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COMPLETELY SUPERHARMONIC MEASURES FOR THE INFINITESIMAL GENERATOR A OF A DIFFUSION SEMI-GROUP AND POSITIVE EIGEN ELEMENTS OF A

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§1. Introduction

Let X be a locally compact Hausdorff space with countable basis. We denote by

$M(X)$ the topological vector space of all real Radon measures in X with the vague topology,

$M_K(X)$ the topological vector space of all real Radon measures in X whose supports are compact with the usual inductive limit topology.

Their subsets of all non-negative Radon measures are denoted by $M^+(X)$ and by $M_K^+(X)$, respectively.

In the paragraph 2, we shall prepare the terminology and the notation which we shall use in the sequel.

A continuous linear operator T from $M_K(X)$ into $M(X)$ is called a diffusion kernel on X if T is positive, i.e., $T\mu \in M^+(X)$ whenever $\mu \in M_K^+(X)$. A semi-group $(T_t)_{t\geqq 0}$ of diffusion kernels on X is called a diffusion semi-group if $T_0 = I$ (the identity) and if, for any $\mu \in M_K(X)$, the mapping $t \to T_t\mu$ is continuous in $M(X)$.

We consider the infinitesimal generator A of a transient and regular diffusion semi-group $(T_t)_{t\geqq 0}$ on X. A Radon measure $\mu \in M(X)$ is said to be A-superharmonic (resp. A-harmonic) if it satisfies $-A\mu \in M^+(X)$ (resp. $A\mu = 0$).

In the paragraph 3, we shall show that every positive A-superharmonic Radon measure is written uniquely as the sum of a V-potential of a non-negative Radon measure and a non-negative A-harmonic measure, where V is the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$, i.e.,

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$$V = \int_0^\infty T_t dt\,. \tag{1.1}$$

By generalizing the classical positive eigen equation with zero conditions on the boundary and by defining that a Radon measure vanishes V-n.e. on the boundary (Definition 21 in § 2), we shall discuss, in the paragraph 4, a positive eigen equation for A with zero conditions in the following setting:

For a positive number $c > 0$,

$$\begin{cases} A\mu = -c\mu \\ \ \mu = 0 \ \ V\text{-n.e. on the boundary.} \end{cases} \tag{1.2}$$

Denote by $E_0(A; c)$ the set of all non-negative solutions of (1.2) and put $E_0(A) = \bigcup_{c \geqq 0} E_0(A; c)$. Under the assumption that A satisfies the condition $(\mathscr{L})$ (Definition 49 in § 4), we shall show that $E_0(A)$ is a Borel measurable set in the metrizable space $M^+(X)$.

By generalizing the notion of the classical complete superharmonicity, we define the complete A-superharmonicity of $\mu \in M(X)$. A Radon measure $\mu \in M(X)$ is said to be completely A-superharmonic if, for any integer $n \geqq 1$, $(-A)^n \mu \in M^+(X)$, where $(-A)^n$ denotes the n-th iterate of $-A$. Let $SC(A)$ be the set of all non-negative completely A-superharmonic measures in X and put

$$SC_0(A) = \{\mu \in SC(A); (-A)^n \mu = 0 \ \ V\text{-n.e. on the boundary for } n = 0, 1, \cdots\}\,. \tag{1.3}$$

Under the condition $(\mathscr{L})$ for A, $SC(A)$ is a closed convex cone in $M^+(X)$ and all extreme rays of $SC(A)$ contained in $SC(A) - SC_0(A)$ are determined whenever all extreme rays of $SC(A)$ contained in $H(A)$ are determined, where $H(A)$ is the convex cone formed by all non-negative A-harmonic measures.

A main purpose of the paragraph 4 is to show that

$$\begin{aligned} SC_0(A) &= \left\{\int \nu d\Phi(\nu) \in M^+(X); \Phi \in M_b^+(E_0(A))\right\} \\ &= \left\{\int_0^\infty \mu_t d\sigma(t) \in M^+(X); \mu_t \in E_0(A; t), \sigma \in M_b^+((0, \infty))\right\}, \end{aligned} \tag{1.4}$$

where $M_b^+(E_0(A))$ and $M_b^+((0, \infty))$ denote the set of all regular Borel non-negative measures Φ on $E_0(A)$ with $\int d\Phi < \infty$ and that of all Borel non-negative measures σ in $(0, \infty)$ with $\int d\sigma < \infty$, respectively. Let $A = d/dx$ in $(0, \infty)$. Then (1.4) implies the Bernstein theorem.

M. V. Noviskiĭ [16] discussed a similar formula as in (1.4) for the infinitesimal generator of a contraction semi-group in a Banach space.

In the paragraph 5, for a given elliptic differential operator L of second order on a subdomain D of an orientable C^∞-manifold, we shall show that the diffusion semi-group defined by the fundamental solution of $\partial/\partial t - L$ is regular if it is transient. Applying our theorem to completely L-superharmonic functions in D, we shall obtain the integral representation of a completely L-superharmonic function in D. This is a generalization of Noviskiĭ's result (see [15]).

§ 2. Basic notation and preliminaries

We denote by

$C(X)$ the Fréchet space of all real-valued continuous functions in X with the topology of compact uniform convergence,

$C_K(X)$ the topological vector space of all real-valued continuous functions in X whose supports are compact with the usual inductive limit topology.

Their subsets of all non-negative functions are also denoted by $C^+(X)$ and $C_K^+(X)$, respectively.

DEFINITION 1. (1) A continuous linear operator T from $M_K(X)$ into $M(X)$ is called a diffusion kernel if T is positive, i.e., $T\mu \in M^+(X)$ whenever $\mu \in M_K^+(X)$.

(2) A linear operator T from $C_K(X)$ into $C(X)$ is called a continuous kernel if T is positive, i.e., $Tf \in C^+(X)$ whenever $f \in C_K^+(X)$.

Remark 2. A continuous kernel T is a continuous mapping from $C_K(X)$ into $C(X)$.

We see easily the following

Remark 3. (1) Let T be a diffusion kernel on X. For $f \in C_K(X)$, we put

$$T^*f(x) = \int f dT\varepsilon_x , \tag{2.1}$$

where ε_x denotes the Dirac measure at $x \in X$. Then $T^*f \in C(X)$ and $T^*: C_K(X) \ni f \to T^*f \in C(X)$ is a continuous kernel on X.

(2) Let T be a continuous kernel on X. For $\mu \in M_K(X)$, there exists one and only one $T^*\mu \in M(X)$ such that, for any $f \in C_K(X)$,

$$\int f dT^*\mu = \int Tf d\mu , \tag{2.2}$$

and $T^*: M_K(X) \ni \mu \to T^*\mu \in M(X)$ is a diffusion kernel on X.

In (1), T^* is called the dual continuous kernel of T and in (2), T^* is the dual diffusion kernel of T.

Remark 4. Let T be a diffusion kernel or a continuous kernel on X. Then $(T^*)^* = T$.

In the sequel, for a diffusion kernel or a continuous kernel T, its dual kernel is always denoted by T^*. For a diffusion kernel T on X, we put

$$\mathscr{D}(T) = \left\{\mu \in M(X); \int T^*f d|\mu| < \infty \text{ for all } f \in C_K^+(X)\right\} , \tag{2.3}$$

where $|\mu|$ denotes the total variation of μ, and put $\mathscr{D}^+(T) = \mathscr{D}(T) \cap M^+(X)$. Then $\mathscr{D}(T)$ is a linear subspace of $M(X)$ and T can be extended to a positive linear operator from $\mathscr{D}(T)$ into $M(X)$. For $\mu \in \mathscr{D}(T)$, $T\mu$ is called the T-potential of μ.

Let T be a continuous kernel on X. Put

$$\begin{aligned} \mathscr{D}(T) = \Big\{f \in C(X); \int |f| dT^*\mu < \infty \text{ for all } \mu \in M_K^+(X) \text{ and} \\ M_K(X) \ni \mu \to \int f dT^*\mu \text{ is continuous}\Big\} . \end{aligned} \tag{2.4}$$

Then, by the following lemma and Remark 4, we see that $\mathscr{D}(T)$ is a linear subspace of $C(X)$ and that T can be extended to a positive linear operator from $\mathscr{D}(T)$ into $C(X)$ by defining $Tf(x) = \int f dT^*\varepsilon_x$.

LEMMA 5. *Let T and $\mathscr{D}(T)$ be the same as above. If $f \in C(X)$ and $|f| \leqq |g|$ for some $g \in \mathscr{D}(T)$, then $f \in \mathscr{D}(T)$.*

In fact, Lemma 5 follows from the lower semi-continuity of the function $\int h dT^*\varepsilon_x$ of x for all $h \in C^+(X)$.

Let T_j $(j = 1, 2)$ be a diffusion kernel (resp. a continuous kernel) on X. If, for any $\mu \in M_K(X)$ (resp. $f \in C_K(X)$), $T_2\mu \in \mathscr{D}(T_1)$ (resp. $T_2 f \in \mathscr{D}(T_1)$) and if the mapping $\mu \to T_1(T_2\mu)$ (resp. $f \to T_1(T_2 f)$) defines a diffusion kernel (resp. a continuous kernel), it is called the product of T_1 and T_2 and denoted by $T_1 \cdot T_2$.

Remark 6. Let T_j $(j = 1, 2)$ be a diffusion kernel (resp. a continuous kernel) on X. If $T_1 \cdot T_2$ is defined, then $T_2^* \cdot T_1^*$ is defined and $(T_1 \cdot T_2)^* = T_2^* \cdot T_1^*$.

In particular, for a diffusion kernel T (resp. a continuous kernel) on X and a positive integer $n \geqq 2$, we denote by T^n the diffusion kernel (resp. the continuous kernel) defined inductively by $T^{n-1} \cdot T$ provided that it is defined, where $T^1 = T$. In the case of $T \neq 0$, T^0 means the identity I.

DEFINITION 7. A family $(T_t)_{t \geqq 0}$ of diffusion kernels (resp. continuous kernels) on X is called a diffusion semi-group (resp. continuous semi-group) if it satisfies the following three conditions:

$$T_0 = I. \tag{2.5}$$

$$T_t \cdot T_s = T_{t+s} \text{ for any } t \geqq 0,\ s \geqq 0. \tag{2.6}$$

(2.7) For each $\mu \in M_K(X)$ (resp. $f \in C_K(X)$), the mapping $t \to T_t \mu$ $\left(\text{resp. } t \to \int T_t f d\mu\right)$ is continuous in $M(X)$ (resp. continuous for each $\mu \in M_K(X)$).

Evidently, for a diffusion semi-group (resp. a continuous semi-group) $(T_t)_{t \geqq 0}$, $(T_t^*)_{t \geqq 0}$ is a continuous semi-group (resp. a diffusion semi-group).

Let $(T_t)_{t \geqq 0}$ be a diffusion semi-group (resp. a continuous semi-group) on X. Putting

$$\mathscr{D}((T_t)_{t \geqq 0}) = \left\{\mu \in \bigcap_{t \geqq 0} \mathscr{D}(T_t);\ t \longrightarrow T_t \mu \text{ is continuous in } M(X)\right\} \tag{2.8}$$

$$\left(\text{resp. } \mathscr{D}((T_t)_{t \geqq 0}) = \left\{f \in \bigcap_{t \geqq 0} \mathscr{D}(T_t);\ t \longrightarrow \int T_t f d\mu \text{ is continuous for each } \mu \in M_K(X)\right\}\right),$$

we call it the domain of $(T_t)_{t \geqq 0}$. We put also $\mathscr{D}^+((T_t)_{t \geqq 0}) = \mathscr{D}((T_t)_{t \geqq 0}) \cap M^+(X)$ (resp. $= \mathscr{D}((T_t)_{t \geqq 0}) \cap C^+(X)$).

DEFINITION 8. Let $(T_t)_{t \geqq 0}$ be a diffusion semi-group (resp. a continuous semi-group) on X. We say that it is transient if the mapping $V: M_K(X) \ni \mu \to \int_0^\infty T_t \mu dt \in M(X)$ (resp. $C_K(X) \ni f \to \int_0^\infty T_t f dt \in C(X)$) is defined as a diffusion kernel (resp. a continuous kernel) on X, where, for any $f \in C_K(X)$, $\int fd\left(\int_0^\infty T_t\mu dt\right) = \int_0^\infty \int fdT_t\mu dt$.

In this case, we denote by

$$V = \int_0^\infty T_t dt \tag{2.9}$$

and call it the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$ (resp. the Hunt continuous kernel for $(T_t)_{t\geqq 0}$).

Evidently we see the following

Remark 9. Let $(T_t)_{t\geqq 0}$ be a diffusion semi-group (resp. a continuous semi-group) on X. Then $(T_t)_{t\geqq 0}$ is transient if and only if $(T_t^*)_{t\geqq 0}$ is transient.

Furthermore, in the case that $(T_t)_{t\geqq 0}$ is transient, we have

$$\left(\int_0^\infty T_t dt\right)^* = \int_0^\infty T_t^* dt\,. \tag{2.10}$$

Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group (resp. a transient continuous semi-group) on X. For any $p \geqq 0$, we put

$$V_p = \int_0^\infty \exp(-pt) T_t dt\,, \tag{2.11}$$

and call $(V_p)_{p\geqq 0}$ the resolvent for $(T_t)_{t\geqq 0}$. In this case, V_p is a diffusion kernel (resp. a continuous kernel, because the Fatou lemma gives that, for any $f \in C_K^+(X)$, $V_p f$ and $Vf - V_p f$ are lower semi-continuous).

In the usual way, we see the following

PROPOSITION 10. (1) *Let $(T_t)_{t\geqq 0}$ and $(T_t')_{t\geqq 0}$ be transient diffusion semi-groups (resp. transient continuous semi-groups) on X. If $\int_0^\infty T_t dt = \int_0^\infty T_t' dt$, then $T_t = T_t'$ for any $t \geqq 0$.*

(2) *Let $(T_t)_{t\geqq t}$ be the same as above and V be the Hunt diffusion kernel (resp. the Hunt continuous kernel) for $(T_t)_{t\geqq 0}$. If a family $(V_p)_{p\geqq 0}$ of diffusion kernels (resp. continuous kernels) satisfies the following*

$$\begin{aligned} &V_p - V_q = (q-p) V_p \cdot V_q \text{ for any } p \geqq 0 \text{ and } q > 0, \text{ and} \\ &\lim_{p\to 0} V_p = V_0 = V\,, \end{aligned} \tag{2.12}$$

then $(V_p)_{p\geqq 0}$ is the resolvent for $(T_t)_{t\geqq 0}$.

We remark here that $\lim_{p\to 0} V_p = V_0$ means that, for any $\mu \in M_K(X)$, $\lim_{p\to 0} V_p\mu = V_0\mu$ in $M(X)$ (resp. for any $f \in C_K(X)$, $\lim_{p\to 0} V_p f = V_0 f$ in $C(X)$). For a transient continuous semi-group $(T_t)_{t\geqq 0}$, the Dini theorem gives that $\lim_{p\to 0} V_p f = V_0 f$ in $C(X)$ if and only if $\lim_{p\to 0} V_p f(x) = V_0 f(x)$ for each $x \in X$. The first equality in (2.12) is called the resolvent equation.

Proof of Proposition 10. We shall show only Proposition 10 for transient diffusion semi-groups, because the proof of the other case is similar. Let $(V_{1,p})_{p\geqq 0}$ and $(V_{2,p})_{p\geqq 0}$ be the resolvent for $(T_t)_{t\geqq 0}$ and that for $(T'_t)_{t\geqq 0}$, respectively. Evidently we have $\lim_{p\to 0} V_{j,p} = V_{j,0}$ $(j = 1, 2)$. For each $p \geqq 0$, we put $H_p(t) = \exp(-pt)$ on $[0, \infty)$ and $= 0$ in $(-\infty, 0)$. Then, for any $p \geqq 0$ and $q > 0$, $H_p - H_q = (q - p)H_p * H_q$. By the Fubini theorem and (2.7), $(V_{j,p})_{p\geqq 0}$ satisfies the resolvent equation. Since, for any $\mu \in M_K(X)$, the mappings $t \to T_t\mu$ and $t \to T'_t\mu$ are continuous in $M(X)$, the above argument and the injectivity of the Laplace transformation show that (2) implies (1). We shall show (2). It suffices to show that, for any $p > 0$ and any integer $n \geqq 1$, $(V_p)^n$ and $(V_{1,p})^n$ are defined and

$$V + \frac{1}{p}I = \frac{1}{p}\Big(I + \sum_{n=1}^{\infty} (pV_p)^n\Big) = \frac{1}{p}\Big(I + \sum_{n=1}^{\infty} (pV_{1,p})^n\Big)\,, \tag{2.13}$$

where $(V_{1,p})_{p\geqq 0}$ is the resolvent for $(T_t)_{t\geqq 0}$, because $(I - pV_p)\cdot(pV + I)\cdot(I - pV_p) = (I - pV_p)\cdot(pV + I)\cdot(I - pV_{1,p})$. By using the resolvent equation, we see that $(V_p)^n$ and $(V_{1,p})^n$ are defined $(n = 1, 2, \cdots)$. We shall show only the first equality in (2.13), because the other is similar. This follows directly from

$$V_q + \frac{1}{p-q}I = \frac{1}{p-q}\Big(I + \sum_{n=1}^{\infty} ((p-q)V_p)^n\Big) \tag{2.14}$$

for any q with $0 < q < p$, because, for any $\mu \in M_K^+(X)$, $V_q\mu \uparrow V\mu$ with $q \downarrow 0$. By the resolvent equation, we have

$$\begin{aligned} &\frac{1}{p-q}\Big(I + \sum_{n=1}^{\infty} ((p-q)V_p)^n\Big) \\ &= \frac{1}{p-q}I + V_q - \lim_{n\to\infty}\Big(\frac{1}{p-q}I + V\Big)\cdot((p-q)V_p)^n \\ &= \frac{1}{p-q}I + V_q\,, \end{aligned} \tag{2.15}$$

because, for any $\mu \in \mathscr{D}^+(V)$,

$$(2.16)\qquad (p-q)^n V(V_p)\mu \leqq \left(\frac{p-q}{p}\right)^n V\mu \,.$$

This completes the proof.

DEFINITION 11. A continuous kernel V on X is said to satisfy the domination principle if, for any $f, g \in C_K^+(X)$, an inequality $Vf(x) \leqq Vg(x)$ on the support of f, $\mathrm{supp}(f)$, implies the same inequality on X.

PROPOSITION 12. *Let $(T_t)_{t\geqq 0}$ be a transient continuous semi-group and V be the Hunt continuous kernel for $(T_t)_{t\geqq 0}$. Then V satisfies the domination principle.*

If X has a structure of an abelian group with which the topology of X is compatible and if, for any $t \geqq 0$, T_t is defined by a positive Radon measure α_t as follows;

$$(2.17)\qquad T_t f(x) = \alpha_t * f(x)\,,$$

then $(T_t)_{t\geqq 0}$ and V are said to be of convolution type. The assertion of Proposition 12 is well-known in the case that $(T_t)_{t\geqq 0}$ is of convolution type (see, for example, [8]). Its proof is also valid in general case.

Proof of Proposition 12. Let $(V_p)_{p\geqq 0}$ be the resolvent for $(T_t)_{t\geqq 0}$ and suppose that, for $f, g \in C_K^+(X)$, $Vf(x) \leqq Vg(x)$ on $\mathrm{supp}(f)$. Let $h \in C_K^+(X)$ such that $h(x) > 0$ on $\mathrm{supp}(f)$. Then, for any $x_0 \in \mathrm{supp}(f)$, there exists $t_0 > 0$ such that $T_t h(x_0) > 0$ for all t with $0 < t < t_0$. Hence $Vh(x_0) > 0$, i.e., $Vh(x) > 0$ on $\mathrm{supp}(f)$. For any integer $n \geqq 1$, there exists $p_0 > 0$ such that, for any $p > p_0$,

$$(2.18)\qquad \left(V + \frac{1}{p}I\right)f(x) \leqq \left(V + \frac{1}{p}I\right)\left(g + \frac{1}{n}h\right)(x) \quad \text{on } \mathrm{supp}(f)\,.$$

Put $u = \inf((V + (1/p)I)f,\ (V + (1/p)I)(g + (1/n)h))$. Then we have

$$(2.19)\qquad (I - pV_p)\left(\left(V + \frac{1}{p}I\right)f - u\right) = pV_p\left(u - \left(V + \frac{1}{p}I\right)f\right) \leqq 0 \quad \text{on } \mathrm{supp}(f)\,.$$

Since $(I - pV_p)(V + (1/p)I)f = (1/p)f$ and $(I - pV_p)u \geqq 0$ on X, we have $(I - pV_p)((V + (1/p)I)f - u) \leqq 0$, which gives that $(V + (1/p)I)f \leqq u$ on X, i.e., $u = (V + (1/p)I)f$ on X. Hence the inequality in (2.18) holds on X. Letting $p \to \infty$ and $n \to \infty$, we obtain that $Vf(x) \leqq Vg(x)$ on X. Thus Proposition 12 is shown.

