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RIEMANNIAN HOMOGENEOUS FOLIATIONS WITHOUT HOLONOMY

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§ 1. Introduction

Let M be a compact connected C^∞ manifold with a smooth Riemannian foliation $\mathscr{F}$. That is, $(M, \mathscr{F})$ admits a bundle-like metric in the sense of [7]. In [4] it is shown that if all leaves of $\mathscr{F}$ are closed without holonomy, then the space of leaves $M/\mathscr{F}$ of the foliation is a manifold and the natural projection $M \to M/\mathscr{F}$ is a locally trivial fibre space. In the present work we show that for certain of the Riemannian homogeneous foliations, holonomy is the only obstruction to the foliation being a fibration.

Let G/K be a simply connected, even dimensional, positively curved Riemannian homogeneous space of a compact connected Lie group G and let $\mathscr{F}$ be a homogeneous G/K-foliation of a compact connected manifold M. For example, $\mathscr{F}$ is a codimension $2q$ elliptic (i.e., homogeneous $SO(2q+1)/SO(2q) \cong S^{2q}$-) foliation. Then $\mathscr{F}$ is cohomologically a fibration in the sense that the base-like cohomology algebra of the foliated manifold $(M, \mathscr{F})$ is isomorphic to the de Rham cohomology algebra of G/K [3]. The main result of this paper asserts that if $\mathscr{F}$ has no holonomy, then it is actually a fibration.

(1.1) THEOREM. *If $\mathscr{F}$ is without holonomy, then M fibres over G/K with the leaves of $\mathscr{F}$ as fibres.*

§ 2. Riemannian Homogeneous Foliations

In this section we prove (1.1) and use its proof to elucidate further properties of Riemannian homogeneous foliations.

Let G/K be a connected homogeneous space on which G acts effectively and let $\mathscr{F}$ be a homogeneous G/K-foliation of a connected manifold M. That is, $\mathscr{F}$ is defined by a G/K-cocycle $\{(U_\alpha, f_\alpha, \lambda_{g_{\alpha\beta}})\}_{\alpha,\beta\in A}$ where

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i) $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M

ii) $f_\alpha : U_\alpha \to G/K$ is a submersion defining $\mathscr{F}|U_\alpha$

iii) $f_\alpha = \lambda_{g_{\alpha\beta}} \circ f_\beta$ on $U_\alpha \cap U_\beta$ where $g_{\alpha\beta} \in G$ and $\lambda_{g_{\alpha\beta}}$ denotes the diffeomorphism of G/K which sends aK into $g_{\alpha\beta}aK$.

To prove the theorem, we recall a construction from [2] which gives a useful realization of the holonomy group of a leaf of $\mathscr{F}$.

Let $P = \{[\lambda_g \circ f_\alpha]_x : x \in U_\alpha,\ \alpha \in A,\ g \in G\}$, where $[\lambda_g \circ f_\alpha]_x$ denotes the germ of $\lambda_g \circ f_\alpha$ at x and let $\pi : P \to M$ be defined by $\pi([\lambda_g \circ f_\alpha]_x) = x$. Then P possesses a well-defined topology and differentiable structure such that $\pi : P \to M$ is a smooth regular covering. Moreover, the evaluation map $F : P \to G/K$ defined by $F([\lambda_g \circ f_\alpha]_x) = \lambda_g(f_\alpha(x))$ is a smooth submersion constant along the leaves of $\pi^{-1}(\mathscr{F})$ where $\pi^{-1}(\mathscr{F})$ denotes the foliation of P whose leaves are the connected components of the inverse images under π of the leaves of $\mathscr{F}$. By choosing a connected component of P, we may assume that P is connected. This connected regular covering gives rise to a homomorphism $\Phi : \pi_1(M, x_0) \to G$ such that $\Gamma = \text{image}(\Phi)$ is the group of covering transformations and such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{F} & G/K \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\lambda_\gamma} \\ P & \xrightarrow{F} & G/K \end{array}$$

is commutative for each $\gamma \in \Gamma$.

Let L be a leaf of $\mathscr{F}$ and choose a leaf L' of $\pi^{-1}(\mathscr{F})$ which projects to L. Then the holonomy group of L is isomorphic to $\Gamma_{L'} = \{\gamma \in \Gamma : \gamma(L') = L'\}$ and thus $\pi|L' : L' \to L$ is a regular covering with the holonomy group of L as its group of covering transformations.

If K is compact, then M admits a bundle-like metric (whence $\mathscr{F}$ is called a Riemannian homogeneous foliation) such that the lifted metric on P is bundle-like with respect to the foliation defined by the submersion F. Thus if M is also compact, we have that $F : P \to G/K$ is a locally trivial fibre space [5]. (We remark that since an isometry of a connected Riemannian manifold is determined by its value and differential at a point, this construction remains valid for a foliation defined by an N-cocycle $\{(U_\alpha, f_\alpha, g_{\alpha\beta})\}_{\alpha,\beta\in A}$ where N is a connected Riemannian manifold and each $g_{\alpha\beta}$ extends to an isometry of N.)

