J. Herzog
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WHEN IS A REGULAR SEQUENCE SUPER REGULAR?

J. HERZOG*)

Let $(B, \mathscr{F})$ be a filtered, noetherian ring. A sequence $x = x_1, \cdots, x_n$ in B is called super regular if the sequence of initial forms

$$\xi_1 = L(x_1), \cdots, \xi_n = L(x_n)$$

is a regular sequence in $gr_{\mathscr{F}}(B)$.

If B is local and the filtration $\mathscr{F}$ is $\mathfrak{A}$-adic then any super regular sequence is also regular, see [6], 2.4.

In [3], Prop. 6 Hironaka shows that in a local ring $(B, \mathfrak{M})$ an element $x \in \mathfrak{M}\backslash\mathfrak{M}^2$ is super regular (with respect to the $\mathfrak{M}$-adic filtration) if and only if x is regular in B and $(x) \cap \mathfrak{M}^{n+1} = (x)\mathfrak{M}^n$ for every integer n.

This result is extended to a more general situation in [6], 1.1. In the present paper we will characterize super regular sequences in a relative case:

Let A be a regular complete local ring, $B = A/I$ an epimorphic image of A and $x = x_1, \cdots, x_n$ a regular sequence in B which is part of a minimal system of generators of the maximal ideal of B. Let $y = y_1, \cdots, y_n$ be a sequence in A which is mapped onto x. Then y is part of a regular system of parameters of A. Therefore y is a super regular sequence in A.

We put $\bar{A} = A/(y)A$, $\bar{I} = I/(y)I$ and $\bar{B} = B/(x)B$. Then $\bar{B} = \bar{A}/\bar{I}$, since x is a B-sequence.

As a consequence of our main result, the following conditions are equivalent:

(a) x is a super regular sequence in B

(b) For all elements $g \in \bar{I}$ there exists $f \in I$, such that

$$\bar{f} = g \quad \text{and} \quad \nu(f) = \nu(g)\,.$$

(Here $\bar{f}$ denotes the image of f in $\bar{I}$ and $\nu(f)$ the degree of the initial form

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of f.)

The equivalence of (a) and (b) can also be expressed in terms of Hironaka's numerical character $\nu^*(J, R)$: x is a super regular sequence in B if and only if $\nu^*(I, A) = \nu^*(\bar{I}, \bar{A})$.

In the applications we will use this characterization to show that the tangent cone of certain algebras is CM (Cohen-Macaulay). Our examples contain some results of J. Sally [4], [5] in a more special case.

§1. Notations and remarks

In the following we fix our notations and recall some basic facts about filtrations. For a more detailed information about filtrations we refer to N. Bourbaki [1].

Let $(A, \mathscr{F})$ be a noetherian filtered ring such that $\mathscr{F}_0 A = A$ and $\mathscr{F}_{i+1} A \subseteq \mathscr{F}_i A$ for $i \geq 0$ and let $(M, \mathscr{G})$ be a filtered $(A, \mathscr{F})$-module. Then $gr_{\mathscr{G}}(M) = \oplus_{i\geq 0} \mathscr{G}_i M / \mathscr{G}_{i+1} M$ is a graded $gr_{\mathscr{F}}(A) = \oplus_{i\geq 0} \mathscr{F}_i A / \mathscr{F}_{i+1} A$-module.

