0$ by the action of ξ_1, ξ_2 and γ_2. By considering (x_1, x_3) instead of (x_1, x_2), it is reduced to the previous cases. We shall see later, by calculating the isotropy subalgebras, that these orbits are different from each other. Q.E.D.

(1) Put $x_0' = (3u_3 \wedge u_4 - u_2 \wedge u_5, u_1 \wedge u_5 - 2u_2 \wedge u_4, 3u_2 \wedge u_3 - u_1 \wedge u_4)$. Then the isotropy subalgebra $\mathfrak{g}_{x_0'}$ is the following standard form.

$$(11.1) \qquad \mathfrak{g}_{x_0'} = \left\{ \begin{bmatrix} 4\alpha & \beta & & & \\ 4\gamma & 2\alpha & 2\beta & & \\ & 3\gamma & 0 & 3\beta & \\ & & 2\gamma & -2\alpha & 4\beta \\ & & & \gamma & -4\alpha \end{bmatrix} \oplus \begin{pmatrix} 2\alpha & \beta & \\ 2\gamma & & 2\beta \\ & \gamma & -2\alpha \end{pmatrix} \right\}$$
$$\cong \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \right\} = \mathfrak{sl}(2).$$

Since $\varLambda_0 = V \times \{0\}$, we have $\operatorname{ord}_{V_{0,30}^{(0)}} f^s = 0$ where f denotes the relative invariant of degree 15 (See [1]).

(2) The isotropy subgroup at x_1 is locally isomorphic to $(GL(1) \times GL(1)) \cdot U(2)$ where $U(2)$ denotes a 2-dimensional unipotent group (See [1]). The conormal vector space $V_{x_1}^*$ is spanned by $(u_3 \wedge u_5, 0, 0) \in S_{21,1}^*$. We have $\dim \varLambda_0 \cap \varLambda_1 = \dim V - 1$; $\operatorname{ord}_{\varLambda_1} f^s = -s - \frac{1}{2}$ and $b_{\varLambda_1}(s)/b_{\varLambda_0}(s) = (s+1)$.

(3) The isotropy subalgebra $\mathfrak{g}_{x_2}$ at x_2 is given as follows.

$$(11.2) \qquad \mathfrak{g}_{x_2} = \left\{ A = \begin{bmatrix} \varepsilon & 0 & 0 & 0 & -\beta \\ \alpha & -2(\varepsilon+\eta) & \beta & 0 & 0 \\ 0 & 0 & \eta & 0 & -\alpha \\ 0 & 0 & -\beta & -2(\varepsilon+\eta) & \gamma \\ 0 & 0 & 0 & 0 & 3(\varepsilon+\eta) \end{bmatrix} \right.$$
$$\left. \oplus \begin{pmatrix} \varepsilon+2\eta & & \\ & 2\varepsilon+\eta & \\ -\beta & -\alpha & -(\varepsilon+\eta) \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(3)\,.$$

The conormal vector space $V_{x_2}^*$ is spanned by $v_1 = (u_3 \wedge u_5, 0, 0)$, and $v_2 = (0, u_1 \wedge u_5, 0)$, and the action $d\rho_{x_2}$ is given by

$$d\rho_{x_2}(A)(v_1, v_2) = (v_1, v_2) \begin{pmatrix} -(4\varepsilon+6\eta) & 0 \\ 0 & -(6\varepsilon+4\eta) \end{pmatrix}$$

i) $V^*_{x_2} - S^*_{x_2} \leftrightarrow v_1 + v_2 = (u_3 \wedge u_5, u_1 \wedge u_5, 0)$, i.e., $v_1 + v_2$ is a generic point of $V^*_{x_2}$, where $S^*_{x_2}$ is the singular set of the P.V. $(G_{x_2}, \rho_{x_2}, V^*_{x_2})$. We use this notation from now in § 11. Put $y = y_1 v_1 + y_2 v_2$.

ii) $(S^*_{x_2})_1 \leftrightarrow d\rho_1(A) = -(4\varepsilon + 6\eta) \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 = (0, u_1 \wedge u_5, 0) \in S^*_{21,1}$, i.e., $(S^*_{x_2})_1 = \{y \in V^*_{x_2}; f_1^*(y) = 0\} = \overline{\rho_{x_2}(G_{x_2}) \cdot v_2}$ and $f_1^*(\rho_{x_2}(g)y) = \rho_1(g) f_1^*(y)$ for $y \in V^*_{x_2}$, $g \in G_{x_2}$. From now on, we use this notation in § 11.

iii) $(S^*_{x_2})_2 \leftrightarrow d\rho_2(A) = -(6\varepsilon + 4\eta) \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 = (u_3 \wedge u_5, 0, 0) \in S^*_{21,1}$

iv) $-\delta\chi = d\rho_1 + d\rho_2$, $\mathrm{tr}_{V^*_{x_2}} = d\rho_1 + d\rho_2$

v) $\mathrm{ord}_{\Lambda_2} f^s = -2s - 2/2$.

Since the Hessian of the localization $f_{x_2}(z) = z_1 z_2$ $(z = z_1 v_1 + z_2 v_2 \in V_{x_2})$ of $f(x)$ is not identically zero, $\Lambda_2 = \Lambda^{(2)}_{2,16}$ is a good holonomic variety. We have $\dim \Lambda_1 \cap \Lambda_2 = \dim V - 1$ and $b_{\Lambda_2}(s)/b_{\Lambda_1}(s) = (s + 1)$.

(4) The isotropy subalgebra $\mathfrak{g}_{x_3}$ is given as follows.

$$\mathfrak{g}_{x_3} = \left\{ A = \begin{bmatrix} \varepsilon & \alpha & 0 & 0 & \beta \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & \gamma & \varepsilon & \beta \\ 0 & 0 & 0 & 0 & -(2\varepsilon+\eta+\xi) \end{bmatrix} \oplus \begin{pmatrix} -(\varepsilon+\eta) & & \\ & -(\varepsilon+\xi) & \\ & & (\varepsilon+\eta+\xi) \end{pmatrix} \right\} \tag{11.3}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(3)\ .$$

The conormal vector space $V^*_{x_3}$ is spanned by $v_1 = (u_3 \wedge u_5, 0, 0)$, $v_2 = (0, u_2 \wedge u_5, 0)$ $v_3 = (0, 0, u_2 \wedge u_3)$, and

$$d\rho_{x_3}(A)(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} 3\varepsilon+2\eta & & \\ & 3\varepsilon+2\xi & \\ & & -(\varepsilon+2\eta+2\xi) \end{pmatrix}$$

i) $V^*_{x_3} - S^*_{x_3} \leftrightarrow v_1 + v_2 + v_3 = (u_3 \wedge u_5, u_2 \wedge u_5, u_2 \wedge u_3) \in S^*_{15,3}$

ii) $(S^*_{x_3})_1 \leftrightarrow d\rho_1(A) = 3\varepsilon + 2\eta \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 + v_3 = (0, u_2 \wedge u_5, u_2 \wedge u_3)$ $\in S^*_{16,2}$

iii) $(S^*_{x_3})_2 \leftrightarrow d\rho_2(A) = 3\varepsilon + 2\xi \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 + v_3 = (u_3 \wedge u_5, 0, u_2 \wedge u_3)$ $\in S^*_{16,2}$

iv) $(S^*_{x_3})_3 \leftrightarrow d\rho_3(A) = -(\varepsilon + 2\eta + 2\xi) \leftrightarrow f_3^*(y) = y_3 \leftrightarrow v_1 + v_2 = (u_3 \wedge u_5, u_2 \wedge u_5, 0) \in S^*_{16,2}$

v) $-\delta\chi = d\rho_1 + d\rho_2 + d\rho_3$, $\mathrm{tr}_{V^*_{x_3}} = d\rho_1 + d\rho_2 + d\rho_3$.

Since the localization $f_{x_3}(z) = z_1 z_2 z_3$ $(z = \sum z_i v_i \in V_{x_3})$ is non-degenerate, $\Lambda_{3,15}$ is a good holonomic variety and $\mathrm{ord}_{\Lambda_{3,15}} f^s = -3s - \frac{3}{2}$. We have $\dim \Lambda_2 \cap \Lambda_{3,15} = \dim V - 1$ and $b_{\Lambda_{3,15}}(s)/b_{\Lambda_2}(s) = (s+1)$.

(5) The isotropy subalgebra $\mathfrak{g}_{x'_3}$ at x'_3 is given as follows.

$$\mathfrak{g}_{x'_3} = \left\{ A = \begin{bmatrix} -2(\varepsilon+\eta) & \alpha & \beta & 0 & 0 \\ 0 & \varepsilon & \gamma & 0 & -2\alpha \\ 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & -\alpha & -(3\varepsilon+\eta) & \delta \\ 0 & 0 & 0 & 0 & 4\varepsilon+2\eta \end{bmatrix} \oplus \begin{pmatrix} \varepsilon+2\eta & 0 & 0 \\ -\gamma & 2\varepsilon+\eta & 0 \\ \beta & -\alpha & -\varepsilon-\eta \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(4)\,. \tag{11.4}$$

The conormal vector space $V^*_{x_3'}$ is spanned by $v_1 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_2 = (u_3 \wedge u_4, 0, 0)$, $v_3 = (u_3 \wedge u_5, 0, 0)$, and

$$d\rho_{x'_3}(A)(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} -(6\varepsilon+4\eta) & 0 & 0 \\ 0 & 2(\varepsilon-\eta) & 0 \\ -2\gamma & -\delta & -5(\varepsilon+\eta) \end{pmatrix}$$

i) $V^*_{x_3'} - S^*_{x_3'} \leftrightarrow v_1 + v_2 = (u_2 \wedge u_5 + u_3 \wedge u_4, -u_3 \wedge u_5, 0) \in S^*_{13,3}$

ii) $(S^*_{x_3'})_1 \leftrightarrow d\rho_1(A) = -(6\varepsilon + 4\eta) \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 \in S^*_{21,1}$

iii) $(S^*_{x_3'})_2 \leftrightarrow d\rho_2(A) = 2(\varepsilon - \eta) \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 \in S^*_{16,2}$

iv) $-\delta\chi = 2d\rho_1 + d\rho_2 = -10(\varepsilon + \eta)$, $\mathrm{tr}_{V^*_{x_3}} = 2d\rho_1 + \frac{3}{2} d\rho_2 = -9\varepsilon - 11\eta$.

Since $\dim \Lambda_{3,13} \cap \Lambda_2 = \dim \Lambda_{3,13} \cap \Lambda_1 = \dim V - 1$ and they intersect G_0-prehomogeneously, $\Lambda_{3,13}$ is a good holonomic variety and $\mathrm{ord}_{\Lambda_{3,13}} f^s = -3s - \frac{4}{2}$. The intersection exponent of $\Lambda_{3,13}$ and Λ_1 is $(1:0)$. We have $b_{\Lambda_{3,13}}(s)/b_{\Lambda_1}(s) = (s+1)(s+\frac{3}{2})$ and $b_{\Lambda_{3,13}}(s)/b_{\Lambda_2}(s) = (s + \frac{3}{2})$.

(6) The isotropy subalgebra $\mathfrak{g}_{x_4}$ at x_4 is given as follows.

$$\mathfrak{g}_{x_4} = \left\{ A = \begin{bmatrix} \varepsilon & \alpha & \gamma & 0 & 0 \\ 0 & \eta & \beta & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 \\ 0 & 0 & -\alpha & \varepsilon-\eta+\xi & \delta \\ 0 & 0 & 0 & 0 & -2(\varepsilon+\xi) \end{bmatrix} \oplus \begin{pmatrix} -(\varepsilon+\eta) & 0 & 0 \\ -\beta & -(\varepsilon+\xi) & 0 \\ 0 & 0 & (\varepsilon+\eta+\xi) \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(4)\,. \tag{11.5}$$

The conormal vector space $V_{x_4}^*$ is spanned by $v_1 = (u_3 \wedge u_4, 0, 0)$, $v_2 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_3 = (u_3 \wedge u_5, 0, u_2 \wedge u_3)$, $v_4 = (0, 0, u_2 \wedge u_3)$, and

$$d\rho_{x_4}(A)(v_1, \cdots, v_4) = (v_1, \cdots, v_4)\begin{bmatrix} 2(\eta-\varepsilon) & 0 & 0 & 0 \\ 0 & 3\varepsilon+2\eta & 0 & 0 \\ \alpha & -2\beta & 3\varepsilon+\eta+\xi & 0 \\ 0 & 0 & 0 & -(\varepsilon+2\eta+2\xi) \end{bmatrix}$$

i) $V_{x_4}^* - S_{x_4}^* \leftrightarrow v_1 + v_2 + v_4 = (u_3 \wedge u_4 + u_2 \wedge u_5, -u_3 \wedge u_5, u_2 \wedge u_3) \in S_{11,4}^*$

ii) $(S_{x_4}^*)_1 \leftrightarrow d\rho_1(A) = 2(\eta - \xi) \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 + v_4$
$= (u_2 \wedge u_5, -u_3 \wedge u_5, u_2 \wedge u_3) \in S_{15,3}^*$

iii) $(S_{x_4}^*)_2 \leftrightarrow d\rho_2(A) = 3\varepsilon + 2\xi \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 + v_4$
$= (u_3 \wedge u_4, 0, u_2 \wedge u_3) \in S_{16,2}^*$

iv) $(S_{x_4}^*)_3 \leftrightarrow d\rho_3(A) = -(\varepsilon + 2\eta + 2\xi) \leftrightarrow f_3^*(y) = y_4 \leftrightarrow v_1 + v_2$
$= (u_2 \wedge u_5 + u_3 \wedge u_4, -u_3 \wedge u_5, 0) \in S_{13,3}^*$

v) $-\delta\chi = d\rho_1 + 2d\rho_2 + d\rho_3$, $\mathrm{tr}_{V_{x_4}^*} = \frac{3}{2}d\rho_1 + 2d\rho_2 + d\rho_3$.

The conormal bundle Λ_4 is a good holonomic variety with $\mathrm{ord}_{\Lambda_4} f^s = -4s - \frac{5}{2}$. We have $b_{\Lambda_4}(s)/b_{\Lambda_2}(s) = (s+1)(s+\frac{3}{2})$, $b_{\Lambda_4}(s)/b_{\Lambda_{3,13}}(s) = (s+1)$ and $b_{\Lambda_4}(s)/b_{\Lambda_{3,15}}(s) = (s+\frac{3}{2})$. Note that these intersections are regular and G_0-prehomogeneous.

