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# HOMOTOPY CLASSIFICATION OF CONNECTED SUMS OF SPHERE BUNDLES OVER SPHERES, I

***Dedicated to Professor A. Komatu on his 70th birthday***

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## Introduction

In the classification problems of manifolds, the connected sums of sphere bundles over spheres appear frequently. In fact, the manifolds with certain tangential and homotopy properties come to such connected sums (cf. Tamura [15], [16], Ishimoto [6], [8], [9]). Motivated by those, in this paper and the subsequent paper, we classify connected sums of sphere bundles over spheres up to homotopy equivalence by extending the results of I. M. James and J. H. C. Whitehead [10], [11], which correspond to the case that the connected sums of the above happen to be single sums. We also use Wall [17] in the case when the fibres and the base spaces of bundles are same dimensional.

In this paper*⁾, we treat with the case that bundles admit cross-sections, and in the subsequent paper, we discuss the general case.

Let $A$ be a $p$-sphere bundle over a $q$-sphere $(p, q > 1)$ which admits a cross-section, and consider the diagram

$$\begin{array}{ccccc} & \overset{\partial}{\nearrow} & \pi_{q-1}(SO_p) & \xrightarrow{i_*} & \pi_{q-1}(SO_{p+1}) \\ \pi_q(S^p) & & \downarrow J & & \downarrow J \\ & \underset{P}{\searrow} & \pi_{p+q-1}(S^p) & \xrightarrow{E} & \pi_{p+q}(S^{p+1}) \end{array},$$

where $\partial$, $i_*$ belong to the homotopy exact sequence of the fibering $SO_p \to SO_{p+1} \to S^p = SO_{p+1}/SO_p$, $P = [\ ,\iota_p]$ (the Whitehead product with the orientation generator $\iota_p$ of $\pi_p(S^p)$), $E$ is the suspension homomorphism, and $J$ means the $J$-homomorphism. The diagram commutes up to sign,

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that is, $P = -J \circ \partial$, $E \circ J = -J \circ i_*$. We denote the characteristic element of $A$ by $\alpha(A)$. Since $A$ has a cross-section, $\alpha(A) = i_* \xi$ for some $\xi \in \pi_{q-1}(SO_p)$. Then, $\{J\xi\} \in J\pi_{q-1}(SO_p)/P\pi_q(S^p)$ does not depend on the choice of $\xi$. We denote it by $\lambda(A)$ (James-Whitehead [10]).

Let $A_i$, $i = 1, 2, \cdots, r$, be $p$-sphere bundles over $q$-spheres which admit cross-sections. It is understood that each $A_i$ also denotes the total space of the bundle and has the differentiable structure induced from those of the fibre and the base space. $\#_{i=1}^r A_i$ means the connected sum $A_1 \# A_2 \# \cdots \# A_r$ of the total spaces. When given another set of such bundles $A'_i$, $i = 1, 2, \cdots, r'$, if $\#_{i=1}^{r'} A'_i$ has the homotopy type of $\#_{i=1}^r A_i$, then $r'$ must be equal to $r$ by those homological aspect.

THEOREM 1. *Let $A_i, A'_i$, $i = 1, 2, \cdots, r$, be $p$-sphere bundles over $q$-spheres which admit cross-sections, and assume that $2p > q$, $q > 1$, $p \neq q$. Then, the connected sums $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ are of the same homotopy type if and only if there exists a unimodular $(r \times r)$-matrix $L$ of integer components such that*

$$\begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix} = L \begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix},$$

*where the abelian group $J\pi_{q-1}(SO_p)/P\pi_q(S^p)$ is considered as a left $Z$-module.*

Let $H$ be a free abelian group of finite rank. The homomorphisms $f, f' : H \to J\pi_{q-1}(SO_p)/P\pi_q(S^p)$ are said to be *equivalent* if and only if there exists an isomorphism $h : H \to H$ such that $f = f' \circ h$. Then, the following is an equivalent form of Theorem 1 and is easily verified.