If $x \in M$ we define $\nu(x) = \sup\{n / x \in \mathscr{G}_n M\}$ to be the degree of x and call

$$L(x) = x + \mathscr{G}_{n+1} M \text{ the initial form of } x .$$

Let $\varphi: M \to N$ be a homomorphism of filtered modules then φ induces a homogeneous homomorphism

$$gr(\varphi): gr(M) \to gr(N) .$$

If φ is an epimorphism, we always will assume that N admits the canonical filtration induced from the filtration of M. Then

$$\operatorname{Ker}(gr(\varphi)) = \{L(x) / x \in \operatorname{Ker} \varphi\} .$$

We call a sequence $(x_1, \cdots, x_n)$, $x_i \in \operatorname{Ker} \varphi$ a *standard base of* $\operatorname{Ker} \varphi$ if

$$\operatorname{Ker}(gr(\varphi)) = (L(x_1), \cdots, L(x_n)) .$$

In the particular case that $\varphi: A \to B$ is an epimorphism of filtered rings, we now give a slightly different but useful description of a standard base: Corresponding to a sequence $(x_1, \cdots, x_n)$, $x_i \in \operatorname{Ker} \varphi$, we define a filtration on A^n:

$$\mathscr{F}_i A^n = \{(a_1, \cdots, a_n) / a_j \in \mathscr{F}_{i-\nu(x_j)} A\} .$$

Now

$$(1) \qquad A^n \xrightarrow[(x_1, \cdots, x_n)]{} A \xrightarrow[\varphi]{} B \longrightarrow 0$$

is a complex of filtered A-modules inducing a complex of $gr(A)$-modules

$$(2) \qquad gr(A^n) \xrightarrow[(L(x_1), \cdots, L(x_n))]{} gr(A) \xrightarrow[gr(\varphi)]{} gr(B) \longrightarrow 0$$

and $(x_1, \cdots, x_n)$ is a standard base of $\operatorname{Ker}\varphi$ if and only if the complex (2) is exact.

If A is complete and separated then any standard base of $\operatorname{Ker}\varphi$ is also a base of $\operatorname{Ker}\varphi$. However the converse is false in general. Consider the following case:
Let $B = A/xA$, where x is not a zero-divisor on A and let $\varphi: A \to B$ be the canonical epimorphism and $\xi = L(x)$.

LEMMA. (a) *If x is super regular then*

$$(*) \qquad gr(A) \xrightarrow{\xi} gr(A) \xrightarrow{gr(\varphi)} gr(B) \longrightarrow 0$$

is exact, i.e. (x) is a standard base of $\operatorname{Ker}\varphi = (x)$.

(b) *If A is complete and separated and the sequence $(*)$ is exact then x is super regular.*

The lemma shows that a non-zero-divisor x in a complete separated ring forms a standard base of (x) if and only if it is super regular.

Proof of the lemma. (a) Let $\alpha \neq 0$ be a homogeneous element of $\operatorname{Ker}(gr(\varphi))$. Then $\alpha = L(xa)$ for some $a \in A$. Since $\xi L(a) \neq 0$, we have $\xi L(a) = L(xa) = \alpha$.

(b) Let $\alpha \in gr(A)$ be a homogeneous element such that $\xi\alpha = 0$.

We construct a convergent series (a_n) such that for all $n \geq 1$ we have $L(a_n) = \alpha$ and $\nu(xa_n) \geq \nu(x) + \nu(a_1) + n$.

Let $a = \lim a_n$, then $\alpha = L(a)$ and $xa \in \bigcap \mathscr{F}_i A = \{0\}$. Therefore $a = 0$ and consequently $\alpha = 0$. Construction of the sequence (a_n) by induction on n:

Let $a_1 \in A$ such that $\alpha = L(a_1)$. Since $\xi\alpha = 0$ we have $\nu(xa_1) \geq \nu(x) + \nu(a_1) + 1$.

Suppose we have already constructed $a_1, \cdots, a_n$. By induction hypothesis we have $\nu(xa_n) \geq \nu(x) + \nu(a_1) + n$. Since $L(xa_n) \in \operatorname{Ker}(gr(\varphi))$ and since we suppose that $(*)$ is exact we find a homogeneous element γ_n such that $\xi\gamma_n = L(xa_n)$.