(7) The isotropy subalgebra $\mathfrak{g}_{x_5}$ at x_5 is given as follows.

$$\mathfrak{g}_{x_5} = \left\{ A = \begin{bmatrix} \varepsilon & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & 2\varepsilon+4\eta & \gamma_5 & 0 & \gamma_6 \\ 0 & 0 & \eta & 0 & \gamma_3+\gamma_5 \\ 0 & 0 & -\gamma_1 & -(\varepsilon+3\eta) & -\gamma_2 \\ 0 & 0 & 0 & 0 & -2(\varepsilon+\eta) \end{bmatrix} \right. \tag{11.6}$$

$$\left. \oplus \begin{pmatrix} -(3\varepsilon+4\eta) & 0 & 0 \\ \gamma_3-\gamma_5 & -(\varepsilon+\eta) & 0 \\ -\gamma_6 & -\gamma_5 & \varepsilon+2\eta \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(6) .$$

Then $V_{x_5}^*$ is spanned by $v_1 = (u_1 \wedge u_5 - u_3 \wedge u_4, u_4 \wedge u_5, 0)$, $v_2 = (u_2 \wedge u_3, -u_2 \wedge u_5, u_3 \wedge u_5)$, $v_3 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_4 = (u_4 \wedge u_5, 0, 0)$, $v_5 = (u_3 \wedge u_5, 0, 0)$, and

$$d\rho_{x_5}(A)(v_1, \cdots, v_5) = (v_1, \cdots, v_5)\begin{bmatrix} 4\varepsilon+6\eta & 0 & 0 & 0 & 0 \\ 0 & \varepsilon-\eta & 0 & 0 & 0 \\ -\gamma_1 & -\gamma_5 & 3\varepsilon+2\eta & 0 & 0 \\ -3\gamma_3 & 0 & 0 & 6\varepsilon+9\eta & 0 \\ -2\gamma_5 & 2\gamma_6 & \gamma_3-2\gamma_5 & \gamma_1 & 5(\varepsilon+\eta) \end{bmatrix} \tag{11.7}$$

i) $V^*_{x_5} - S^*_{x_5} \leftrightarrow v_1 + v_2 = (u_1 \wedge u_5 - u_3 \wedge u_4 + u_2 \wedge u_3, u_4 \wedge u_5$
$- u_2 \wedge u_5, u_3 \wedge u_5) \in S^*_{8,5}$

ii) $(S^*_{x_5})_1 \leftrightarrow d\rho_1(A) = 4\varepsilon + 6\eta \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 + v_4 \in S^*_{11,4}$

iii) $(S^*_{x_5})_2 \leftrightarrow d\rho_2(A) = \varepsilon - \eta \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 \in S^*_{13,3}$

iv) $-\delta\chi = 3d\rho_1 + 3d\rho_2(= 15\varepsilon + 15\eta), \operatorname{tr}_{V^*_{x_5}} = 4d\rho_1 + 3d\rho_2(= 19\varepsilon + 21\eta)$.

The conormal bundle Λ_5 is a good holonomic variety with $\operatorname{ord}_{\Lambda_5} f^s = -6s - \frac{9}{2}$. We have $b_{\Lambda_5}(s)/b_{\Lambda_4}(s) = (s+\frac{4}{3})(s+\frac{5}{3})$ and $b_{\Lambda_5}(s)/b_{\Lambda_{3,13}}(s) = (s+1)(s+\frac{4}{3})(s+\frac{5}{3})$. Note that the intersection exponent of Λ_5 and Λ_4 is $(2:1)$. The intersection of Λ_5 and $\Lambda_{3,13}$ is regular and G_0-prehomogeneous.

(8) The isotropy subalgebra $\mathfrak{g}_{x_6}$ at x_6 is given as follows.

$$(11.8) \quad \mathfrak{g}_{x_6} = \left\{ \tilde{A} = \left(\begin{array}{c|c|c} -4\varepsilon & \gamma_1 \quad \gamma_2 & 0 \\ \hline 0 & \varepsilon I_2 + A & 0 \\ \hline 0 & 0 & \varepsilon I_2 + B \end{array}\right) \oplus \left(\begin{array}{c|c} 3\varepsilon I_2 - {}^tA & 0 \\ \hline \gamma_2 - \gamma_1 & -2\varepsilon \end{array}\right); A, B \in \mathfrak{sl}(2) \right\}$$
$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(2)\,.$$

Then $V^*_{x_6}$ is spanned by $v_1 = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (u_3 \wedge u_4, 0, 0)$, $v_4 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_5 = (0, u_2 \wedge u_5, 0)$, $v_6 = (u_3 \wedge u_5, 0, 0)$, and $(G_{x_6}, \rho_{x_6}, V^*_{x_6}) \cong (GL(1) \times SL(2) \times SL(2), 5\Lambda_1 \otimes 2\Lambda_1 \otimes \Lambda_1, V(1) \otimes V(3) \otimes V(2)) \cong (SO(3) \times GL(2), \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(2))$ and hence Λ_6 is a good holonomic variety.

i) $V^*_{x_6} - S^*_{x_6} \leftrightarrow v_2 + v_6 = (u_3 \wedge u_5, u_2 \wedge u_4, 0) \in S^*_{12,6}$

ii) $(S^*_{x_6})_1 \leftrightarrow v_2 + v_4 = (u_2 \wedge u_5, u_2 \wedge u_4 - u_3 \wedge u_5, 0) \in S^*_{13,3}$

iii) $-\delta\chi = d\rho_1, \operatorname{tr}_{V^*_{x_6}} = \frac{3}{2} d\rho_1$

We have $\operatorname{ord}_{\Lambda_6} f^s = -4s - \frac{6}{2}$ and $b_{\Lambda_6}(s)/b_{\Lambda_{3,13}}(s) = (s+\frac{3}{2})$.

(9) The isotropy subalgebra $\mathfrak{g}_{x_7}$ at x_7 is given as follows.

$$(11.9) \quad \mathfrak{g}_{x_7} = \left\{ \tilde{A} = \left(\begin{array}{c|c|c} -2(\varepsilon+\eta) & F & 0 \\ \hline 0 & \varepsilon I_2 + A & 0 \\ \hline 0 & 0 & \eta I_2 + B \end{array}\right) \oplus \left(\begin{array}{c|c} (\varepsilon + 2\eta) I_2 - {}^tA & 0 \\ \hline 0 & -2\eta \end{array}\right); \right.$$
$$\left. A, B \in \mathfrak{sl}(2), {}^tF \in \boldsymbol{C}^2 \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(2)\,.$$

Then $V^*_{x_7}$ is spanned by $v_1 = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (u_3 \wedge u_4, 0, 0)$, $v_4 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_5 = (0, u_2 \wedge u_5, 0)$, $v_6 = (u_3 \wedge u_5, 0, 0)$, $v_7 = (0, 0, u_2 \wedge u_3)$, and $(G_{x_7}, \rho_{x_7}, V^*_{x_7}) \cong (GL(1) \times GL(1) \times SL(2) \times SL(2), (2\Lambda_1^* \otimes 3\Lambda_1^* \otimes 2\Lambda_1 \otimes \Lambda_1) \oplus (2\Lambda_1^* \otimes 2\Lambda_1 \otimes 1 \otimes 1), V(6) \oplus V(1))$.

i) $V^*_{x_7} - S^*_{x_7} \leftrightarrow v_2 + v_6 + v_7 = (u_3 \wedge u_5, u_2 \wedge u_4, u_2 \wedge u_3) \in S^*_{9,7}$

ii) $(S^*_{x_7})_1 \leftrightarrow d\rho_1(\tilde{A}) = -8\varepsilon - 12\eta \leftrightarrow f_1^*(y)(\deg f_1^* = 4) \leftrightarrow v_2 + v_4 + v_7 \in S^*_{11,4}$

iii) $(S_{x_7}^*)_2 \leftrightarrow d\rho_2(\tilde{A}) = -2\varepsilon + 2\eta \leftrightarrow f_2^*(y) = y_7 \leftrightarrow v_2 + v_6 \in S_{12,6}^*$

iv) $-\delta\chi = d\rho_1 + d\rho_2, \mathrm{tr}_{V_{x_7}^*} = \frac{3}{2}d\rho_1 + d\rho_2.$

Then by Corollary 1–7 conormal bundle $\Lambda_{7,9}$ is a good holonomic variety with $\mathrm{ord}_{\Lambda_{7,9}} f^s = -5s - \frac{7}{2}$. We have $\dim \Lambda_{7,9} \cap \Lambda_4 = \dim \Lambda_{7,9} \cap \Lambda_6 = \dim V - 1$, $b_{\Lambda_{7,9}}(s)/b_{\Lambda_4}(s) = (s + \frac{3}{2})$, and $b_{\Lambda_{7,9}}(s)/b_{\Lambda_6}(s) = (s + 1)$.

(10) The isotropy subalgebra $\mathfrak{g}_{x_7'}$ at x_7' is given as follows.

$$\mathfrak{g}_{x_7'} = \left\{ A = \left[\begin{array}{cc|ccc} 3\varepsilon+\alpha & \beta & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma & 3\varepsilon-\alpha & \gamma_4 & \gamma_5 & \gamma_6 \\ \hline & & -2\varepsilon & -2\gamma & -2\beta \\ \multicolumn{2}{c|}{0} & -\beta & -2\varepsilon+2\alpha & 0 \\ & & -\gamma & 0 & -2\varepsilon-2\alpha \end{array}\right] \oplus \left(\begin{array}{c|cc} -6\varepsilon & \multicolumn{2}{c}{0} \\ \hline \gamma_2-\gamma_4 & -\varepsilon-\alpha & \gamma \\ \gamma_1-\gamma_6 & \beta & -\varepsilon+\alpha \end{array}\right) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus V(6) . \tag{11.10}$$

Then $V_{x_7'}^*$ is spanned by $v_1 = (u_1 \wedge u_4, 0, u_3 \wedge u_4)$, $v_2 = (u_1 \wedge u_3 - u_2 \wedge u_4, - u_3 \wedge u_4, - 2u_4 \wedge u_5)$, $v_3 = (u_1 \wedge u_5 - u_2 \wedge u_3, 2u_4 \wedge u_5, - u_3 \wedge u_5)$, $v_4 = (- u_2 \wedge u_5, u_3 \wedge u_5, 0)$, $v_5 = (u_4 \wedge u_5, 0, 0)$, $v_6 = (u_3 \wedge u_4, 0, 0)$, $v_7 = (u_3 \wedge u_5, 0, 0)$ and the action $d\rho_{x_7'}$ of $\mathfrak{g}_{x_7'}$ on $V_{x_7'}^*$ is given by

$$d\rho_{x_7'}(A)(v_1, \cdots, v_7) = (v_1, \cdots, v_7)\left(\begin{array}{c|c} 5\varepsilon I_4 + A_1 & 0 \\ \hline C & 10\varepsilon I_3 + A_2 \end{array}\right)$$

where

$$(C, A_2) = \begin{pmatrix} \gamma_3 & 2\gamma_1-\gamma_6 & 3\gamma_2-2\gamma_4 & -\gamma_5 & 0 & -2\beta & 2\gamma \\ \gamma_6-2\gamma_1 & 2\gamma_2 & \gamma_5 & 0 & -\gamma & -2\alpha & 0 \\ 0 & \gamma_3 & 2\gamma_6 & \gamma_2-2\gamma_4 & \beta & 0 & 2\alpha \end{pmatrix}$$

and

$$A_1 = \begin{bmatrix} -3\alpha & 3\gamma & 0 & 0 \\ \beta & -\alpha & -2\gamma & 0 \\ 0 & -2\beta & \alpha & \gamma \\ 0 & 0 & 3\beta & 3\alpha \end{bmatrix}$$

i) $V_{x_7'}^* - S_{x_7'}^* \leftrightarrow v_1 + v_4 = (u_1 \wedge u_4 - u_2 \wedge u_5, u_3 \wedge u_5, u_3 \wedge u_4) \in S_{7,7}^{(2)*}$

ii) $(S_{x_7'}^*)_1 \leftrightarrow d\rho_1(A) = 20\varepsilon \leftrightarrow f_1^*(y_1, \cdots, y_4)$: the discriminant of binary cubic forms $\leftrightarrow v_2 \in S_{8,5}^*$

iii) $-\delta\chi = 2d\rho_1, \mathrm{tr}_{V_{x_7'}^*} = \frac{5}{2}d\rho_1.$

The conormal bundle $\Lambda_{7,7}^{(1)}$ is a good holonomic variety with $\mathrm{ord}_{V_{7,7}^{(1)}} f^s = -8s - \frac{13}{2}$. We have $\dim \Lambda_5 \cap \Lambda_{7,7}^{(1)} = \dim V - 1$ and $b_{\Lambda_{7,7}^{(1)}}(s)/b_{\Lambda_5}(s) = (s + \frac{5}{4})(s + \frac{7}{4})$. The intersection is regular and G_0-prehomogeneous.

(11) The isotropy subalgebra $\mathfrak{g}_{x_7''}$ at x_7'' is given as follows.

$$(11.11) \quad \mathfrak{g}_{x_7''} = \left\{ \tilde{A} = \left(\begin{array}{c|c|c} -2(\varepsilon+\eta) & 0 & C \\ \hline 0 & \varepsilon I_2 + A & D \\ \hline 0 & 0 & \eta I_2 - {}^tA \end{array}\right) \oplus \left(\begin{array}{c|c} (\varepsilon+2\eta)I_2 - {}^tA & 0 \\ \hline C & -(\varepsilon+\eta) \end{array}\right); \right.$$

$$\left. A \in \mathfrak{sl}(2) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(5) .$$

Then $V_{x_7''}^*$ is spanned by $v_1 = (u_3 \wedge u_4, 0, 0)$, $v_2 = (u_2 \wedge u_4 - u_3 \wedge u_5, -u_3 \wedge u_4, 0)$, $v_3 = (u_2 \wedge u_5, u_2 \wedge u_4 - u_3 \wedge u_5, 0)$, $v_4 = (0, u_2 \wedge u_5, 0)$, $v_5 = (u_1 \wedge u_5, -u_1 \wedge u_4, u_4 \wedge u_5)$, $v_6 = (u_4 \wedge u_5, 0, 0)$, $v_7 = (0, u_4 \wedge u_5, 0)$, and

$$d\rho_{x_7''}(\tilde{A})(v_1, \cdots, v_7) = (v_1, \cdots, v_7)\left(\begin{array}{c|c|c} -(2\varepsilon+3\eta)I_4 + 3\Lambda_1(A) & 0 & 0 \\ \hline 0 & \varepsilon-\eta & 0 \\ \hline * & * & -(\varepsilon+4\eta)I_2 + A \end{array}\right).$$

i) $V_{x_7''}^* - S_{x_7''}^* \leftrightarrow v_1 + v_4 + v_5 \in S_{7,7}^{(1)*}$

ii) $(S_{x_7''}^*)_1 \leftrightarrow d\rho_1(\tilde{A}) = -8\varepsilon - 12\eta \leftrightarrow f_1^*(y_1, \cdots, y_4)$: the discriminant of binary cubic forms $\leftrightarrow v_3 + v_4 + v_5 \in S_{8,5}^*$

iii) $(S_{x_7''}^*)_2 \leftrightarrow d\rho_2(\tilde{A}) = \varepsilon - \eta \leftrightarrow f_2^*(y) = y_5 \leftrightarrow v_1 + v_4 \in S_{12,6}^*$

iv) $-\delta\chi = d\rho_1 + 3d\rho_2$, $\mathrm{tr}_{V x_7''} = \frac{3}{2}d\rho_1 + 3d\rho_2$.