THEOREM 1′. *Let $2p > q$, $q > 1$, $p \neq q$, and let $r$ be fixed. Then, homotopy equivalence classes of connected sums consisting of $r$ $p$-sphere bundles over $q$-spheres which admit cross-sections correspond bijectively with equivalence classes of homomorphisms of $H$ to $J\pi_{q-1}(SO_p)/P\pi_q(S^p)$, where $H$ is a free abelian group of rank $r$.*

The case $p = q$ is complicated and we can't apply the technique used in the proof of Theorem 1. But, in this case, the manifolds which we are considering are $(p-1)$-connected and $2p$-dimensional. So that, Wall [17] is applicable. We have the following analogue of Theorem 1.

THEOREM 2. *Let $A_i, A'_i$, $i = 1, 2, \cdots, r$, be $p$-sphere bundles over $p$-spheres and assume that $p > 2$ and $p \neq 4, 8$. Then, the connected sums*

$\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ *are of the same homotopy type if and only if there exists a unimodular* $(r \times r)$*-matrix* $L$ *of integer components such that*

$$\begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix} = L \begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix}.$$

In the case $p = q$, $\lambda(A_i)$, $\lambda(A'_i)$, $i = 1, 2, \cdots, r$, belong to

$$J\pi_{p-1}(SO_p)/P\pi_p(S^p) \cong J\pi_{p-1}(SO) \cong Z/mZ,$$

where

$$m = \begin{cases} 1 & \text{if } p = 3, 5, 6, 7 \pmod 8, \\ 2 & \text{if } p = 1, 2 \pmod 8, \\ m(2s) & \text{if } p = 4s \ (s > 0), \end{cases}$$

and $m(2s)$ is the denominator of $B_s/4s$ (Adams [1]). Represent $\lambda(A_i)$, $\lambda(A'_i)$ by the integers $\lambda_i$, $\lambda'_i$ respectively such that $0 \leqq \lambda_i$, $\lambda'_i \leqq m - 1$, $i = 1, 2, \cdots, r$. Then, it is easily seen that the following is an equivalent form of Theorem 2 for $r > 1$.

THEOREM 2$'$. *Let* $A_i$, $A'_i$, $i = 1, 2, \cdots, r$ $(r > 1)$, *be p-sphere bundles over p-spheres and assume that* $p > 2$ *and* $p \neq 4, 8$. *Then, the connected sums* $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ *are of the same homotopy type if and only if* G.C.D. $(\lambda_1, \cdots, \lambda_r, m) =$ G.C.D. $(\lambda'_1, \cdots, \lambda'_r, m)$. *Especially, if* $p = 1, 2 \pmod 8$, *then* $m = 2$ *and therefore, the connected sums are of the same homotopy type if and only if they have simultaneously non-trivial bundles or only trivial bundles.*

(Note: Originally, Theorem 2 was a little more complicated in formulation. The above simple forms which are equivalent to the original one were remarked to the author by J. Yoshida.)

*Remark* 1. In Theorem 2 and 2$'$, if $p = 3, 5, 6, 7 \pmod 8$, the theorems hold trivially since $\pi_{p-1}(SO_{p+1}) = 0$.

*Remark* 2. In Theorem 2 and 2$'$, even if $p = 4, 8$, the theorems hold if all $\alpha(A_i)$, $\alpha(A'_i)$ are even, that is, $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ are of type II (cf. Milnor [14]). In the case that there are odd numbers in $\alpha(A_i)$, $\alpha(A'_i)$, $i = 1, 2, \cdots, r$, that is, $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ are of type I (cf. Milnor [14]), we can also mention the necessary and sufficient condition for $\#_{i=1}^r A_i$, $\#_{i=1}^r A'_i$ to be homotopy equivalent using the invariants $\lambda(A_i)$, $\lambda(A'_i)$, $i = 1, 2, \cdots, r$, though it is complicated.

*Remark* 3. In the above theorems, $\lambda(A_i)$ can be replaced by $-J(\alpha(A_i))$ since the sequence

$$\pi_q(S^p) \xrightarrow{P} \pi_{p+q-1}(S^p) \xrightarrow{E} \pi_{p+q}(S^{p+1})$$

is exact if $2p > q - 1$. (cf. James-Whitehead [10, (1.10)] and (7.7) of James [12]).