Choose $g_n \in A$ such that $\gamma_n = L(g_n)$, then $\nu(g_n) = \nu(xa_n) - \nu(x) \geq \nu(a_1) + n$ and $\nu(x(a_n - g_n)) \geq \nu(x) + \nu(a_1)(n+1)$. The element $a_{n+1} = a_n - g_n$ is the next member of the sequence.

§ 2. The main result

Let $\varepsilon: A \to B$ be an epimorphism of complete and separated filtered rings. As before we assume that B admits the induced filtration. Then $\operatorname{Ker} \varepsilon$ is a closed ideal of A.

Suppose we are given a super regular sequence $y = y_1, \cdots, y_n$ on A and let $x_i = \varepsilon(y_i)$. Suppose that $x = x_1, \cdots, x_n$ is a regular sequence on B and that

$$\nu(x_i) = \nu(y_i) > 0$$

for $i = 1, \cdots, n$.

Let $\bar{A} = A/(y)A$, $\bar{B} = B/(x)B$, $I = \operatorname{Ker} \varepsilon$ and $\bar{I} = I/(y)I$. We have $\bar{I} \subset \bar{A}$ and $\bar{B} = \bar{A}/\bar{I}$, since x is a regular sequence on B. If f is an element of A or of B we denote its image in $\bar{A}$ or $\bar{B}$ by $\bar{f}$.

THEOREM 1. 1) *The following properties are equivalent*:

a) *For each $g \in \bar{I}$ there exists $f \in I$ such that $\bar{f} = g$ and $\nu(f) = \nu(g)$.*

b) *There exists a standard base $g_1, \cdots, g_m \in \bar{I}$ and elements $f_i \in I$ such that $\bar{f}_i = g_i$ and $\nu(f_i) = \nu(g_i)$ for $i = 1, \cdots, m$.*

c) *x is a super regular sequence.*

2) *If the equivalent conditions of* 1) *hold and the f_i are chosen as in* b), *then $(f_1, \cdots, f_m)$ is a standard base of I.*

Proof. It is sufficient to consider the case that the sequence x consists only of one element. The general case follows by induction on the length of the sequence.

1) a) $\Rightarrow$ b): is obvious

b) $\Rightarrow$ c): Let $(g_1, \cdots, g_m)$ be a standard base of $\bar{I}$ and $f_i \in I$ be such that $\bar{f}_i = g_i$ and $\nu(f_i) = \nu(g_i)$.

We define on A^n and $\bar{A}^n$ filtrations

$$\mathscr{F}_i A^n = \{(a_1, \cdots, a_n)/a_j \in \mathscr{F}_{i-\nu(f_j)} A\}$$
$$\mathscr{F}_i \bar{A}^n = \{(\bar{a}_1, \cdots, \bar{a}_n)/\bar{a}_j \in \mathscr{F}_{i-\nu(g_j)} \bar{A}\}$$

and obtain a commutative diagram of filtered modules

$$
(1)\qquad
\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A^n & \xrightarrow[(f_1,\cdots,f_n)]{} & A & \longrightarrow & B & \longrightarrow & 0 \\
y\downarrow & & \downarrow y & & \downarrow x & & \\
A^n & \xrightarrow[(f_1,\cdots,f_n)]{} & A & \xrightarrow[\varepsilon]{} & B & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\bar{A}^n & \xrightarrow[(g_1,\cdots,g_n)]{} & \bar{A} & \xrightarrow[\bar{\varepsilon}]{} & \bar{B} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

inducing a commutative diagram of graded modules

$$
(2)\qquad
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
gr(A^n) & \longrightarrow & gr(A) & \longrightarrow & gr(B) & \longrightarrow & 0 \\
\downarrow \eta & & \downarrow \eta & & \downarrow \xi & & \\
gr(A^n) & \xrightarrow[\varphi]{} & gr(A) & \xrightarrow[gr(\varepsilon)]{} & gr(B) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \sigma & & \\
gr(\bar{A}^n) & \longrightarrow & gr(\bar{A}) & \xrightarrow[gr(\bar{\varepsilon})]{} & gr(\bar{B}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

$\xi = L(x)$, $\eta = L(y)$.