Remark 13. Let V be the same as above. If, for $f, g \in \mathscr{D}^+(V)$, $Vf \leqq Vg$ on $\mathrm{supp}(f)$, then the same inequality holds on X.

In fact, for any $f' \in C_K^+(X)$ with $f' \leqq f$, there exists $h \in C_K^+(X)$ such that $Vh(x) > 0$ on $\mathrm{supp}(f')$. Hence, for any integer $n \geqq 1$, there exists $g_n \in C_K^+(X)$ such that $g_n \leqq g$ and $Vf' \leqq Vg_n + (1/n)Vh$ on $\mathrm{supp}(f')$. Proposition 12 gives that $Vf' \leqq Vg_n + (1/n)Vh \leqq Vg + (1/n)Vh$ on X. Letting $f' \uparrow f$ and $n \uparrow \infty$, we have $Vf \leqq Vg$ on X.

Similarly as in Definition 11, we define the domination principle for a diffusion kernel.

DEFINITION 14. A diffusion kernel V on X is said to satisfy the domination principle if, for any $\mu, \nu \in M_K^+(X)$, $V\mu \leqq V\nu$ in a certain neighborhood of $\mathrm{supp}(\mu)$ implies that the same inequality holds on X[1].

PROPOSITION 15. *Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. Then V satisfies the domination principle.*

Proof. Assume that, for $\mu, \nu \in M_K^+(X)$, $V\mu \leqq V\nu$ in a certain open neighborhood ω of $\mathrm{supp}(\mu)$. Choose a relatively compact open set ω_1 in X such that $\mathrm{supp}(\mu) \subset \omega_1 \subset \bar{\omega}_1 \subset \omega$. Let $(V_p)_{p\geqq 0}$ be the resolvent for $(T_t)_{t\geqq 0}$, and put $\mu_p = pV_p\mu$ in ω_1 and $\mu_p = 0$ on $C\omega_1$ $(p > 0)$. Since $\lim_{p\to\infty} pV_p\mu = \mu$, $\lim_{p\to\infty} \mu_p = \mu$ in $M_K(X)$. Hence $\lim_{p\to\infty} V\mu_p = V\mu$ in $M(X)$. By $p(V + (1/p)I)\cdot V_p = V$, we have $(V + (1/p)I)\mu_p \leqq V\nu$ in ω. Put

$$\lambda = \frac{1}{2}\left(V\nu + \left(V + \frac{1}{p}I\right)\mu_p - \left|V\nu - \left(V + \frac{1}{p}I\right)\mu_p\right|\right)$$
$$\left(= \inf\left(V\nu, \left(V + \frac{1}{p}I\right)\mu_p\right)\right).$$

Since $(V + (1/p)I)\mu_p \geqq pV_p\lambda$ and $V\nu \geqq pV_p\lambda$, we have

$$\lambda \geqq pV_p\lambda \text{ and } \lambda = p\left(V + \frac{1}{p}I\right)(\lambda - pV_p\lambda)\,. \tag{2.20}$$

Since

$$\begin{aligned}(I - pV_p)&\left(\lambda - \left(V + \frac{1}{p}I\right)\mu_p\right)\\ &= pV_p\left(\left(V + \frac{1}{p}I\right)\mu_p - \lambda\right) \leqq 0 \quad \text{in } \omega\,,\end{aligned} \tag{2.21}$$

1) We denote also by $\mathrm{supp}(\mu)$ the support of μ.

we have $\lambda \geqq (V + (1/p)I)\mu_p$ on X, i.e., $\lambda = (V + (1/p)I)\mu_p$, so that

$$\left(V + \frac{1}{p}I\right)\mu_p \leqq V\nu \text{ on } X . \tag{2.22}$$

Letting $p \to \infty$, we have $V\mu \leqq V\nu$ on X. This completes the proof.

Propositions 12, 15 and the Choquet-Deny theorem[2] implies the following

PROPOSITION 16. *Let $(T_t)_{t \geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t \geqq 0}$. For any $\mu \in \mathscr{D}^+(V)$ and any relatively compact open set ω in X, there exists one and only one $\mu'_\omega \in M_K^+(X)$ such that:*

$$\operatorname{supp}(\mu'_\omega) \subset \bar{\omega} . \tag{2.23}$$

$$V\mu'_\omega \leqq V\mu \text{ on } X . \tag{2.24}$$

$$V\mu'_\omega = V\mu \text{ in } \omega . \tag{2.25}$$

(2.26) *If $\nu \in M_K^+(X)$ satisfies $V\nu \geqq V\mu$ in ω, then $V\nu \geqq V\mu'_\omega$ on X.*

Proof. First we assume that $\mu \in M_K^+(X)$. Choose an exhaustion $(\omega_n)_{n=1}^\infty$ of ω[3]. The Choquet-Deny theorem[2] (see [4]) and Proposition 12 give that there exists $\mu'_n \in M_K^+(X)$ such that $\operatorname{supp}(\mu'_n) \subset \bar{\omega}_n$, $V\mu'_n \leqq V\mu$ on X and $V\mu'_n = V\mu$ in ω_n. By Proposition 15, $(V\mu'_n)_{n=1}^\infty$ is increasing. Since, for any compact K in X, there exists $h \in C_K^+(X)$ such that $V^*h(x) > 0$ on K, $(\mu'_n)_{n=1}^\infty$ is vaguely bounded, and hence we may assume that it converges vaguely to $\mu'_\omega \in M_K^+(X)$ as $n \to \infty$. We shall show that μ'_ω is a required measure. Evidently μ'_ω satisfies (2.23), (2.24) and (2.25), because $V\mu'_\omega = \lim_{n\to\infty} V\mu'_n$. Let $\nu \in M_K^+(X)$ satisfy $V\nu \geqq V\mu$ in ω. Then, for any $n \geqq 1$, Proposition 15 gives that $V\mu'_n \leqq V\nu$ on X, so that $V\mu'_\omega \leqq V\nu$ on X, i.e., μ'_ω is a required measure.

In general, we assume that $\mu \in \mathscr{D}^+(V)$. We can write $\mu = \sum_{n=1}^\infty \mu_n$, where $\mu_n \in M_K^+(X)$. Let $\mu'_{n,\omega}$ the non-negative Radon measure obtained above for μ_n. Then $\sum_{n=1}^\infty \mu'_{n,\omega}$ converges vaguely. Putting $\mu'_\omega = \sum_{n=1}^\infty \mu'_{n,\omega}$, we see easily that μ'_ω is a required measure.

2) This shows that V^* satisfies the domination principle if and only if, for any $\mu \in M_K^+(X)$ and any relatively compact open set ω in X, there exists $\mu' \in M_K^+(X)$ satisfying (2.23), (2.24) and (2.25) in Proposition 16.

3) For an open set ω in X, $(\omega_n)_{n=1}^\infty$ is called an exhaustion of ω if, for each $n \geqq 1$, ω_n is a relatively compact open set in ω, $\bar{\omega}_n \subset \omega_{n+1}$ $(n = 1, 2, \cdots)$ and $\bigcup_{n=1}^\infty \omega_n = \omega$.

Finally we show the unicity of μ'_ω. Let μ''_ω be another non-negative Radon measure satisfying the required four conditions. Then $V\mu'_\omega = V\mu''_\omega$. By virtue of the resolvent equation, we have, for any $p > 0$, $V_p\mu'_\omega = V_p\mu''_\omega$. By remarking that mappings $t \to T_t\mu'_\omega$ and $t \to T_t\mu''_\omega$ are vaguely continuous and that the Laplace transformation is injective, we obtain that, for any $t \geqq 0$, $T_t\mu'_\omega = T_t\mu''_\omega$, i.e., $\mu'_\omega = \mu''_\omega$. Thus the unicity of μ'_ω is shown. This completes the proof.

The above non-negative Radon measure μ'_ω is called the V-balayaged measure of μ on ω. In general, the above assertion does not hold if ω is not relatively compact. Proposition 16 gives the following

COROLLARY 17. *Let $(T_t)_{t\geqq 0}$ and V be the same as above. The mapping $V: \mathscr{D}(V) \ni \mu \to V\mu \in M(X)$ is injective.*

Proof. Assume that, for $\mu_j \in \mathscr{D}^+(V)$ $(j = 1, 2)$, $V\mu_1 = V\mu_2$. Let $(\omega_n)_{n=1}^\infty$ be an exhaustion of X. Put $\mu_{j,n} = \mu_j$ in ω_n and $\mu_{j,n} = 0$ on $C\omega_n$ $(j = 1, 2,;$ $n = 1, 2, \cdots)$. We denote by $\mu''_{j,n}$ the V-balayaged measure of $\mu_j - \mu_{j,n}$ on ω_n. Then $\mu_{j,n} + \mu''_{j,n}$ is the V-balayaged measure of μ_j on ω_n $(j = 1, 2;$ $n = 1, 2, \cdots)$. Evidently we have $V(\mu_{1,n} + \mu''_{1,n}) = V(\mu_{2,n} + \mu''_{2,n})$ for all $n \geqq 1$. In the same manner as above, we have

$$\mu_{1,n} + \mu''_{1,n} = \mu_{2,n} + \mu''_{2,n} \quad (n = 1, 2, \cdots) . \tag{2.27}$$

Since $V\mu''_{j,n} \leqq V(\mu_j - \mu_{j,n})$ and $\lim_{n\to\infty} V(\mu_j - \mu_{j,n}) = 0$, we have $\lim_{n\to\infty} V\mu''_{j,n} = 0$ (vaguely), and hence $\lim_{n\to\infty} \mu''_{j,n} = 0$ (vaguely) for $j = 1, 2$. Letting $n \to \infty$ in (2.27), we obtain that $\mu_1 = \mu_2$. This completes the proof.

By generalizing the notion of associated families (see [7]), we define the following

DEFINITION 18. Let $(T_t)_{t\geqq 0}$ be a transient continuous semi-group on X and V be the Hunt continuous kernel for $(T_t)_{t\geqq 0}$. We say that $(T_t)_{t\geqq 0}$ satisfies the condition (D) if, for any $f \in C_K^+(X)$, there exists an associated family of f with respect to $(T_t)_{t\geqq 0}$.

Here, an associated family $(f_n)_{n=1}^\infty$ of f with respect to $(T_t)_{t\geqq 0}$ is, by definition, a sequence in $\mathscr{D}^+((T_t)_{t\geqq 0}) \cap \mathscr{D}^+(V)$ satisfying the following two conditions:

$$Vf - Vf_n \in C_K^+(X) \quad (n = 1, 2, \cdots) . \tag{2.28}$$

(2.29) $(Vf_n)_{n=1}^\infty$ converges decreasingly to 0 as $n \uparrow \infty$.

By the Dini theorem, the convergence in (2.29) is that in the sense of $C(X)$.

DEFINITION 19. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X. We say that $(T_t)_{t\geqq 0}$ satisfies the condition (D^*) if $(T_t^*)_{t\geqq 0}$ satisfies the condition (D).

We denote by $\mathfrak{N}(x)$ the totality of compact neighborhoods of $x \in X$.

PROPOSITION 20. *Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. Assume that $(T_t)_{t\geqq 0}$ satisfies the condition (D^*). Then, for any $\mu \in \mathscr{D}^+(V)$ and any $x \in X$,*

$$\bigcap_{N \in \mathfrak{N}(x)} P_{CN}(V; V\mu) = \{0\}\,, \tag{2.30}$$

where $P_{CN}(V; V\mu)$ denotes the vague closure of the set

$$\{V\nu; \nu \in M_K^+(X), \operatorname{supp}(\nu) \subset CN,\ V\nu \leqq V\mu \text{ in } CN\}\,. \tag{2.31}$$

Proof. Let $N \in \mathfrak{N}(x)$ and choose an exhaustion $(\omega_n)_{n=1}^\infty$ of CN. Let μ_n' be the V-balayaged measure of μ on ω_n. Since $(V\mu_n')_{n=1}^\infty$ is increasing and $V\mu_n' \leqq V\mu$ on X $(n = 1, 2, \cdots)$,

$$\eta_{CN} = \lim_{n\to\infty} V\mu_n' \quad \text{(vaguely)} \tag{2.32}$$

exists. Proposition 15 gives that η_{CN} does not depend on the choice of $(\omega_n)_{n=1}^\infty$ and that, for any $\eta \in P_{CN}(V; V\mu)$, $\eta \leqq \eta_{CN}$ on X. Choose a sequence $(N_n)_{n=1}^\infty \subset \mathfrak{N}(x)$ such that $N_n \subset \mathring{N}_{n+1}$ and $\bigcup_{n=1}^\infty N_n = X$, where $\mathring{N}_{n+1}$ denotes the interior of N_{n+1}. Proposition 15 gives that $(\eta_{CN_n})_{n=1}^\infty$ is also decreasing. Put

$$\eta_0 = \lim_{n\to\infty} \eta_{CN_n}\,. \tag{2.33}$$

Then $\eta_0 \in \bigcap_{N\in\mathfrak{N}(x)} P_{CN}(V; V\mu)$ and, for any $\eta' \in \bigcap_{N\in\mathfrak{N}(x)} P_{CN}(V; V\mu)$, $\eta' \leqq \eta_0$ on X. Let $(\omega_{n,k})_{k=1}^\infty$ be an exhaustion of CN_n and $\mu_{n,k}'$ be the V-balayaged measure of μ on $\omega_{n,k}$ $(n = 1, 2, \cdots; k = 1, 2, \cdots)$. For any $f \in C_K^+(X)$ and any associated family $(f_m)_{m=1}^\infty$ of f with respect to $(T_t^*)_{t\geqq 0}$, we have, for any $m \geqq 1$,

$$\begin{aligned} 0 \leqq \int f d\eta_0 &= \lim_{n\to\infty} \int (f - f_m) d\eta_{CN_n} + \lim_{n\to\infty} \int f_m d\eta_{CN_n} \\ &\leqq \lim_{n\to\infty}\lim_{k\to\infty} \int (f - f_m) dV\mu_{n,k}' + \int f_m dV\mu \end{aligned} \tag{2.34}$$

$$= \lim_{n\to\infty}\lim_{k\to\infty}\int (V^*f - V^*f_m)d\mu'_{n,k} + \int V^*f_m d\mu \leqq \int V^*f_m d\mu \,.$$

Since $V^*f_m \leqq V^*f$, (2.29) gives that $\lim_{m\to\infty}\int V^*f_m d\mu = 0$, which implies that $\int f d\eta_0 = 0$. Thus $\eta_0 = 0$, and hence our required equality (2.30) holds. This completes the proof.

Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. For $\lambda \in M(X)$ and an open set ω in X, we put

$$(2.35) \qquad P_\omega(V;\lambda) = \overline{\{V\nu; \nu \in M_K^+(X),\ \mathrm{supp}(\nu) \subset \omega,\ V\nu \leqq |\lambda| \text{ in } \omega\}}\,,$$

where the closure is in the sense of vague topology.

DEFINITION 21. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq t}$. We say that $\lambda \in M(X)$ vanishes V-n.e. on the boundary of X if, for any $x \in X$,

$$(2.36) \qquad \bigcap_{N \in \mathfrak{N}(x)} P_{CN}(V;\lambda) = \{0\}$$

and if there exists $\mu \in \mathscr{D}^+(V)$ such that $|\lambda| \leqq V\mu$.

Evidently, for any $x \in X$, (2.36) holds if and only if there exists an $x \in X$ satisfying (2.36).

DEFINITION 22. A transient diffusion semi-group $(T_t)_{t\geqq 0}$ on X is said to be weakly regular if, for each $\mu \in M_K^+(X)$, $V\mu$ vanishes V-n.e. on the boundary of X, where V is the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$.

PROPOSITION 23. *Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. Then the following two statements are equivalent:*

(1) *$(T_t)_{t\geqq 0}$ is weakly regular.*

(2) *For any $\mu \in \mathscr{D}^+(V)$ and any open set ω in X, there exists one and only one V-balayaged measure μ'_ω of μ on ω*[4]. *Furthermore we have, for any $x \in X$,*

$$(2.37) \qquad \lim_{\substack{N \uparrow X \\ N \in \mathfrak{N}(x)}} V\mu'_{CN} = 0 \ (vaguely)\,.$$

4) This means also a positive Radon measure satisfying the analogous conditions to (2.23)–(2.26).

Proof. It suffices to show that (1) $\Rightarrow$ (2), because the domination principle for V implies that, for any $N\in\mathfrak{N}(x)$ and any $\eta\in P_{CN}(V;V\mu)$, $\eta\leqq V\mu'_{CN}$ on X, and (2.37) gives that $\bigcap_{N\in\mathfrak{N}(x)}P_{CN}(V;V\mu)=\{0\}$.

Let $x\in X$ and choose a suquence $(N_n)_{n=1}^{\infty}\subset\mathfrak{N}(x)$ such that $N_n\subset\mathring{N}_{n+1}$ and $\bigcup_{n=1}^{\infty}N_n=X$. Then $(\eta_{CN_n})_{n=1}^{\infty}$ is decreasing. Since $\eta_{CN_n}\in P_{CN_n}(V;V\mu)$, the weak regularity of V gives that $\lim_{n\to\infty}\eta_{CN_n}=0$ (vaguely). Similarly as in Proposition 16, it suffices to assume that $\mu\in M_K^+(X)$. Let $(\omega_n)_{n=1}^{\infty}$ be an exhaustion of ω and μ'_n be the V-balayaged measure of μ on ω_n. Then $(V\mu'_n)_{n=1}^{\infty}$ is increasing and $V\mu'_n\leqq V\mu$ on X $(n=1,2,\cdots)$. Put

$$\eta_\omega=\lim_{n\to\infty}V\mu'_n\,. \tag{2.38}$$

Then $\eta_\omega\in P_\omega(V;V\mu)$ and η_ω does not depend on the choice of $(\omega_n)_{n=1}^{\infty}$. Since $(\mu'_n)_{n=1}^{\infty}$ is vaguely bounded, we may assume that it converges vaguely to $\mu'_\omega\in M^+(X)$ as $n\to\infty$. Evidently $\eta_\omega\geqq V\mu'_\omega$ on X. We shall show the inverse inequality. Let $\varphi_k\in C_K^+(X)$ such that $0\leqq\varphi_k\leqq 1$, $\varphi_k=1$ on N_k and $\operatorname{supp}(\varphi_k)\subset\mathring{N}_{k+1}$ $(k=1,2,\cdots)$. Then, for any $n\geqq 1$, $V((1-\varphi_{k+1})\mu'_n)\in P_{CN_k}(V;V\mu)$ $(k=1,2,\cdots)$, and hence $V((1-\varphi_{k+1})\mu'_n)\leqq\eta_{CN_k}$ on X. Therefore, for any $f\in C_K^+(X)$,

$$\int fdV\mu'_\omega\geqq\int fdV(\varphi_{k+1}\mu'_\omega)=\lim_{n\to\infty}\int fdV(\varphi_{k+1}\mu'_n)\geqq\int fd\eta_\omega-\int fd\eta_{CN_k} \quad (k=1,2,\cdots)\,. \tag{2.39}$$

Letting $k\to\infty$, we obtain that $V\mu'_\omega\geqq\eta_\omega$ on X. Thus $\eta_\omega=V\mu'_\omega$. Similarly as in Proposition 16, μ'_ω is a required measure. Its unicity follows directly from Corollary 17.

Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be a Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. Put

$$R(V^*)=\{V^*f;f\in\mathscr{D}((T_t^*)_{t\geqq 0})\cap\mathscr{D}(V^*)\}\,, \tag{2.40}$$

$R^+(V^*)=R(V^*)\cap C^+(X)$, $R_K(V^*)=R(V^*)\cap C_K(X)$ and $R_K^+(V^*)=R(V^*)\cap C_K^+(X)$. Then $R_K(V^*)$ is a linear subspace of $C_K(X)$ and $R_K^+(V^*)$ is a convex cone. Put

$$\mathscr{D}^0=\left\{\mu\in M(X);\int|f|\,d|\mu|<\infty \text{ for any } V^*f\in R_K(V^*)\right\} \tag{2.41}$$

and, for each $\mu\in\mathscr{D}^0$, define the linear functional $A\mu$ on $R_K(V^*)$ by

$$A\mu(V^*f)=-\int fd\mu \text{ for any } V^*f\in R_K(V^*)\,. \tag{2.42}$$

Precisely we write $\mathscr{D}^0(A) = \mathscr{D}^0$. Then we have easily the following

Remark 24. Let $(T_t)_{t\geqq 0}$ and V be the same as above. Assume that $R_K^+(V^*)$ is total in $C_K(X)$[5]. Then, for $\mu \in \mathscr{D}^0$, a continuous extension of $A\mu$ to $C_K(X)$ is uniquely determined if it exists. Furthermore if, for $\mu \in \mathscr{D}^0$, $-A\mu$ is non-negative, i.e., $-A\mu(g) \geqq 0$ if $g \in R_K^+(V^*)$, then a positive linear extension of $-A\mu$ to $C_K(X)$ exists.

DEFINITION 25. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. If $R_K^+(V^*)$ is total in $C_K(X)$, then $(T_t)_{t\geqq 0}$ is said to satisfy the condition (C^*).

For a transient diffusion semi-group on X satisfying the condition (C^*), we denote by $\mathscr{D}(A)$ the set of all $\mu \in \mathscr{D}^0(A)$ such that a continuous linear extension to $C_K(X)$ exists. For $\mu \in \mathscr{D}(A)$, we can write again $A\mu$ its continuous linear extension to $C_K(X)$ without confusion (see Remark 24). Evidently $\mathscr{D}(A)$ is a linear subspace of $M(X)$ and the linear operator $A: \mathscr{D}(A) \ni \mu \rightarrow A\mu \in M(X)$ is defined.

DEFINITION 26. The above linear operator A is called the infinitesimal generator of $(T_t)_{t\geqq 0}$.

DEFINITION 27. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X. If $(T_t)_{t\geqq 0}$ satisfies the conditions (D^*) and (C^*), it is said to be regular.

If a transient diffusion semi-group $(T_t)_{t\geqq 0}$ is of convolution type, it is always regular (see, for example, [7] and [8]).

Remark 28. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and $(V_p)_{p\geqq 0}$ be the resolvent for $(T_t)_{t\geqq 0}$. Let $p > 0$ and put

$$T_{p,t} = \exp(-pt)\Big(I + \sum_{n=1}^{\infty} \frac{(pt)^n}{n!}(pV_p)^n\Big) \ (t > 0) \text{ and } T_{p,0} = I\,. \tag{2.43}$$

Then $(T_{p,t})_{t\geqq 0}$ is a transient diffusion semi-group on X and $V + (1/p)I = \int_0^\infty T_{p,t}dt$, where $V_0 = V$. Furthermore, if $(T_t)_{t\geqq 0}$ is regular (resp. weakly regular), then so is $(T_{p,t})_{t\geqq 0}$ for any $p > 0$.

In fact, (2.13) gives directly the first part. Assume that $(T_t)_{t\geqq 0}$ is regular. Since $p(V^* + (1/p)I)\cdot(I - pV_p^*) = I$, $C_K(X) = R_K(V^* + (1/p)I)$, and hence $(T_{p,t})_{t\geqq 0}$ satisfies the condition (C^*). Let $f \in C_K^+(X)$ and $(f_n)_{n=1}^{\infty}$ be an

5) This means that $R_K^+(V^*) \subset C_K(X)$ and, for any $x \in X$ and any neighborhood U of x, there exists an $f \neq 0 \in R_K^+(V^*)$ such that $\operatorname{supp}(f) \subset U$.

associated family of f with respect to $(T_t^*)_{t\geqq 0}$. Then $pV_p^*f_n \in \mathscr{D}((T_{p,t}^*)_{t\geqq 0}) \cap \mathscr{D}(V^* + (1/p)I)$ and $(V^* + (1/p)I)(pV_p^*f_n) = V^*f_n$. Thus we see that $(pV_p^*f_n)_{n=1}^{\infty}$ is an associated family of f with respect to $(T_{p,t}^*)_{t\geqq 0}$. Hence $(T_{p,t})_{t\geqq 0}$ is regular for any $p > 0$. Next we assume that V is weakly regular. Let $p > 0$ be fixed and $\mu \in M_K^+(X)$. For any $x \in X$ and any $N \in \mathfrak{N}(x)$ with $\mathring{N} \supset \operatorname{supp}(\mu)$, we have, in the same manner as in Proposition 15,

$$\left(V + \frac{1}{p}I\right)\nu \leqq V\mu'_{CN} \text{ on } X \tag{2.44}$$

whenever $(V + (1/p)I)\nu \in P_{CN}(V + (1/p)I; (V + (1/p)I)\mu)$, where μ'_{CN} is the V-balayaged measure of μ on CN. By Proposition 23 and (2.44), $V + (1/p)I$ is weakly regular.

Remark 29. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X satisfying the condition (C^*), V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$ and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. Then, for any $\mu \in \mathscr{D}(V)$, $V\mu \in \mathscr{D}(A)$ and $A(V\mu) = -\mu$.

In fact, we may assume that μ is non-negative. For any $V^*f \in R_K^+(V^*)$,

$$\lim_{t\to 0} \frac{1}{t}(I - T_t^*)(V^*f) = \lim_{t\to 0} \frac{1}{t}\int_0^t T_s^* f ds = f \text{ (pointwise)}. \tag{2.45}$$

Since $\operatorname{supp}(f^+) \subset \operatorname{supp}(V^*f)$,

$$\int |f|\, dV\mu \leqq 2\int f^+ dV\mu < \infty\ , \tag{2.47}$$

which gives that $V\mu \in \mathscr{D}^0(A)$, because, for any $V^*f \in R_K(V^*)$, there exists $V^*g \in R_K^+(V^*)$ such that $V^*g \geqq |V^*f|$. Since, for any $V^*f \in R_K(V^*)$, $\int V^*f d\mu = \int f dV\mu$, our assertion holds.

§ 3. The Riesz decomposition theorem

We begin by the following two lemmas:

LEMMA 30. *Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. For a given positive Radon measure μ in X, there exists $h \in \mathscr{D}^+((T_t^*)_{t\geqq 0}) \cap \mathscr{D}^+(V^*)$ such that $V^*h(x) > 0$ on X and $\int h d\mu < \infty$.*

Proof. Let $(\omega_n)_{n=1}^{\infty}$ be an exhaustion of X. Then, for any n, there exists $h_n \in C_K^+(X)$ such that $V^*h_n > 0$ in ω_n. We choose also $g_n \in C_K^+(X)$ satisfying $V^*g_n \geqq h_n$ on X. Since, for any $t > 0$,

$$0 \leqq T_t^*h_n \leqq T_t^*(V^*g_n) = \int_t^\infty T_s^*g_n ds \leqq V^*g_n \text{ on } X, \tag{3.1}$$

there exists a constant $c_n > 0$ such that

$$c_n V^*h_n \leqq \frac{1}{2^n}, c_n T_t^*h_n \leqq \frac{1}{2^n} \text{ on } \bar{\omega}_n \ (0 \leqq t < \infty) \quad \text{and } c_n \int h_n d\mu < \frac{1}{2^n}. \tag{3.2}$$

Then $h = \sum_{n=1}^\infty c_n h_n$ is a required function.

LEMMA 31. *Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X satisfying the condition (D^*) and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. For any $f \in \mathscr{D}^+((T_t^*)_{t\geqq 0}) \cap \mathscr{D}^+(V^*)$, there exists also an associated family of f with respect to $(T_t^*)_{t\geqq 0}$.*

Proof. Choose a sequence $(f_n)_{n=1}^\infty \subset C_K^+(X)$ such that $f = \sum_{n=1}^\infty f_n$ and an exhaustion $(\omega_n)_{n=1}^\infty$ of X. Let $(f_{n,m})_{m=1}^\infty$ be an associated family of f_n with respect to $(T_t^*)_{t\geqq 0}$. We may assume that, for any $m \geqq 1$ and any k with $1 \leqq k \leqq m$, $V^*f_{k,m} \leqq 1/m^2$ on $\bar{\omega}_m$. Put

$$g_n = \sum_{k=1}^n f_{k,n} + \sum_{k=n+1}^\infty f_k \ (n = 1, 2, \cdots), \tag{3.3}$$

then $g_n \in \mathscr{D}((T_t^*)_{t\geqq 0}) \cap \mathscr{D}(V^*)$. We see easily that $(g_n)_{n=1}^\infty$ is a required associated family of f with respect to $(T_t^*)_{t\geqq 0}$.

DEFINITION 32. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X satisfying the condition (C^*) and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. A real Radon measure μ in X is said to be A-superharmonic (resp. A-harmonic) if $\mu \in \mathscr{D}(A)$ and $-A\mu \in M^+(X)$ (resp. $A\mu = 0$).

Clearly this is equivalent to $\mu \in \mathscr{D}^0(A)$ and $\int f d\mu \geqq 0$ $\left(\text{resp. } \int f d\mu = 0\right)$ for all $V^*f \in R_K^+(V^*)$, because $R_K^+(V^*)$ is total in $C_K(X)$ and forms a convex cone.

DEFINITION 33. Let $(T_t)_{t\geqq 0}$ be a diffusion semi-group on X. A real Radon measure μ in X is said to be excessive (resp. invariant) with respect to $(T_t)_{t\geqq 0}$ if, for any $t \geqq 0$, $\mu \in \mathscr{D}(T_t)$ and $\mu \geqq T_t\mu$ (resp. $\mu = T_t\mu$).

Remark 34. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group satisfying the condition (C^*) and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. If $\mu\in M^+(X)$ is excessive with respect to $(T_t)_{t\geqq 0}$, then μ is A-superharmonic.

In fact, for $g = V^*f\in R_K^+(V^*)$ and $t>0$, we put $f_t^+ = 1/t(g - T_t^*g)^+$ and $f_t^- = 1/t(g - T_t^*g)^-$. Then $\operatorname{supp}(f_t^+)\subset \operatorname{supp}(g)$ for all $t>0$, and hence the Lebesgue theorem gives that $\lim_{t\to 0}\int f_t^+ d\mu = \int f^+ d\mu$. By the Fatou lemma and $\lim_{t\to 0} f_t^-(x) = f^-(x)$ for all $x\in X$,

$$
\begin{aligned}
0 &\leqq \varliminf_{t\to 0}\frac{1}{t}\int gd(I - T_t)\mu = \varliminf_{t\to 0}\frac{1}{t}\int (I - T_t^*)gd\mu \\
&= \varliminf_{t\to 0}\int (f_t^+ - f_t^-)d\mu \leqq \int f^+d\mu - \int f^-d\mu = \int fd\mu\,,
\end{aligned}
\tag{3.4}
$$

which implies that μ is A-superharmonic.

The main theorem of this paragraph is the following Riesz decomposition theorem.

THEOREM 35. *Let $(T_t)_{t\geqq 0}$ be a transient and regular diffusion semi-group on X, V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$ and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. Then every non-negative A-superharmonic measure μ in X can be written uniquely as*

$$
\mu = V\nu + \mu_h
\tag{3.5}
$$

where $\nu\in\mathscr{D}^+(V)$ and μ_h is a non-negative A-harmonic measure in X. Furthermore $\nu = -A\mu$.

First we prepare the following two lemmas.

LEMMA 36. *Let $(T_t)_{t\geqq 0}$, V and A be the same as above, and let μ be a positive A-superharmonic measure. Then, for any $f\in\mathscr{D}^+((T_t^*)_{t\geqq 0})\cap\mathscr{D}^+(V^*)$ with $\int fd\mu<\infty$ and an associated family $(f_n)_{n=1}^\infty$ of f with respect to $(T_t^*)_{t\geqq 0}$, $\left(\int f_nd\mu\right)_{n=1}^\infty$ is decreasing, $\int f_nd\mu\leqq\int fd\mu$ $(n = 1, 2, \cdots)$ and $\lim_{n\to\infty}\int f_nd\mu$ does not depend on the choice of $(f_n)_{n=1}^\infty$.*

Proof. Since, for any $n\geqq 1$, $V^*(f - f_n)\in R_K^+(V^*)$, $\int f_nd\mu\leqq\int fd\mu$ and $\left(\int f_nd\mu\right)_{n=1}^\infty$ is decreasing. Let $(g_n)_{n=1}^\infty$ be another associated family of f with respect to $(T_t^*)_{t\geqq 0}$. We choose $h\in\mathscr{D}^+((T_t^*)_{t\geqq 0})\cap\mathscr{D}^+(V^*)$ satisfying $V^*h>0$

on X and $\int hd\mu < \infty$ (see Lemma 30) and an associated family $(h_n)_{n=1}^{\infty}$ of h with respect to $(T_t^*)_{t\geqq 0}$. For any integer $m \geqq 1$ and any positive number δ, there exists an integer $n_0 \geqq 1$ such that, for all $n \geqq n_0$,

$$\delta V^*(h - h_n) + V^*f - V^*f_n \geqq V^*f - V^*g_m \text{ on } X , \tag{3.6}$$

which implies that

$$\int (\delta(h - h_n) + g_m - f_n)d\mu \geqq 0 . \tag{3.7}$$

Letting $n \to \infty$ and next $\delta \to 0$, $m \to \infty$, we obtain that

$$\lim_{m\to\infty} \int g_m d\mu \geqq \lim_{n\to\infty} \int f_n d\mu . \tag{3.8}$$

In the same manner, we see the inverse inequality. Thus $\lim_{n\to\infty} \int f_n d\mu$ does not depend on $(f_n)_{n=1}^{\infty}$, and hence the proof is achieved.

LEMMA 37. *Let $(T_t)_{t\geqq 0}$, V, A and μ be the same as above. Assume that, for any $f \in \mathscr{D}^+(V^*)$ with $\int fd\mu < \infty$ and any associated family $(f_n)_{n=1}^{\infty}$ of f with respect to $(T_t^*)_{t\geqq 0}$, $\lim_{n\to\infty} \int f_n d\mu = 0$. Then, for any $V^*g \in R^+(V^*)$, $\int gd\mu \geqq 0$ whenever $\int g^+ d\mu < \infty$.*

Proof. It suffices to show that for any $f \in C_K^+(X)$ with $f \leqq g^-$, $\int g^+ d\mu \gtreqq \int fd\mu$. Let $(g_n)_{n=1}^{\infty}$ and $(f_n)_{n=1}^{\infty}$ be an associated family of g^+ with respect to $(T_t^*)_{t\geqq 0}$ and that of f with respect to $(T_t^*)_{t\geqq 0}$, respectively. Let h and $(h_n)_{n=1}^{\infty}$ be the same as in the above proof. Similarly as in Lemma 36, for any integer $n \geqq 1$ and any number $\delta > 0$, there exists an integer $m_0 \gtreqq 1$ such that, for all $m \geqq m_0$,

$$\delta(V^*h - V^*h_m) + V^*g^+ - V^*g_m \geqq V^*f - V^*f_n \text{ on } X , \tag{3.9}$$

and hence

$$\int \delta(h - h_m)d\mu + \int (g^+ - g_m + f_n - f)d\mu \geqq 0 . \tag{3.10}$$

Letting $m \to \infty$ and next $\delta \to 0$, $n \to \infty$, we obtain that $\int g^+ d\mu \geqq \int fd\mu$.

Thus Lemma 37 is shown.

Proof of Theorem 35. By Lemma 36, there exists one and only one $\mu_h \in M^+(X)$ such that, for any $f \in C_K^+(X)$,

$$\int f d\mu_h = \lim_{n\to\infty} \int f_n d\mu , \tag{3.11}$$

where $(f_n)_{n=1}^{\infty}$ is an associated family of f with respect to $(T_t^*)_{t\geqq 0}$. Put $\mu_p = \mu - \mu_h$. Then we shall show the following two statements:

(a) μ_h is A-harmonic.

(b) There exists $\nu \in \mathscr{D}^+(V)$ such that $\mu_p = V\nu$.

We begin by the proof of (a). Let $V^*f \in R_K^+(V^*)$. Then $|f| \in \mathscr{D}((T_t^*)_{t\geqq 0}) \cap \mathscr{D}(V^*)$ and $\mathrm{supp}(f^+)$ is compact (see the proof of Remark 34). Let $(f_n)_{n=1}^{\infty}$ be an associated family of f^- with respect to $(T_t^*)_{t\geqq 0}$. Then it is also an associated family of f^+ with respect to $(T_t^*)_{t\geqq 0}$. Hence (a) follows from the equality

$$\int g d\mu_h = \lim_{n\to\infty} \int g_n d\mu \tag{3.12}$$

for any $g \in \mathscr{D}^+((T_t^*)_{t\geqq 0}) \cap \mathscr{D}^+(V^*)$ with $\int g d\mu < \infty$, where $(g_n)_{n=1}^{\infty}$ is an associated family of g with respect to $(T_t^*)_{t\geqq 0}$. We remark that $\int g d\mu_h \leqq \int g d\mu$, because, for any $g' \in C_K^+(X)$ with $g' \leqq g$, $\int g' d\mu_h \leqq \int g' d\mu \leqq \int g d\mu$. Let h and $(h_n)_{n=1}^{\infty}$ be the same as in the proof of Lemma 36, and let $(f_n)_{n=1}^{\infty}$ be an increasing sequence $\subset C_K^+(X)$ with $\lim_{n\to\infty} f_n = g$ in $C(X)$. Then $(V^*f_n)_{n=1}^{\infty}$ converges increasingly to V^*g as $n \uparrow \infty$, i.e., $\lim_{n\to\infty} V^*f_n = V^*g$ in $C(X)$. For any integer $n \geqq 1$ and any number $\delta > 0$, there exists an integer $m_0 \geqq 1$ such that, for all $m \geqq m_0$,

$$\delta V^*h + V^*f_m > V^*g - V^*g_n \text{ on } X . \tag{3.13}$$

Let $(f_{n,k})_{k=1}^{\infty}$ be an associated family of f_n with respect to $(T_t^*)_{t\geqq 0}$. By (3.13), for any $m \geqq m_0$, there exists $k_m \geqq 1$ such that, for all $k \geqq k_m$,

$$\delta V^*(h - h_k) + V^*(f_m - f_{m,k}) \geqq V^*g - V^*g_n \text{ on } X . \tag{3.14}$$

This implies that

$$\delta \int (h - h_k) d\mu + \int (f_m - f_{m,k}) d\mu \geqq \int (g - g_n) d\mu . \tag{3.15}$$

Letting $k \to \infty$, $m \to \infty$, $\delta \to 0$ and $n \to \infty$, we obtain that

$$\int g d\mu_h \leqq \lim_{n\to\infty} \int g_n d\mu . \tag{3.16}$$

On the other hand, for any integer $n \geqq 1$, $k \geqq 1$ and any positive number $\delta > 0$, there exists an integer $m_0 \geqq 1$ such that, for all $m \geqq m_0$,

$$\delta(V^*h - V^*h_m) + V^*(g - g_m) \geqq V^*(f_n - f_{n,k}) \text{ on } X . \tag{3.17}$$

This gives that the inverse inequality of (3.16) holds, i.e., (3.12) holds. Consequently (a) is shown.