Our results make sense for $r > 1$, but if $r = 1$, coincide with those of James-Whitehead [10] except precise conditions on $p$, $q$. The proofs of the above theorems are given in the following sections.

## 1. Cell structures of connected sums

Let $A_i$, $i = 1, 2, \cdots, r$, be $p$-sphere bundles over $q$-spheres $(p, q > 1)$ which admit cross-sections. Let $\alpha(A_i) = i_*(\xi_i)$ for some given $\xi_i \in \pi_{q-1}(SO_p)$, $i = 1, 2, \cdots, r$, and let $S_i^q$ be the cross-section of $A_i$ determined by $\xi_i$ as an orbit of the north pole of the $p$-sphere. Let $S_i^p$ be a fibre of $A_i$ suitably chosen, $i = 1, 2, \cdots, r$. For each $i$, $S_i^p \cap S_i^q$ is a single point $e_i^o$. We assume that $S_i^p$, $S_i^q$, and $A_i$, $i = 1, 2, \cdots, r$, are oriented compatibly.

Since $A_i - (S_i^p \cup S_i^q)$ is an open $(p + q)$-cell, $A_i$ has the cell structure $A_i = (S_i^p \cup S_i^q) \bigcup_{\varphi_i} D_i^{p+q}$ and the interior of $D_i^{p+q}$ may be assumed to be imbedded differentiably. By (3.7) of James-Whitehead [10], the homotopy class of the attaching map $\varphi_i : \partial D_i^{p+q} = S_i^{p+q-1} \to S_i^p \cup S_i^q$ is given by

$$\{\varphi_i\} = \iota_p^i \circ \eta_i + [\iota_q^i, \iota_p^i] , \tag{1}$$

where $\eta_i = J\xi_i$ and $\iota_p^i$, $\iota_q^i$ are the orientation generators of $\pi_p(S_i^p)$, $\pi_q(S_i^q)$ respectively. We adopt $e_i = (0, \cdots, 0, 1)$ as the base point of $S_i^{p+q-1}$, and assume that $\varphi_i(e_i) = e_i^o$.

Let $r = 2$. $A_1 \# A_2$ can be regarded as the union of $A_1 - \mathrm{Int}(\frac{1}{2}D_1^{p+q})$ and $A_2 - \mathrm{Int}(\frac{1}{2}D_2^{p+q})$ identified at $\frac{1}{2}S_1^{p+q-1} = \partial(\frac{1}{2}D_1^{p+q})$ and $\frac{1}{2}S_2^{p+q-1} = \partial(\frac{1}{2}D_2^{p+q})$ orientation reversingly, i.e. by the map $r(x_1, \cdots, x_{p+q}) = (-x_1, x_2, \cdots, x_{p+q})$, where $\frac{1}{2}D_i^{p+q}$ is the $(p + q)$-disk of radius $\frac{1}{2}$ with center $o$ of $D_i^{p+q}$, $i = 1, 2$. Let $b_i = (0, \cdots, 0, \frac{1}{2})$ be the base point of $\frac{1}{2}S_i^{p+q}$ and join $b_i$ to the base point $e_i$ of $S_i^{p+q-1}$ by a segment $w_i$, where $i = 1, 2$. Then, $e_1^o$ is joined to $e_2^o$ by the path $w = w_1^{-1} \cdot w_2$ in $A_1 \# A_2$. $D_i^{p+q} - \mathrm{Int}(\frac{1}{2}D_i^{p+q})$ can be considered as the product $S_i^{p+q-1} \times [\frac{1}{2}, 1]$ where $w_i$ corresponds to $e_i \times [\frac{1}{2}, 1]$. Hence, there is a canonical map $\Psi_i : C_i \times [\frac{1}{2}, 1] \to D_i^{p+q} - \mathrm{Int}(\frac{1}{2}D_i^{p+q})$, where $C_i$ is a $(p + q - 1)$-disk, which carries $C_i \times \frac{1}{2}$, $\partial C_i \times [\frac{1}{2}, 1]$, and $C_i \times 1$ to $\frac{1}{2}S_i^{p+q-1}$, $w_i$, and $S_i^{p+q-1}$ respectively and is a homeomorphism from $(C_i)^o \times (\frac{1}{2}, 1)$