The lowest row is exact since $(g_1, \cdots, g_n)$ is a standard base. Also the middle column is exact since y is super regular.

By diagram chasing we find, that also the sequence

$$gr(B) \xrightarrow{\xi} gr(B) \longrightarrow gr(\bar{B}) \longrightarrow 0 \text{ is exact.}$$

By the lemma it follows that x is super regular. c) $\Rightarrow$ a): Let $g \in \bar{I}$, then we can find an element $f \in A$ such that $\bar{f} = g$ and $\nu(f) = \nu(g)$.

However we would like to find such an element f in I. To do this we consider

$$\sigma gr(\varepsilon)(L(f)) = gr(\bar{\varepsilon})(L(g)) = 0 .$$

Since we assume that x is super regular it follows from the lemma that $gr(\varepsilon)(L(f)) = \beta\xi$. Therefore $L(f) = \alpha\eta + \gamma$, where α, γ are homogeneous and $\gamma \in \operatorname{Ker}(gr(\varepsilon))$.

Hence we can choose $a_1, f_1 \in A$ and $h_1 \in I$ such that

$$f = a_1 y + h_1 + f_1 ,$$
$$\nu(f) = \nu(a_1 y) = \nu(h_1) < \nu(f_1) .$$

From this we obtain $g = \bar{f} = \bar{h}_1 + \bar{f}_1 \in \bar{I}$, hence $\bar{f}_1 \in \bar{I}$. Repeating the same reasoning for f_1, we can find $a_2, f_2 \in A$ and $h_2 \in I$ such that

$$f_1 = a_2 y + h_2 + f_2 ,$$
$$\nu(f_1) = \nu(a_2 y) = \nu(h_2) < \nu(f_2) .$$

This time it may happen that $\nu(f_1) < \nu(\bar{f}_1)$, but that doesn't matter and we can take $h_2 = 0$ in that case. Proceeding that way we construct sequences (a_i), (h_i) and (f_i) such that $h_i \in I$ and

$$f_i = a_i y + h_i + f_{i+1}$$
$$\nu(f_i) = \nu(a_i y) = \nu(h_i) < \nu(f_{i+1}) .$$

Put $a = \sum_{i=1}^{\infty} a_i$, $h = \sum_{i=1}^{\infty} h_i$. Then $f = ay + h$, $h \in I$ and $\nu(h) = \nu(h_1) = \nu(f) = \nu(g)$. Thus h is the desired element.

2) Consider again the diagram (2). We have to show that if $\alpha \in \operatorname{Ker}(gr(\varepsilon))$ is a homogeneous element then there exists $\gamma \in gr(A^n)$ such that $\varphi(\gamma) = \alpha$. We prove this by induction on the degree of α. If $\deg \alpha < 0$, then $\alpha = 0$. Thus suppose that $\deg \alpha > 0$. By assumption all columns and the lowest row are exact. By diagram chasing we can find homogeneous elements β, δ such that

$$\alpha = \beta\eta + \delta ,$$

where $\delta \in \operatorname{Im} \varphi$ and $\beta \in \operatorname{Ker}(gr(\varepsilon))$. Since by assumption $\deg \eta > 0$, we have $\deg \beta < \deg \alpha$. From the induction hypothesis the assertion follows.

§ 3. Some applications

(a) Let $B = k[[x_1, \cdots, x_n]]/I$ be a 1-dimensional complete algebra over an algebraically closed field k. In the following we consider only the m_B-adic filtration of B.

Suppose that the residue class x_1 of X_1 is not a zero-divisor and a superficial element of B, then $gr(B)$ is a CM-ring (Cohen-Macaulay) if and only if x_1 is super regular on B.