Next we shall show (b). By (a) and (3.12), μ_p is a positive A-superharmonic measure and the assumption in Lemma 37 is satisfied. For any $f \in C_K^+(X)$ and any $t > 0$, $V^*(I - T_t^*)f = \int_0^t T_s^* f ds \in R^+(V^*)$ and $\int ((I - T_t^*)f)^+ d\mu < \infty$. Hence Lemma 37 gives that

$$0 \leqq \int (I - T_t^*) f d\mu_p = \int f d(I - T_t)\mu_p , \tag{3.18}$$

and hence, $(I - T_t)\mu_p \in M^+(X)$ for any $t > 0$. For any $f \in C_K^+(X)$, we choose $g \in C_K^+(X)$ such that $f \leqq V^*g$ on X. Since, for any $t > 0$,

$$\begin{aligned} \frac{1}{t}\int f d(I - T_t)\mu_p &\leqq \frac{1}{t}\int V^*g d(I - T_t)\mu_p \\ &= \frac{1}{t}\iint_0^t T_s^* g ds d\mu_p \leqq \int g d\mu_p , \end{aligned} \tag{3.19}$$

$(1/t(I - T_t)\mu_p)_{t>0}$ is vaguely bounded. Let $\nu \in M^+(X)$ be its vaguely cluster point as $t \to 0$ and choose a sequence $(t_n)_{n=1}^\infty$ of positive numbers such that $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} 1/t_n(I - T_{t_n})\mu_p = \nu$ (vaguely). By remarking (3.19) and $\lim_{t\to 0} T_t = I$, we have $\nu \in \mathscr{D}^+(V)$ and $\mu_p \geqq V\nu$. On the other hand, let $f \in C_K^+(X)$ and $(f_n)_{n=1}^\infty$ be its associated family with respect to $(T_t^*)_{t\geqq 0}$. Then, for any $k \geqq 1$,

$$\begin{aligned} \int f dV\nu &= \int V^* f d\nu \geqq \int V^*(f - f_k) d\nu \\ &= \lim_{n\to\infty} \int V^*(f - f_k) d\Big(\frac{1}{t_n}(I - T_{t_n})\mu_p\Big) \\ &= \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \Big(\int (f - f_k) dT_s\mu_p\Big) ds \geqq \int f d\mu_p - \int f_k d\mu_p , \end{aligned} \tag{3.20}$$

because the vague boundedness of $(1/t(I - T_t)\mu_p)_{t>0}$ leads to $\lim_{t\to 0} T_t\mu_p = \mu_p$

(vaguely). Letting $k \to \infty$ in (3.20), we obtain that $\int f dV\nu \geqq \int f d\mu_p$, i.e., $V\nu \geqq \mu_p$. Thus we have $\mu_p = V\nu$. We have also $\lim_{t\to 0} 1/t(I - T_t)\mu_p = \nu$ (vaguely), by the injectivity of V. Consequently we have $\mu = V\nu + \mu_h$. Let $\mu = V\nu' + \mu'_h$ be another decomposition satisfying our required conditions. Then Remark 29 implies that $-A\mu = \nu = \nu'$, and so $\mu_h = \mu'_h$. Thus we see the unicity of the decomposition of μ and $\nu = -A\mu$. This completes the proof.

DEFINITION 38. The above $V\nu$ and μ_h are called the potential part of μ and the harmonic part of μ, respectively. The decomposition of μ in Theorem 35 is called the Riesz decomposition of μ.

Theorem 35 gives directly the following

COROLLARY 39. *Let $(T_t)_{t\geqq 0}$, V and A be the same as in Theorem 35. Then we have;*

(1) *If $\mu \in M^+(X)$ is invariant with respect to $(T_t)_{t\geqq 0}$, then μ is A-harmonic.*

(2) *Let $\mu \in M^+(X)$ be A-superharmonic. The harmonic part of μ is the greatest A-harmonic minorant of μ.*

Evidently (1) holds. Let $\nu \in M^+(X)$ be an A-harmonic measure satisfying $\mu \geqq \nu$. Applying Theorem 35 to $\mu - \nu$, we see that $\mu_h \geqq \nu$, where μ_h is the harmonic part of μ.

Now we consider A^*-superharmonic functions and A^*-harmonic functions.

DEFINITION 40. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X satisfying the condition (C^*), V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$ and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. Let Ω be an open set in X. A real-valued Borel function u in X is said to be A^*-superharmonic (resp. A^*-harmonic) in Ω if $\int |u|\, d|A\mu| < \infty$ and $-\int u dA\mu \geqq 0$ $\left(\text{resp. } \int u dA\mu = 0\right)$ for any $\mu \in \mathscr{D}_K^+(A; \Omega)$, where

$$(3.21) \qquad \mathscr{D}_K^+(A; \Omega) = \{V\mu \in M_K^+(X); \mu \in \mathscr{D}(V) \text{ and } \operatorname{supp}(V\mu) \subset \Omega\}\,.$$

LEMMA 41. *Let $(T_t)_{t\geqq 0}$ be a transient and weakly regular diffusion semi-group on X and V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. Let $\mu \in \mathscr{D}^+(V)$ and F be a closed set in X. For an exhaustion $(\omega_n)_{n=1}^\infty$ of CF, we denote by μ'_n the V-balayaged measure of μ on $C\overline{\omega}_n$. Then $(\mu'_n)_{n=1}^\infty$ converges vaguely* *and its limit does not depend on the choice of* $(\omega_n)_{n=1}^{\infty}$.

Proof. Evidently $(V\mu_n')_{n=1}^{\infty}$ is decreasing and $V\mu_n' \leqq V\mu$. This implies also that $(\mu_n')_{n=1}^{\infty}$ is vaguely bounded. Let μ_F' be its vaguely cluster point as $n \to \infty$. Similarly as in Proposition 23, we have

$$V\mu_F' = \lim_{n\to\infty} V\mu_n' \quad \text{(vaguely).} \tag{3.22}$$

By Corollary 17, $(\mu_n')_{n=1}^{\infty}$ converges vaguely to μ_F' as $n \to \infty$. Let $(\omega_n')_{n=1}^{\infty}$ be another exhaustion of CF and μ_n'' be the V-balayaged measure of μ on $C\bar{\omega}_n'$. Then it is easily seen that $\lim_{n\to\infty} V\mu_n' = \lim_{n\to\infty} V\mu_n''$. By using Corollary 17 again, we have $\mu_F' = \lim_{n\to\infty} \mu_n''$. Thus Lemma 41 is shown.

The above measure μ_F' is also called the V-balayaged measure of μ on F.

PROPOSITION 42. *Let* $(T_t)_{t\geqq 0}$, V *and* A *be the same as in Definition* 40, *and let* Ω *be an open set in* X. *Assume that* $(T_t)_{t\geqq 0}$ *is weaklyr egular. For* $f \in C_K(X)$, *we put*

$$u_f(x) = \int f d\varepsilon_{x,C\Omega}' \ \text{in } X\,, \tag{3.23}$$

where $\varepsilon_{x,C\Omega}'$ *is the V-balayaged measure of* ε_x *on* $C\Omega$. *Then* u_f *is* A^**-harmonic in* Ω.

Proof. First we shall show that u_f is Borel measurable in X. By Lemma 41, it is sufficient to show that, for any open set ω, the function $\int f d\varepsilon_{x,\omega}'$ of x is Borel measurable, where $\varepsilon_{x,\omega}'$ is the V-balayaged measure of ε_x on ω. Let $V^*g \in R_K(V^*)$. Then $\int |g|\, d\varepsilon_{x,\omega}' < \infty$ and $\int V^*g d\varepsilon_{x,\omega}' = \int g dV\varepsilon_{x,\omega}'$. Since $R_K(V^*)$ is dence in $C_K(X)$, it suffices to show that, for any $g \in C_K^+(X)$, the function $\int g dV\varepsilon_{x,\omega}'$ of x is Borel measurable. Let $x \in X$ and $(x_n)_{k=1}^{\infty}$ be a sequence $\subset X$ with $\lim_{n\to\infty} x_n = x$. We choose a subsequence $(x_{n(k)})_{k=1}^{\infty}$ such that $\varepsilon_{x_{n(k)},\omega}'$ converges vaguely and

$$\varliminf_{n\to\infty} \int g dV\varepsilon_{x_n,\omega}' = \lim_{k\to\infty} \int g dV\varepsilon_{x_{n(k)},\omega}'\ . \tag{3.24}$$

Put $\nu = \lim_{k\to\infty} \varepsilon_{x_{n(k)},\omega}'$. Then $\text{supp}(\nu) \subset \bar{\omega}$ and, similarly as in Proposition 23, we have

(3.25) $$V\nu = \lim_{k\to\infty} V\varepsilon'_{x_{n(k)},\omega} \quad \text{(vaguely)}$$

i.e., $V\nu = V\varepsilon_x$ in ω. By the definition of V-balayaged measures, we have $V\nu \geqq V\varepsilon'_{x,\omega}$, which implies that the function $\int g dV\varepsilon'_{x,\omega}$ of x is lower semi-continuous in X. Thus we see that u_f is Borel measurable in X. Let $V\mu \in \mathfrak{D}_K^+(A;\Omega)$. Choose $h \in C_K^+(X)$ such that $V^*h(x) > 0$ on $\mathrm{supp}(f)$ and that $\int h dV|\mu| < \infty$ (see Lemma 30). Since $R_K(V^*)$ is dense in $C_K(X)$, there exists a sequence $(V^*g_n)_{n=1}^\infty \subset R_K(V^*)$ such that $|f(x) - V^*g_n(x)| \leqq (1/n)V^*h(x)$ on X. Then we have

(3.26) $$\begin{aligned} \left|\int (u_f(x) - u_{V^*g_n}(x))d\mu(x)\right| &\leqq \frac{1}{n}\int u_{V^*h}(x)d|\mu|(x) \\ &\leqq \frac{1}{n}\int V^*h(x)d|\mu|(x)\,, \end{aligned}$$

where $u_{V^*g_n}$ and u_{V^*h} are defined analogously to u_f. Consequently, it suffices to show that, for any $V^*g \in R_K(V^*)$,

(3.27) $$\int u_{V^*g}d\mu = 0\,.$$

By remarking the first part of this proof, we have

(3.28) $$\begin{aligned} \int u_{V^*g}(x)d\mu(x) &= \iint V^*g(y)d\varepsilon'_{x,C\Omega}(y)d\mu(x) \\ &= \int V^*g(y)d\Big(\int \varepsilon'_{x,C\Omega}d\mu(x)\Big)(y) = \int g(y)dV\Big(\int \varepsilon'_{x,C\Omega}d\mu(x)\Big)(y)\,. \end{aligned}$$

Let $(\omega_n)_{n=1}^\infty$ be an exhaustion of Ω, and put $\mu_1 = \mu^+$, $\mu_2 = \mu^-$. We denote by $\mu'_{j,n}$ the V-balayaged measure of μ_j on $C\bar{\omega}_n$ $(j = 1, 2)$. Then, by virtue of the domination principle for V and by Proposition 16,

(3.29) $$V\mu'_{j,n+1} \leqq V\Big(\int \varepsilon'_{x,C\bar{\omega}_n}d\mu_j(x)\Big) \leqq V\mu'_{j,n-1} \ (j = 1, 2;\ n = 2, 3, \cdots)\,,$$

where $\varepsilon'_{x,C\bar{\omega}_n}$ is the V-balayaged measure of ε_x on $C\bar{\omega}_n$. This shows that $\int \varepsilon'_{x,C\Omega}d\mu_j(x)$ is the V-balayaged measure of μ_j on $C\Omega$ $(j = 1, 2)$. Since $V\mu_1 = V\mu_2$ in a certain neighborhood of $C\Omega$, we have

(3.30) $$\int \varepsilon'_{x,C\Omega}d\mu_1(x) = \int \varepsilon'_{x,C\Omega}d\mu_2(x)\,,$$

which implies (3.27). This completes the proof.

This implies the following

COROLLARY 43. *Let $(T_t)_{t\geqq 0}$, V and A be the same as above, Ω be an open set in X, and let $g\in C^+(X)$ and $f\in C_K^+(X)$ with* $\operatorname{supp}(f)\subset\Omega$. *Assume that there exists $\varphi\in\mathcal{D}^+(V^*)$ such that $V^*\varphi\geqq g$ on X. If g is A^*-superharmonic in Ω and if $f=-A^*g$, i.e., for any $V\mu\in\mathcal{D}_K^+(A;\Omega)$, $\int gd\mu=\int fdV\mu$, then*

$$g(x)=\int fd(V\varepsilon_x-V\varepsilon'_{x,C\Omega})+h(x) \tag{3.31}$$

on X, where $\varepsilon'_{x,C\Omega}$ is the same as above and h is an A^*-harmonic function in Ω. In this case,

$$h(x)=\int g(y)d\varepsilon'_{x,C\Omega}(y) \text{ on } X. \tag{3.32}$$

Proof. Let $(\omega_n)_{n=1}^\infty$ be an exhaustion of Ω and $\varepsilon'_{x,C\bar\omega_n}$ be the same as above. Then, for any $x\in X$ and any $n\geqq 1$, $V\varepsilon_x-V\varepsilon'_{x,C\bar\omega_n}\in\mathcal{D}_K^+(A;\Omega)$. This implies that $g(x)\geqq\int g(y)d\varepsilon'_{x,C\Omega}(y)$ on X. Let h be the function defined in (3.32). By Proposition 42, h is A^*-harmonic in Ω. By our assumption, for any $x\in X$ and any $n\geqq 1$,

$$g(x)-\int g(y)d\varepsilon'_{x,C\bar\omega_n}(y)=\int fd(V\varepsilon_x-V\varepsilon'_{x,C\bar\omega_n})\,. \tag{3.33}$$

Since $\lim_{n\to\infty}\varepsilon'_{x,C\bar\omega_n}=\varepsilon'_{x,C\Omega}$ (vaguely), we have

$$\begin{aligned}&\varliminf_{n\to\infty}\int gd\varepsilon'_{x,C\bar\omega_n}\geqq\int gd\varepsilon'_{x,C\Omega}\text{ and}\\ &\varliminf_{n\to\infty}\int(V^*\varphi-g)d\varepsilon'_{x,C\bar\omega_n}\geqq\int(V^*\varphi-g)d\varepsilon'_{x,C\Omega}\,.\end{aligned} \tag{3.34}$$

Remarking that $(V\varepsilon'_{x,C\bar\omega_n})_{n=1}^\infty$ converges decreasingly to $V\varepsilon'_{x,C\Omega}$ as $n\uparrow\infty$, we have

$$\lim_{n\to\infty}\int V^*\varphi d\varepsilon'_{x,C\bar\omega_n}=\int V^*\varphi d\varepsilon'_{x,C\Omega}\,. \tag{3.35}$$

By combining (3.33), (3.34) and (3.35), we see the required equality.

§ 4. Positive eigen elements for A and completely A-superharmonic measures

We begin by the following

DEFINITION 44. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X satisfying the condition (C^*), V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$ and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$.

(1) Given a non-negative number c, the set of all non-negative solutions of the equation

$$-A\mu = c\mu \tag{4.1}$$

is denoted by $E(A; c)$ and called the eigen cone of c. Put $E(A) = \bigcup_{c\geqq 0} E(A; c)$. We call $\mu \in E(A)$ a non-negative eigen element of A.

(2) Given a non-negative number c, the set of all non-negative solutions of the equations

$$\begin{cases} -A\mu = c\mu \\ \quad \mu = 0 \ \ V\text{-n.e. on the boundary of } X \end{cases} \tag{4.2}$$

is denoted by $E_0(A; c)$ and called the eigen cone of c with zero conditions. Put $E_0(A) = \bigcup_{c\geqq 0} E_0(A; c)$. We call $\mu \in E_0(A)$ a non-negative eigen element of A with zero conditions.

Now we denote by $H(A)$ the set of all non-negative A-harmonic measures in X.

PROPOSITION 45. *Let $(T_t)_{t\geqq 0}$, V, A, $E(A; c)$ and $E_0(A; c)$ be the same as above. Furthermore we assume that $(T_t)_{t\geqq 0}$ is regular. Then, $\mu \in E_0(A; c)$ if and only if*

$$\mu = cV\mu\,, \tag{4.3}$$

and we have

$$E(A; c) = E_0(A; c) \oplus H(A)\,, \tag{4.4}$$

where $\oplus$ denotes the direct sum.

In fact, Remark 29, Theorem 35 and Corollary 39 give the first equivalence, and (4.3) and Theorem 35 give (4.4).

DEFINITION 46. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X satisfying the condition (C^*) and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. A Radon measure $\mu \in M^+(X)$ is called a completely A-superharmonic if, for all $n = 0, 1, 2, \cdots$, $(-A)^n\mu \in \mathscr{D}(A)$ and $(-A)^{n+1}\mu \in M^+(X)$, where $(-A)^0 = I$, $(-A)^1 = -A$ and $(-A)^{n+1}\mu = -A((-A)^n\mu)$. In particular, a completely A-superharmonic measure μ is said to be with zero conditions if, for all $n = 0, 1, \cdots$, $(-A)^n\mu$ vanishes V-n.e. on the boundary of X, where V is the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$.

We denote by $SC(A)$ the set of all completely A-superharmonic measures in X and by $SC_0(A)$ the set of all completely A-superharmonic measures in X with zero conditions.

Evidently $SC(A)$ and $SC_0(A)$ are convex cones in $M^+(X)$, and $SC(A) \supset E(A)$ and $SC_0(A) \supset E_0(A)$.

PROPOSITION 47. *Let $(T_t)_{t\geqq 0}$ be a transient and regular diffusion semigroup on X, V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$ and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. Assume that, for all $n = 1, 2, \cdots$, V^n is defined as a diffusion kernel on X. Then, for any $\mu \in SC(A)$, we have the following unique representation*:

$$\mu = \sum_{n=0}^{\infty} V^n\mu_n + \mu_\infty \,, \tag{4.5}$$

where $\mu_n \in H(A)$ $(n = 0, 1, \cdots)$ and $\mu_\infty \in SC_0(A)$.

Proof. By Theorem 35, we have inductively, for any $k \geqq 0$ and any $n \geqq k$,

$$(-A)^k\mu = \mu_k + V\mu_{k+1} + \cdots + V^{n-k-1}\mu_{n-1} + V^{n-k}((-A)^n\mu) \,, \tag{4.6}$$

where $\mu_k, \cdots, \mu_{n-1} \in H(A)$. This implies that $(V^{n-k}((-A)^n\mu))_{n=k+1}^{\infty}$ is decreasing. Put

$$\mu_{\infty,k} = \lim_{n\to\infty} V^{n-k}((-A)^n\mu) \,. \tag{4.7}$$

Then we have $\mu_{\infty,0} = V^k\mu_{\infty,k}$. Putting $\mu_\infty = \mu_{\infty,0}$, then $\mu_\infty \in SC_0(A)$. Putting $k = 0$ and letting $n \to \infty$ in (4.6), we obtain a required representation of μ. By virtue of the unicity of the Riesz decomposition of $(-A)^k\mu$ $(k = 0, 1, \cdots)$, we see the unicity of the representation (4.5) of μ. This completes the proof.

Now we denote by $S(A)$ the set of all non-negative A-superharmonic measures in X.

Remark 48. Let $(T_t)_{t\geqq 0}$ and A be the same as in Proposition 47. Then $S(A)$ is a vaguely closed convex cone in $M^+(X)$.

In fact, let V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$. For any $V^*f \in R_K^+(V^*)$, $\operatorname{supp}(f^+) \subset \operatorname{supp}(V^*f)$, and hence, for any vaguely cluster point μ of $S(A)$, we have $\int f d\mu \geqq 0$. This gives that $\overline{S(A)} = S(A)$.

But, in order to discuss the closedness of $SC(A)$ and that of $E(A)$, we need the following

DEFINITION 49. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group on X satisfying the condition (C^*) and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. We say that A satisfies the condition $(\mathscr{L})$ if, for any $(\mu_n)_{n=1}^{\infty} \subset S(A)$,

(4.8) $$\lim_{n\to\infty} \mu_n = \mu \in S(A) \text{ (vaguely) implies } \lim_{n\to\infty} A\mu_n = A\mu \text{ (vaguely).}$$

PROPOSITION 50. *Let $(T_t)_{t\geqq 0}$ and A be the same as in Proposition 47. If A satisfies the condition $(\mathscr{L})$, then, for any constant $c \geqq 0$, $H(A)$, $E(A; c)$, $E(A)$ and $SC(A)$ are vaguely closed convex cones in $M^+(X)$.*

Proof. It is easy to see the vague closedness of $H(A)$ and that of $E(A; c)$. We remark here $H(A) = E(A; 0)$. Let $(\mu_n)_{n=1}^{\infty}$ be a sequence in $E(A)$ tending vaguely to $\mu \in M^+(X)$ as $n \to \infty$. Then there exists a sequence of non-negative numbers $(c_n)_{n=1}^{\infty}$ such that $-A\mu_n = c_n\mu_n$. By $E(A) \supset H(A)$, we may assume that $-A\mu \neq 0$. The condition $(\mathscr{L})$ for A gives that $(c_n\mu_n)_{n=1}^{\infty}$ converges vaguely to $-A\mu$ as $n \to \infty$. Hence $(c_n)_{n=1}^{\infty}$ converges to a non-negative number c as $n \to \infty$, which implies that $\mu \in E(A; c) \subset E(A)$. Thus we see the vague closedness of $E(A)$. Let $(\mu_n)_{n=1}^{\infty}$ be a sequence of $SC(A)$ tending vaguely to $\mu \in M^+(X)$ as $n \to \infty$. Inductively we have, for any integer $k \geqq 0$,

(4.9) $$\lim_{k\to\infty} (-A)^k\mu_n = (-A)^k\mu \in M^+(X) \text{ (vaguely),}$$

which implies that $\mu \in SC(A)$, and hence the vague closedness of $SC(A)$ is shown. This completes the proof.