The conormal bundle $\Lambda_{7,7}^{(2)}$ is a good holonomic variety with $\mathrm{ord}_{\Lambda_{7,7}^{(2)}} f^s = -7s - \frac{11}{2}$. We have $\dim \Lambda_5 \cap \Lambda_{7,7}^{(2)} = \dim \Lambda_6 \cap \Lambda_{7,7}^{(2)} = \dim V - 1$, $b_{\Lambda_{7,7}^{(2)}}(s)/b_{\Lambda_5}(s) = (s + \frac{3}{2})$ and $b_{\Lambda_{7,7}^{(2)}}(s)/b_{\Lambda_6}(s) = (s+1)(s + \frac{4}{3})(s + \frac{5}{3})$. The intersections are regular and G_0-prehomogeneous.

(12) We shall calculate the isotropy subalgebra at $\tilde{x}_8 = (u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, 0)$ instead of x_8.

$$(11.12) \quad \mathfrak{g}_{\tilde{x}_8} = \left\{ \tilde{A} = \left[\begin{array}{cc|ccc} 3\varepsilon+\alpha & \alpha_{12} & \gamma_1 & \gamma_2 & \gamma_3 \\ \alpha_{21} & 3\varepsilon-\alpha & \gamma_3 & \gamma_1 & \gamma_4 \\ \hline & & -2\varepsilon & -2\alpha_{12} & -2\alpha_{21} \\ & 0 & -\alpha_{21} & -2\varepsilon-2\alpha & \\ & & -\alpha_{12} & & -2\varepsilon+2\alpha \end{array}\right] \right.$$

$$\left. \oplus \left(\begin{array}{cc|c} -\varepsilon+\alpha & \alpha_{12} & \gamma_5 \\ \alpha_{21} & -\varepsilon-\alpha & \alpha_6 \\ \hline \multicolumn{2}{c|}{0} & \eta \end{array}\right) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus V(6) .$$

Then $V^*_{\tilde{x}_8}$ is spanned by $v_1 = (0, 0, u_1 \wedge u_2)$, $v_2 = (0, 0, u_1 \wedge u_5)$, $v_3 = (0, 0, u_1 \wedge u_3 - u_2 \wedge u_5)$, $v_4 = (0, 0, u_1 \wedge u_4 - u_2 \wedge u_3)$, $v_5 = (0, 0, u_2 \wedge u_4)$, $v_6 = (0, 0, u_4 \wedge u_5)$, $v_7 = (0, 0, u_3 \wedge u_5)$, $v_8 = (0, 0, u_3 \wedge u_4)$ and the action $d\rho_{\tilde{x}_8}$ of $\mathfrak{g}_{\tilde{x}_8}$ on $V^*_{\tilde{x}_8}$ is given by

$$d\rho_{\tilde{x}_8}(\tilde{A})(v_1, \cdots, v_8) = (v_1, \cdots, v_8)\left(\begin{array}{c|c|c} -6\varepsilon-\eta & 0 & 0 \\ \hline C_1 & -(\varepsilon+\eta)I_4+A_1 & 0 \\ \hline 0 & C_2 & (4\varepsilon-\eta)I_3+A_2 \end{array}\right)\begin{array}{l}\}1\\ \}4\\ \}3\end{array}$$

where

$$(C_2, A_2) = \begin{pmatrix} -\gamma_2 & \gamma_1 & \gamma_3 & \gamma_4 & 0 & 2\alpha_{12} & -2\alpha_{21} \\ -\gamma_1 & 2\gamma_3 & -\gamma_4 & 0 & \alpha_{21} & -2\alpha & 0 \\ 0 & \gamma_2 & -2\gamma_1 & -\gamma_3 & -\alpha_{12} & 0 & 2\alpha \end{pmatrix}$$

and

$$(C_1, A_1) = \begin{bmatrix} -\gamma_4 & -3\alpha & 3\alpha_{21} & 0 & 0 \\ -\gamma_3 & \alpha_{12} & -\alpha & 2\alpha_{21} & 0 \\ -\gamma_1 & 0 & 2\alpha_{12} & \alpha & -\alpha_{21} \\ \gamma_2 & 0 & 0 & -3\alpha_{12} & 3\alpha \end{bmatrix}.$$

i) $V^*_{\tilde{x}_8} - S^*_{\tilde{x}_8} \leftrightarrow v_1 + v_6 = (0, 0, u_1 \wedge u_2 + u_4 \wedge u_5) \in S^*_{18,8}$

ii) $(S^*_{\tilde{x}_8})_1 \leftrightarrow d\rho_1(\tilde{A}) = -4\varepsilon - 4\eta \leftrightarrow f_1^*(y_2, \cdots, y_5)$: the discriminant of binary cubic forms $\leftrightarrow v_1 + v_7 = (0, 0, u_1 \wedge u_2 + u_3 \wedge u_5) \in S^*_{18,8}$

iii) $(S^*_{\tilde{x}_8})_2 \leftrightarrow d\rho_2(\tilde{A}) = -6\varepsilon - \eta \leftrightarrow f_2^*(y) = y_1 \leftrightarrow v_2 + v_5 = (0, 0, u_1 \wedge u_5 + u_2 \wedge u_4) \in S^*_{18,8}$

iv) $-\delta\chi = 10\varepsilon - 5\eta = -3d\rho_1 + 2d\rho_2$, $\mathrm{tr}_{V^*_{\tilde{x}_8}} = 2\varepsilon - 8\eta = 2d\rho_1 + \frac{5}{2}d\rho_2$.

The conormal bundle $\Lambda_{8,18}$ is a good holonomic variety with $\mathrm{ord}_{\Lambda_{8,18}} f^s = -5s - \frac{8}{2}$. The conormal vector space $(G_{x_8}, \rho_{x_8}, V^*_{x_8})$ is a regular P.V. In fact, for $z = \sum_{i=1}^8 z_i v_i \in V_{\tilde{x}_8}$, the localization $f_{\tilde{x}_8}(z)$ of $f(x)$ is given by $f_{\tilde{x}_8}(z) = z_1z_6^4 + z_1z_7^2z_8^2 + 2z_1z_6^2z_7z_8 + z_2^2z_8^3 + z_3^2z_6^2z_8 + 2z_2z_3z_6z_8^2 + z_2z_5z_6^3 + z_2z_4z_6^2z_8 + z_3z_4z_6^3 + 3z_2z_5z_6z_7z_8 + z_3z_5z_6^2z_7 - z_2z_4z_7z_8^2 - z_3z_4z_6z_7z_8 - z_3z_5z_7^2z_8 - z_5^2z_7^3 - z_4^2z_6^2z_7 - 2z_4z_5z_6z_7^2$, and hence its Hessian is not identically zero. Now we shall show that $\dim \Lambda_{7,7}^{(1)} \cap \Lambda_{8,18} = \dim V - 1$. From iii) above, $\Lambda = \overline{G(\tilde{x}_8, v_2 + v_5)}$ is one-codimensional and $\Lambda \subset \Lambda_{8,18}$. It is enough to show $(\tilde{x}_8, v_2 + v_5) = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, 0), (0, 0, u_1 \wedge u_5 + u_2 \wedge u_4)\} \in \Lambda_{7,7}^{(1)}$. Put $z = \{(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_2 \wedge u_3 + u_1 \wedge u_5), (u_1 \wedge u_4 - u_2 \wedge u_5, u_3 \wedge u_5, u_3 \wedge u_4)\}$. Then $z \in \Lambda_{7,7}^{(1)}$ (See (10)). Then for $\varepsilon>0$, put

$$g_\varepsilon = \left[\begin{array}{ccc|cc} -\varepsilon^3 & & & & \\ & \varepsilon^3 & & & \\ & & \varepsilon^{-2} & & \\ \hline & & & & -\varepsilon^{-2} \\ & & & -\varepsilon^{-2} & \end{array}\right] \times \begin{pmatrix} & & \varepsilon^{-1} \\ & -\varepsilon^{-1} & \\ \varepsilon^{-1} & & \end{pmatrix} \in G = SL(5) \times GL(3).$$

Then $g_\varepsilon \cdot z = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, -\varepsilon^5 u_1 \wedge u_2), (-\varepsilon^5 u_3 \wedge u_5, \varepsilon^5 u_3 \wedge u_4, u_1 \wedge u_5 + u_2 \wedge u_4)\}$. Since $g_\varepsilon \cdot z \in \Lambda^{(1)}_{7,7}$ and $\Lambda^{(1)}_{7,7}$ is closed, we have $(\tilde{x}_8, v_2 + v_5) = \lim_{\varepsilon\to 0} g_\varepsilon \cdot z \in \Lambda^{(1)}_{7,7}$ and hence $\dim \Lambda^{(1)}_{7,7} \cap \Lambda_{8,18} = \dim V - 1$. One can see easily that their intersection is regular and G_0-prehomogeneous. We have $b_{\Lambda^{(1)}_{7,7}}(s)/b_{\Lambda_{8,18}}(s) = (s+1)(s+\frac{4}{3})(s+\frac{5}{3})$. Next we shall show that $\dim \Lambda_{8,18} \cap \Lambda_{3,13} = \dim V - 1$. From ii) above, $\Lambda = \overline{G(\tilde{x}_8, v_1 + v_7)}$ is one-codimensional and $\Lambda \subset \Lambda_{8,18}$, where $(\tilde{x}_8, v_1 + v_7) = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, 0), (0, 0, u_1 \wedge u_2 + u_3 \wedge u_5)\}$. Put $w = \{(u_1 \wedge u_2, u_1 \wedge u_3 + u_2 \wedge u_4, u_2 \wedge u_3 + u_4 \wedge u_5), (u_2 \wedge u_5 + u_3 \wedge u_4, -u_3 \wedge u_5, 0)\}$. Then $w \in \Lambda_{3,13}$ (See (5)). For $\varepsilon > 0$, put

$$g_\varepsilon = \begin{pmatrix} & & -\sqrt{-1}\varepsilon^{-4} & & \\ & & & \varepsilon & \\ & -\sqrt{-1}\varepsilon & & & \\ \varepsilon^6 & & & & \\ & & & & \varepsilon^{-4} \end{pmatrix} \times \begin{pmatrix} 0 & -\sqrt{-1}\varepsilon^{-2} & 0 \\ 0 & 0 & \varepsilon^3 \\ \sqrt{-1}\varepsilon^3 & 0 & 0 \end{pmatrix}$$

$$\in G = SL(5) \times GL(3).$$

Then $g_\varepsilon \cdot w = \{(u_2 \wedge u_3 + u_1 \wedge u_4, u_1 \wedge u_3 + u_2 \wedge u_5, -\varepsilon^{10} u_3 \wedge u_4), (\varepsilon^{10} u_3 \wedge u_5, 0, u_1 \wedge u_2 + u_3 \wedge u_5)\}$. Since $g_\varepsilon \cdot w \in \Lambda_{3,13}$ and $\Lambda_{3,13}$ is closed, we have $(\tilde{x}_8, v_1 + v_7) = \lim_{\varepsilon\to 0} g_\varepsilon \cdot w \in \Lambda_{3,13}$, i.e., $\Lambda \subset \Lambda_{3,13} \cap \Lambda_{8,18}$. Hence we have $\dim \Lambda_{3,13} \cap \Lambda_{8,18} = \dim V - 1$. The intersection is G_0-prehomogeneous and regular, and hence we have $b_{\Lambda_{8,18}}(s)/b_{\Lambda_{3,13}}(s) = (s+\frac{5}{4})(s+\frac{7}{4})$.

(13) The isotropy subalgebra $\mathfrak{g}_{x'_8}$ at x'_8 is given as follows.

$$(11.13)\quad \mathfrak{g}_{x'_8} = \left\{ A = \left(\begin{array}{cc|cc|c} \varepsilon+\alpha+\beta & \alpha_{12} & & \beta_{12} & \gamma_1 \\ \alpha_{21} & \varepsilon-\alpha+\beta & -\beta_{12} & & \gamma_2 \\ \hline & \beta_{21} & \varepsilon-\alpha-\beta & & \gamma_3 \\ -\beta_{21} & & & \varepsilon+\alpha-\beta & \gamma_4 \\ \hline & & 0 & & -4\varepsilon \end{array}\right) \right.$$

$$\left. \oplus \left(\begin{array}{c|c|c} -2\varepsilon-2\beta & & -\beta_{21} \\ & -2\varepsilon+2\beta & \beta_{12} \\ 2\beta_{12} & -2\beta_{21} & -2\varepsilon \end{array}\right) \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(4).$$

Then $V^*_{x'_8}$ is spanned by $v_1 = (u_3 \wedge u_5, 0, 0)$, $v_2 = (u_2 \wedge u_5, 0, -u_3 \wedge u_5)$, $v_3 = (0, u_3 \wedge u_5, -u_2 \wedge u_5)$, $v_4 = (0, u_2 \wedge u_5, 0)$, $v_5 = (u_4 \wedge u_5, 0, 0)$, $v_6 = (u_1 \wedge u_5, 0, u_4 \wedge u_5)$, $v_7 = (0, u_4 \wedge u_5, u_1 \wedge u_5)$, $v_8 = (0, u_1 \wedge u_5, 0)$. We have $(G_{x'_8}, \rho_{x'_8}, V^*_{x'_8}) \cong (GL(1) \times SL(2) \times SL(2), 5\Lambda_1 \otimes 3\Lambda_1 \otimes \Lambda_1, V(1) \otimes V(4) \otimes V(2))$. Since $\dim \rho_{x'_8}(G_{x'_8}) = 7 < \dim V^*_{x'_8} = 8$, this is not a P.V. The dual of the orbit

$S_{8,14}$ is $S_{14,8}^*$ in V^*, i.e., $\Lambda_{8,14} = \Lambda_{14,8}^*$.

(14) The isotropy subalgebra $\mathfrak{g}_{x_8''}$ at x_8'' is given as follows.

$$(11.14)\quad \mathfrak{g}_{x_8''} = \left\{ A = \begin{bmatrix} \varepsilon & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ 0 & \eta & \gamma_5 & \gamma_6 & \gamma_7 \\ 0 & 0 & -2(\eta+\xi) & 0 & \gamma_8 \\ 0 & 0 & 0 & \xi & -\gamma_1 \\ 0 & 0 & 0 & 0 & -\varepsilon+\eta+\xi \end{bmatrix} \oplus \begin{pmatrix} -(\varepsilon+\eta) & 0 & 0 \\ -\gamma_5 & -\varepsilon+2\eta+2\xi & 0 \\ \gamma_3-\gamma_7 & -\gamma_8 & -\eta-\xi \end{pmatrix} \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(8)\,.$$

Then $V_{x_8''}^*$ is spanned by $v_1 = (u_1 \wedge u_5 - u_2 \wedge u_4, 2u_3 \wedge u_4, u_4 \wedge u_5)$, $v_2 = (0, u_2 \wedge u_5, 0)$, $v_3 = (u_2 \wedge u_3, 0, u_3 \wedge u_5)$, $v_4 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_5 = (0, u_4 \wedge u_5, 0)$, $v_6 = (u_3 \wedge u_4, 0, 0)$, $v_7 = (u_4 \wedge u_5, 0, 0)$, $v_8 = (u_3 \wedge u_5, 0, 0)$.