onto the complement of $S_i^{p+q-1} \cup w_i \cup \frac{1}{2}S_i^{p+q-1}$. In fact, let $\pi_i: C_i \to S_i^{p+q-1} = C_i/\partial C_i$ be the canonical map. Then $\pi_i \times 1: C_i \times [\frac{1}{2}, 1] \to S_i^{p+q-1} \times [\frac{1}{2}, 1]$ will correspond to $\Psi_i$.

Let $E_i^{p+q}$, $i = 1, 2$, be copies of the unit $(p+q)$-disk and let $T_i^{p+q-1} = \partial E_i^{p+q} = U_i^+ \cup U_i^-$, $i = 1, 2$, where $U_i^+$ $(U_i^-)$ is the right (left) hemisphere of $T_i^{p+q-1}$, $i = 1, 2$, i.e. $U_i^+(U_i^-) = \{(x_1, \cdots, x_{p+q}) \in T_i^{p+q-1};\ x_1 \geqq 0\ (x_1 \leqq 0)\}$. Let $V_i^+$ $(V_i^-)$ be the subspace of $U_i^+$ $(U_i^-)$ such that $x_1 \geqq 1/\sqrt{2}$ $(x_1 \leqq -1/\sqrt{2})$ and $W_i^+$ $(W_i^-)$ be the subspace of $U_i^+$ $(U_i^-)$ such that $x_1 \leqq 1/\sqrt{2}$ $(x_1 \geqq -1/\sqrt{2})$. Then, there are canonical homeomorphisms $\Theta_i: E_i^{p+q} \to C_i \times [\frac{1}{2}, 1]$, $i = 1, 2$, such that $\Theta_1$ $(\Theta_2)$ maps $U_1^-$ $(U_2^+)$, $W_1^+$ $(W_2^-)$, and $V_1^+$ $(V_2^-)$ respectively to $C_1 \times 1$ $(C_2 \times 1)$, $\partial C_1 \times [\frac{1}{2}, 1]$ $(\partial C_2 \times [\frac{1}{2}, 1])$, and $C_1 \times \frac{1}{2}$ $(C_2 \times \frac{1}{2})$.

Let $\Phi_i = \Psi_i \circ \Theta_i$, $i = 1, 2$, and let $E^{p+q} = E_1^{p+q} \cup E_2^{p+q}$ identified at $V_1^+$ and $V_2^-$ orientation reversingly. Then, $E^{p+q}$ is a $(p+q)$-cell and a map $\Phi: E^{p+q} \to D_1^{p+q} \# D_2^{p+q}$ is defined by $\Phi = \Phi_1 \cup \Phi_2$, where $D_1^{p+q} \# D_2^{p+q}$ is the union of $D_i^{p+q} - \operatorname{Int}(\frac{1}{2}D_i^{p+q})$, $i = 1, 2$, identified at $\frac{1}{2}S_1^{p+q-1}$ and $\frac{1}{2}S_2^{p+q-1}$ orientation reversingly. $\Phi$ maps the interior of $E^{p+q}$ onto the complement of $S_1^{p+q-1} \cup w_1^{-1} \cdot w_2 \cup S_2^{p+q-1}$ homeomorphically, and carries $U_1^-$, $W_1^+ \cup W_2^-$, and $U_2^+$, respectively to $S_1^{p+q-1}$, $w_1^{-1} \cdot w_2$, and $S_2^{p+q-1}$.