The above proposition gives the following

PROPOSITION 51. *Let $(T_t)_{t\geqq 0}$, V and A be the same as above. Assume that A satisfies the condition $(\mathscr{L})$ and that, for all $n = 1, 2, \cdots$, V^n is defined as a diffusion kernel on X. Then, for any number $c \geqq 0$, $SC_0(A)$, $E_0(A)$ and $E_0(A; c)$ are Borel measurable convex cones in the metrizable space $M^+(X)$.*

Proof. Since X is with countable basis, $M^+(X)$ is metrizable. Choose

$(f_n)_{n=1}^{\infty} \subset C_K^+(X)$ such that $(f_n)_{n=1}^{\infty}$ is total in $C_K(X)$. For each integer $m \geqq 0$, $n \geqq 1$ and $p \geqq 1$, we put

$$B_{m,n,p} = \left\{\mu \in SC(A); \int f_n d\mu_{h,m} \geqq \frac{1}{p}\right\}, \tag{4.10}$$

where $\mu_{h,m} = (-A)^m\mu - V((-A)^{m+1}\mu)$. The condition $(\mathscr{L})$ for A gives that $B_{m,n,p}$ is vaguely closed. Since

$$SC_0(A) = \bigcap_{m=0}^{\infty}\bigcap_{n=1}^{\infty}\bigcap_{p=1}^{\infty}(CB_{m,n,p} \cap SC(A)), \tag{4.11}$$

$SC_0(A)$ is Borel measurable. Remarking that $E_0(A) = E(A) \cap SC_0(A)$ and $E_0(A;c) = E(A;c) \cap SC_0(A)$, we see that $E_0(A)$ and $E_0(A;c)$ are Borel measurable. Their convexities are evident, so we achieve the proof.

The following remark shows that the condition $(\mathscr{L})$ for A does not always imply the compactness of the support of A^*, where A^* denotes the dual operator of A.

Remark 52. Let $(T_t)_{t\geqq 0}$ and A be the same as in Proposition 47.

(1) If A^* is with compact support, i.e., if, for any $V^*f \in R_K(V^*)$, $\mathrm{supp}(f)$ is compact, then A satisfies the condition $(\mathscr{L})$.

(2) Assume that $(T_t)_{t\geqq 0}$ be of convolution type and A satisfies the condition $(\mathscr{L})$. For a positive number p, let A_p be the infinitesimal generator of the semi-group $(T_{p,t})_{t\geqq 0}$ defined in (2.43). Then A_p also satisfies the condition $(\mathscr{L})$.

In fact, clearly we have (1). We shall show (2). Denote by $(V_p)_{p\geqq 0}$ the resolvent for $(T_t)_{t\geqq 0}$. Then, for any $p > 0$, $\mathscr{D}(A_p) \supset M_K(X)$ and $A_p = p(I - pV_p)$. Let $(\mu_n)_{n=1}^{\infty}$ be a sequence in $S(A_p)$ satisfying $\lim_{n\to\infty}\mu_n = \mu \in S(A_p)$ (vaguely). By Theorem 35, we have

$$\begin{aligned}\mu_n &= \left(V + \frac{1}{p}I\right)\nu_n + \mu_{n,h} \ (n = 1, 2, \cdots) \text{ and} \\ \mu &= \left(V + \frac{1}{p}I\right)\nu + \mu_h,\end{aligned} \tag{4.12}$$

where $\nu_n = p(I - pV_p)\mu_n$, $\nu = p(I - pV_p)\mu$, $\mu_{n,h} \in H(A_p)$ and $\mu_h \in H(A_p)$. Since $\mu_{n,h} = pV_p\mu_{n,h}$, the resolvent equation gives that, for any $q > 0$, $\mu_{n,h} = qV_q\mu_{n,h}$, which implies that $\mu_{n,h}$ is invariant with respect to $(T_t)_{t\geqq 0}$. Similarly μ_h is also invariant with respect to $(T_t)_{t\geqq 0}$. Since $(V\nu_n + \mu_{n,h})_{n=1}^{\infty}$ is vaguely bounded, we may assume that it converges vaguely. By Theorem

35, its limit is of the form $V\lambda + \mu'_h$, where $\lambda \in \mathscr{D}^+(V)$ and $\mu'_h \in H(A)$. The condition ($\mathscr{L}$) for A implies that $\lim_{n\to\infty} \nu_n = \nu$ (vaguely). Hence

$$\left(V + \frac{1}{p}I\right)\nu + \mu_h = \left(V + \frac{1}{p}I\right)\lambda + \mu'_h . \tag{4.13}$$

Since $(T_t)_{t\geqq 0}$ is of convolution type, it is known that μ'_h is also invariant with respect to $(T_t)_{t\geqq 0}$ (see [8], p. 343). By virtue of the unicity of the Riesz decomposition of μ, we have $\nu = \lambda$ and $\mu_h = \mu'_h$. Thus (2) is shown.

Hereafter in this paragraph, for any nonzero element μ of $M^+(X)$, we choose a fixed $f_\mu \in C^+(X)$ such that $f_\mu(x) > 0$ on X and $\int f_\mu d\mu < \infty$. For a transient and regular diffusion semi-group $(T_t)_{t\geqq 0}$ on X and its infinitesimal generator A, we put, for $\mu \in M^+(X)$,

$$SC(A;\mu) = \left\{\nu \in SC(A); \int f_\mu d\nu \leqq 1\right\} . \tag{4.14}$$

It is easily seen that if A satisfies the condition ($\mathscr{L}$), then $SC(A;\mu)$ is vaguely compact convex set in $M^+(X)$.

In general, for a convex set C in a locally convex space, we denote by ex C the set of all extreme points of C and, for a convex cone K in a locally convex space, we denote by $\widetilde{\text{exr}}\, K$ the set of all extreme rays in K[6].

Our main theorem is the following

THEOREM 53. *Let $(T_t)_{t\geqq t}$ be a transient and regular diffusion semi-group on X, V be the Hunt diffusion kernel for $(T_t)_{t\geqq 0}$ and A be the infinitesimal generator of $(T_t)_{t\geqq 0}$. Assume that, for all integer $n \geqq 1$, V^n is defined as a diffusion kernel and that A satisfies the condition ($\mathscr{L}$). Then we have:*

(1) *The set of all extreme rays in $SC(A)$ is represented as follows:*

$$\widetilde{\text{exr}}\, SC(A) = \left(\bigcup_{n=0}^{\infty} V^n((\widetilde{\text{exr}}\, H(A)) \cap \mathscr{D}(V^n))\right) \cup \left(\bigcup_{t\geqq 0} \widetilde{\text{exr}}\, E_0(A;t)\right) , \tag{4.15}$$

where $V^n((\widetilde{\text{exr}}\, H(A)) \cap \mathscr{D}(V^n)) = \{V^n\rho; \rho \in (\widetilde{\text{exr}}\, H(A)) \cap \mathscr{D}(V^n)\}$ and $V^n\rho = \{\lambda V^n\nu; \lambda \in R^+\}$ with nonzero element ν of ρ, and $SC(A)$ is the closed convex

6) A ray ρ in K is a set of the form $\{\lambda x; \lambda \in R^+\}$, where $0 \neq x \in K$, and we say that ρ is an extreme ray if, for any $x \in \rho$ and any $y, z \in K$, $y, z \in \rho$ whenever $x = \lambda y + (1-\lambda)z$ for $\lambda > 0$. We denote here by R^+ the totality of all non-negative numbers.

hull of $\widetilde{\text{exr}}\, SC(A)$[7].

(2) *For any* $\mu \in SC_0(A)$, *there exists a regular Borel non-negative measure* Φ *on* $E_0(A)$ *with* $\int d\Phi < \infty$ *carried by* $\bigcup_{t\geqq 0} \widetilde{\text{exr}}\, E_0(A;t)$[8] *such that*

$$(4.16) \qquad \mu = \int \lambda d\Phi(\lambda) \left(\text{i.e., } \int f d\mu = \int \left(\int f d\lambda\right) d\Phi(\lambda) \text{ for all } f \in C_K(X)\right).$$

Furthermore, for any $\mu \in SC_0(A)$, *there exists a Borel non-negative measure* σ *in* $(0, \infty)$ *with finite total mass and a bounded* σ*-measurable mapping* $(0, \infty) \ni t \to \mu_t \in E_0(A)$ *with* $\mu_t \in E_0(A;t)$[9] *such that*

$$(4.17) \qquad \mu = \int_0^\infty \mu_t d\sigma(t) \left(\text{i.e., } \int f d\mu = \int_0^\infty \left(\int f d\mu_t\right) d\sigma(t) \text{ for all } f \in C_K(X)\right).$$

To prove our main theorem, we use the following three Choquet theorems.

PROPOSITION 54 (see [17], p. 7 and p. 19). *Let* C *be a metrizable compact convex subset of a locally convex space. Then* $\text{ex}\, C$ *forms a* G_δ*-set and, for any* $x \in C$, *there exists a regular Borel probability measure* μ *on* C *carried by* $\text{ex}\, C$ *which represents* x[10].

PROPOSITION 55 (see [17], p. 88–89). *Let* K *be a closed convex cone in a locally convex space and suppose that* K *is union of its caps*[11]. *Then* K *is the closed convex hull of* $\widetilde{\text{exr}}\, K$.

PROPOSITION 56 (see [17], p. 88). *Let* K *be a closed convex cone in a locally convex space and* C *be its cap. Then every extreme points of* C *lies on an extreme ray in* K.

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7) In this case, $\widetilde{\text{exr}}\, SC(A)$ means $\{y \in \rho;\, \rho \in \widetilde{\text{exr}}\, SC(A)\}$ and $\widetilde{\text{exr}}\, E_0(A;t)$ means the analogous set.

8) We say that a regular Borel measure Φ on $E_0(A)$ is carried by a set $Y \subset E_0(A)$ if, there exists a Borel set B such that $B \subset Y$ and $\Phi(CB) = 0$.

9) We say that $t \to \mu_t$ is σ-measurable if, for any $f \in C_K(X)$, the function $\int f d\mu_t$ of t is σ-measurable and that is bounded if, for any $f \in C_K(X)$, $\int f d\mu_t$ is bounded in $(0, \infty)$.

10) *A* point $x \in C$ is said to be represented by μ if, for any continuous linear functional f,

$$f(x) = \int f(y) d\mu(y).$$

11) *A* non-empty subset C of K is called a cap of K if C is a compact convex subset and if $K - C$ is also convex.

Proof of Theorem 53. (a) First we shall show that, for any $\mu_0 \neq 0 \in M^+(X)$,

$$(4.18) \qquad (\mathrm{ex}\, SC(A;\mu_0)) \cap SC_0(A) \subset E_0(A)\,.$$

Let $0 \neq \mu \in SC(A;\mu_0) \cap SC_0(A)$. Theorem 35 and Corollary 39 give that $\mu = V(-A\mu)$. Let $t > 0$. Remarking that $T_t(-A\mu) \leqq -A\mu$ and $V\cdot T_t = T_t \cdot V$, we obtain that $T_t\mu \in \mathscr{D}^+(A)$ and $-A(T_t\mu) = T_t(-A\mu)$. Hence we have

$$(4.19) \qquad (-A)^n(T_t\mu) = T_t((-A)^n\mu) \in M^+(X) \ (n = 0, 1, \cdots)\,,$$

because $\mu = V^n((-A)^n\mu)$. This implies that $T_t\mu \in SC(A)$. Since $(I - T_t)\mu = \int_t^\infty T_s(-A\mu)ds$, we have also $(I - T_t)\mu \in SC(A)$. Let $0 \neq \mu \in (\mathrm{ex}\, SC(A;\mu_0)) \cap SC_0(A)$ and put

$$(4.20) \qquad c_{1,t} = \int f_{\mu_0} dT_t\mu \ \text{ and } \ c_{2,t} = \int f_{\mu_0} d(I - T_t)\mu\,.$$

Then $c_{j,t} > 0$ $(j = 1, 2)$, because $-A\mu \neq 0$, and $\int f_{\mu_0} d\mu = 1$. From $T_t\mu \in SC(A;\mu_0)$, $(I - T_t)\mu \in SC(A;\mu_0)$,

$$(4.21) \qquad \mu = c_{1,t}\Big(\frac{T_t\mu}{c_{1,t}}\Big) + c_{2,t}\Big(\frac{(I - T_t)\mu}{c_{2,t}}\Big) \ \text{ and } \ c_{1,t} + c_{2,t} = 1\,,$$

it follows that, with a constant $0 < c_t < 1$,

$$(4.22) \qquad \mu = c_t T_t \mu\,,$$

which implies that, with a constant $a > 0$,

$$(4.23) \qquad -A\mu = \lim_{t\to 0} \frac{\mu - T_t\mu}{t} = \lim_{t\to 0} \Big(\frac{1 - c_t}{t}\Big)\mu = a\mu\,.$$

Thus we see (4.18).

(b) Let $0 \neq \mu_0 \in M^+(X)$. We shall show that, for any $\mu \in SC(A;\mu_0) \cap SC_0(A)$, there exists a regular Borel probability measure Φ on $E_0(A)$ carried by $(\mathrm{ex}\, SC(A;\mu_0)) \cap SC_0(A)$ such that the analogous equality to (4.16) holds. Put, for each integer $n \geqq 1$,

$$(4.24) \qquad H_n(A) = \{V^n\mu;\, \mu \neq 0 \in \mathscr{D}^+(V^n) \cap H(A)\}$$

and $H_0(A) = H(A)$. The condition $(\mathscr{L})$ for A implies that, for any $n \geqq 0$, $\oplus_{k=0}^n H_k(A)$ is vaguely closed and, similarly as in Proposition 51, we see that $H_n(A)$ is Borel measurable. Remarking that $(H_n(A))_{n=1}^\infty$ and $SC_0(A) - \{0\}$ are mutually disjoint, we have

$$\text{(4.25)}\quad \begin{aligned}&\operatorname{ex} SC(A;\mu_0)\\ &= \left(\bigcup_{n=0}^{\infty}(\operatorname{ex} SC(A;\mu_0)) \cap H_n(A)\right) \cup ((\operatorname{ex} SC(A;\mu_0)) \cap SC_0(A))\,,\end{aligned}$$

and $(\operatorname{ex} SC(A;\mu_0)) \cap H_n(A)$ $(n = 0, 1, \cdots)$ and $(\operatorname{ex} SC(A;\mu_0)) \cap (SC_0(A) - \{0\})$ are mutually disjoint Borel measurable sets (see Propositions 51 and 54). By Proposition 54, there exists a regular Borel probability measure on $\operatorname{ex} SC(A;\mu_0)$ such that $\mu = \int \lambda d\Phi(\lambda)$. Put

$$\text{(4.26)}\quad \Phi_n = \begin{cases}\Phi \text{ on } (\operatorname{ex} SC(A;\mu_0)) \cap H_n(A) \ (n \geqq 0)\\ 0 \text{ otherwise}\end{cases} \quad \text{and} \quad \Phi_\infty = \Phi - \sum_{n=0}^{\infty} \Phi_n\,.$$

Then we have

$$\text{(4.27)}\quad \mu = \sum_{n=0}^{\infty} \int \lambda d\Phi_n(\lambda) + \int \lambda d\Phi_\infty(\lambda)\,.$$

By (a), Φ_∞ is a regular Borel non-negative measure on $E_0(A)$ carried by $(\operatorname{ex} SC(A;\mu_0)) \cap SC_0(A)$. For any $n \geqq 0$, the closedness of $\oplus_{k=0}^{n} H_k(A)$ implies that $\sum_{k=0}^{n} \int \lambda d\Phi_k(\lambda) \in \oplus_{k=0}^{n} H_k(A)$, and hence Proposition 47 gives that $\int \lambda d\Phi_n(\lambda) = 0$. Hence we may assume that $\Phi = \Phi_\infty$, which gives our assertion.

(c) We shall show that, for any nonzero element μ_0 of $M^+(X)$,

$$\text{(4.28)}\quad (\operatorname{ex} SC(A;\mu_0)) \cap SC_0(A) = \bigcup_{t \geqq 0} \operatorname{ex}(E_0(A;t) \cap SC(A;\mu_0))\,.$$

Evidently we have the inclusion $\subset$, and so we shall show the inverse inclusion. Let $0 \neq \mu \in \operatorname{ex}(E_0(A;c) \cap SC(A;\mu_0))$. Then $c \neq 0$. Assume that, for $\mu_j \in SC(A;\mu_0)$ $(j = 1, 2)$, $\mu = 1/2(\mu_1 + \mu_2)$. Then $\mu_j \in SC_0(A)$ $(j = 1, 2)$. By (b), there exists a regular Borel probability measure Φ_j on $E_0(A)$ carried by $(\operatorname{ex} SC(A;\mu_0)) \cap SC_0(A)$ such that $\mu_j = \int \lambda d\Phi_j(\lambda)$ $(j = 1, 2)$. By using Propositions 50 and 51, we see that $E_0(A;c)$, $\bigcup_{c>t\geqq 0} E_0(A;t)$ and $\bigcup_{t>c} E_0(A;t)$ are Borel measurable, because, similarly as in Proposition 50, we see that, for any $s > 0$, $\bigcup_{t\geqq s} E(A;t)$ is closed in $M^+(X)$ and that $(\bigcup_{t\geqq s} E(A;t)) \cap SC_0(A) = \bigcup_{t\geqq s} E_0(A;t)$. Put, for $j = 1, 2$ and $k = 0, 1, 2$,

$$\text{(4.29)}\quad \Phi_{0,j} = \begin{cases}\Phi_j \text{ on } E_0(A;c)\\ 0 \text{ otherwise,}\end{cases} \quad \Phi_{1,j} = \begin{cases}\Phi_j \text{ on } (\bigcup_{c>t\geqq 0} E_0(A;t)) - \{0\}\\ 0 \text{ otherwise,}\end{cases}$$

$$\Phi_{2,j} = \Phi_j - \Phi_{0,j} - \Phi_{1,j} \quad \text{and} \quad \Phi'_k = \frac{1}{2}(\Phi_{k,1} + \Phi_{k,2})\,.$$

For any integer $n \geqq 1$, we have, by the condition ($\mathscr{L}$) for A,

$$\mu = \left(-\frac{1}{c}A\right)^n \mu = \int \left(-\frac{1}{c}A\right)^n \lambda d\Phi_0'(\lambda) + \int \left(-\frac{1}{c}A\right)^n \lambda d\Phi_1'(\lambda)$$

$$(4.30) \qquad + \int \left(-\frac{1}{c}\right)^n \lambda d\Phi_2'(\lambda)$$

$$= \int \lambda d\Phi_0'(\lambda) + \int \left(-\frac{c_\lambda}{c}\right)^n \lambda d\Phi_1'(\lambda) + \int \left(-\frac{c_\lambda}{c}\right)^n \lambda d\Phi_2'(\lambda)\,,$$

where c_λ is a positive constant satisfying $-A\lambda = c_\lambda \lambda$. We remark here that the mapping $(E_0(A) - \{0\}) \ni \lambda \to c_\lambda$ is continuous. By letting $n \to \infty$ in (4.30), we see that $\mu = \int \lambda d\Phi_0'(\lambda)$. This implies that $\mu_j = \int \lambda d\Phi_{0,j}(\lambda)$ ($j = 1, 2$). Since $\int f_{\mu_0} d\mu = 1$, we have $\mu = \mu_j$ ($j = 1, 2$), Thus we see that (4.28) holds.