The action $d\rho_{x_8''}$ of $\mathfrak{g}_{x_8''}$ on $V_{x_8''}^*$ is given by

$$d\rho_{x_8''}(A)(v_1, \cdots, v_8) = (v_1, \cdots, v_8) \left[\begin{array}{cccc|cccc} A_1 & & & & & & & \\ & A_2 & & & & & & \\ & & A_3 & & & 0 & & \\ -2\gamma_1 & \gamma_5 & -\gamma_8 & A_4 & & & & \\ \hline 3\gamma_8 & -\gamma_6 & & & A_5 & & & \\ 3\gamma_5 & & \gamma_6 & & & A_6 & & \\ -2\gamma_3 & & & -\gamma_6 & \gamma_5 & \gamma_8 & A_7 & \\ -\gamma_2 & & 2\gamma_7-\gamma_3 & -2\gamma_5 & & \gamma_1 & & A_8 \end{array}\right]$$

where $A_1 = \varepsilon - \xi$, $A_2 = 2\varepsilon - 4\eta - 3\xi$, $A_3 = \varepsilon + 2\eta + 2\xi$, $A_4 = 2\varepsilon - \eta - \xi$, $A_5 = 2\varepsilon - 3\eta - 4\xi$, $A_6 = \varepsilon + 3\eta + \xi$, $A_7 = 2\varepsilon - 2\xi$ and $A_8 = 2\varepsilon + 2\eta + \xi$.

i) $V_{x_8''}^* - S_{x_8''}^* \leftrightarrow v_1 + v_2 + v_3 \in S_{5,8}^*$

ii) $(S_{x_8''}^*)_1 \leftrightarrow d\rho_1(A) = \varepsilon - \xi \leftrightarrow f_1^*(y) = y_1(y = \sum y_i v_i) \leftrightarrow v_2 + v_3 + v_5 \in S_{9,7}^*$

iii) $(S_{x_8''}^*)_2 \leftrightarrow d\rho_2(A) = 2\varepsilon - 4\eta - 3\xi \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 + v_3 \in S_{7,7}^{(2)*}$

iv) $(S_{x_8''}^*)_3 \leftrightarrow d\rho_3(A) = \varepsilon + 2\eta + 2\xi \leftrightarrow f_3^*(y) = y_3 \leftrightarrow v_1 + v_2 \in S_{7,7}^{(1)*}$

v) $-\delta\chi = 6d\rho_1 + d\rho_2 + 2d\rho_3$, $\mathrm{tr}_{V_{x_8''}^*} = \frac{15}{2}d\rho_1 + \frac{3}{2}d\rho_2 + \frac{5}{2}d\rho_3$.

Since the intersection of $\Lambda_{8,5}^{(3)}$ and $\Lambda_{7,7}^{(1)}$ is G_0-prehomogeneous, the conormal bundle $\Lambda_{8,5}^{(3)}$ is a good holonomic variety by Proposition 1–5. The order is given by $\mathrm{ord}_{\Lambda_{8,5}} f^s = -9s - \frac{15}{2}$. We have $\dim \Lambda_{8,5}^{(3)} \cap \Lambda_{7,9} = \dim \Lambda_{8,5}^{(3)} \cap \Lambda_{7,7}^{(i)} = \dim V - 1$ $(i = 1, 2)$, $b_{\Lambda_{8,5}^{(3)}}(s)/b_{\Lambda_{7,7}^{(1)}}(s) = s + \frac{3}{2}$ and $b_{\Lambda_{8,5}^{(3)}}(s)/b_{\Lambda_{7,7}^{(2)}}(s) =$

$(s+\frac{5}{4})(s+\frac{7}{4})$. In (15), we shall prove that $\dim \Lambda_{8,5}^{(3)} \cap \Lambda_{7,9}^{(2)} \cap \Lambda_{5,8}^{(2)} = \dim \Lambda_{8,5}^{(3)} \cap \Lambda_{9,7}^{(4)} \cap \Lambda_{5,8}^{(2)} = \dim V - 1$.

(15) The isotropy subalgebra $\mathfrak{g}_{x_9}$ at x_9 is given as follows.

$$\mathfrak{g}_{x_9} = \left\{ A = \begin{bmatrix} \varepsilon_1 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ 0 & \varepsilon_2 & \gamma_4 & \gamma_5 & \gamma_6 \\ 0 & 0 & \varepsilon_3 & 0 & \gamma_7 \\ 0 & 0 & 0 & \varepsilon_4 & \gamma_8 \\ 0 & 0 & 0 & 0 & -(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) \end{bmatrix} \oplus \begin{pmatrix} -\varepsilon_1-\varepsilon_2 & 0 & 0 \\ -\gamma_4 & -\varepsilon_1-\varepsilon_3 & 0 \\ \gamma_2 & 0 & -\varepsilon_2-\varepsilon_4 \end{pmatrix} \right\} \tag{11.15}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1)) \oplus \mathfrak{u}(8)\ .$$

Then $V_{x_9}^*$ is spanned by $v_1 = (0, u_2 \wedge u_5, 0)$, $v_2 = (0, 0, u_1 \wedge u_5)$, $v_3 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_4 = (u_1 \wedge u_5, 0, u_4 \wedge u_5)$, $v_5 = (0, u_4 \wedge u_5, 0)$, $v_6 = (0, 0, u_3 \wedge u_5)$, $v_7 = (u_3 \wedge u_4, 0, 0)$, $v_8 = (u_3 \wedge u_5, 0, 0)$, $v_9 = (u_4 \wedge u_5, 0, 0)$.

The action $d\rho_{x_9}$ of $\mathfrak{g}_{x_9}$ on $V_{x_9}^*$ is given by

$$d\rho_{x_9}(A)(v_1, \cdots, v_9) = (v_1, \cdots, v_9) \begin{bmatrix} A_1 & & & & & & & & \\ & A_2 & & & & & & & \\ \gamma_4 & & A_3 & & & & & & \\ & -\gamma_2 & & A_4 & & & & & \\ -\gamma_5 & & & & A_5 & & & & \\ & -\gamma_1 & & & & A_6 & & & \\ & & & & & & A_7 & & \\ & & -2\gamma_4 & -\gamma_1 & & -\gamma_2 & -\gamma_8 & A_8 & \\ & & -\gamma_5 & -2\gamma_2 & \gamma_4 & & \gamma_7 & & A_9 \end{bmatrix}$$

where $A_1 = 2\varepsilon_1 + 2\varepsilon_3 + \varepsilon_4$, $A_2 = 2\varepsilon_2 + \varepsilon_3 + 2\varepsilon_4$, $A_3 = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$, $A_4 = \varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4$, $A_5 = 2\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3$, $A_6 = \varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_4$, $A_7 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4$, $A_8 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_4$ and $A_9 = 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$.

i) $V_{x_9}^* - S_{x_9}^* \leftrightarrow v_1 + v_2 + v_7 = (u_3 \wedge u_4, u_2 \wedge u_5, u_1 \wedge u_5) \in S_{7,9}^*$

ii) $(S_{x_9}^*)_1 \leftrightarrow d\rho_1(A) = 2\varepsilon_1 + 2\varepsilon_3 + \varepsilon_4 \leftrightarrow f_1^*(y) = y_1 \leftrightarrow v_2 + v_3 + v_5 + v_7 \in S_{8,5}^*$

iii) $(S_{x_9}^*)_2 \leftrightarrow d\rho_2(A) = 2\varepsilon_2 + \varepsilon_3 + 2\varepsilon_4 \leftrightarrow f_2^*(y) = y_2 \leftrightarrow v_1 + v_4 + v_6 + v_7 \in S_{8,5}^*$

iv) $(S_{x_9}^*)_3 \leftrightarrow d\rho_3(A) = \varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 \leftrightarrow f_3^*(y) = y_7 \leftrightarrow v_1 + v_2 + v_8 + v_9 \in S_{14,8}^*$

v) $-\delta\chi = 3d\rho_1 + 3d\rho_2 + 4d\rho_3$, $\mathrm{tr}_{V_{x_9}^*} = 4d\rho_1 + 4d\rho_2 + 5d\rho_3$.

The conormal bundle $\Lambda_{9,7}$ is a good holonomic variety with $\mathrm{ord}_{\Lambda_{9,7}} f^s = -10s - \frac{17}{2}$.

Put $p = (u_1 \wedge u_2, u_1 \wedge u_3, u_2 \wedge u_4; u_1 \wedge u_5 + u_3 \wedge u_4, u_2 \wedge u_5, u_3 \wedge u_5 + u_4 \wedge u_5)$. Then by iii), we have $p \in \Lambda_{9,7}^{(4)} \cap \Lambda_{5,8}^{(2)}$ and $\dim G\cdot p = \dim V - 1$. We shall prove that $p \in \Lambda_{8,5}^{(3)}$, i.e., $\dim \Lambda_{5,8}^{(2)} \cap \Lambda_{8,5}^{(3)} \cap \Lambda_{9,7}^{(4)} = \dim V - 1$. By (14), for any $\varepsilon > 0$, we have $(u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_5 + u_2 \wedge u_4; u_1 \wedge u_5 - u_2 \wedge u_4 + (1/\varepsilon) u_2 \wedge u_3 + (1/\varepsilon^2) u_3 \wedge u_4, 2u_3 \wedge u_4 + u_2 \wedge u_5, u_4 \wedge u_5 + (1/\varepsilon) u_3 \wedge u_5) \in (x_8'', V_{x_8''}^*) \subset \Lambda_{8,5}^{(3)}$. Therefore, by the action of $g_\varepsilon = \begin{bmatrix} \varepsilon &&&& \\ & 1 &&& \\ && \varepsilon^{-1} && \\ &&& 1 & \\ &&&& 1 \end{bmatrix} \times \begin{pmatrix} \varepsilon^{-1} && \\ & 1 & \\ && 1 \end{pmatrix}$ $\in G = SL(5) \times GL(3)$, we have $p_\varepsilon = (u_1 \wedge u_2, u_1 \wedge u_3, \varepsilon u_1 \wedge u_5 + u_2 \wedge u_4; u_1 \wedge u_5 - \varepsilon u_2 \wedge u_4 + \varepsilon u_2 \wedge u_3 + u_3 \wedge u_4, 2\varepsilon u_3 \wedge u_4 + u_2 \wedge u_5, u_3 \wedge u_5 + u_4 \wedge u_5) \in \Lambda_{8,5}^{(3)}$. Hence, we have $p = \lim_{\varepsilon \to 0} p_\varepsilon \in \Lambda_{8,5}^{(3)}$. Since $\Lambda_{5,8}^{(2)} = \Lambda_{8,5}^{(3)*}$, $\Lambda_{8,5}^{(3)} = \Lambda_{5,8}^{(2)*}$, $\Lambda_{9,7}^{(4)} = \Lambda_{7,9}^{(2)*}$ and $(G, \rho, V) \cong (G, \rho^*, V^*)$, we have also $\dim \Lambda_{5,8}^{(2)} \cap \Lambda_{8,5}^{(3)} \cap \Lambda_{7,9}^{(2)} = \dim V - 1$.

(16) The isotropy subalgebra $\mathfrak{g}_{x_{10}}$ at x_{10} is given as follows.

$$(11.16) \qquad \mathfrak{g}_{x_{10}} = \left\{ A = \left[\begin{array}{cc|cc|c} \eta & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ & -4\varepsilon & 0 & 0 & \gamma_5 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \varepsilon+\alpha & \alpha_{12} & \gamma_3 \\ && \alpha_{21} & \varepsilon-\alpha & -\gamma_2 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 2\varepsilon-\eta \end{array}\right] \oplus \left(\begin{array}{cc|c} 4\varepsilon-\eta & 0 & \gamma_6 \\ -\gamma_5 & -2\varepsilon & \gamma_7 \\ \hline \multicolumn{2}{c|}{0} & \xi \end{array}\right) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(7)\ .$$

Then $V_{x_{10}}^*$ is spanned by $v_1 = (0, 0, u_1 \wedge u_3)$, $v_2 = (0, 0, u_1 \wedge u_4)$, $v_3 = (0, 0, u_1 \wedge u_5 - u_3 \wedge u_4)$, $v_4 = (0, 0, u_2 \wedge u_3)$, $v_5 = (0, 0, u_2 \wedge u_4)$, $v_6 = (u_3 \wedge u_5, 0, 0)$, $v_7 = (u_4 \wedge u_5, 0, 0)$, $v_8 = (0, 0, u_3 \wedge u_5)$, $v_9 = (0, 0, u_4 \wedge u_5)$, $v_{10} = (0, 0, u_2 \wedge u_5)$.

$$V_{x_{10}}^* - S_{x_{10}}^* \leftrightarrow v_1 + v_5 + v_7 = (u_4 \wedge u_5, 0, u_1 \wedge u_3 + u_2 \wedge u_4) = y_{10}.$$

Let A_0 be an element of $\mathfrak{g}_{x_{10}}$ with $\alpha = -\frac{1}{3} - 5\varepsilon$, $\eta = \frac{2}{3} + 6\varepsilon$, $\xi = -\frac{4}{3} - 2\varepsilon$, all remaining parts zero in (5.16). Then $d\rho(A_0)x_{10} = 0$ and $d\rho^*(A_0)y_{10} = y_{10}$. Since $-\delta\chi(A_0) = 10(1 + 3\varepsilon)$ is not definite, the conormal bundle $\Lambda_{10,10}$ is *not* a good holonomic variety.

(17) The isotropy subalgebra $\mathfrak{g}_{x_{11}}$ at x_{11} is given as follows.

$$(11.17) \qquad \mathfrak{g}_{x_{11}} = \left\{ A = \left[\begin{array}{c|cc|c|c} \varepsilon+\eta & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \hline \multirow{2}{*}{0} & \varepsilon+\alpha & \alpha_{12} & \gamma_5 & \gamma_6 \\ & \alpha_{21} & \varepsilon-\alpha & \gamma_7 & \gamma_8 \\ \hline \multirow{2}{*}{0} & \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \varepsilon-\eta & \gamma_9 \\ & && n_0 & -4\varepsilon \end{array}\right] \oplus \left(\begin{array}{c|c|c} -2\varepsilon-\eta-\alpha & -\alpha_{21} & \multirow{2}{*}{0} \\ -\alpha_{12} & -2\varepsilon-\eta+\alpha & \\ \hline \gamma_2-\gamma_5 & -\gamma_1-\gamma_7 & -2\varepsilon \end{array}\right) \right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(9)\ .$$

The conormal vector space $V^*_{x_{11}}$ is spanned by $v_1 = (0, u_1 \wedge u_5, - u_2 \wedge u_5)$, $v_2 = (u_1 \wedge u_5, 0, u_3 \wedge u_5)$, $v_3 = (u_2 \wedge u_5, 0, - u_4 \wedge u_5)$, $v_4 = (u_2 \wedge u_4, - u_3 \wedge u_4, 0)$, $v_5 = (0, u_2 \wedge u_4, 0)$, $v_6 = (u_3 \wedge u_4, 0, 0)$, $v_7 = (u_2 \wedge u_5, - u_3 \wedge u_5, 0)$, $v_8 = (0, u_2 \wedge u_5, 0)$, $v_9 = (u_3 \wedge u_5, 0, 0)$, $v_{10} = (0, u_4 \wedge u_5, 0)$, $v_{11} = (u_4 \wedge u_5, 0, 0)$.