Applying Theorem 1 we find:
$gr(B)$ is a CM-ring if and only if for all $F \in I$ there exists $G \in k[[X_1, \cdots, X_n]]$ such that

$$F(0, X_2, \cdots, X_n) + GX_1 \in I \quad \text{and} \quad \nu(G) \geq \nu(F(0, X_2, \cdots, X_n)) - 1 .$$

Next we restrict our attention to the more special case that B is a monomial ring:
Let $H \subset N$ be a numerical semigroup generated minimally by $n_1 < n_2 < \cdots < n_l$, see [2].

To H belongs the monomial ring $B = k[t^{n_1}, \cdots, t^{n_l}]$, whose maximal ideal is $m_B = (t^{n_1}, \cdots, t^{n_l})$. We want to describe in terms of the semigroup when $gr_{m_B}(B)$ is a CM-ring.
t^{n_1} is a superficial element of B. Let $\bar{B} = B/t^{n_1}B \simeq k[[X_2, \cdots, X_l]]/\bar{I}$. It is easy to see that a standard base of $\bar{I}$ can be chosen such that the elements of the base are either monomials $X_2^{\nu_2} \cdots X_l^{\nu_l}$ or differences of monomials

$$X_2^{\mu_2} \cdots X_1^{\mu_1} - X_2^{\mu_2^*} \cdots X_l^{\mu_l^*}$$

with

$$\sum_{i=2}^{l} \mu_i n_i = \sum_{i=2}^{l} \mu_i^* n_i .$$

Let $n_1 + H = \{n_1 + h/h \in H\}$. A monomial $X_2^{\nu_2}, \cdots, X_l^{\nu_l}$ is an element of $\bar{I}$ if and only if

$$\sum_{i=2}^{l} \nu_i n_i \in n_1 + H .$$

Thus we find:
$gr(B)$ is a CM-ring if and only if for all integers $\nu_2 \geq 0, \nu_3 \geq 0, \cdots, \nu_l \geq 0$ such that

$$\sum_{i=2}^{l} \nu_i n_i \in n_1 + H ,$$

there exist $\nu_1^* > 0, \nu_2^* \geq 0, \cdots, \nu_l^* \geq 0$ such that

$$\sum_{i=2}^{l} \nu_i n_i = \sum_{i=1}^{l} \nu_i^* n_i \quad \text{and} \quad \sum_{i=2}^{l} \nu_i \leq \sum_{i=1}^{1} \nu_i^* .$$

It is not difficult to see that it suffices to consider only such ν_i with the extra condition that $n_i > \nu_i$. Therefore only a finite number of conditions are to be checked.

If in addition $\bar{I}$ is generated only by monomials, then there is a unique minimal system of generators of $\bar{I}$ consisting of monomials $M_1, \cdots, M_k$. These monomials form a standard base of $\bar{I}$.

Thus $gr(B)$ is a CM-ring if and only if to each such monomial

$$M_i = X_2^{\nu_2} \cdots X_l^{\nu_1}$$

we can find

$$F_i = X_2^{\nu_2} \cdots X_l^{\nu_l} - X_1^{\nu_1^*} \cdots X_l^{\nu_l^*} \in I ,$$

with

$$\nu_1^* > 0 \quad \text{and} \quad \sum_{i=2}^{l} \nu_i \leq \sum_{i=1}^{l} \nu_i^* .$$

In particular if $gr(B)$ is a CM-ring then $F_1, \cdots, F_k$ forms a standard base of I and also a minimal base of I.

We now discuss in more detail monomial rings of embedding dimension 3. These examples were first studied by G. Valla and R. Robbiano in [7] and communicated to me, when I was visiting Genova. Using different methods they are able to construct in all cases a standard base. Here we restrict ourselves to the question whether $gr(B)$ is a CM-ring.