Let $\rho_1: (T_1^{p+q-1}, U_1^+) \to (S_1^{p+q-1}, e_1)$, $\rho_2: (T_2^{p+q-1}, U_2^-) \to (S_2^{p+q-1}, e_2)$ be the maps of degree 1, and put $\tilde{\varphi}_1 = \varphi_1 \circ \rho_1$, $\tilde{\varphi}_2 = \varphi_2 \circ \rho_2$. Since $\rho_i: T_i^{p+q-1} \to S_i^{p+q-1}$, $i = 1, 2$, are respectively homotopic to the identity, $\tilde{\varphi}_i$, $i = 1, 2$, are homotopic to $\varphi_i$, $i = 1, 2$, respectively. Now, $A_1 \# A_2$ has the cell structure $(S_1^p \cup S_1^q) \cup w_1^{-1} \cdot w_2 \cup (S_2^p \cup S_2^q) \cup E^{p+q}$, where $E^{p+q}$ is attached by the map $\varphi$ which is equal to $\varphi_1 \circ \Phi = \tilde{\varphi}_1$ on $U_1^-$, $\varphi_2 \circ \Phi = \tilde{\varphi}_2$ on $U_2^+$, and $\Phi$ on $W_1^+ \cup W_2^-$. Hence, if we shrink the 1-cell $w = w_1^{-1} \cdot w_2$ to a point, $E^{p+q}$ may be regarded as the unit disk $D^{p+q}$ and the homotopy class of the attaching map $\varphi$ comes to $\{\varphi\} = \{\tilde{\varphi}_1\} + \{\tilde{\varphi}_2\} = \{\varphi_1\} + \{\varphi_2\}$. Thus, we know that $A_1 \# A_2$ has the homotopy type of $\{\bigvee_{i=1}^2 (S_i^p \vee S_i^q)\} \bigcup_\varphi D^{p+q}$ and by (1) the homotopy class of the attaching map $\varphi$ is given by

$$\{\varphi\} = \sum_{i=1}^{2} (\iota_p^i \circ \eta_i + [\iota_q^i, \iota_p^i]) , \tag{2}$$

where $\pi_{p+q-1}(S_i^p \vee S_i^q)$, $i = 1, 2$, are considered as direct summands of $\pi_{p+q-1}(\bigvee_{i=1}^2 (S_i^p \vee S_i^q))$.

Similar arguments hold for $r > 2$. Thus we have

LEMMA 1.1. *Let $A_i$, $i = 1, 2, \cdots, r$, be p-sphere bundles over q-spheres $(p, q > 1)$ with characteristic elements $\alpha(A_i) = i_*\xi_i$ for some given $\xi_i \in$*

*$\pi_{p-1}(SO_q)$, $i=1,2,\cdots,r$. Then, $\#_{i=1}^{r} A_i$ has the homotopy type of $\{\bigvee_{i=1}^{r}(S_i^p \vee S_i^q)\} \bigcup_\varphi D^{p+q}$ and the homotopy class of the attaching map $\varphi$ is given by*

$$\{\varphi\} = \sum_{i=1}^{r} (\iota_p^i \circ \eta_i + [\iota_q^i, \iota_p^i]) ,$$

*where $\iota_p^i$, $\iota_q^i$ are the orientation generators of $\pi_p(S_i^p)$, $\pi_q(S_i^q)$ respectively, $\eta_i = J\xi_i$, and $\pi_{p+q-1}(S_i^p \vee S_i^q)$, $i=1,2,\cdots,r$, are considered as direct summands of $\pi_{p+q-1}(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q))$.*