The action $d\rho_{x_{11}}$ of $\mathfrak{g}_{x_{11}}$ on $V^*_{x_{11}}$ is given by

$$d\rho_{x_{11}}(A)(v_1, \cdots, v_{11}) \begin{bmatrix} A_1 & & & & \\ B_1 & A_2 & & & \\ & & A_3 & & \\ B_2 & A_6 & B_3 & A_4 & \\ \underbrace{B_4}_{2} & \underbrace{B_5}_{1} & \underbrace{B_6}_{3} & \underbrace{B_7}_{3} & \underbrace{A_5}_{2} \end{bmatrix} \begin{matrix} \}\,2 \\ \}\,1 \\ \}\,3 \\ \}\,3 \\ \}\,2 \end{matrix}$$

where $(B_1, A_2) = (- \gamma_5, \gamma_7, 5\varepsilon + \eta)$, $B_3 = - \gamma_9 \cdot I_3$, $B_4 = - \gamma_3 \cdot I_2$, $A_1 = 5\varepsilon I_2 + A'$, $A_5 = (5\varepsilon + 2\eta)I_2 + A'$ with $A' = \begin{pmatrix} -\alpha & \alpha_{21} \\ \alpha_{12} & \alpha \end{pmatrix}$,

$$(B_5, B_6, B_7) = \left(\begin{array}{c|ccc|ccc} -\gamma_1-\gamma_7 & -\gamma_8 & \gamma_6 & 0 & -\gamma_7 & -\gamma_5 & 0 \\ \gamma_2-2\gamma_5 & \gamma_6 & 0 & \gamma_8 & -\gamma_5 & 0 & -\gamma_7 \end{array} \right),$$

$$A_3 = 2\eta \cdot I_3 + A'', \ A_4 = (5\varepsilon + \eta)I_3 + A'' \ \text{ with } \ A'' = \begin{pmatrix} 0 & \alpha_{12} & -\alpha_{21} \\ 2\alpha_{21} & -2\alpha & 0 \\ -2\alpha_{12} & 0 & 2\alpha \end{pmatrix},$$

$$(B_2, A_6) = \begin{pmatrix} -\gamma_2 & -\gamma_1-\gamma_7 & \\ -2\gamma_1-\gamma_7 & & \alpha_{21} \\ 0 & \gamma_5-2\gamma_2 & -\alpha_{12} \end{pmatrix}.$$

Note that A_6 will disappear if we take $v_3 - \frac{1}{2}v_7$ instead of v_3.

i) $V^*_{x_{11}} - S^*_{x_{11}} \leftrightarrow v_1 + v_2 + v_4 = (u_1 \wedge u_5 + u_2 \wedge u_4, u_1 \wedge u_5 - u_3 \wedge u_4, u_3 \wedge u_5 - u_2 \wedge u_5) \in S^{(3)*}_{4,11}$

ii) $(S^*_{x_{11}})_1 \leftrightarrow v_1 + v_4 + v_9 = (u_2 \wedge u_4 + u_3 \wedge u_5, u_1 \wedge u_5 - u_3 \wedge u_4, - u_2 \wedge u_5) \in S^{(2)*}_{5,8} \leftrightarrow d\rho_1 = 10\varepsilon + 2\eta$

iii) $(S^*_{x_{11}})_2 \leftrightarrow v_2 + v_5 = (u_1 \wedge u_5, u_2 \wedge u_4, u_3 \wedge u_5) \in S^{(2)*}_{7,9} \leftrightarrow d\rho_2 = 4\eta \leftrightarrow f^*_2(y) = y^2_4 + y_5 y_6$

iv) $- \delta\chi = 30\varepsilon + 10\eta = 3d\rho_1 + d\rho_2$, $\mathrm{tr}_{V^*_{x_{11}}} = 40\varepsilon + 14\eta = 4d\rho_1 + \frac{3}{2}d\rho_2$.

Remark for calculation of $d\rho_1$. Let f^*_1 be the relative invariant on $V^*_{x_{11}}$ corresponding to $d\rho_1$. Since $f^*_1(v_2 + v_5) \neq 0$, the restriction of $d\rho_1$ to the isotropy subalgebra of $\mathfrak{g}_{x_{11}}$ at $v_2 + v_5$ should be zero. Hence $d\rho_1$ must be of the form $d\rho_1 = 5\lambda\varepsilon + \lambda\eta$ for some λ. Take an element A_0 in $\mathfrak{g}_{x_{11}}$ satisfying $d\rho^*(A_0)x^*_{11} = x^*_{11}$ where $x^*_{11} = v_1 + v_2 + v_4 \in V^*_{x_{11}} - S^*_{x_{11}}$. Then, by the Euler's identity, we have $(\deg f^*_1) \cdot f^*_1(x^*_{11}) = \langle d\rho^*(A_0)y, D_y \rangle f^*_1(y)|_{y=x^*_{11}} =$

$d\rho_1(A_0)f_1^*(x_{11})$ and hence $\deg f_1^* = d\rho_1(A_0) = \frac{3}{2}\lambda \in N$. Therefore $d\rho_1 = (10\varepsilon + 2\eta)\mu$ where μ is a natural number. Since $-\delta\chi$ is a linear combination of $d\rho_1$ and $d\rho_2$ with coefficients in Z, we have $\mu = 1$ or 3. On the other hand, $2\,\mathrm{tr}_{V^*_{x_{11}}}$ is also a linear combination of $d\rho_1$ and $d\rho_2$ with coefficients in Z, μ is a divisor of 8, and hence $\mu = 1$, i.e., $d\rho_1 = 10\varepsilon + 2\eta$.

(18) The isotropy subalgebra $\mathfrak{g}_{x_{12}}$ at x_{12} is given as follows.

$$\mathfrak{g}_{x_{12}} = \left\{ A = \left[\begin{array}{cc|cc|c} \varepsilon+\alpha & \alpha_{12} & \multicolumn{2}{c|}{0} & \gamma_1 \\ \alpha_{21} & \varepsilon-\alpha & & & \gamma_2 \\ \hline \multicolumn{2}{c|}{0} & \eta+\beta & \beta_{12} & \gamma_3 \\ & & \beta_{21} & \eta-\beta & \gamma_4 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & -2(\varepsilon+\eta) \end{array}\right] \oplus \left(\begin{array}{cc|c} -2\varepsilon & 0 & \gamma_5 \\ 0 & -2\eta & \gamma_6 \\ \hline \multicolumn{2}{c|}{0} & \xi \end{array}\right) \right\} \tag{11.18}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)) \oplus V(6)\,.$$

The conormal vector space $V^*_{x_{12}}$ is spanned by $v_1 = (0, 0, u_1 \wedge u_5)$, $v_2 = (0, 0, u_2 \wedge u_5)$, $v_3 = (0, 0, u_3 \wedge u_5)$, $v_4 = (0, 0, u_4 \wedge u_5)$, $v_5 = (0, u_1 \wedge u_5, 0)$, $v_6 = (0, u_2 \wedge u_5, 0)$, $v_7 = (u_3 \wedge u_5, 0, 0)$, $v_8 = (u_4 \wedge u_5, 0, 0)$, $v_9 = (0, 0, u_1 \wedge u_3)$, $v_{10} = (0, 0, u_1 \wedge u_4)$, $v_{11} = (0, 0, u_2 \wedge u_3)$, $v_{12} = (0, 0, u_2 \wedge u_4)$.

The action $d\rho_{x_{12}}$ of $\mathfrak{g}_{x_{12}}$ on $V^*_{x_{12}}$ is given by

$$d\rho_{x_{12}}(A)(v_1, \cdots, v_{12}) = (v_1, \cdots, v_{12}) = \begin{array}{c} \left[\begin{array}{ccccc} A_1 & & B_1 & & B_2 \\ & A_2 & & B_3 & B_4 \\ & & A_3 & & \\ & & & A_4 & \\ & & & & A_5 \end{array}\right] \begin{array}{l} \}2 \\ \}2 \\ \}2 \\ \}2 \\ \}4 \end{array} \\ \begin{array}{ccccc} \underbrace{\quad}_{2} & \underbrace{\quad}_{2} & \underbrace{\quad}_{2} & \underbrace{\quad}_{2} & \underbrace{\quad}_{4} \end{array} \end{array}$$

where $B_1 = -\gamma_6 I_2$, $B_3 = -\gamma_5 I_2$, $A_1 = (\varepsilon + 2\eta - \xi)I_2 + A'$, $A_2 = (2\varepsilon + \eta - \xi)I_2 + B'$, $A_3 = (\varepsilon + 4\eta)I_2 + A'$, $A_4 = (4\varepsilon + \eta)I_2 + B'$ with $A' = \begin{pmatrix} -\alpha & -\alpha_{21} \\ -\alpha_{12} & \alpha \end{pmatrix}$, $B' = \begin{pmatrix} -\beta & -\beta_{21} \\ -\beta_{12} & \beta \end{pmatrix}$, $A_5 = -(\varepsilon + \eta + \xi)I_4 + \begin{pmatrix} -\alpha I_2 + B' & -\alpha_{21}I_2 \\ -\alpha_{12}I_2 & \alpha I_2 + B' \end{pmatrix}$, $B_2 = \begin{pmatrix} -\gamma_3 & -\gamma_4 & 0 & 0 \\ 0 & 0 & -\gamma_3 & -\gamma_4 \end{pmatrix}$ and $B_4 = \begin{pmatrix} \gamma_1 & 0 & \gamma_2 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 \end{pmatrix}$.

i) $V^*_{x_{12}} - S^*_{x_{12}} \leftrightarrow v_6 + v_7 + v_{10} + v_{11} = (u_3 \wedge u_5, u_2 \wedge u_5, u_1 \wedge u_4 + u_2 \wedge u_3)$
$\in S^{(1)*}_{6,12}$

ii) $(S^*_{x_{12}})_1 \leftrightarrow v_6 + v_8 + v_9 = (u_4 \wedge u_5, u_2 \wedge u_5, u_1 \wedge u_3) \in S^{(2)*}_{7,9} \leftrightarrow d\rho_1$
$= -2(\varepsilon + \eta + \xi) \leftrightarrow f_1^*(y) = y_{10}y_{11} - y_9 y_{12}$

iii) $(S^*_{x_{12}})_2 \leftrightarrow v_5 + v_7 + v_{10} + v_{11} = (u_3 \wedge u_5, u_1 \wedge u_5, u_2 \wedge u_3 + u_2 \wedge u_4)$
$\in S^{(2)*}_{7,7} \leftrightarrow d\rho_2 = 4(\varepsilon + \eta) - \xi \leftrightarrow \deg f_2^*(y) = 3$

iv) $-\delta\chi = 10(\varepsilon + \eta) - 5\xi = d\rho_1 + 3d\rho_2$, $\mathrm{tr}_{V^*_{x_{12}}} = 12(\varepsilon + \eta) - 8\xi = 2d\rho_1 + 4d\rho_2$.

(19) The isotropy subalgebra $\mathfrak{g}_{x_{13}}$ at x_{13} is given as follows.

$$(11.19)\qquad \mathfrak{g}_{x_{13}} = \left\{ A = \left[\begin{array}{cc|cc|c} \varepsilon+\alpha & \alpha_{12} & \gamma_1 & \gamma_2 & \gamma_5 \\ \alpha_{21} & \varepsilon-\alpha & \gamma_3 & \gamma_4 & \gamma_6 \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -(\varepsilon+\eta)-\alpha & -\alpha_{21} & \gamma_7 \\ & & -\alpha_{12} & -(\varepsilon+\eta)+\alpha & \gamma_8 \\ \hline \multicolumn{2}{c|}{0} & \multicolumn{2}{c|}{0} & 2\eta \end{array}\right] \right.$$

$$\left. \oplus \begin{pmatrix} -2\varepsilon & 0 & \delta_1 \\ \gamma_2-\gamma_3 & \eta & \delta_2 \\ 0 & 0 & \xi \end{pmatrix} \right\} \cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2)) \oplus \mathfrak{u}(10)\ .$$

The conormal vector space $V_{x_{13}}^*$ is spanned by $v_1 = (0, 0, -u_4 \wedge u_5)$, $v_2 = (0, 0, u_3 \wedge u_5)$, $v_3 = (0, 0, u_1 \wedge u_5)$, $v_4 = (0, 0, u_2 \wedge u_5)$, $v_5 = (0, 0, u_3 \wedge u_4)$, $v_6 = (0, 0, -u_1 \wedge u_4)$, $v_7 = (0, 0, u_1 \wedge u_3 - u_2 \wedge u_4)$, $v_8 = (0, 0, u_2 \wedge u_3)$, $v_9 = (-u_4 \wedge u_5, 0, 0)$, $v_{10} = (u_3 \wedge u_5, 0, 0)$, $v_{11} = (u_3 \wedge u_4, 0, 0)$, $v_{12} = (u_1 \wedge u_5, u_4 \wedge u_5, 0)$, $v_{13} = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$.

The action $d\rho_{x_{13}}$ of $\mathfrak{g}_{x_{13}}$ on $V_{x_{13}}^*$ is given by

$$d\rho_{x_{13}}(A)(v_1, \cdots, v_{13}) = (v_1, \cdots, v_{13}) \begin{array}{c} \left[\begin{array}{ccccccc} A_1 & B_1 & B_2 & B_3 & B_4 & & B_5 \\ & A_2 & & B_6 & & & B_7 \\ & & A_3 & B_8 & & B_9 & \\ & & & A_4 & & & \\ & & & & A_5 & B_{10} & B_{11} \\ & & & & & A_6 & \\ & & & & & & A_7 \end{array}\right] \begin{array}{l} \}2 \\ \}2 \\ \}1 \\ \}3 \\ \}2 \\ \}1 \\ \}2 \end{array} \\ \begin{array}{ccccccc} \underbrace{}_{2} & \underbrace{}_{2} & \underbrace{}_{1} & \underbrace{}_{3} & \underbrace{}_{2} & \underbrace{}_{1} & \underbrace{}_{2} \end{array} \end{array}$$

where $A_3 = 2\varepsilon + 2\eta - \xi$, $B_8 = (\gamma_1, \gamma_2 + \gamma_3, \gamma_4)$, $B_9 = -\delta_1$, $A_6 = 4\varepsilon + 2\eta$, $B_4 = B_7 = -\delta_1 I_2$, $B_5 = \delta_2 I_2$, $A_1 = (\varepsilon - \eta - \xi)I_2 + A'$ with $A' = \begin{pmatrix} -\alpha & -\alpha_{21} \\ -\alpha_{12} & \alpha \end{pmatrix}$, $(B_1, B_2, B_3) = \begin{pmatrix} \gamma_2 & \gamma_4 & -\gamma_7 & \gamma_5 & \gamma_6 & 0 \\ -\gamma_1 & -\gamma_3 & -\gamma_8 & 0 & \gamma_5 & \gamma_6 \end{pmatrix}$, $B_6 = \begin{pmatrix} \gamma_8 & -\gamma_7 & \\ & \gamma_8 & -\gamma_7 \end{pmatrix}$, $(B_{10}, B_{11}) = \begin{pmatrix} -\gamma_7 & 2\gamma_2-\gamma_3 & \gamma_4 \\ -\gamma_8 & -\gamma_1 & \gamma_2-2\gamma_3 \end{pmatrix}$, $A_2 = -(\varepsilon + 2\eta + \xi)I_2 + A'$, $A_5 = (3\varepsilon - \eta)I_2 + A'$, $A_7 = (\varepsilon - 2\eta)I_2 + A'$ and $A_4 = (\eta - \xi)I_3 + \begin{pmatrix} -2\alpha & -2\alpha_{21} & 0 \\ -\alpha_{12} & 0 & -\alpha_{21} \\ 0 & -2\alpha_{12} & 2\alpha \end{pmatrix}$.

i) $V_{x_{13}}^* - S_{x_{13}}^* \leftrightarrow v_7 + v_{11} + v_{12} + v_{13} \in S_{3,13}^{(2)*}$

ii) $(S_{x_{13}}^*)_1 \leftrightarrow v_7 + v_{12} + v_{13} \in S_{6,12}^{(1)*} \leftrightarrow d\rho_1 = 4\varepsilon + 2\eta \leftrightarrow f_1^*(y) = y_{11}$

iii) $(S_{x_{13}}^*)_2 \leftrightarrow v_8 + v_{11} + v_{12} \in S_{4,11}^{(3)*} \leftrightarrow d\rho_2 = 2(\eta - \xi) \leftrightarrow f_2^*(y) = y_7^2 - y_6 y_8$

iv) $(S_{x_{13}}^*)_3 \leftrightarrow v_7 + v_{11} + v_{12} \in S_{5,8}^{(2)*} \leftrightarrow d\rho_3 = 2\varepsilon - 3\eta - \xi \leftrightarrow \deg f_2^*(y) = 3$

v) $-\delta\chi = d\rho_1 + d\rho_2 + 3d\rho_3$, $\mathrm{tr}_{V_{x_{13}}^*} = \frac{3}{2}d\rho_1 + 2d\rho_2 + 4d\rho_3$.