Let $B = k[[t^{n_1}, t^{n_2}, t^{n_3}]]$ and assume first that B is not a complete intersection. In [2] it is shown that $I = (F_1, F_2, F_3)$ with

$$\begin{aligned} F_1 &= X_1^{c_1} - X_2^{r_{12}} \cdot X_3^{r_{13}} \\ F_2 &= X_2^{c_2} - X_1^{r_{21}} \cdot X_3^{r_{23}} \\ F_3 &= X_3^{c_3} - X_1^{r_{31}} \cdot X_2^{r_{32}} \end{aligned}$$

where $r_{ij} > 0$ and $c_1 = r_{21} + r_{31}$, $c_2 = r_{12} + r_{32}$ and $c_3 = r_{13} + r_{23}$. It follows that $\bar{I}$ is generated by monomials and therefore $gr(B)$ is a CM-ring if and only if

$$\begin{aligned} c_1 &\geq r_{12} + r_{13} \\ c_2 &\leq r_{21} + r_{23} \\ c_3 &\leq r_{31} + r_{32} . \end{aligned}$$

The first inequality is always satisfied since

$$c_1 n_1 = r_{12} n_2 + r_{13} n_3$$

and

$$n_1 < n_2 < n_3 \ .$$

Similarly the third inequality is always true. Our final result is therefore: $gr(B)$ is a CM-ring if and only if $c_2 \leq r_{21} + r_{23}$.

n_1	n_2	n_3	c_2	r_{21}	r_{23}	CM
3	4	5	2	1	1	Yes
5	6	13	3	1	1	No

We now assume that $B = k[[t^{n_1}, t^{n_2}, t^{n_3}]]$ is a complete intersection. Then I can be generated by two elements F_1, F_2. We have to distinguish several case:

Case	F_1, F_2	Example
α)	$X_1^{c_1} - X_2^{c_2}, X_1^{c_1} - X_3^{c_3}$	6, 10, 15
β)	$X_2^{c_2} - X_3^{c_3}, X_1^{c_1} - X_2^{r_{12}} \cdot X_3^{r_{13}}$	7, 9, 12
γ)	$X_1^{c_1} - X_3^{c_3}, X_2^{c_2} - X_1^{r_{21}} \cdot X_3^{r_{23}}$	4, 5, 6
δ)	$X_1^{c_1} - X_2^{c_2}, X_3^{c_3} - X_1^{r_{31}} \cdot X_2^{r_{32}}$	4, 6, 7
	all $r_{ij} > 0$	

Case α). $\bar{I} = (X_2^{c_2}, X_3^{c_3})$ is generated by monomials. Since $c_1 > c_2$ and $c_1 > c_3$, it follows that B is a strict complete intersection.

Case β). $\bar{I} = (X_2^{c_2} - X_3^{c_3}, X_2^{r_{12}} \cdot X_3^{r_{13}})$.
We want to find a standard base of $\bar{I}$:

$$X_2^{c_2+r_{12}}, X_3^{c_3}, X_2^{r_{12}} \cdot X_3^{r_{13}}$$

are relations of $gr(\bar{B})$. We easily compute the length l of

$$k[[X_2, X_3]]/(X_2^{c_2+r_{12}}, X_3^{c_3}, X_2^{r_{12}} \cdot X_3^{r_{13}})$$

to be

$$l = r_{12}c_3 + r_{13}c_2 \ .$$

On the other hand we have

$$n_2 = c_3c_1, \ n_3 = c_2c_1$$

and

$$c_1n_1 = r_{12}n_2 + r_{13}n_3 \ ,$$

therefore

$$n_1 = r_{12}c_3 + r_{12}c_2 = l\ .$$

Since

$$n_1 = 1(B/t^{n_1}B) = l(gr(B/t^{n_1}B))\ ,$$

it follows that

$$X_2^{c_2+r_{12}},\ X_2^{c_2} - X_3^{c_3},\ X_2^{r_{12}}X_3^{r_{13}}$$

is a standard base of $\bar{I}$.