Assume now that M, $\mathscr{F}$, and G/K satisfy the hypotheses of (1.1). Fix an orientation on G/K invariant under the action of G. Let $\gamma \in \Gamma$. Then λ_γ is an orientation-preserving isometry of G/K and hence has a fixed point y [6]. It is here that we have used the assumption in Theorem (1.1) that G/K is an even dimensional, positively curved Riemannian homogeneous space of the compact connected Lie group G. Since $F: P \to G/K$ is a fibration over a simply connected manifold, we have that the space of leaves of $\pi^{-1}(\mathscr{F})$ is diffeomorphic to G/K and hence there exists a unique leaf L_0' of $\pi^{-1}(\mathscr{F})$ such that $F(L_0') = y$. Now $F(\gamma(L_0')) = \lambda_\gamma(F(L_0')) = \lambda_\gamma(y) = y = F(L_0')$ and hence $\gamma(L_0') = L_0'$. Since $\Gamma_{L_0'}$ is isomorphic to the holonomy group of the leaf $L_0 = \pi(L_0')$ of $\mathscr{F}$, it follows that γ is the identity transformation. Hence Γ is trivial and so M is diffeomorphic to P whence $F: M \to G/K$ is a fibration with the leaves of $\mathscr{F}$ as fibres.

(2.1) COROLLARY. *Let G/K be a simply connected, even dimensional, positively curved Riemannian homogeneous space of a compact connected Lie group G and let $\mathscr{F}$ be a one-dimensional homogeneous G/K-foliation of a compact connected manifold M. Then $\mathscr{F}$ has a compact leaf and $\pi_1(M)$ has polynomial growth of degree ≤ 1.*

Proof. If $\mathscr{F}$ is without holonomy, then by (1.1) the leaves of $\mathscr{F}$ are the fibres of a fibration $M \to G/K$ and hence all the leaves are circles. Since G/K is simply connected, the exact homotopy sequence of this fibration gives a surjection $\pi_1(S^1) \to \pi_1(M)$ whence $\pi_1(M)$ has polynomial growth of degree ≤ 1. If $\mathscr{F}$ has a leaf L whose holonomy group is non-trivial, then L is diffeomorphic to S^1. Moreover, since L is compact, the image of its fundamental group in $\pi_1(M)$ is a subgroup of finite index [2] and hence $\pi_1(M)$ has polynomial growth of degree ≤ 1 [1].

(2.2) PROPOSITION. *Let $\mathscr{F}$ be a codimension 2 transversely Euclidean (homogeneous $SO(2)\cdot \boldsymbol{R}^2/SO(2) \cong \boldsymbol{R}^2$-) foliation of a compact connected manifold M. If $\mathscr{F}$ is without holonomy, then M fibres over T^2.*

Proof. In this case Γ is a subgroup of $SO(2)\cdot \boldsymbol{R}^2$, the group of rigid motions of the plane. Since $\mathscr{F}$ is without holonomy, it follows that Γ acts freely on $\boldsymbol{R}^2$ and hence is a group of translations. If (x, y) denotes the standard coordinates on $\boldsymbol{R}^2$, then dx and dy are linearly independent translation-invariant one-forms and hence there exist smooth linearly independent one-forms ω_1 and ω_2 on M such that $\pi^*\omega_1 = F^*(dx)$, $\pi^*\omega_2 = F^*(dy)$. Moreover, ω_1 and ω_2 are closed and so M fibres over T^2 [8].

We now apply the above construction to study the existence of compact leaves for a class of Riemannian homogeneous foliations. Let G/K be a compact simply connected Riemannian symmetric space with nonzero Euler characteristic where $G = I_0(G/K)$.

(2.3) PROPOSITION. *Let M be a compact manifold with solvable fundamental group. Then every homogeneous G/K-foliation of M has a compact leaf.*

Proof. The image Γ of the homomorphism $\Phi : \pi_1(M, x_0) \to G$ is a solvable subgroup of the compact Lie group G and so its closure is a compact solvable Lie subgroup of G. Let H be the connected component of the identity of $\bar{\Gamma}$. Then H is a toral subgroup. Since the Euler characteristic of G/K is nonzero, G and K have the same rank [9] and hence H is contained in some conjugate of K. Thus there exists a point y of G/K fixed under the action of H and hence, since H is a subgroup of $\bar{\Gamma}$ of finite index, the orbit of y under the action of Γ is a finite set of points $y = y_1, y_2, \cdots, y_r$. For each $i = 1, \cdots, r$ let L_i be the unique leaf of $\pi^{-1}(\mathscr{F})$ such that $F(L_i) = y_i$. Then $L_1, \cdots, L_r$ all project via π to the same leaf L of $\mathscr{F}$ and, since $\pi^{-1}(L) = \bigcup_{i=1}^{r} L_i$ is a closed subset of P, it follows that L is compact.

Recall that by a codimension q elliptic foliation of a manifold M we mean a homogeneous G/K-foliation of M where $G = SO(q+1)$, $K = SO(q)$, $G/K \cong S^q$.

(2.4) COROLLARY. *Every codimension $2q$ elliptic foliation of a compact manifold with solvable fundamental group has a compact leaf.*

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