(20) The isotropy subalgebra $\mathfrak{g}_{x_{14}}$ at x_{14} is given as follows.

$$(11.20)\quad \mathfrak{g}_{x_{14}} = \left\{ A = \left(\begin{array}{c|c|c} \varepsilon & * & * \\ \hline 0 & \eta I_3 + X & * \\ \hline 0 & 0 & -\varepsilon - 3\eta \end{array}\right) \oplus (-(\varepsilon + \eta) I_3 - {}^t X);\ X \in \mathfrak{sl}(3) \right\}$$
$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(3)) \oplus \mathfrak{u}(7)\ .$$

The conormal vector space $V^*_{x_{14}}$ is spanned by $v_1 = (u_3 \wedge u_4, 0, 0)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (0, 0, u_2 \wedge u_3)$, $v_4 = (0, u_2 \wedge u_3, -u_2 \wedge u_4)$, $v_5 = (u_2 \wedge u_3, 0, u_3 \wedge u_4)$, $v_6 = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_7 = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_8 = (0, u_3 \wedge u_5, -u_4 \wedge u_5)$, $v_9 = (0, u_2 \wedge u_5, 0)$, $v_{10} = (0, 0, u_3 \wedge u_5)$, $v_{11} = (u_4 \wedge u_5, 0, 0)$, $v_{12} = (u_3 \wedge u_5, 0, 0)$, $v_{13} = (0, u_4 \wedge u_5, 0)$, $v_{14} = (0, 0, u_2 \wedge u_5)$.

Since $\dim \rho_{x_{14}} (G_{x_{14}}) = 13$ and $\dim V^*_{x_{14}} = 14$, the conormal vector space $(G_{x_{14}}, \rho_{x_{14}}, V^*_{x_{14}})$ is not a P.V. Note that it is also obtained from the fact that $\Lambda_{8,14} = \Lambda^*_{14,8}$ is not G-prehomogeneous (See (13)).

(21) The isotropy subalgebra $\mathfrak{g}_{x_{15}}$ at x_{15} is given as follows.

$$(11.21)\quad \mathfrak{g}_{x_{15}} = \left\{ A = \left(\begin{array}{c|c} 2\varepsilon I_3 + X & Z \\ \hline 0 & -3\varepsilon I_2 + Y \end{array}\right) \oplus (-4\varepsilon I_3 + S^{-1} X S);\ X \in \mathfrak{sl}(3),\right.$$
$$\left. Y \in \mathfrak{sl}(2),\ Z \in V(6),\ S = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \right\}$$
$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{sl}(3) \oplus \mathfrak{sl}(2)) \oplus V(6)\ .$$

The conormal vector space $V^*_{x_{15}}$ is spanned by $v_1 = (0, 0, u_1 \wedge u_4)$, $v_2 = (0, u_2 \wedge u_4, 0)$, $v_3 = (u_3 \wedge u_4, 0, 0)$, $v_4 = (0, u_1 \wedge u_4, -u_2 \wedge u_4)$, $v_5 = (u_2 \wedge u_4, -u_3 \wedge u_4, 0)$, $v_6 = (u_1 \wedge u_4, 0, u_3 \wedge u_4)$, $v_7 = (0, 0, u_1 \wedge u_5)$, $v_8 = (0, u_2 \wedge u_5, 0)$, $v_9 = (u_3 \wedge u_5, 0, 0)$, $v_{10} = (0, u_1 \wedge u_5, -u_2 \wedge u_5)$, $v_{11} = (u_2 \wedge u_5, -u_3 \wedge u_5, 0)$, $v_{12} = (u_1 \wedge u_5, 0, u_3 \wedge u_5)$, $v_{13} = (0, 0, u_4 \wedge u_5)$, $v_{14} = (0, u_4 \wedge u_5, 0)$, $v_{15} = (u_4 \wedge u_5, 0, 0)$. Then the action $d\rho_{x_{15}}$ of $\mathfrak{g}_{x_{15}}$ on $V^*_{x_{15}}$ is given by

$$(v_1, \cdots, v_{15}) \mapsto (v_1, \cdots, v_{15}) \left[\begin{array}{c|c} \begin{matrix} GL(1) \times SL(3) \times SL(2) \\ 5\Lambda_1 \otimes 2\Lambda_1^* \otimes \Lambda_1^* \end{matrix} & 0 \\ \hline * & \begin{matrix} GL(1) \times SL(3) \\ 10\Lambda_1 \otimes \Lambda_1^* \end{matrix} \end{array}\right]$$

i) $V^*_{x_{15}} - S^*_{x_{15}} \leftrightarrow v_1 + v_2 + v_7 + v_9 = (u_3 \wedge u_5, u_2 \wedge u_4, u_1 \wedge u_4 + u_1 \wedge u_5)$
$\in S^{(3)*}_{3,15}$

ii) $(S^*_{x_{15}})_1 \leftrightarrow v_4 + v_{11} + v_{12} \in S^{(3)*}_{4,11} \leftrightarrow d\rho_1 = 60\varepsilon$

iii) $-\delta\chi = 60\varepsilon = d\rho_1$, $\mathrm{tr}_{V^*_{x_{15}}} = 90\varepsilon = \frac{3}{2} d\rho_1$.

(22) The isotropy subalgebra $\mathfrak{g}_{x_{16}}$ at x_{16} is given as follows.

$$(11.22)\qquad \mathfrak{g}_{x_{16}}=\left\{A=\left[\begin{array}{c|cc|cc} -2(\varepsilon+\eta) & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4\\ \hline 0 & \varepsilon+\alpha_1 & \alpha_{12} & \gamma_5 & \gamma_6\\ & \alpha_{21} & \varepsilon-\alpha_1 & \gamma_7 & \gamma_8\\ \hline 0 & 0 & & \eta+\beta_1 & \beta_{12}\\ & & & \beta_{21} & \eta-\beta_1 \end{array}\right]\right.$$

$$\left.\oplus\left(\begin{array}{cc|c} \varepsilon+2\eta-\alpha_1 & -\alpha_{21} & \gamma_9\\ -\alpha_{12} & \varepsilon+2\eta+\alpha_1 & \gamma_{10}\\ \hline \multicolumn{2}{c|}{0} & \xi \end{array}\right)\right\}$$

$$\cong(\mathfrak{gl}(1)\oplus\mathfrak{gl}(1)\oplus\mathfrak{gl}(1)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2))\oplus\mathfrak{u}(10)\,.$$

The conormal vector space $V_{x_{16}}^*$ is spanned by $v_1=(0,0,u_4\wedge u_5)$, $v_2=(0,-u_4\wedge u_5,0)$, $v_3=(u_4\wedge u_5,0,0)$, $v_4=(0,0,u_2\wedge u_4)$, $v_5=(0,0,u_2\wedge u_5)$, $v_6=(0,0,u_3\wedge u_4)$, $v_7=(0,0,u_3\wedge u_5)$, $v_8=(0,0,u_2\wedge u_3)$, $v_9=(0,0,u_1\wedge u_4)$, $v_{10}=(0,0,u_1\wedge u_5)$, $v_{11}=(0,-u_2\wedge u_4,0)$, $v_{12}=(u_2\wedge u_4,-u_3\wedge u_4,0)$, $v_{13}=(u_3\wedge u_4,0,0)$, $v_{14}=(0,-u_2\wedge u_5,0)$, $v_{15}=(u_2\wedge u_5,-u_3\wedge u_5,0)$, $v_{16}=(u_3\wedge u_5,0,0)$.

The action $d\rho_{x_{16}}$ of $\mathfrak{g}_{x_{16}}$ on $V_{x_{16}}^*$ is given by

$$d\rho(A)(v_1,\cdots,v_{16})=(v_1,\cdots,v_{16})\begin{array}{c@{}l} \left[\begin{array}{cccccc} A_1 & B_1 & B_2 & & B_3 & \\ & A_2 & & & & B_4\\ & & A_3 & B_5 & B_6 & B_7\\ & & & A_4 & & \\ & & & & A_5 & \\ & & & & & A_6 \end{array}\right] & \begin{array}{l}\}1\\ \}2\\ \}4\\ \}1\\ \}2\\ \}6\end{array}\\ \begin{array}{cccccc}\underbrace{\quad}_{1} & \underbrace{\quad}_{2} & \underbrace{\quad}_{4} & \underbrace{\quad}_{1} & \underbrace{\quad}_{2} & \underbrace{\quad}_{6}\end{array} & \end{array}$$

where $(A_1,B_1,B_2)=(-2\eta-\xi,\gamma_{10},-\gamma_9,\gamma_6,-\gamma_5,\gamma_8,-\gamma_7)$, $B_3=(\gamma_4,-\gamma_3)$, $A_4=-2\varepsilon-\xi$, $A_2=-(\varepsilon+4\eta)I_2+\begin{pmatrix}-\alpha_1 & -\alpha_{21}\\ -\alpha_{12} & \alpha_1\end{pmatrix}$, $B_4=\begin{pmatrix}\gamma_6 & \gamma_8 & 0 & -\gamma_5 & -\gamma_7 & 0\\ 0 & \gamma_6 & \gamma_8 & 0 & -\gamma_5 & -\gamma_7\end{pmatrix}$, $A_5=(2\varepsilon+\eta-\xi)I_2+B$ with $B=\begin{pmatrix}-\beta_1 & -\beta_{21}\\ -\beta_{12} & \beta_1\end{pmatrix}$, $A_3=-(\varepsilon+\eta+\xi)I_4+\left(\begin{array}{c|c}-\alpha_1I_2+B & -\alpha_{21}I_2\\ \hline -\alpha_{12}I_2 & \alpha_1I_2+B\end{array}\right)$, $A_6=-(2\varepsilon+3\eta)I_6+\left(\begin{array}{c|c}-\beta_1I_3+A' & -\beta_{21}I_3\\ \hline -\beta_{12}I_3 & \beta_1I_3+A'\end{array}\right)$ with $A'=\begin{pmatrix}-2\alpha_1 & -2\alpha_{21} & 0\\ -\alpha_{12} & 0 & -\alpha_{21}\\ 0 & -2\alpha_{12} & 2\alpha_1\end{pmatrix}$ and

$$(B_5,B_6,B_7)=\left[\begin{array}{c|cc|cccccc} -\gamma_7 & -\gamma_1 & & \gamma_{10} & -\gamma_9 & & & & \\ -\gamma_8 & & -\gamma_1 & & & & \gamma_{10} & -\gamma_9 & \\ \gamma_5 & -\gamma_2 & & & \gamma_{10} & -\gamma_9 & & & \\ \gamma_6 & & -\gamma_2 & & & & & \gamma_{10} & -\gamma_9 \end{array}\right]$$

i) $V^*_{x_{16}} - S^*_{x_{16}} \leftrightarrow v_8 + v_9 + v_{11} + v_{15} + v_{16} = (u_2 \wedge u_5 + u_3 \wedge u_5,$
$- u_2 \wedge u_4 - u_3 \wedge u_5, u_2 \wedge u_3 + u_1 \wedge u_4) \in S^{(2)*}_{2,16}$

ii) $(S^*_{x_{16}})_1 \leftrightarrow v_9 + v_{10} + v_{11} + v_{16} = (u_3 \wedge u_5, - u_2 \wedge u_4, u_1 \wedge u_4 + u_1 \wedge u_5)$
$\in S^{(3)*}_{3,15} \leftrightarrow d\rho_1 = - 2\varepsilon - \xi \leftrightarrow f_1^*(y) = y_8$

iii) $(S^*_{x_{16}})_2 \leftrightarrow v_8 + v_9 + v_{10} + v_{11} + v_{15} = (u_2 \wedge u_5, - u_2 \wedge u_4 - u_3 \wedge u_5,$
$u_2 \wedge u_3 + u_1 \wedge u_4 + u_1 \wedge u_5) \in S^{(2)*}_{3,13} \leftrightarrow d\rho_2 = - 8\varepsilon - 12\eta$

$$\leftrightarrow f_2^*(y) = \det\begin{pmatrix} y_{11} & y_{14} \\ y_{13} & y_{16} \end{pmatrix}^2 - 4\det\begin{pmatrix} y_{12} & y_{15} \\ y_{13} & y_{16} \end{pmatrix} \cdot \det\begin{pmatrix} y_{11} & y_{14} \\ y_{12} & y_{15} \end{pmatrix}$$

iv) $(S^*_{x_{16}})_3 \leftrightarrow v_8 + v_{10} + v_{11} + v_{16} = (u_3 \wedge u_5, - u_2 \wedge u_4, u_1 \wedge u_5 + u_2 \wedge u_3)$
$\in S^{(3)*}_{4,11} \leftrightarrow d\rho_3 = - 4\eta - 2\xi$

$$\leftrightarrow f_3^*(y) = \det\begin{pmatrix} y_9 & y_{12} \\ y_{10} & y_{15} \end{pmatrix}^2 - \det\begin{pmatrix} y_9 & y_{11} \\ y_{10} & y_{14} \end{pmatrix} \cdot \det\begin{pmatrix} y_9 & y_{13} \\ y_{10} & y_{16} \end{pmatrix}$$

v) $- \delta\chi = d\rho_1 + d\rho_2 + 2d\rho_3$, $\mathrm{tr}_{V^*_{x_{16}}} = 2d\rho_1 + \frac{3}{2}d\rho_2 + 3d\rho_3$.