## 2. Exchange of the representation

Let $A_i$, $i=1,2,\cdots,r$, be $p$-sphere bundles over $q$-spheres $(p, q > 1, p \neq q)$ which admit cross-sections and let $\bar{A}_i$, $i=1,2,\cdots,r$, be $(p+1)$-disk bundles over $q$-spheres associated with $A_i$, $i=1,2,\cdots,r$, respectively. We assume that $A_i$ and $\bar{A}_i$ are oriented compatibly for each $i$. Put $W = \bar{A}_1 \natural \cdots \natural \bar{A}_r$, the boundary connected sum of $\bar{A}_i$, $i=1,2,\cdots,r$. Then, $\partial W = A_1 \# \cdots \# A_r$. $W$ is considered to be a handlebody of $\mathscr{H}(n, r, q)$, $n = p+q+1$. Since each $A_i$ has a cross-section, the inclusion map $i: \partial W \to W$ induces the isomorphism $i_*: H_q(\partial W) \to H_q(W)$ $(p \neq q)$. Let $g_i$, $i=1,2,\cdots,r$, be the basis of $H_q(W)$ represented by zero cross-sections of $\bar{A}_i$, $i=1,2,\cdots,r$, and let $f_i$, $i=1,2,\cdots,r$, be the basis of $H_q(\partial W)$ determined by cross-sections of $A_i$, $i=1,2,\cdots,r$. Since $f_i = i_*^{-1}(g_i)$, $i=1,2,\cdots,r$, those are independent of the choice of cross-sections. Let $e_i$, $i=1,2,\cdots,r$, be the basis of $H_p(\partial W)$ respesented by fixed fibres of $A_i$, $i=1,2,\cdots,r$. The bases $\{e_1,\cdots,e_r\}$ and $\{f_1,\cdots,f_r\}$ are called to be *fibre and sectional* with respect to the representation $\partial W = A_1 \# \cdots \# A_r$. They satisfy $e_i \cdot f_j = \delta_{ij}$ for each $i$, $j$.

LEMMA 2.1. *Let $2p \geqq q+1$, $q > 1$, and $p \neq q$. For any bases $\{\tilde{e}_1,\cdots,\tilde{e}_r\}$ of $H_p(\partial W)$ and $\{\tilde{f}_1,\cdots,\tilde{f}_r\}$ of $H_q(\partial W)$ such that $\tilde{e}_i \cdot \tilde{f}_j = \delta_{ij}$, $i,j=1,2,\cdots,r$, there exists a representation $\partial W = \tilde{A}_1 \# \cdots \# \tilde{A}_r$ by $p$-sphere bundles over $q$-spheres which admit cross-sections, such that the bases $\{\tilde{e}_1,\cdots,\tilde{e}_r\}$ and $\{\tilde{f}_1,\cdots,\tilde{f}_r\}$ are fibre and sectional with respect to it.*

*Proof.* Let $\phi: H_q(W) \times H_q(W) \to \pi_q(S^{p+1})$ be Wall's pairing[1)] [18]. Then, $\phi(g_i, g_j) = 0$ if $i \neq j$. Since each $A_i$ admits a cross-section, $\phi(g_i, g_i) = (E \circ \pi_*)(\alpha_i) = 0$ for all $i$ by Theorem 1 of [18], where $\alpha_i$ is the characteristic

1) In the original paper of Wall, the symbol $\lambda$ is used. But, we have already used it in this paper.

element of $A_i$, $i = 1, 2, \cdots, r$, $\pi_*: \pi_{q-1}(SO_{p+1}) \to \pi_{q-1}(S^p)$ is the homomorphism induced from the projection $\pi: SO_{p+1} \to SO_{p+1}/SO_p = S^p$, and $E: \pi_{q-1}(S^p) \to \pi_q(S^{p+1})$ is the suspension homomorphism. Thus, $\phi$ is trivial since it is bilinear. So that, any representation of $W$ by a basis of $H_q(W)$ comes to a boundary connected sum of $r$ $(p+1)$-disk bundles over $q$-spheres. In fact, since $\phi$ is trivial, the imbedded $q$-spheres which represent the given basis elements of $H_q(W)$ can be taken to be disjoint in the interior of $W$ (cf. Ishimoto [7]). Then, by tying the thin closed neighborhoods of the imbedded spheres with bands in Int $W$, we obtain a boundary connected sum of $(p+1)$-disk bundles over $q$-spheres which is diffeomorphic to $W$.