There is only one way to lift these equations:

$$X_2^{c_2+r_{12}} - X_3^{c_3-r_{13}}\cdot X_1^{c_1},\ X_2^{c_2} - X_3^{c_3},\ X_1^{c_1} - X_2^{r_{12}}\cdot X_3^{r_{13}}\ .$$

Since $c_1 \geq r_{12} + r_{13}$, we find that $gr(B)$ is a CM-ring if and only if

$$c_2 + r_{12} \leq c_3 - r_{13} + c_1\ .$$

However B is never a strict complete intersection.

γ) $\bar{I} = (X_3^{c_3}, X_2^{c_2})$ is generated by monomials. Thus B is a strict complete intersection if and only if $c_2 \leq r_{21} + r_{23}$.

δ) $\bar{I} = (X_2^{c_2}, X_3^{c_3})$ is generated by monomials and $c_3 < r_{31} + r_{32}$, therefore B is always a strict complete intersection.

THEOREM 2. *Let $B = k[[X_1, \cdots, X_n]]/I$ be a complete k-algebra and suppose that I admits a standard base $F_1, \cdots, F_m$ such that:*

1) *$\nu(F_i) = 2$ for $i = 1, \cdots, m$.*

2) *For each homomorphism $\varphi: I/I^2 \rightarrow B$ the elements $\varphi(F_i + I^2)$, $i = 1, \cdots, m$ are not units in B (equivalently, B is not a direct summand of I/I^2).*

Then for any complete algebra $\tilde{B} = k\ [[Y_1, \cdots, Y_k]]/J$ and any regular $\tilde{B}$-sequence $t_1, \cdots, t_k$ such that $\tilde{B}/(t_1, \cdots, t_k)\tilde{B} = B$ it follows that $(t_1, \cdots, t_k)$ is a super regular sequence on $\tilde{B}$.

Proof. We may write

$$\tilde{B} \simeq k[[X_1, \cdots, X_n, T_1, \cdots, T_k]]/J$$

such that $t_i = T_i + J$, $i = 1, \cdots, k$. Then $J = (G_1, \cdots, G_m)$ with

$$G_i = F_i + \sum_{j=1}^{k} F_i^{(j)} T_j + H_i\ ,$$

$H_i \in (T_1, \cdots, T_k)^2$ and $F_i^{(j)} \in k[[X_1, \cdots, X_n]]$. Since $t_1, \cdots, t_k$ is a regular $\tilde{B}$-sequence, we obtain B-module homomorphisms

$$\varphi_j : I/I^2 \to B, \qquad j = 1, \cdots, m$$
$$F_i + I^2 \mapsto F_i^{(j)} + I$$

By assumption 2) it follows that $\nu(F_i^{(j)}) \geq 1$ and by assumption 1) it follows that $\nu(G_i) = \nu(F_i)$ for $i = 1, \cdots, m$.
From our criterion of section 2 the assertion follows.

We use this theorem to derive two results of J. Sally in a slightly more special case.

We introduce the following notations: $e(B)$ = embedding dimension of B, $d(B)$ = Krull dimension of B and $m(B)$ = multiplicity of B.

THEOREM 3 ([4], [5]). *Let $B \simeq k[[X_1, \cdots, X_n]]/I$ be a complete CM-algebra and suppose that either*

α) $m(B) \leq e(B) - d(B) + 1$
or
β) $m(B) \leq e(B) - d(B) + 2$ *and B is a Gorenstein ring*

then $gr(B)$ is a CM-ring.

Proof. We may assume that k is algebraically closed.

α) There exists a regular sequence $(t_1, \cdots, t_d)$ such that

$$l(B/(t_1, \cdots, t_d)B) = m(B) .$$

This sequence is part of a minimal system of generators of m_B. Let $\bar{B} = B/(t_1, \cdots, t_d)B$, then $e(\bar{B}) = e(B) - d(B) = m(B) - 1 = l(\bar{B}) - 1$. It follows that $m_{\bar{B}}^2 = 0$, and $\bar{B} = k[[X_1, \cdots, X_m]]/\bar{I}$ with $\bar{I} = (X_1, \cdots, X_m)^2$. We may assume that $m \geq 2$ and show that $\bar{B}$ satisfies the conditions of Theorem 2.