(23) The isotropy subalgebra $\mathfrak{g}_{x_{18}}$ at x_{18} is given as follows.

$$(11.23) \quad \mathfrak{g}_{x_{18}} = \left\{ A = \left(\begin{array}{c|c} \varepsilon I_4 + X & * \\ \hline 0 & -4\varepsilon \end{array}\right) \oplus \left(\begin{array}{c|c} -2\varepsilon & * \\ \hline 0 & Y \end{array}\right); X \in \mathfrak{sp}(2), Y \in \mathfrak{gl}(2) \right\}$$
$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sp}(2) \oplus \mathfrak{sl}(2)) \oplus V(6) .$$

The conormal vector space $V^*_{x_{18}}$ is spanned by $v_1 = (0, u_1 \wedge u_3 - u_2 \wedge u_4, 0)$, $v_2 = (0, u_1 \wedge u_2, 0)$, $v_3 = (0, u_1 \wedge u_4, 0)$, $v_4 = (0, u_3 \wedge u_4, 0)$, $v_5 = (0, u_2 \wedge u_3, 0)$, $v_6 = (0, 0, u_1 \wedge u_3 - u_2 \wedge u_4)$, $v_7 = (0, 0, u_1 \wedge u_2)$, $v_8 = (0, 0, u_1 \wedge u_4)$, $v_9 = (0, 0, u_3 \wedge u_4)$, $v_{10} = (0, 0, u_2 \wedge u_3)$, $v_{11} = (0, u_1 \wedge u_5, 0)$, $v_{12} = (0, u_2 \wedge u_5, 0)$, $v_{13} = (0, u_3 \wedge u_5, 0)$, $v_{14} = (0, u_4 \wedge u_5, 0)$, $v_{15} = (0, 0, u_1 \wedge u_5)$, $v_{16} = (0, 0, u_2 \wedge u_5)$, $v_{17} = (0, 0, u_3 \wedge u_5)$, $v_{18} = (0, 0, u_4 \wedge u_5)$. The action $d\rho_{x_{18}}$ of $\mathfrak{g}_{x_{18}}$ on $V^*_{x_{18}}$ is given by

$$(v_1, \cdots, v_{18})$$
$$\mapsto (v_1, \cdots, v_{18}) \left(\begin{array}{c|c} \begin{matrix} GL(1) \times GL(1) \times Sp(2) \times SL(2) \\ 2\Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_2 \otimes \Lambda_1^* \end{matrix} & 0 \\ \hline * & \begin{matrix} GL(1) \times GL(1) \times Sp(2) \times SL(2) \\ 3\Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1 \otimes \Lambda_1^* \end{matrix} \end{array}\right)$$

i) $V^*_{x_{18}} - S^*_{x_{18}} \leftrightarrow v_2 + v_4 + v_8 + v_{10} + v_{15} \in S^*_{8,18}$

ii) $(S^*_{x_{18}})_1 \leftrightarrow v_1 + v_8 + v_{10} + v_{11} \in S^*_{8,18} \leftrightarrow d\rho_1 = - 2\varepsilon - 6\eta \leftrightarrow \deg f_1^* = 6$

iii) $(S^*_{x_{18}})_2 \leftrightarrow v_1 + v_7 + v_{17} + v_{18} \in S^*_{8,18} \leftrightarrow d\rho_2 = - 8\varepsilon - 4\eta \leftrightarrow \deg f_2^* = 4$

iv) $- \delta\chi = 3d\rho_1 - 2d\rho_2$, $\mathrm{tr}_{V^*_{x_{18}}} = 4d\rho_1 - \frac{3}{2}d\rho_2$.

(24) The isotropy subalgebra $\mathfrak{g}_{x_{21}}$ at x_{21} is given as follows.

$$(11.24)\quad \mathfrak{g}_{x_{21}} = \left\{A = \left(\begin{array}{c|c} 3\varepsilon I_3 + X & * \\ \hline 0 & -2\varepsilon I_3 + Y \end{array}\right) \oplus \left(\begin{array}{c|c} -6\varepsilon & * \\ \hline 0 & \eta I_2 + Z \end{array}\right);\right.$$

$$\left. X, Z \in \mathfrak{sl}(2),\ Y \in \mathfrak{sl}(3)\right\}$$

$$\cong (\mathfrak{gl}(1) \oplus \mathfrak{gl}(1) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(3)) \oplus V(8)\,.$$

Then $V_{x_{21}}^*$ is spanned by $v_1 = (0, u_4 \wedge u_5, 0)$, $v_2 = (0, u_3 \wedge u_5, 0)$, $v_3 = (0, u_3 \wedge u_4, 0)$, $v_4 = (0, 0, u_4 \wedge u_5)$, $v_5 = (0, 0, u_3 \wedge u_5)$, $v_6 = (0, 0, u_3 \wedge u_4)$, $v_7 = (u_4 \wedge u_5, 0, 0)$, $v_8 = (u_3 \wedge u_5, 0, 0)$, $v_9 = (u_3 \wedge u_4, 0, 0)$, $v_{10} = (0, u_1 \wedge u_3, 0)$, $v_{11} = (0, u_1 \wedge u_4, 0)$, $v_{12} = (0, u_1 \wedge u_5, 0)$, $v_{13} = (0, u_2 \wedge u_3, 0)$, $v_{14} = (0, u_2 \wedge u_4, 0)$, $v_{15} = (0, u_2 \wedge u_5, 0)$, $v_{16} = (0, 0, u_1 \wedge u_3)$, $v_{17} = (0, 0, u_1 \wedge u_4)$, $v_{18} = (0, 0, u_1 \wedge u_5)$, $v_{19} = (0, 0, u_2 \wedge u_3)$, $v_{20} = (0, 0, u_2 \wedge u_4)$, $v_{21} = (0, 0, u_2 \wedge u_5)$. The action $d\rho_{x_{21}}$ of $\mathfrak{g}_{x_{21}}$ on $V_{x_{21}}^*$ is given by

$$(v_1, \cdots, v_{21})$$

$$\mapsto (v_1, \cdots, v_{21}) \left(\begin{array}{c|c|c} \begin{array}{c} GL(1) \times GL(1) \times SL(2) \\ \times SL(3) \\ 4\Lambda_1 \otimes \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1 \end{array} & * & * \\ \hline 0 & \begin{array}{c} GL(1) \times SL(3) \\ (10\Lambda_1) \otimes \Lambda_1 \end{array} & 0 \\ \hline 0 & 0 & \begin{array}{c} GL(1) \times GL(1) \times SL(2) \\ \times SL(2) \times SL(3) \\ \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1^* \otimes \Lambda_1^* \end{array} \end{array}\right)$$

i) $V_{x_{21}}^* - S_{x_{21}}^* \leftrightarrow v_7 + v_{10} + v_{14} + v_{18} + v_{19} = (u_4 \wedge u_5, u_1 \wedge u_3 + u_2 \wedge u_4,$
$u_1 \wedge u_5 + u_2 \wedge u_3) \in S_{1,21}^{(2)*}$

ii) $(S_{x_{21}}^*)_1 \leftrightarrow v_8 + v_{10} + v_{14} + v_{18} + v_{19} = (u_3 \wedge u_5, u_1 \wedge u_3 + u_2 \wedge u_4,$
$u_1 \wedge u_5 + u_2 \wedge u_3) \in S_{3,13}^{(2)*} \leftrightarrow d\rho_1 = 18\varepsilon - 2\eta \leftrightarrow \deg f_1^* = 4$

iii) $(S_{x_{21}}^*)_2 \leftrightarrow v_8 + v_9 + v_{10} + v_{14} + v_{18} = (u_3 \wedge u_4 + u_3 \wedge u_5, u_1 \wedge u_3$
$+ u_2 \wedge u_4, u_1 \wedge u_5) \in S_{2,16}^{(2)*} \leftrightarrow d\rho_2 = -6\varepsilon - 6\eta \leftrightarrow \deg f_2^* = 6$

iv) $-\delta\chi = 2d\rho_1 + d\rho_2$, $\mathrm{tr}_{V_{x_{21}}^*} = 3d\rho_1 + 2d\rho_2$.

(25) The isotropy subalgebra $\mathfrak{g}_{x_{30}}$ at $x_{30} = 0$ is $\mathfrak{g}$ itself.

This is a good holonomic variety and $\mathrm{ord}_\Lambda f^s = -15s - \frac{30}{2}$. Thus we obtain the holonomy diagram (Figure 11–1). From this diagram, we obtain the b-function $b(s) = ((s+1)(s+\frac{3}{2})(s+2))^3 \cdot ((s+\frac{4}{3})(s+\frac{5}{3}))^2 \cdot (s+\frac{5}{4})(s+\frac{7}{4})$.

Remark. Let $\Lambda_0 = \overline{G(x_0, y_0)}$ and $\Lambda_1 = \overline{G(x_1, y_1)}$ be good holonomic varieties satisfying $(x_0, y_1) \in \Lambda_0 \cap \Lambda_1$ and $\dim G(x_0, y_1) = \dim V - 1$. Then we can calculate β by Proposition 1–4. It is known that if β depends on the choice of A_1, then (x_0, y_1) is not contained in other Λ_i $(i \neq 0, 1)$, i.e., there are no three Λ_i's which intersect at (x_0, y_1) with codimension one. (If more

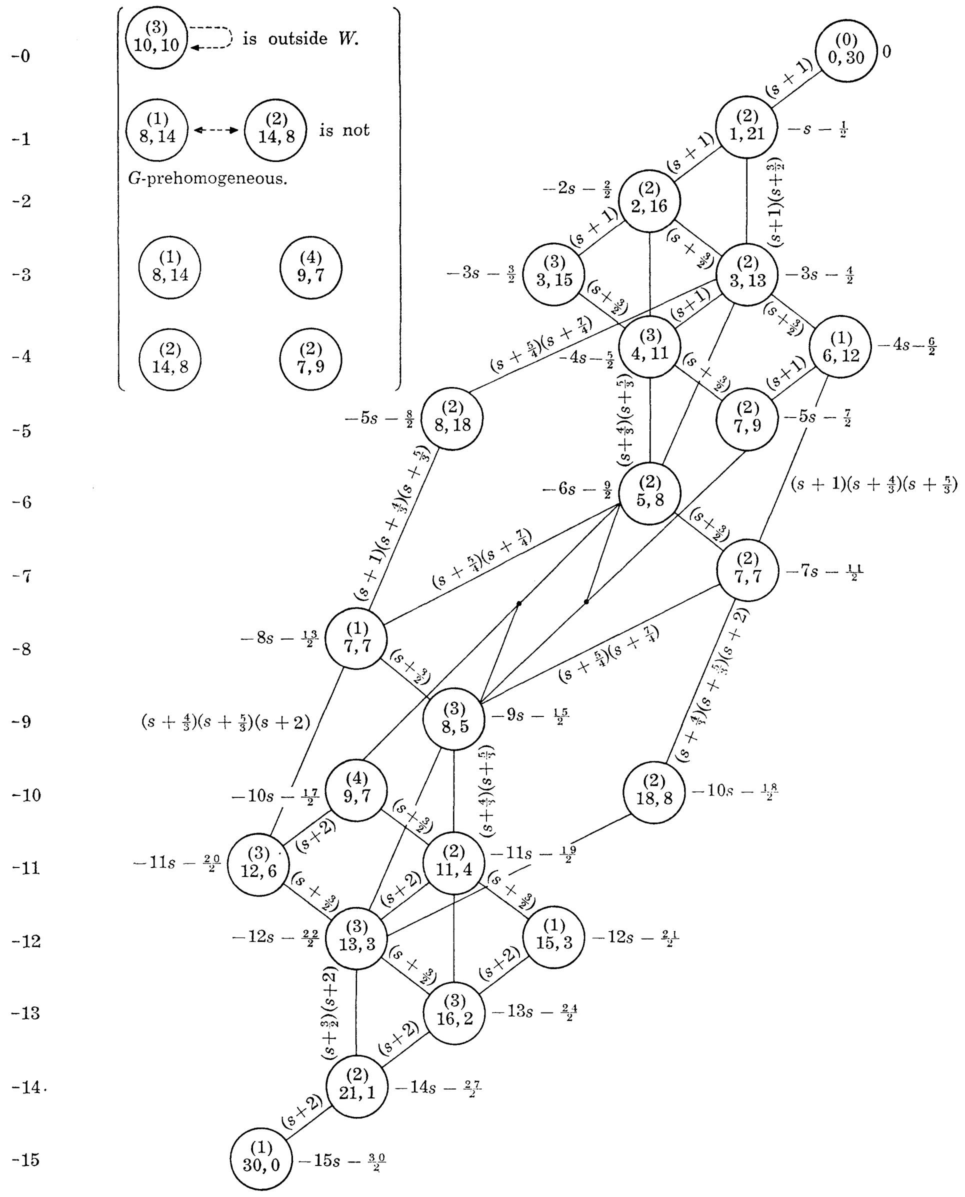


Fig. 11-1. Holonomy diagram of $(SL(5)\times GL(3),\ \Lambda_2\otimes\Lambda_1,\ V(10)\otimes V(3))$ where $\overset{(k)}{i\,j}$ denotes the conormal bundles of the orbit $S_{ij}^{(k)}$ in Proposition 6-1.

than two Λ_i's intersect with codimension one, then $\beta = 1$ and it does not depend on A_1.) All one-codimensional intersections obtained from (1) ~ (14) satisfy this condition except $\Lambda_{5,8}^{(2)} \cap \Lambda_{4,11}^{(3)}$ and $\Lambda_{8,5}^{(3)} \cap \Lambda_{7,9}^{(2)}$. In general, if $(x_0, y_1) \in \overline{G(x_2, y_2)}$, then we have $\operatorname{codim}_V \rho(G)x_0 \geqq \operatorname{codim}_V \rho(G)x_1$ and $\operatorname{codim}_{V^*} \rho^*(G)y_1 \geqq \operatorname{codim}_{V^*} \rho^*(G)y_2$. From this, there are no other Λ's satisfying $\dim \Lambda \cap \Lambda_{5,8}^{(2)} \cap \Lambda_{4,11}^{(3)} = \dim V - 1$. For $\Lambda_{8,5}^{(3)} \cap \Lambda_{7,9}^{(2)}$, it is enough to check $\Lambda_{7,7}^{(1)}$, $\Lambda_{7,7}^{(2)}$ and $\Lambda_{5,8}^{(2)}$. By using the duality, i.e., $(G, \rho, V) \cong (G, \rho^*, V^*)$, we get all one-codimensional intersections of three good holonomic varieties.

§ 12. Table of the *b*-functions of irreducible reduced regular P.V.'s

(1) $(G \times GL(m), \rho \otimes \Lambda_1, V(m) \otimes V(m))$ where $\rho : G \to GL(V(m))$ is an m-dimensional irreducible representation of a connected semi-simple algebraic group G (or $G = \{1\}$ and $m = 1$).