Let $\tilde{g}_i = i_*(\tilde{f}_i)$, $i = 1, 2, \cdots, r$, be the basis of $H_q(W)$ and represent $W$ by it. Then, $\partial W$ has a representation $\partial W = \tilde{A}_1 \# \cdots \# \tilde{A}_r$. Here, each $\tilde{A}_i$ is a $p$-sphere bundle over a $q$-sphere and admits a cross-section since $E\pi_*(\tilde{\alpha}_i) = \phi(\tilde{g}_i, \tilde{g}_i) = 0$, where $\tilde{\alpha}_i$ is the characteristic element of $\tilde{A}_i$. Then, up to orientation, any cross-section of $\tilde{A}_i$ represents $\tilde{f}_i$, and a fibre $\tilde{S}_i^p$ of $\tilde{A}_i$ with canonical orientation represents $\tilde{e}_i$, since $(\tilde{S}_i^p)\cdot\tilde{f}_j = \delta_{ij} = \tilde{e}_i\cdot\tilde{f}_j$ for each $i, j$. This completes the proof.

Let $A'_i$, $i = 1, 2, \cdots, r$, be another set of $p$-sphere bundles over $q$-spheres which admit cross-sections $(p, q > 1,\ p \neq q)$. $\overline{A}'_i$, $i = 1, 2, \cdots, r$, $W' = \overline{A}'_i \natural \cdots \natural \overline{A}'_r$, and then $\partial W' = A'_1 \# \cdots \# A'_r$, are similar as in the above, and those are oriented canonically. The isomorphism $i'_*: H_q(\partial W') \to H_q(W')$, the basis $\{g'_1, \cdots, g'_r\}$ of $H_q(W')$, and the fibre and sectional bases $\{e'_1, \cdots, e'_r\}$ of $H_p(\partial W')$, $\{f'_1, \cdots, f'_r\}$ of $H_q(\partial W')$ are respectively defined similarly.

By Lemma 1.1, $\partial W = A_1 \# \cdots \# A_r$ and $\partial W' = A'_1 \# \cdots \# A'_r$ have the homotopy type of $CW$-complexes $X = \{\bigvee_{i=1}^r (S_i^p \vee S_i^q)\} \bigcup_\varphi e^{p+q}$ and $X' = \{\bigvee_{i=1}^r (S_i'^p \vee S_i'^q)\} \bigcup_{\varphi'} e'^{p+q}$ respectively, associated with the representations. Suppose that $\partial W$ has the homotopy type of $\partial W'$ by an orientation preserving homotopy equivalence $g: \partial W \to \partial W'$. Then, $g$ induces a homotopy equivalence $f: X \to X'$ of degree 1 and we may assume that $f$ is a cellular map. We consider $\pi_p(S_i^p)$ and $\pi_q(S_i^q)$ as direct summands of $\pi_p(\bigvee_{i=1}^r (S_i^p \vee S_i^q))$ and $\pi_q(\bigvee_{i=1}^r (S_i^p \vee S_i^q))$ respectively, for each $i$. Similar identifications are made about $\pi_p(S_i'^p)$, $\pi_q(S_i'^q)$, $i = 1, 2, \cdots, r$. Note that $\pi_n(\bigvee_{i=1}^r (S_i^p \vee S_i^q)) \cong \pi_n(X)$, $\pi_n(\bigvee_{i=1}^r (S_i^p \vee S_i^q), \bigvee_{i=1}^r S_i^p) \cong \pi_n(X, \bigvee_{i=1}^r S_i^p)$ if $n < p + q - 1$, and the situation is quite similar about $X'$, $(X', \bigvee_{i=1}^r S_i'^p)$.

LEMMA 2.2. *If $g_*(e_i) = e'_i$, $g_*(f_j) = f'_j$ for all $i, j = 1, 2, \cdots, r$, then $f$ satisfies the following properties: Let $1 < p < q$.*

(i) $f_*\iota_p^i = \iota_p'^i$, $i = 1, 2, \cdots, r$, *where*

$$f_*: \pi_p\left(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q)\right) \to \pi_p\left(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q)\right),$$

*and* $\iota_p^i$, $\iota_p'^i$ *are the orientation generators of* $\pi_p(S_i^p)$, $\pi_p(S_i'^p)$ *respectively.*