(d) Since $SC(A) = \bigcup_{0 \neq \mu \in SC(A)} SC(A; \mu)$, Proposition 55 gives that $SC(A)$ is the closed convex hull of $\widetilde{\operatorname{exr}}\, SC(A)$. Evidently we have

$$\widetilde{\operatorname{exr}}\, SC(A) \subset \left(\bigcup_{n=0}^{\infty} V^n ((\widetilde{\operatorname{exr}}\, H(A)) \cap \mathscr{D}(V^n))\right) \cup \left(\bigcup_{t \geqq 0} \widetilde{\operatorname{exr}}\, E_0(A; t)\right)$$

and

$$\widetilde{\operatorname{exr}}\, SC(A) \supset \bigcup_{n=0}^{\infty} V^n ((\widetilde{\operatorname{exr}}\, H(A)) \cap \mathscr{D}(V^n))$$

by Proposition 47. Let $t > 0$ and $\rho \in \widetilde{\operatorname{exr}}\, E_0(A; t)$. We choose a nonzero element μ of ρ. Then $\mu \in \operatorname{ex}(E_0(A; t) \cap SC(A; \mu))$, and hence (c) implies that $\mu \in (\operatorname{ex} SC(A; \mu)) \cap SC_0(A)$. By Proposition 56, we have $\rho \in \widetilde{\operatorname{exr}}\, SC(A)$. This implies that (4.15) holds. Proposition 56, (b) and (c) give also (4.16).

(e) Finally, we shall show (4.17). Let $\mu \in SC_0(A)$ and Φ be a regular Borel non-negative measure with $\int d\Phi < \infty$ defined by (4.16). By (b) and (c), Φ is carried by $(\operatorname{ex} SC(A; \mu)) \cap (\bigcup_{t \geqq 0} E_0(A; t))$. For any $t > 0$, we put

$$(4.31) \qquad \Phi_t = \begin{cases} \Phi \text{ on } \bigcup_{t \geqq s \geqq 0} E_0(A; s) \\ 0 \text{ otherwise} \end{cases} \quad \text{and} \quad v(t) = \int d\Phi_t = \int \left(\int f_\mu d\lambda\right) d\Phi_t(\lambda)\,,$$

because $\int f_\mu d\lambda = 1$ for any nonzero element λ of $\operatorname{ex} SC(A; \mu)$. Then $v(t)$ is a bounded non-negative increasing function on $(0, \infty)$. Let σ be a non-negative Borel measure in $(0, \infty)$ such that $v(t) = \int_0^t d\sigma$. Then $\int_0^\infty d\sigma < \infty$. For $f \in C_K(X)$, we put

$$v_f(t) = \int \left(\int f d\lambda \right) d\Phi_t(\lambda) . \tag{4.32}$$

Then there exists a real Borel measure σ_f in $(0, \infty)$ such that $v_f(t) = \int_0^t d\sigma_f$. We have also $\int_0^\infty d|\sigma_f| < \infty$. Since $|f| \leqq c_f f_\mu$ on X for some positive number c_f, we have, for any $t > s > 0$,

$$|v_f(t) - v_f(s)| \leqq c_f(v(t) - v(s)) , \tag{4.33}$$

which shows that σ_f is absolutely continuous with respect to σ. By the Radon-Nikodym theorem, there exists a σ-integrable function $\tilde{f}$ on $(0, \infty)$ such that $d\sigma_f = \tilde{f} d\sigma$. We have also $|\tilde{f}| \leqq c_f$ σ-a.e.. By (4.32), we have, for any $f, g \in C_K^+(X)$, and any constants a, b,

$$\widetilde{af + bg} = a\tilde{f} + b\tilde{g} \quad \sigma\text{-a.e.}. \tag{4.34}$$

We choose a countable set of continuous functions $(f_n)_{n=1}^\infty \subset C_K^+(X)$ such that $(f_n)_{n=1}^\infty$ is total in $C_K(X)$. By (4.34), there exists a Borel set F in $(0, \infty)$ such that $\sigma(CF) = 0$ and that, for any $t \in F$, any rational number r and any integers $n \geqq 1$ and $m \geqq 1$,

$$(r\tilde{f}_n)(t) = t\tilde{f}_n(t), \ \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_t^{t+\delta} \tilde{f}_n d\sigma = \tilde{f}_n(t) \quad \text{and} \quad (\widetilde{f_n + f_m})(t) = \tilde{f}_n(t) + \tilde{f}_m(t) . \tag{4.35}$$

For any $t \in CF$, the mapping $f_n \to \tilde{f}_n(t)$ can be extended to a positive linear form on $C_K(X)$ in the usual way, and hence there exists a uniquely determined non-negative Radon measure μ_t in X such that $\tilde{f}_n(t) = \int f_n d\mu_t$ for all $n \geqq 1$. By defining $\mu_t = 0$ for all $t \in CF$, we see that $(0, \infty) \ni t \to \mu_t \in M^+(X)$ is σ-measurable. Since $\int f_\mu d\mu_t \leqq 1$ for all $t \in (0, \infty)$, $(0, \infty) \ni t \to \mu_t \in M^+(X)$ is bounded. Furthermore we have

$$\mu = \int_0^\infty \mu_t d\sigma(t) . \tag{4.36}$$

The condition $(\mathscr{L})$ for A and the second equality in (4.35) give that

$\mu_t \in E(A;t)$ for all $t \in (0,\infty)$. By Theorem 35 and (4.36), we may assume that $\mu_t \in E_0(A;t)$. This completes the proof.

Now we notice the following equality:

$$SC_0(A) = \left\{\int_0^\infty \mu_t d\sigma(t); \sigma \in M_b^+((0,\infty)),\ t \to \mu_t \in E_0(A;t)\text{: bounded and } \sigma\text{-measurable}\right\}, \tag{4.37}$$

where $M_b^+((0,\infty))$ denotes the totality of all non-negative Borel measures in $(0,\infty)$ with finite total mass. In fact, let $\sigma \in M_b^+((0,\infty))$ and $(0,\infty) \ni t \to \mu_t \in E_0(A;t)$ be a bounded σ-measurable mapping. Put $\sigma_n = \sigma$ on $[1/n, n]$ and $\sigma_n = 0$ otherwise $(n = 1, 2, \cdots)$. Then the condition $(\mathscr{L})$ for A gives that, for all $n = 1, 2, \cdots$ and $m = 0, 1, 2, \cdots$,

$$\begin{aligned}(-A)^m \int_0^\infty \mu_t d\sigma_n(t) &= \int_0^\infty t^m \mu_t d\sigma_n(t) \quad \text{and} \\ \int_0^\infty \mu_t d\sigma_n(t) &= V^m\left(\int_0^\infty t^m \mu_t d\sigma_n(t)\right).\end{aligned} \tag{4.38}$$

By letting $n \to \infty$ in (4.38) and using the condition $(\mathscr{L})$ for A, we have, for any $m \geqq 0$, $\int_0^\infty t^m \mu_t d\sigma(t) \in M^+(X)$ and

$$\begin{aligned}(-A)^m \int_0^\infty \mu_t d\sigma(t) &= \int_0^\infty t^m \mu_t d\sigma(t) \quad \text{and} \\ \int_0^\infty \mu_t d\sigma(t) &= V^m\left(\int_0^\infty t^m \mu_t d\sigma(t)\right).\end{aligned} \tag{4.39}$$

By combining Theorem 53 and (4.39), we have (4.37).

For $\mu \in M(X)$, we write $\rho(\mu) = \{c\mu; c \in R^+\}$. In particular, we have the following

PROPOSITION 57. *Let X be a locally compact abelian group with countable basis and ξ be a Haar measure on X. Let $(T_t)_{t\geqq 0}$ be a transient diffusion semi-group of convolution type on X and α_t be the non-negative Radon measure on X defining T_t (see (2.17)). Assume that the infinitesimal generator A of $(T_t)_{t\geqq 0}$ satisfies the condition $(\mathscr{L})$ and let* Exp(X) *be the totality of all positive continuous exponential functions on X*[12]*. Then we have:*

12) A real-valued function φ on X is said to be exponential if, for any $x, y \in X$, $\varphi(x+y) = \varphi(x)\cdot\varphi(y)$.

(1) $\widetilde{\operatorname{exr}}\, H(A) \subset \left\{\rho(\varphi\xi); \varphi \in \operatorname{Exp}(X), \int \varphi d\alpha_t = 1 \textit{ for all } t \geqq 0\right\} \subset H(A)$[13)]

(2) *For any* $c > 0$, $\widetilde{\operatorname{exr}}\, E_0(A; c) \subset \left\{\rho(\varphi\xi); \varphi \in \operatorname{Exp}(X), c\int_0^\infty \left(\int \varphi d\alpha_t\right) dt = 1\right\}$ $\subset E_0(A; c)$.

Proof. It is known that

$$(4.40)\qquad \begin{aligned} H(A) &= \{\mu \in M^+(X); \mu = \mu * \alpha_t \text{ for all } t \geqq 0\} \\ &= \{\mu \in M^+(X); \mu = \mu * \alpha_{t_0} \text{ for some } t_0 > 0\} \end{aligned}$$

(see [8], p. 343). This implies the second inclusion in (1). By the Choquet-Deny theorem (see [5])[13)], we see the first inclusion in (1). Similarly we see the assertion (2). Lastly in this paragraph, we shall discuss the Bernstein theorem. Put

$$(4.41)\qquad T_t : M_K((0, \infty)) \ni \mu \to \text{the restriction of } \tau_{-t}\mu \text{ to } (0, \infty) \in M((0, \infty))$$

for all $t \geqq 0$, where τ_{-t} is the translation of $-t$. Then $(T_t)_{t\geqq 0}$ is transient and regular diffusion semi-group on $(0, \infty)$, and its infinitesimal generator A is equal to d/dt. Denote by dt the Lebesgue measure in $(0, \infty)$. Since the Hunt diffusion kernel V for $(T_t)_{t\geqq 0}$ satisfies

$$(4.42)\qquad V\mu = \left(\int_t^\infty d\mu\right) dt \text{ for all } \mu \in M_K((0, \infty))$$

and

$$(4.43)\qquad H\left(\frac{d}{dt}\right) = \rho(dt) \text{ and } E_0\left(\frac{d}{dt}; c\right) = \rho(\exp(-ct)dt) \text{ for all } c > 0\,.$$

Hence, our main theorem implies the Bernstein theorem. We remark here that

$$(4.44)\qquad V^n\mu = \left(\int_t^\infty \frac{1}{(n-1)!}(x-t)^{n-1} d\mu(x)\right) dt \quad \text{for all } \mu \in M_K((0, \infty)) \text{ and } n = 1, 2, \cdots,$$

and that

13) This shows that, for a non-negative Radon measure σ in X, the solution μ of the convolution equation $\mu = \mu * \sigma$ is of form

$$\mu = \left(\int \varphi d\lambda(\varphi)\right)\xi\,,$$

where λ is a regular Borel measure with finite total mass on $\left\{\varphi \in \operatorname{Exp}(X); \int \varphi d\sigma = 1\right\}$.

(4.45) $$dt \notin \mathscr{D}^+(V^n) \text{ for all } n = 1, 2, \cdots .$$

§ 5. Application to elliptic differential operators

In this paragraph, we consider the same setting as in S. Itô's paper [10]. Let D be a subdomain of an orientable N-dimensional C^∞-manifold ($N \geqq 2$) and L be an elliptic differential operator of the form:

(5.1)
$$Lu(x) = \sum_{i,j=1}^{N} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i}\left(\sqrt{a(x)} \cdot a^{ij}(x) \frac{\partial u}{\partial x^j}(x)\right) + \sum_{i=1}^{N} b^i(x) \frac{\partial u}{\partial x^i}(x) + c(x)u(x)$$

for $u \in C^2(D)$[14] and $x = (x^1, \cdots, x^N) \in D$, where $(a^{ij}(x))_{i,j=1}^N$ is a contravariant tensor of class C^∞ in D and is symmetric and strictly positive-definite for each $x \in D$, $a(x) = \det(a_{ij}(x)) = \det(a^{ij}(x))^{-1}$, $(b^i(x))_{i=1}^N$ is a contravariant vector of class C^∞ in D and $c(x)$ is a non-positive function of class C^∞ in D. We shall denote by dx the volume element with respect to the Riemannian metric defined by the tensor $(a_{ij}(x))_{i,j=1}^N$. The formally adjoint operator L^* of L is defined by

(5.2)
$$L^*v(x) = \sum_{i,j=1}^{N} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i}\left(\sqrt{a(x)} \cdot a^{ij}(x) \frac{\partial v}{\partial x^j}(x)\right) - \sum_{i=1}^{N} \frac{1}{\sqrt{a(x)}} \frac{\partial}{\partial x^i}(\sqrt{a(x)} \cdot b^i(x) \cdot v(x)) + c(x)v(x)$$

for $v \in C^2(D)$.

Evidently we have the following

Remark 58. Let u and v be in $C^2(D)$. If $u \in C_K^2(D)$ or $v \in C_K^2(D)$, then we have

(5.3) $$\int Lu(x)v(x)dx = \int u(x)L^*v(x)dx .$$

DEFINITION 59 (see [10]). Let Ω be a subdomain of D. We say that Ω satisfies the condition (S) if its closure $\bar{\Omega}$ is contained in D and its boundary $\partial\Omega$ consists of finite number of simple closed hypersurfaces of class C^3.

PROPOSITION 60 (see [9], Theorem 1). *Let Ω be a subdomain of D*

14) We denote by $C^n(D) = \{f \in C(D); f \text{ is of class } C^n \text{ in } D\}$ for $n \geqq 1$ and by $C^\infty(D) = \cap_{n=1}^\infty C^n(D)$. We write also $C_K^n(D) = C^n(D) \cap C_K(D)$ and $C_K^\infty(D) = C^\infty(D) \cap C_K(D)$.

satisfying the condition (S). Then there exists one and only one fundamental solution $U_\Omega(t, x, y)$ of the initial-boundary value problem:

Given $u_0 \in C(\bar{\Omega})$ and $\varphi \in C((0, \infty) \times \partial\Omega)$,

$$(5.4) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x) = Lu(t, x) \text{ for each } (t, x) \in (0, \infty) \times \Omega \\ u(0, x) = u_0(x) \text{ for each } x \in \bar{\Omega} \\ u(t, x) = \varphi(t, x) \text{ for each } (t, x) \in (0, \infty) \times \partial\Omega \,. \end{cases}$$

Furthermore $U_\Omega(t, x, y)$ satisfies the following five conditions:

(5.5) *$U_\Omega(t, x, y)$ is a non-negative finite continuous function on $(0, \infty) \times \bar{\Omega} \times \bar{\Omega}$ and $U_\Omega(t, x, y) = 0$ if and only if $x \in \partial\Omega$ or $y \in \partial\Omega$.*

(5.6) *$\int U_\Omega(t, x, y)dy \leqq 1$ for any $(t, x) \in (0, \infty) \times \bar{\Omega}$.*

(5.7) *$\int U_\Omega(t, x, y)U_\Omega(s, y, z)dy = U_\Omega(t + s, x, z)$ for any $t > 0$, $s > 0$ and any $(x, z) \in \bar{\Omega} \times \bar{\Omega}$.*

(5.8) *For any $u_0 \in C(\bar{\Omega})$, we put $u(t, x) = \int U_\Omega(t, x, y)u_0(y)dy$. Then $u(t, x)$ is the unique solution of* (5.4) *with $\varphi = 0$.*

(5.9) *For any $u_0 \in C(\bar{\Omega})$, we put $u^*(t, x) = \int U_\Omega(t, y, x)u_0(y)dy$. Then $u^*(t, x)$ is the unique solution of the initial-boundary value problem:*

$$(5.10) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x) = L^*u(t, x) \text{ for each } (t, x) \in (0, \infty) \times \Omega \\ u(0, x) = u_0(x) \text{ for each } x \in \bar{\Omega} \\ u(t, x) = 0 \text{ for each } (t, x) \in (0, \infty) \times \partial\Omega \,. \end{cases}$$

The following remark is elementary.

Remark 61. Let Ω be a subdomain of D. Then there exists a sequence $(\Omega_n)_{n=1}^\infty$ of subdomains in Ω satisfying the condition (S) such that $\bar{\Omega}_n \subset \Omega_{n+1}$, $\bigcup_{n=1}^\infty \Omega_n = \Omega$.

We call $(\Omega_n)_{n=1}^\infty$ a regular exhaustion of Ω.

PROPOSITION 62 (see [9], Lemma 5.4). *Let Ω and $(\Omega_n)_{n=1}^\infty$ be the same as above. Then $(U_{\Omega_n}(t, x, y))_{n=1}^\infty$ converges increasingly to a continuous func-*

tion $U_\Omega(t, x, y)$ *in* $(0, \infty) \times \Omega \times \Omega$[15].

We remark here that $U_{\Omega_n}(t, x, y) \to U_\Omega(t, x, y)$ *in* $C((0, \infty) \times \Omega \times \Omega)$ *as* $n \to \infty$ *and that* $U_\Omega(t, x, y)$ *does not depend on the choice of* $(\Omega_n)_{n=1}^\infty$.

COROLLARY 63. *Let* Ω *and* $U_\Omega(t, x, y)$ *be the same as above. Then we have*:

(1) *For* $t > 0$, $s > 0$ *and* $(x, z) \in \Omega \times \Omega$,

$$\int U_\Omega(t, x, y) U_\Omega(s, y, z) dy = U_\Omega(t + s, x, z) . \tag{5.11}$$

(2) *For any* $f \in C_K(\Omega)$, *we put*

$$u(t, x) = \begin{cases} \int U_\Omega(t, x, y) f(y) dy \text{ in } (0, \infty) \times \Omega \\ f(x) \text{ on } \{0\} \times \Omega \end{cases} \tag{5.12}$$

and

$$u^*(t, x) = \begin{cases} \int U_\Omega(t, y, x) f(y) dy \text{ in } (0, \infty) \times \Omega \\ f(x) \text{ on } \{0\} \times \Omega . \end{cases} \tag{5.13}$$

Then $u(t, x)$ *and* $u^*(t, x)$ *are finite continuous in* $[0, \infty) \times \Omega$.

Proof. Since $U_{\Omega_n}(t, x, y) \uparrow U_\Omega(t, x, y)$ as $n \uparrow \infty$, (5.7) gives (5.11). To show (2), we may assume that f is non-negative. Put

$$u_n(t, x) = \begin{cases} \int U_{\Omega_n}(t, x, y) f(y) dy \text{ in } (0, \infty) \times \Omega \\ f(x) \text{ on } \{0\} \times \Omega . \end{cases} \tag{5.14}$$

Then u_n is finite continuous on $[0, \infty) \times \Omega$. Since $(u_n(t, x))_{n=1}^\infty$ converges increasingly to $u(t, x)$ as $n \to \infty$, u is lower semi-continuous on $[0, \infty) \times \Omega$. Evidently $u(t, x)$ is finite continuous in $(0, \infty) \times \Omega$. Let t_0 be a fixed positive number. Then there exists a constant $c > 0$ such that $c \int U_\Omega(t_0, x, y) f(y) dy \geqq f(x)$ on Ω. Hence $cu(t_0 + t, x) - u(t, x)$ is also lower semi-continuous on $[0, \infty) \times \Omega$. This implies that $u(t, x)$ is finite continuous on $[0, \infty) \times \Omega$. By the similar argument, we see that $u^*(t, x)$ is also finite continuous on $[0, \infty) \times \Omega$. This completes the proof.

15) We may assume that $U_{\Omega_n}(t,x,y)$ is a finite continuous function in $(0,\infty) \times \Omega \times \Omega$, by defining that $U_{\Omega_n}(t,x,y) = 0$ if $x \in C\Omega_n$ or $y \in C\Omega_n$.

Let Ω be a subdomain of D. For any $t>0$, we define linear operators $T_{L,\Omega,t}$ and $T_{L^*,\Omega,t}$ from $M_K(\Omega)$ into $M(\Omega)$ as follows:

$$T_{L,\Omega,t}\mu = \left(\int U_\Omega(t, x, y)d\mu(y)\right)dx \quad\text{and}\quad T_{L^*,\Omega,t}\mu = \left(\int U_\Omega(t, y, x)d\mu(y)\right)dx . \tag{5.15}$$

By Corollary 63, we have the following

Remark 64. Putting $T_{L,\Omega,0} = T_{L^*,\Omega,0} = I$, we see that $(T_{L,\Omega,t})_{t\geqq 0}$ and $(T_{L^*,\Omega,t})_{t\geqq 0}$ are diffusion semi-groups on Ω.

For the sake of simplicity, we write $T_{L,t} = T_{L,D,t}$ and $T_{L^*,t} = T_{L^*,D,t}$ $(t \geqq 0)$.