$b(s) = (s + 1)(s + 2) \cdots (s + m)$ (See Figure 2–1 and 2–4).

(2) $(GL(n), 2\Lambda_1, V(\frac{1}{2}n(n + 1)))$ $(n \geqq 2)$

$$b(s) = \prod_{\nu=1}^{n} \left(s + \frac{\nu + 1}{2}\right) = (s + 1)\left(s + \frac{3}{2}\right) \cdots \left(s + \frac{n + 1}{2}\right)$$

(See Figure 2–2 and 2–4).

(3) $(GL(2m), \Lambda_2, V(m(2m - 1)))$ $(m \geqq 3)$

$$b(s) = \prod_{k=1}^{m} (s + 2k - 1) = (s + 1)(s + 3) \cdots (s + 2m - 1)$$

(See Figure 2–3 and 2–4).

(4) $(GL(2), 3\Lambda_1, V(4))$

$b(s) = (s + 1)^2(s + \frac{5}{6})(s + \frac{7}{6})$ (See [2]).

(5) $(GL(6), \Lambda_3, V(20))$

$b(s) = (s + 1)(s + \frac{5}{2})(s + \frac{7}{2})(s + 5)$ (See Figure 8–1).

(6) $(GL(7), \Lambda_3, V(35))$

$b(s) = (s + 1)(s + 2)(s + \frac{5}{2})(s + \frac{7}{2})(s + 3)(s + 4)(s + 5)$

(See Figure 10–1).

(7) $(GL(8), \Lambda_3, V(56))$

$b(s) = (s+1)(s+\frac{3}{2})^2(s+\frac{11}{6})(s+2)^3(s+\frac{13}{6})(s+\frac{7}{3})(s+\frac{5}{2})^3(s+\frac{8}{5})(s+3)^2(s+\frac{7}{2})$

(See [10]).

(8) $(SL(3) \times GL(2),\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(2))$

$b(s) = \{(s + 1)^2(s + \frac{5}{6})(s + \frac{7}{6})(s + \frac{3}{4})(s + \frac{5}{4})\}^2$ (See [12]).

(9) $(SL(6) \times GL(2), \Lambda_2 \otimes \Lambda_1, V(15) \otimes V(2))$

$b(s) = (s + 1)^2(s + \frac{5}{6})(s + \frac{7}{6})(s + \frac{3}{2})^2(s + 2)^2(s + \frac{5}{2})^2(s + \frac{7}{3})(s + \frac{8}{3})$

(See [12]).

(10) $(SL(5) \times GL(3), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(3))$
$b(s) = ((s+1)(s+\frac{3}{2})(s+2))^3 \cdot ((s+\frac{4}{3})(s+\frac{5}{3}))^2 \cdot (s+\frac{5}{4})(s+\frac{7}{4})$
(See Figure 11–1).

(11) $(SL(5) \times GL(4), \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$ (See [11]).

(12) $(SL(3) \times SL(3) \times GL(2), \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(3) \otimes V(3) \otimes V(2))$
$b(s) = (s+1)^4(s+\frac{3}{2})^4(s+\frac{4}{3})(s+\frac{5}{3})(s+\frac{5}{6})(s+\frac{7}{6})$ (See [12]).

(13) $(Sp(n) \times GL(2m), \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$ $(n \geqq 2m \geqq 2)$
$$b(s) = \prod_{k=1}^{m} (s+2k-1) \prod_{\ell=0}^{m-1} (s+2n-2\ell)$$
$$= (s+1)(s+3)\cdots(s+2m-1)(s+2n)(s+2n-2)\cdots$$
$(s+2n-2m+2)$ (See Figure 3–1 and 3–2).

(14) $(GL(1) \times Sp(3), \square \otimes \Lambda_3, V(1) \otimes V(14))$
$b(s) = (s+1)(s+2)(s+\frac{5}{2})(s+\frac{7}{2})$ (See Figure 9–1).

(15) $(SO(n) \times GL(m), \Lambda_1 \otimes \Lambda_1, V(n) \otimes V(m))$ $\left(n \geq 3, \frac{n}{2} \geqq m \geqq 1\right)$
$$b(s) = \prod_{k=1}^{m} \left(s + \frac{k+1}{2}\right) \prod_{\ell=1}^{m} \left(s + \frac{n-\ell+1}{2}\right)$$
$$= (s+1)\left(s+\frac{3}{2}\right)\cdots\left(s+\frac{m+1}{2}\right)\left(s+\frac{n}{2}\right)\left(s+\frac{n-1}{2}\right)\cdots$$
$\left(s + \frac{n-m+1}{2}\right)$ (See [2]).

(16) $(GL(1) \times \mathrm{Spin}\,(7), \square \otimes$ spin rep., $V(1) \otimes V(8))$
$b(s) = (s+1)(s+4)$ (See Remark in § 5).

(17) Spin (7) × $GL(2)$, spin rep. $\otimes \Lambda_1$, $V(8) \otimes V(2))$
$b(s) = (s+1)(s+\frac{3}{2})(s+4)(s+\frac{7}{2})$ (See Remark in § 5).

(18) (Spin(7) × $GL(3)$, spin rep. $\otimes \Lambda_1$, $V(8) \otimes V(3))$
$b(s) = (s+1)(s+\frac{3}{2})(s+2)(s+4)(s+\frac{7}{2})(s+3)$ (See Remark in § 5).

(19) $(GL(1) \times \mathrm{Spin}\,(9), \square \otimes$ spin rep., $V(1) \otimes V(16))$
$b(s) = (s+1)(s+8)$ (See Remark in § 5).

(20) (Spin (10) × $GL(2)$, half-spin rep. $\otimes \Lambda_1$, $V(16) \otimes V(2))$
$b(s) = (s+1)(s+4)(s+5)(s+8)$ (See Figure 4–1).

(21) (Spin (10) × $GL(3)$, half-spin rep. $\otimes \Lambda_1$, $V(16) \otimes V(3))$
$b(s) = (s+1)(s+\frac{3}{2})(s+2)(s+3)(s+\frac{7}{2})(s+4)(s+\frac{5}{3})(s+\frac{6}{3})(s+\frac{7}{3}) \times$
$\times (s+\frac{8}{3})(s+\frac{9}{3})(s+\frac{10}{3})$ (See [15]).

(22) $(GL(1) \times \mathrm{Spin}\,(11), \square \otimes$ spin rep., $V(1) \otimes V(32))$
$b(s) = (s+1)(s+\frac{7}{2})(s+\frac{11}{2})(s+8)$ (See Remark in § 5).

(23) $(GL(1) \times \mathrm{Spin}\,(12), \square \otimes$ half-spin rep., $V(1) \otimes V(32))$
$b(s) = (s+1)(s+\frac{7}{2})(s+\frac{11}{2})(s+8)$ (See Figure 5–1).

(24) $(GL(1) \times \mathrm{Spin}\,(14),\ \square \otimes \text{half-spin rep.},\ V(1) \otimes V(64))$
$b(s) = (s+1)(s+\frac{5}{2})(s+\frac{7}{2})(s+4)(s+5)(s+\frac{11}{2})(s+\frac{13}{2})(s+8)$
(See Appendix).

(25) $(GL(1) \times (G_2),\ \square \otimes \Lambda_2,\ V(1) \otimes V(7))$
$b(s) = (s+1)(s+\frac{7}{2})$ (See Remark in § 5).

(26) $((G_2) \times GL(2),\ \Lambda_2 \otimes \Lambda_1,\ V(7) \otimes V(2))$
$b(s) = (s+1)(s+\frac{3}{2})(s+\frac{7}{2})(s+3)$ (See Remark in § 5).

(27) $(GL(1) \times E_6,\ \square \otimes \Lambda_1,\ V(1) \otimes V(27))$
$b(s) = (s+1)(s+5)(s+9)$ (See Figure 6–1).

(28) $(E_6 \times GL(2),\ \Lambda_1 \otimes \Lambda_1,\ V(27) \otimes V(2))$
$b(s) = (s+1)^2(s+\frac{5}{6})(s+\frac{7}{6})(s+\frac{5}{2})^2(s+3)^2(s+\frac{9}{2})^2(s+\frac{13}{3})(s+\frac{14}{3})$
(See [12]).

(29) $(GL(1) \times E_7,\ \square \otimes \Lambda_6,\ V(1) \otimes V(56))$
$b(s) = (s+1)(s+\frac{11}{2})(s+\frac{19}{2})(s+14)$ (See Figure 7–1).

We can obtain the b-functions of all irreducible regular P.V.'s, except for those in the castling class of (11), from the Table above and the following theorem due to T. Shintani.

THEOREM (T. Shintani). *Let* (G', ρ', V') *be a castling transform of an irreducible regular P.V.* (G, ρ, V), *i.e., there exists a triplet* $(\tilde{G}, \tilde{\rho}, V(m))$ *and a positive number* n *with* $m > n \geqq 1$ *such that*

$$(G, \rho, V) \cong (\tilde{G} \times GL(n), \tilde{\rho} \otimes \Lambda_1, V(m) \otimes V(n))$$
$$(G', \rho', V') \cong (\tilde{G} \times GL(m-n), \tilde{\rho}^* \otimes \Lambda_1, V(m)^* \otimes V(m-n)) .$$

Then the b-functions $b(s)$ *and* $b'(s)$ *of them satisfy*

$$\begin{aligned} b(s) \prod_{i=1}^{d} (ds-i)(ds-i+1)\cdots(ds-i+m-n-1) \\ = b'(s) \prod_{i=1}^{d} (ds-i)(ds-i+1)\cdots(ds-i+n-1) \end{aligned}$$

where $\deg f = dm$ *and* $\deg f' = d(m-n)$. *Here* f *and* f' *are the basic relative invariants of* (G, ρ, V) *and* (G', ρ', V') *respectively.*

Appendix with I. Ozeki

Here we consider the regular irreducible P.V. $(GL(1) \times \mathrm{Spin}\,(14),$ $\square \otimes \text{half-spin rep.},\ V(1) \otimes V(64))$. The orbital decomposition of this space has been done by the author and I. Ozeki ([7]), by Popov ([9]), by V. Gatti and E. Viniberghi ([10]). There exist ten orbits, and the conormal bundle of each orbit is a good Lagrangian variety. The relative invariant of this space is of degree eight ([1]), and its b-function is given by $b(s) = (s+1)(s+\frac{5}{2})(s+\frac{7}{2})(s+4)(s+5)(s+\frac{11}{2})(s+\frac{13}{2})(s+8)$. Its holonomy diagram is given by Figure A, where we denote by ⓜ the conormal bundle Λ of the m-codimensional orbit.

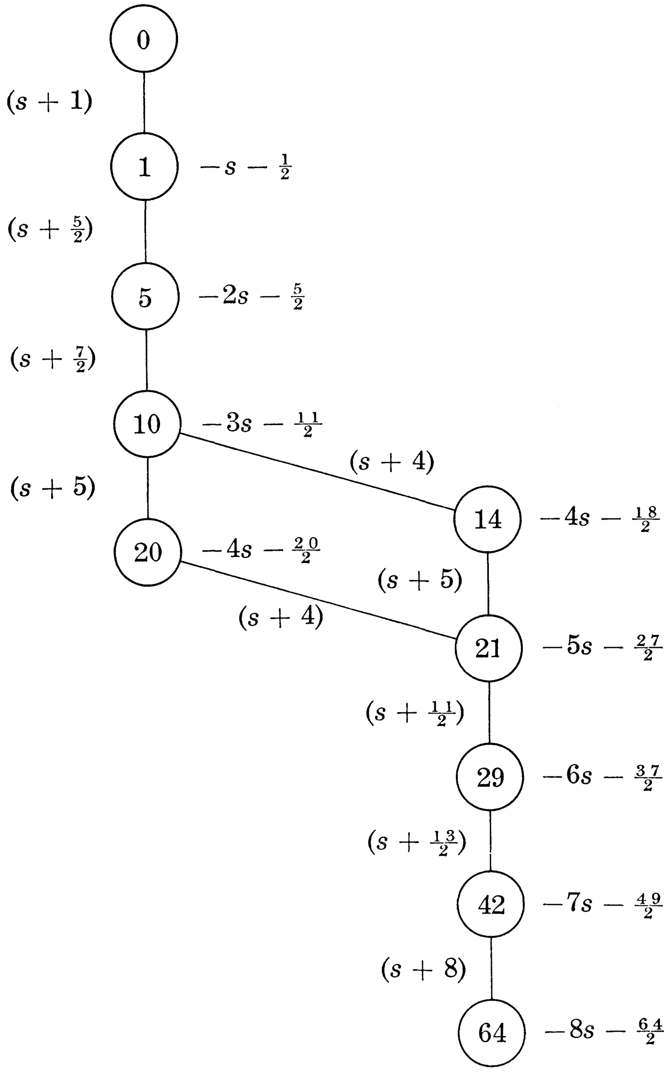


Figure A. Holonomy diagram of $(GL(1) \times \mathrm{Spin}(14)$, $\square \otimes$ half-spin rep., $V(1) \otimes V(64))$

REFERENCES

[1] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J., **65** (1977), 1–155.

[2] M. Sato, M. Kashiwara, T. Kimura and T. Oshima, Micro-local analysis of prehomogeneous vector spaces, Inv. math., **62** (1980), 117–179.

[3] J-I. Igusa, A classification of spinors up to dimension twelve, Amer. J. of Math., **92**, No. 4 (1970), 997–1028.

[4] C. Chevalley, The Algebraic Theory of Spinors, Columbia University Press, 1954.

[5] S. J. Haris, Some irreducible representation of exceptional algebraic groups, Amer. J. of Math., **93** (1971).

[6] G. B. Gurevich, Foundations of the Theory of Algebraic Invariants, P. Noordhoff LTD-Groningen, 1964.

[7] T. Kimura, Study of irreducible prehomogeneous vector spaces, Master Thesis in Japanese, University of Tokyo, 1973, 373 pp.

[8] H. Kawahara, Prehomogeneous vector spaces related with the spin group, Master Thesis in Japanese, University of Tokyo, 1974.

[9] V. L. Popov, A classification of spinors of dimension fourteen, Uspechi Math. Nauk, **32**, (1977).

[10] V. Gatti and E. Viniberghi, Spinors of 13-dimensional space, Adv. Math., **30**, No. 2 (1978).

[11] I. Ozeki, On the micro-local structure of the regular prehomogeneous vector space associated with $SL(5) \times GL(4)$, I, Proc. Japan Acad., **55**, Ser. A, (1979).

[12] T. Kimura and M. Muro, On some series of regular irreducible prehomogeneous vector spaces, Proc. Japan Acad., **55**, Ser A, No. 10 (1979), 384–389.

[13] I. Ozeki, On the micro-local structure of a regular prehomogeneous vector space associated with $GL(8)$, Proc. Japan Acad., **56**, Ser. A, No. 1 (1980), 18–21.

[14] T. Kimura, Remark on some combinatorial construction of relative invariants, Tsukuba J. of Math., **5**, No. 1 (1981).

[15] T. Kimura and I. Ozeki, On the micro-local structure of regular prehomogenous vector space associated with Spin $(10) \times GL(3)$. (in preparation).

Institute of Mathematics
University of Tsukuba