Condition 1) is obviously satisfied since $\bar{I}$ is generated by the monomials X_iX_j of degree 2, which form a standard base of $\bar{I}$.

Suppose there exists a $\bar{B}$-module homomorphism $\varphi : \bar{I}/\bar{I}^2 \to \bar{B}$ and integers i, j such that $\varphi(X_iX_j + \bar{I}^2)$ is a unit.

1*st Case.* If $i = j$, then for any $k \neq i$ we have

$$x_k\varphi(X_i^2 + \bar{I}^2) + x_i\varphi(X_iX_k + \bar{I}^2) ,$$

a contradiction since $(x_1, \cdots, x_m)$ is a minimal base of $m_{\bar{B}}$.

2nd Case. If $i \neq j$, then $x_i\varphi(X_iX_j + \bar{I}^2) = x_j\varphi(X_i^2 + \bar{I}^2)$, again a contradiction.

β) As in the case α) we can reduce B to an algebra $\bar{B}$ such that $l(\bar{B}) = e(\bar{B}) + 2$. It follows that $m_{\bar{B}}^3 = 0$ and that $\bar{B}$ is a graded ring with Hilbert function $1 + e(\bar{B})t + t^2$. Let σ be generator of $\bar{B}_2$. The multiplication on $\bar{B}$ induces a non singular quadratic form $\mathfrak{q}: \bar{B}_1 \times \bar{B}_1 \to k$ defined by

$$\mathfrak{q}(v, w)\sigma = v \cdot w$$

Since we assume that k is algebraically closed we can choose a k-vectorspace base $x_1, \cdots, x_m$ of $\bar{B}_1$ such that $x_i^2 = \sigma$ for $i = 1, \cdots, m$ and $x_ix_j = 0$ for $i \neq j$.

We treat the case $m = 2$ separately, since in that case $\bar{B}$ is a complete intersection and Theorem 2 is not applicable. However then we have $B = k[[X_1 \cdots X_n]]/(F_1, F_2)$ with $\bar{F}_1 = X_1^2 - X_2^2$, $\bar{F}_2 = X_1X_2$. If $\nu(F_1) = \nu(F_2) = 2$, then the assertion follows from Theorem 1. Otherwise, say $\nu(F_1) = 1$, then B is a hypersurface ring and the assertion follows again.

Now if $m > 2$ we apply Theorem 2: Again the first condition is satisfied. We check condition 2):

1st Case. Suppose there exists a $\bar{B}$-module homomorphism $\varphi: \bar{I}/\bar{I}^2 \to \bar{B}$ such that $\varphi(X_1^2 - X_i^2 + \bar{I}^2)$ is a unit, then

$$\sigma\varphi(X_1^2 - X_i^2 + \bar{I}^2) = x_1^2\varphi(X_1^2 - X_i^2 + \bar{I}^2) = \varphi(X_1^4 - X_1^2X_i^2 + \bar{I}^2) = 0\,,$$

since $X_1^4 - X_1^2X_i^2 \in \bar{I}^2$. This is a contradiction.

2nd Case. Suppose there exists a $\bar{B}$-module homomorphism $\varphi: \bar{I}/\bar{I}^2 \to \bar{B}$ such that $\varphi(X_iX_j + I^2)$ is a unit, then $\sigma\varphi(X_iX_j + I^2) = x_1^2\varphi(X_iX_j + I^2) = \varphi((X_1X_i)(X_1X_j) + I^2) = 0$ since $(X_1X_i)(X_1X_j) \in I^2$. This is again a contradiction.

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Universität Essen – Gesamthochschule
Fachbereich 6
Universitätsstr. 3