PROPOSITION 65. *The diffusion semi-group $(T_{L,t})_{t\geqq 0}$ on D is transient if and only if the Green function $G(x,y)$ of L on D*[16] *exists. If $G(x,y)$ exists, then $G(x,y) = \int_0^\infty U_D(t, x, y)dt$.*

This follows from the following

PROPOSITION 66. *The Green function $G(x, y)$ of L on D exists if and only if there exists a non-constant lower semi-continuous and locally integrable function f satisfying $0 \leqq f \leqq \infty$, $f \not\equiv \infty$ and $-Lf \geqq 0$ in the sense of distributions in D. Furthermore, if $G(x,y)$ exists, we have $G(x,y) = \int_0^\infty U_D(t, x, y)dt$. For any $y \in D$, the functions $G(x, y)$ and $G(y, x)$ of x belong to $C^\infty(D - \{y\})$, and for any $f \in C_K^\infty(D)$, $Gf(x) = \int G(x, y)f(y)dy \in C^\infty(D)$ and*

$$LGf = G(Lf) = -f . \tag{5.16}$$

S. Itô shows the above assertion in the case of $c(x) \equiv 0$ (see [10]). In the case that $c(x) \not\equiv 0$, we see, in the same manner as in [10], that there exists the Green function of L on D (see also [9] and [12]).

Remark 67 (see [9], § 10 and [10]). If $G(x, y)$ exists, then $G^*(x, y) = G(y, x) = \int_0^\infty U_D(t, y, x)dt$ is the Green function of L^* on D and, for any

16) For an open set Ω in D, the Green function $G_\Omega(x,y)$ of L on Ω means a non-negative continuous function in $\Omega\times\Omega$ in the extended sense satisfying the following conditions:

(a) $G_\Omega(x,y) < \infty$ if $x \neq y$.

(b) $L_x G_\Omega(x,y) = -\varepsilon_y$ in the sense of distributions.

(c) For any $y \in \Omega$ and any non-negative function $h \in C^2(\Omega)$ with $Lh = 0$ in Ω, $G_\Omega(x,y) \geqq h(x)$ in Ω implies $h \equiv 0$.

$f \in C_K^\infty(D)$, $G^*f(x) = \int G^*(x, y)f(y)dy \in C^\infty(D)$ and

$$L^*G^*f = G^*(L^*f) = -f . \tag{5.17}$$

Proof of Proposition 65. We remark that, if $(T_{L,t})_{t \geqq 0}$ is transient, then, for any nonzero element μ of $M_K^+(D)$, $\int_0^\infty \int U_D(t, x, y)d\mu(y)dt$ is a non-constant lower semi-continuous and locally integrable function in D satisfying $-L\left(\int_0^\infty \int U_D(t, x, y)d\mu(y)dt\right) \geqq 0$ in the sense of distributions in D. If $G(x, y)$ exists, Proposition 66 and Remark 67 give that, for any $f \in C_K^+(D)$, $\int_0^\infty T_{L,t}^* f dt$ is a non-negative lower semi-continuous function in D and that, for any $f \in C_K^\infty(D)$, $\int_0^\infty T_{L,t}^* f dt = G^*f \in C^\infty(D)$, and hence $(T_{L,t})_{t \geqq 0}$ is transient.

Hereafter, we shall always assume that the Green function $G(x, y)$ of L on D exists. Define the linear operators V_L and V_{L^*} from $M_K(D)$ into $M(D)$ as follows:

$$V_L\mu = (G\mu)dx \quad \text{and} \quad V_{L^*}\mu = (G^*\mu)dx , \tag{5.18}$$

where $G\mu(x) = \int G(x, y)d\mu(y)$ and $G^*\mu(x) = \int G^*(x, y)d\mu(y)$. Then V_L and V_{L^*} respectively are the Hunt diffusion kernel for $(T_{L,t})_{t \geqq 0}$ and that for $(T_{L^*,t})_{t \geqq 0}$.

Remark 68. Let $\mu \in M_K(D)$. Then

$$LG\mu = -\mu \quad \text{and} \quad L^*G^*\mu = -\mu \tag{5.19}$$

in the sense of distributions in D.

In fact, V_L and V_{L^*} are defined, so that $G\mu$ and $G^*\mu$ are locally integrable. The two equalities in (5.19) follow from (5.16) and (5.17). The two equalities (5.16) and (5.17) imply also the following

Remark 69. We have $R_K(V_L^*) \supset C_K^\infty(D)$ and $R_K(V_{L^*}^*) \supset C_K^\infty(D)$, i.e., $(T_{L,t})_{t \geqq 0}$ and $(T_{L^*,t})_{t \geqq 0}$ satisfy the condition (C^*). Let A_L and A_{L^*} be the infinitesimal generator of $(T_{L,t})_{t \geqq 0}$ and that of $(T_{L^*,t})_{t \geqq 0}$, respectively. Then, for any $\mu \in \mathscr{D}(A_L)$ (resp. $\mu \in \mathscr{D}(A_{L^*})$),

$$A_L\mu = L\mu \text{ (resp. } A_{L^*}\mu = L^*\mu) \tag{5.20}$$

in the sense of distributions.

Let Ω be a subdomain of D satisfying the condition (S). It is well-known that, for any $y \in \Omega$, there exists the V_L-balayaged measure $\varepsilon'_{y,C\Omega}$ (resp. V_{L^*}-balayaged measure $\varepsilon''_{y,C\Omega}$) of ε_y on $C\Omega$. We have $\operatorname{supp}(\varepsilon'_{y,C\Omega}) \subset \partial\Omega$, $\operatorname{supp}(\varepsilon''_{y,C\Omega}) \subset \partial\Omega$,

$$\text{(5.21)} \qquad \begin{gathered} \int_0^\infty U_\Omega(t,x,y)dt = G(x,y) - G\varepsilon'_{y,C\Omega}(x) \quad \text{and} \\ \int_0^\infty U_\Omega(t,y,x)dt = G^*(x,y) - G\varepsilon''_{y,C\Omega}(x) \end{gathered}$$

(see, for example, [11], p. 333). Put $G_\Omega(x,y) = \int_0^\infty U_\Omega(t,x,y)dt$. Then $G_\Omega(x,y)$ is the Green function of L on Ω. In this case,

$$\text{(5.22)} \qquad \lim_{y\to\partial\Omega} G_\Omega(x,y) = \lim_{y\to\partial\Omega} G_\Omega(y,x) = 0 \text{ for all } x \in \Omega\,.$$

To apply our main theorem to L, we need the following

THEOREM 70. *Two diffusion semi-groups $(T_{L,t})_{t\geqq 0}$ and $(T_{L^*,t})_{t\geqq 0}$ are regular.*

Proof. We shall show only that $(T_{L,t})_{t\geqq 0}$ is regular, because the other is proved similarly. By Remark 69, it suffices to show that $(T_{L,t})_{t\geqq 0}$ satisfies the condition (D^*). By Proposition 62, Remark 61 and (5.21), $(T_{L,t})_{t\geqq 0}$ is weakly regular. Let $(D_n)_{n=1}^\infty$ be a regular exhaustion of D and put $T_{n,t} = T_{L,D_n,t}$ $(t \geqq 0;\ n = 1, 2, \cdots)$. Since, for any $\mu \in M_K^+(D)$, $T_{n,t}\mu \leqq T_{L,t}\mu$ in D_n, $(T_{n,t})_{t\geqq 0}$ is also a transient and weakly regular diffusion semi-group on D_n. Let $V_{L,n}$ the Hunt diffusion kernel for $(T_{n,t})_{t\geqq 0}$. Then $V_{L,n}\mu = (G_{D_n}\mu)dx$ for any $n \geqq 1$. First we shall show that if, for any $n \geqq 1$, $(T_{n,t})_{t\geqq 0}$ satisfies the condition (D^*), then so is $(T_{L,t})_{t\geqq 0}$. For each $f \in C_K^+(D)$, we choose an integer $n_f \geqq 1$ such that $f \in C_K^+(D_n)$ for all $n \geqq n_f$. Let $(f_{n,m})_{m=1}^\infty$ be an associated family of f with respect to $(T_{n,t}^*)_{t\geqq 0}$ $(n \geqq n_f)$. By Proposition 62, we have

$$\text{(5.23)} \qquad V_{L,n}^*f \leqq V_{L,n+1}^*f \text{ in } D \text{ and } \lim_{n\to\infty} V_{L,n}^*f = V_L^*f \text{ in } C(D)^{17)}\,.$$

Hence we can choose inductively a sequence $(f_{n_k,m_k})_{k=1}^\infty$ satisfying the following conditions (5.24), (5.25) and (5.26), where $n_1 \geqq n_f$ and $n_k < n_{k+1}$:

$$\text{(5.24)} \qquad V_L^*f - V_{L,n_k}^*f < \frac{1}{k} \text{ on } \overline{D}_{n_{k-1}}\,,$$

17) We put $V_{L,n}^*f = 0$ on CD_n. Then $V_{L,n}^*f \in C_K^+(D)$ by (5.21) and (5.22).

$$V^*_{L,n_k} f_{n_k,m_k} < \frac{1}{k} \text{ on } \bar{D}_{n_{k-1}}, \tag{5.25}$$

$$V^*_{L,n_{k-1}} f - V^*_{L,n_{k-1}} f_{n_{k-1},m_{k-1}} \leqq V^*_{L,n_k} f - V^*_{L,n_k} f_{n_k,m_k} \text{ in } D. \tag{5.26}$$

We shall show that $(f_{n_k,m_k})_{k=1}^{\infty}$ is an associated family of f with respect to $(T^*_{L,t})_{t\geqq 0}$. Since, for any $n \geqq n_f$ and any $m \geqq 1$, $L^* V^*_{L,n}(f - f_{n,m}) = -f + f_{n,m}$ in the sense of distributions in D_n, $f_{n,m} \in C_K^+(D_n)$, and hence we may assume that $f_{n,m} \in C_K^+(D)$. We have

$$V^*_L f - V^*_L f_{n_k,m_k} = V^*_{L,n_k} f - V^*_{L,n_k} f_{n_k,m_k} \ (k \geqq 1), \tag{5.27}$$

because $L^*(V^*_{L,n_k} f - V^*_{L,n_k} f_{n_k,m_k}) = f - f_{n_k,m_k}$ in the sense of distributions in D. This implies that $V^*_L f \geqq V^*_L f_{n_k,m_k}$ and $\mathrm{supp}(V^*_L f - V^*_L f_{n_k,m_k})$ is compact. By (5.24), (5.25), (5.26) and (5.27), we have $V^*_L f_{n_{k-1},m_{k-1}} \geqq V^*_L f_{n_k,m_k}$ in D and $V^*_L f_{n_k,m_k} \leqq 2/k$ on $\bar{D}_{n_{k-1}}$. Thus we see that $(f_{n_k,m_k})_{k=1}^{\infty}$ is an associated family of f with respect to $(T^*_{L,t})_{t\geqq 0}$. Consequently, it suffices to show that, for any subdomain Ω of D satisfying the condition (S), $(T_{L,\Omega,t})_{t\geqq 0}$ satisfies the condition (D^*). For a fixed $y_0 \in C\Omega$, we put $h(x) = G^*(x, y_0)$ for each $x \in \Omega$. Then $\inf_{x\in\Omega} h(x) > 0$, $h \in C^\infty(\Omega)$ and $L^* h = 0$ in Ω. Let $f \in C_K^+(\Omega)$, and put $G^*_\Omega(x, y) = G_\Omega(y, x)$ and

$$a = \min_{x\in \mathrm{supp}(f)} \frac{G^*_\Omega f(x)}{h(x)} > 0. \tag{5.28}$$

We choose a sequence $(\varphi_n)_{n=1}^{\infty} \subset C_K^\infty(R^1)$ such that, for each $n \geqq 1$, $\mathrm{supp}(\varphi_n) \subset (a/(n+2), a/(n+1))$ and $\int \varphi_n(r)dr = 1$. For any $0 < r < a$, we put

$$\Omega_r = \{x \in \Omega; G^*_\Omega f(x) > rh(x)\}. \tag{5.29}$$

Then Ω_r is an open set with $\bar{\Omega}_r \subset \Omega$, because $G^*_\Omega f(x) \to 0$ as $x \to \partial\Omega$. Let $V_{L,\Omega}$ and $A_{L,\Omega}$ be the Hunt diffusion kernel for $(T_{L,\Omega,t})_{t\geqq 0}$ and the infinitesimal generator of $(T_{L,\Omega,t})_{t\geqq 0}$, respectively. Then, for any $V_{L,\Omega}\mu \in \mathscr{D}_K^+(A_{L,\Omega}; \Omega_r)$,

$$\int (G^* f - rh)^+ d\mu = \int f dV_{L,\Omega}\mu - r\int G(y_0, x)d\mu(x) = \int f dV_{L,\Omega}\mu, \tag{5.30}$$

because $\mathrm{supp}(\mu) \subset \Omega_r$. Hence Corollary 43 and (5.21) give that

$$(G^* f - rh)^+(x) = \int fd(V_{L,\Omega}\varepsilon_x - V_{L,\Omega}\varepsilon'_{x,C\Omega_r}) = G^*_\Omega f(x) - G^*_\Omega f''_{C\Omega_r}(x) \text{ in } \Omega, \tag{5.31}$$

where $\varepsilon'_{x,C\Omega_r}$ is the $V_{L,\Omega}$-balayaged measure of ε_x on $C\Omega_r$ and $f''_{C\Omega_r}$ is the $V_{L^*,\Omega}$-balayaged measure of fdx on $C\Omega_r$. Put

$$g_n(x) = \int G^* f''_{C\Omega_r}(x)\varphi_n(r)dr \quad (n = 1, 2, \cdots)\ . \tag{5.32}$$

Then we have

$$g_n(x) = G^*_\Omega f(x) - h(x)\varphi_n * \psi\left(\frac{G^*f(x)}{h(x)}\right) \text{ in } \Omega\ , \tag{5.33}$$

where $\psi(t) = t$ in $(0, \infty)$ and $\psi(t) = 0$ in $(-\infty, 0]$. By (5.32), $g_n \in C^\infty(\Omega_{a/2})$ and, by (5.33), $g_n \in C^\infty(\Omega - \operatorname{supp}(f))$, i.e., $g_n \in C^\infty(\Omega)$ $(n = 1, 2, \cdots)$. By (5.32), $(g_n)_{n=1}^\infty$ converges decreasingly to 0 as $n \to \infty$. Since $\operatorname{supp}(G^*_\Omega f - g_n) \subset \bar{\Omega}_{a/(n+1)}$, $G^*f - g_n$ is with compact support in Ω. Since, for any $x \in \Omega$, the function $G^*_\Omega f''_{C\Omega_r}(x)$ of r is finite continuous in $(0, a)$, (5.17) gives that $(0, a) \ni r \to f''_{C\Omega_r}$ is vaguely continuous, and hence $\int f''_{C\Omega_r}\varphi_n(r)dr$ is defined. Putting $f_n = -L^* g_n$, we see that $f_n \in C^+_K(\Omega)$ and $f_n = \int f''_{C\Omega_r}\varphi_n(r)dr$ in the sense of distributions. Thus $(f_n)_{n=1}^\infty$ is an associated family of f with respect to $(T^*_{L,\Omega,t})_{t\geqq 0}$. This completes the proof.

In the usual way, we define the L-superharmonicity and the L-harmonicity.

DEFINITION 71. A function u in D is said to be L-superharmonic (resp. L-harmonic) if u satisfies the following three conditions:

(5.34) u is lower semi-continuous (resp. continuous).

(5.35) $-\infty < u \leqq \infty$, $u \not\equiv \infty$ (resp. $-\infty < u < \infty$).

(5.36) u is a locally integrable function in D and $-L\mu \geqq 0$ (resp. $Lu = 0$) in the sense of distributions.

Similarly we define the L^*-superharmonicity and the L^*-harmonicity.

PROPOSITION 72. *Let u be a lower semi-continuous function in D satisfying $-\infty < u \leqq \infty$ and $u \not\equiv \infty$. Then the following three conditions are equivalent:*

(1) *u is L-superharmonic.*

(2) *If Ω is a relatively compact subdomain in D and if v is continuous on $\bar{\Omega}$, L-harmonic in Ω and satisfies $v(x) \leqq u(x)$ on $\partial\Omega$, then $v(x) \leqq u(x)$ in Ω.*

(3) *For any relatively compact subdomain Ω in D and any $x \in \Omega$,*

(5.37) $$u(x) \geqq \int u(y) d\varepsilon''_{x,C\Omega}(y) ,$$

where $\varepsilon''_{x,C\Omega}$ *is the* V_{L^*}*-balayaged measure of* ε_x *on* $C\Omega$.

Proof. The equivalence between (1) and (2) is shown by S. Itô (see, [12], Theorem 2).

(2)⇨(3). Let $(\Omega_n)_{n=1}^{\infty}$ be a regular exhaustion of Ω such that $\Omega_1 \ni x$. It is well-known that, for any $f \in C(\partial\Omega_n)$, the function $\int f d\varepsilon''_{x,C\Omega_n}$ of x is L-harmonic in Ω_n (see, for example, [11]). In particular, if $f \leqq u$ on $\partial\Omega_n$, then (2) gives that $u(x) \geqq \int f d\varepsilon''_{x,C\Omega_n}$. By letting $f \uparrow u$ and $n \to \infty$, we obtain the required inequality.

The implication (3)⇨(2) is directly followed from Proposition 42 and Corollary 43. This completes the proof.

COROLLARY 73. *Let* u *and* v *be L-superharmonic functions in* D. *If* $u = v$ dx*-a.e. in* D, *then* $u = v$ *everywhere.*

Proof. First we remark that, for any $x \in D$, $G(x, x) = \infty$. Let Ω be a subdomain of D satisfying the condition (S). For a fixed $y \in C\Omega$, put $h(x) = G^*(x, y)$ on Ω. For any $x_0 \in \Omega$ and $r > 0$, we denote by Ω_r the connected component of $\{x \in \Omega; G^*(x_0, x) > rh(x)\}$ with $\Omega_r \ni x_0$ and choose $\varphi_n \in C_K^\infty(R^1)$ such that $\varphi_n \geqq 0$, $\int \varphi_n(r) dr = 1$ and $\mathrm{supp}(\varphi_n) \subset (n, n+1)$ $(n = 1, 2, \cdots)$. Similarly as in Theorem 70, $\int \varepsilon''_{x_0,C\Omega_r} \varphi_n(r) dr \in C_K^\infty(\Omega)$ in the sense of distributions, and hence

(5.38) $$\int \left(\int u d\varepsilon''_{x_0,C\Omega_r} \right) \varphi_n(r) dr = \int \left(\int v d\varepsilon''_{x_0,C\Omega_r} \right) \varphi_n(r) dr .$$

Since $\left(\int \varepsilon''_{x_0,C\Omega_r} \varphi_n(r) dr \right)_{n=1}^{\infty}$ converges vaguely to ε_{x_0} as $n \to \infty$, the lower semi-continuity of u, that of v and (3) in Proposition 72 imply that $u(x_0) = v(x_0)$. The subdomain Ω and x_0 being arbitrary, we see Corollary 73.

By the above corollary, we obtain the following

PROPOSITION 74. *Let* $\mu \in M(D)$. *If* μ *is* A_L*-superharmonic (resp.* A_{L^*}*-superharmonic), then there exists one and only one L-superharmonic (resp. L*-superharmonic) function* u *in* D *such that* $\mu = udx$.

Conversely, for an L-superharmonic (resp. L-superharmonic) function*

u in D, udx is A_L-superharmonic (resp. A_{L^*}-superharmonic).

In order to prove Proposition 74, we use the following known lemma.

LEMMA 75 (see [18], p. 143). *Let Ω be a domain in the N-dimensional Euclidean space R^N ($N \geqq 1$) and L be an elliptic differential operator of the analogous form to* (5.1). *If, for $\mu \in M(\Omega)$, $L\mu \in C^\infty(\Omega)$ in the sense of distributions, then $\mu \in C^\infty(\Omega)$ in the sense of distributions. In particular, $L\mu = 0$ in Ω implies $\mu \in C^\infty(\Omega)$ in the sense of distributions.*

Proof of Proposition 74. Let $\mu \in M(D)$ be A_L-superharmonic. Then Remark 69 gives that $-L\mu \geqq 0$ in the sense of distributions. Let ω be a subdomain of D satisfying the condition (S) and λ_ω be the restriction of the positive measure $-L\mu$ to ω. Put $\lambda = \mu - (G\lambda_\omega)dx$ in ω. Then $L\lambda = 0$ in ω, and hence $\lambda = \varphi dx$ in ω by Lemma 75, where $\varphi \in C^\infty(\omega)$. The subdomain ω being arbitrary, we obtain that $\mu = udx$, where u is an L-superharmonic function in D. By Corollary 73, u is uniquely determined. Let u be an L-superharmonic function in D and put $\mu = udx$. Since $-L\mu \geqq 0$ in the sense of distributions in D, Remark 69 gives that μ is A_L-superharmonic if $\mu \in \mathscr{D}^0(A_L)$. Let $V_L^* f \in R_K(A_L)$. Then supp(f) is compact, and hence $\int |f|\, d\mu < \infty$, which implies $\mu \in \mathscr{D}^0(A_L)$. Thus μ is A_L-superharmonic. The rest of proof is similar. This completes the proof.

This implies evidently the following

COROLLARY 76. *The infinitesimal generators A_L and A_{L^*} satisfy the condition ($\mathscr{L}$).*

We denote by $S(L)$ the convex cone of all non-negative L-superharmonic functions in D and by $H(L)$ the convex cone of all non-negative L-harmonic functions in D.

By Theorem 35, Corollary 73 and Proposition 74, we obtain the well-known Riesz decomposition theorem.

Remark 77. For each $u \in S(L)$, there exists uniquely $(\nu, h) \in M^+(D) \times H(L)$ such that $\mu = G\nu + h$.

Now we discuss the Martin compactification of D for L.

PROPOSITION 78. *The Martin compactification D^* of D for L is defined. Let $\mathfrak{S}_1$ be the essential part of the Martin boundary $\Gamma = D^* - D$*[18] *and*

18) $\mathfrak{S}_1 = \{\xi \in \Gamma;$ the harmonic function $K(x, \xi)$ of x is minimal$\}$. A positive harmonic function u in D is said to be minimal if, for any positive harmonic function v in D, $v = cu$ with a positive constant c whenever $u \geqq v$ in D.

$K(x, \xi)$ be the Martin kernel on $D \times \Gamma$. If h is positive L-harmonic in D, then there exists one and only one regular Borel positive measure μ on $\mathfrak{S}_1$ with $\int d\mu < \infty$ such that

$$h(x) = \int_{\mathfrak{S}_1} K(x, \xi) d\mu(\xi) \text{ in } D . \tag{5.39}$$

In the case of $c(x) \equiv 0$, the same assertion is obtained by S. Itô (see, [11], Theorem 5.3). Similarly we can prove Proposition 79 (see also [6], Chapter 11 and [18]).

For a constant $c > 0$, we discuss non-negative solution of the following ideal boundary value problem:

$$\begin{cases} -Lu(x) = cu(x) \text{ for any } x \in D \\ \underline{\lim}_{\substack{y \to \xi \\ y \in D}} u(y) = 0 \ \lambda_{x_0} - \text{a.e. on } \Gamma , \end{cases} \tag{5.40}$$

where λ_{x_0} is the harmonic measure for a certain $x_0 \in D$.

Denote by $E_0(L; c)$ the set of non-negative functions of class C^∞ in D satisfying (5.40) and by $E_0(L) = \bigcup_{c \geq 0} E_0(L; c)$.

PROPOSITION 79. *Let c be a non-negative constant. For each $\mu \in E_0(A_L; c)$, there exists one and only one $u \in E_0(L; c)$ such that $\mu = udx$. Conversely, for any $u \in E_0(L; c)$, we have $udx \in E_0(A_L; c)$.*

Proof. Since $E_0(A_L; 0) = \{0\}$ and $E_0(L; 0) = \{0\}$, it suffices to show our conclusion in the case $c > 0$. Let μ be a nonzero element of $E_0(A_L; c)$. Then, by Propositions 45, 74, Corollary 73 and Remark 77, there exists one and only one $u \in S(L)$ such that $\mu = udx$ and $u = cGu$. Since the function

$$\int \underline{\lim}_{\substack{y \to \xi \\ y \in D}} u(y) \frac{K(x, \xi)}{K(x_0, \xi)} d\lambda_{x_0}(\xi)$$

of x is L-harmonic and $\leq u$ in D, the second equality in (5.40) holds. Hence it suffices to show that $u \in C^\infty(D)$. We put inductively $G^{n+1}(x, y) = \int G^n(x, z) G(z, y) dz$ and $G^n u(x) = \int G^n(x, y) u(y) dy$ for $n = 1, 2, \cdots$, where $G^1(x, y) = G(x, y)$. Then we have $u = c^n G^n u$. Let Ω be a relatively compact subdomain of D. When we consider L as a differential operator in Ω, L is uniformly elliptic and all coefficients of L are of class C^∞ on $\bar{\Omega}$.

Hence, for any $n \geqq N/2 + 1$, $G_\Omega^n(x, y)$ is finite continuous in $\Omega \times \Omega$ (see, for example, [15], p. 1288), where the function $G_\Omega^n(x, y)$ is defined analogously to $G^n(x, y)$. Let Ω_1 be another subdomain of D such that $\bar{\Omega}_1 \subset \Omega$ and f be in $C_K^+(D)$ such that $0 \leqq f \leqq 1$, $f(x) = 1$ on $\bar{\Omega}_1$ and $\mathrm{supp}(f) \subset \Omega$. Put $u_1 = fu$ and $u_2 = (1 - f)u$. Then $G_\Omega^n u_1$ is finite continuous in Ω whenever $n \geqq N/2 + 1$. By remarking that, for any $k \geqq 1$,

$$G^{k+1}u_1 - G_\Omega^{k+1}u_1 = G(G^k u_1 - G_\Omega^k u_1) + G(G_\Omega^k u_1) - G_\Omega(G_\Omega^k u_1) \tag{5.41}$$

and that, for any non-negative locally integrable function g with $g \leqq u$, $Gg - G_\Omega g$ is of class C^∞ in Ω (see Lemma 75 and Corollary 73), we obtain inductively that $G^n u_1 - G_\Omega^n u_1 \in C^\infty(\Omega)$ $(n = 1, 2, \cdots)$. On the other hand, Gu_2 is of class C^∞ in Ω_1 by Lemma 75. Let Ω_2 be a subdomain of D such that $\bar{\Omega}_2 \subset \Omega_1$ and φ be in $C_K^\infty(D)$ such that $0 \leqq \varphi \leqq 1$, $\varphi(x) = 1$ on $\bar{\Omega}_2$ and $\mathrm{supp}(\varphi) \subset \Omega_1$. Then $G((1 - \varphi)Gu_2)$ is of class C^∞ in Ω_2 and $G(\varphi Gu_2) \in C^\infty(D)$, because $\varphi Gu_2 \in C_K^\infty(D)$. The subdomain Ω_2 being arbitrary, $G^2 u_2$ is of class C^∞ in Ω_1. Inductively we see that, for any $n \geqq 1$, $G^n u_2$ is of class C^∞ in Ω_1. Thus $G^n u$ is finite continuous in Ω_1 if $n \geqq N/2 + 1$. The subdomain Ω and Ω_1 being arbitrary, $u \in C(D)$. Since $u_1 \in C_K^+(D)$, $G_\Omega^n u_1 \in C^n(\Omega)$ $(n = 1, 2, \cdots)$, and hence $G^n u_1 \in C^n(\Omega)$. Consequently $G^n u \in C^n(\Omega)$ $(n = 1, 2, \cdots)$, and so $u \in C^\infty(D)$.

Let $u \in E_0(L; c)$. Then, by Remark 77, $u = cGu + h$, where $h \in H(L)$. Since, for any $x \in D$, $\underline{\lim}_{\substack{y \to \xi \\ y \in D}} u(y) = 0$ λ_x-a.e. on Γ, the harmonic part h of u is equal to 0, which implies that $udx \in E_0(A_L; c)$. This completes the proof.

Definition 80. A function u in D is said to be completely L-superharmonic in D if, for any integer $n \geqq 0$, $(-L)^n u$ is L-superharmonic in D, where $(-L)^0 u = u$ and $(-L)^n u$ is in the sense of distributions.

In particular, a completely L-superharmonic function u in D is said to be with zero conditions if $\underline{\lim}_{\substack{y \to x \\ y \in D}} (-L)^n u(y) = 0$ for any $x \in \mathfrak{S}_1$ and any $n = 0, 1, \cdots$.

We denote by $SC(L)$ the convex cone formed by all non-negative completely L-superharmonic functions in D and by $SC_0(L)$ the convex cone formed by all non-negative completely L-superharmonic functions in D with zero conditions.

Similarly as above, we see the following

Proposition 81. *For each $\mu \in SC(A_L)$ (resp. $\in SC_0(A_L)$), there exists one* *and only one* $u \in SC(L)$ *(resp.* $\in SC_0(L)$*) such that* $\mu = udx$*. Conversely, for any* $u \in SC(L)$ *(resp.* $\in SC_0(L)$*),* $udx \in SC(A_L)$ *(resp.* $\in SC_0(A_L)$*).*

Applying Theorem 53 to completely L-superharmonic functions, we obtain the following

THEOREM 82. *We have* $SC(L) \subset C^\infty(D)$ *and the following assertions hold*:

(1) *If there exists an integer* $k \geqq 1$ *such that, for any* n *with* $1 \leqq n \leqq k$, $(V_L)^n$ *is defined as a diffusion kernel in* D *and that* $(V_L)^{k+1}$ *is not defined, then, for each* $u \in SC(L)$*, there exists uniquely a finite family* $(\lambda_j)_{j=0}^{k-1}$ *of non-negative regular Borel measures on* $\mathfrak{S}_1$ *with* $\int d\lambda_j < \infty$ $(j = 0, 1, \cdots, k-1)$ *such that*

$$u(x) = \sum_{n=0}^{k-1} \int_{\mathfrak{S}_1} G^n \cdot K(x, \xi) d\lambda_n(\xi) , \tag{5.42}$$

where $G^0 \cdot K(x, \xi) = K(x, \xi)$ *and* $G^n \cdot K(x, \xi) = \int G^n(x, y) K(y, \xi) dy$.

(2) *If, for any integer* $n \geqq 1$, $(V_L)^n$ *is defined as a diffusion kernel on* D*, then, for each* $u \in SC(L)$*, there exist a sequence* $(\lambda_n)_{n=0}^\infty$ *of non-negative regular Borel measures on* $\mathfrak{S}_1$ *with* $\int d\lambda_n < \infty$ $(n = 0, 1, \cdots)$*, a non-negative Borel measure* σ *on* $(0, \infty)$ *with* $\int d\sigma < \infty$ *and a* σ*-measurable mapping* $(0, \infty) \ni t \to u_t \in C^\infty(D)$ *with* $u_t \in E(L; t)$[19] *such that, for any* $y \in D$,

$$u(y) = \sum_{n=0}^{\infty} \int_{\mathfrak{S}_1} G^n \cdot K(y, \xi) d\lambda_n(\xi) + \int_0^\infty u_t(y) d\sigma(t) . \tag{5.43}$$

Furthermore $(\lambda_n)_{n=0}^\infty$ *is uniquely determined.*

Proof. We first consider the case where the assumption of (1) holds. Let $u \in SC(L)$. Similarly as in Proposition 47, there exist uniquely a finite family $(h_n)_{n=0}^{k-1} \subset H(L)$ and $\nu \in \mathscr{D}^+((V_L)^k)$ such that

$$udx = \sum_{n=0}^{k-1} (V_L)^n (h_n dx) + (V_L)^k \nu . \tag{5.44}$$

Since $\nu \in S(A_L)$, Theorem 35 gives that $\nu = V_L(-A_L\nu) + h_k dx$, where $h_k \in H(L)$. Assume that $\nu \neq 0$. Let $\mu \in M_K^+(D)$ and Ω be a subdomain of D

19) We say that $t \to u_t \in C_\infty(D)$ is σ-measurable if, for any $x \in D$, the function $u_t(x)$ of t is σ-measurable.

satisfying the condition (S) and $\mathrm{supp}(\mu) \subset \Omega$. We denote by $\mu'_{C\Omega}$ the V_L-balayaged measure of μ on $C\Omega$. Then $V_L\mu - V_L\mu'_{C\Omega} \in \mathscr{D}((V_L)^k)$ and, by $\mathrm{supp}(\mu'_{C\Omega}) \subset \partial\Omega$ and the domination principle for V_L, there exists a constant $c > 0$ such that $V_L\mu'_{C\Omega} \leqq c\nu$. Since $\nu \in \mathscr{D}((V_L)^k)$, $V_L\mu \in \mathscr{D}^+((V_L)^k)$, and hence the mapping $M_K(D) \ni \mu \to (V_L)^k(V_L\mu) \in M(D)$ is defined and continuous, i.e., $(V_L)^{k+1}$ is defined as a diffusion kernel, which contradicts our assumption. This, Proposition 78 and (5.44) give (5.42), and (5.42) gives that $SC(L) \subset C^\infty(D)$.

Next we consider the case where the assumption of (2) holds. We remark that, for any $y \in D$, the mapping

$$M^+(D) \supset \{vdx; v \in E_0(L)\} \in vdx \to v(y) \in R^+ \tag{5.45}$$

is lower semi-continuous. This follows from the existence of a sequence $(f_n)_{n=1}^\infty \subset C_K^+(D)$ satisfying $\lim_{n\to\infty} f_n dx = \varepsilon_y$ (vaguely) and $v(y) \geqq \int v(z)f_n(z)dz$ for all $v \in S(L)$ (see the proof of Corollary 73). Let $u \in SC(L)$. By using Theorem 53, there exist a sequence $(h_n)_{n=0}^\infty \subset H(L)$, a non-negative Borel measure σ on $(0, \infty)$ with $\int d\sigma < \infty$ and a bounded σ-measurable mapping $(0, \infty) \ni t \to u_t dx \in E_0(A_L)$ with $u_t \in E_0(L; t)$ such that

$$udx = \sum_{n=0}^\infty (V_L)(h_n dx) + \int_0^\infty (u_t dx)d\sigma(t)\,. \tag{5.46}$$

Hence Corollary 73 and (5.45) give that, for any $x \in D$, $(0, \infty) \ni t \to u_t(x)$ is σ-measurable and that

$$u(x) = \sum_{n=0}^\infty G^n h_n(x) + \int_0^\infty u_t(x)d\sigma(t)\,. \tag{5.47}$$

This fact, Proposition 78 and the unicity of $(h_n)_{n=0}^\infty$ imply the assertion (2). It remains to show $SC(L) \subset C^\infty(D)$ under the assumption of (2). Let n be an integer $\geqq N/2 + 1$ and put $v_n = \int_0^\infty t^n u_t d\sigma(t)$. Then $(-L)^n \left(\int_0^\infty u_t d\sigma(t)dx\right)$ $= v_n dx$ in the sense of distributions in D, i.e., v_n is locally integrable. Similarly as in Proposition 79, $G^n v_n \in C(D)$, and $\int_0^\infty u_t d\sigma(t) = G^n v_n$ (see corollary 73). In the same manner, $(-L)^n u \in C(D)$ in the sense of distributions for all $n \geqq 1$. This implies that $\int_0^\infty u_t d(t) \in C^\infty(D)$, and also, in the same manner as in Proposition 79, $\sum_{n=k}^\infty G^{n-k} h_n(x)$ is finite continuous in

D $(k = 0, 1, \cdots)$, $\sum_{n=0}^{\infty} G^n h_n \in C^\infty(D)$. This completes the proof.

M. V. Noviskiĭ [15] discusses completely L-superharmonic functions in the following setting. Let D be a bounded domain in R^N $(N \geqq 2)$ of class $C^{1,\lambda}$ $(\lambda > 0)$[20] and L be a uniformly elliptic differential operator of the form

$$(5.48) \qquad Lu(x) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x)$$

with coefficients $\in C^\infty(\bar{D})$, for $u \in C^2(D)$ and $x = (x_1, x_2, \cdots, x_N) \in D$, where $a_{ij}(x) = a_{ji}(x)$ and $c(x) \leqq 0$.

Evidently there exists the Green function $G(x, y)$ of L on D and we have $\lim_{\substack{x \to z \\ x \in D}} G(x, y) = \lim_{\substack{x \to z \\ x \in D}} G(y, x) = 0$ for any $y \in D$ and any $z \in \partial D$.

Theorem 82 gives the main theorem of M. V. Noviskiĭ's paper [15].

COROLLARY 83. *Let D be a bounded domain in R^N $(N \geqq 2)$ of class $C^{1,\lambda}$ $(\lambda > 0)$ and L be given in* (5.58). *Denote by φ_1 a first eigen function $\geqq 0, \neq 0$ of L with zero conditions on ∂D. A completely L-superharmonic function u in D*[21] *has the form*

$$(5.49) \qquad u(x) = \sum_{k=0}^{\infty} \int_{\partial D} - \frac{\partial G^{k+1}}{\partial n_y}(x, y) d\mu_k(y) + c\varphi_1(x) ,$$

where $\partial/\partial n_y$ denotes the outer normal derivative on ∂D, μ_k is a non-negative measure on ∂D $(k = 0, 1, \cdots)$ and c is a non-negative constant. Furthermore $(\mu_k)_{k=0}^{\infty}$ and c are uniquely determined.

LEMMA 84 (see, [15], Lemma 3). *Under the same conditions as above, a non-negative L-superharmonic function f in D is integrable if $f \in C^2(D)$.*

Proof of Corollary 83. Similarly as in [11], § 6, we may assume that the kernel $-(\partial/\partial n_y)G(x, y)$ on $D \times \partial D$ is the Martin kernel for L and that ∂D is the essential part of the Martin boundary. We remark that

$$(5.50) \qquad -\frac{\partial G^{k+1}}{\partial n_y}(x, y) = -\int G^k(x, z) \frac{\partial G}{\partial n_y}(z, y) dz \text{ on } D \times \partial D \ (k = 1, 2, \cdots)$$

20) The domain D belongs to the class $C^{k,\lambda}$ $(\lambda > 0)$ if for an arbitrary $x_0 \in \partial D$ there exists a neighborhood of x_0 in which ∂D is specified by an equation $x_i = f(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N)$, where $x = (x_1, x_2, \cdots, x_N) \in \partial D$ and f is a k-times continuously differentiable function, the k-th derivatives of which satisfy a Hölder condition with exponent $\lambda > 0$.

21) By Noviskii's definition, it is an infinitely differentiable function which satisfies the condition $(-L)^n u(x) \geqq 0$, $x \in D$, $n = 0, 1, \cdots$.

and that there exists a first eigen function $\varphi_1 \geqq 0$, $\neq 0$ of L with zero conditions on ∂D (see [13], Theorem 7.10). Hence it suffices to show that $E_0(L) = \{a\varphi_1; a \in R^+\}$. Evidently $E_0(L) \ni \varphi$. By Proposition 79 and Lemma 84, we have, for any $v \in E_0(L)$, $\int v dx < \infty$, so that $G^n v$ is bounded if $n \geqq N/2 + 1$, i.e., v is bounded, and hence $\lim_{\substack{y \to x \\ y \in D}} Gv(y) = 0$ for any $x \in \partial D$, i.e., $\lim_{\substack{y \to x \\ y \in D}} v(y) = 0$ for any $x \in \partial D$. Thus we see that, for any $v \in E_0(L)$, $\int v^2 dx < \infty$. It is also known that there exists a first eigen function $\varphi_1^* \geqq 0$, $\neq 0$ of L^* (see also [13], Theorem 7.10). Evidently $\int (\varphi_1^*)^2 dx < \infty$. Let $c^* > 0$ be the eigen value of φ_1^*. Then $\varphi_1^* = c^* G^* \varphi_1^*$. For any $v \neq 0 \in E_0(L)$, there exists $c > 0$ such that $v = cGv$, which implies that $v > 0$ on D. Since

$$(5.51) \qquad \int \varphi_1^* \cdot v dx = c^* \int G^* \varphi_1^* \cdot v dx = c^* \int \varphi_1^* \cdot G v dx = \frac{c^*}{c} \int \varphi_1^* \cdot v dx \,,$$

we have $c = c^*$, this implies that $E_0(L) = E_0(L; c^*)$. Thus we see that, for any $v \in E_0(L)$ and any real number t, $\varphi_1 - tv$ is also a first eigen function of L with zero conditions on ∂D. By remarking that any first eigen function of L with zero conditions on D takes always non-negative values or non-positive values (see [13]), we obtain that, for any $v \in E_0(L)$ $v = a\varphi_1$ with $a \in R^+$. This completes the proof.

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