(ii) $j'_*(f_*\iota_q^i) = j'_*\iota_q'^i$, $i = 1, 2, \cdots, r$, *where*

$$j'_*: \pi_q\left(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q)\right) \to \pi_q\left(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q), \bigvee_{i=1}^{r} S_i'^p\right),$$

$j'$ *is the inclusion map, and* $\iota_q^i$, $\iota_q'^i$ *are the orientation generators of* $\pi_q(S_i^q)$, $\pi_q(S_i'^q)$ *respectively.*

(iii) $f_*(\partial\sigma) = \partial\sigma'$, *where* $\sigma, \sigma'$ *are the orientation generators of* $\pi_{p+q}(X, \bigvee_{i=1}^{r}(S_i^p \vee S_i^q))$, $\pi_{p+q}(X', \bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q))$ *respectively and* $\partial$ *means the boundary homomorphism.*

*If* $p > q > 1$, *replace* $p$ *and* $q$ *each other in the above.*

*Proof.* (i) $e_i$, $e_i'$ correspond to $(S_i^p)$, $(S_i'^p)$ in $H_p(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q))$, $H_p(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q))$ respectively, which are isomorphic to $\pi_p(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q))$, $\pi_p(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q))$ respectively $(1 < p < q)$. So that $g_*(e_i) = e_i'$ induces $f_*(\iota_p^i) = \iota_p'^i$, where $i = 1, 2, \cdots, r$.

(ii) We have the following commutative diagram:

$$\begin{array}{ccc}
\pi_q\left(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q)\right) & \xrightarrow[\cong]{f_*} & \pi_q\left(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q)\right) \\
j_*\downarrow & & j'_*\downarrow \\
\pi_q\left(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q), \bigvee_{i=1}^{r} S_i^p\right) & \xrightarrow{\bar{f}_*} & \pi_q\left(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q), \bigvee_{i=1}^{r} S_i'^p\right) \\
h\downarrow\cong & & h\downarrow\cong \\
H_q\left(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q), \bigvee_{i=1}^{r} S_i^p\right) & \xrightarrow{\bar{f}_*} & H_q\left(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q), \bigvee_{i=1}^{r} S_i'^p\right) \\
j_*\uparrow\cong & & j'_*\uparrow\cong \\
H_q\left(\bigvee_{i=1}^{r}(S_i^p \vee S_i^q)\right) & \xrightarrow[\cong]{f_*} & H_q\left(\bigvee_{i=1}^{r}(S_i'^p \vee S_i'^q)\right),
\end{array}$$

where $h$ is the Hurewicz homomorphism. $j_*\iota_q^i$, $j'_*\iota_q'^i$ correspond respectively to $(S_i^q)$, $(S_i'^q)$ in each vertical side and $f_*(S_i^q) = (S_i'^q)$ since $g_*(f_i) = f_i'$. So that, we have $j'_*(f_*\iota_q^i) = \bar{f}_*(j_*\iota_q^i) = j'_*\iota_q'^i$ for $i = 1, 2, \cdots, r$.

If $p > q > 1$, similar results are obtained by exchange of $p$ for $q$ in (i), (ii).

(iii) We have the following commutative diagram:

$$\begin{array}{ccc} H_{p+q}(X) & \xrightarrow{\quad f_* \quad} & H_{p+q}(X') \\ \downarrow \cong & & \downarrow \cong \\ H_{p+q}\left(X, \bigvee_{i=1}^{r} (S_i^p \vee S_i^q)\right) & \xrightarrow{\bar{\bar{f}}_*} & H_{p+q}\left(X', \bigvee_{i=1}^{r} (S_i'^p \vee S_i'^q)\right) \\ h \uparrow \cong & & h \uparrow \cong \\ \pi_{p+q}\left(X, \bigvee_{i=1}^{r} (S_i^p \vee S_i^q)\right) & \xrightarrow{\bar{\bar{f}}_*} & \pi_{p+q}\left(X', \bigvee_{i=1}^{r} (S_i'^p \vee S_i'^q)\right) \\ \partial \downarrow & & \partial \downarrow \\ \pi_{p+q-1}\left(\bigvee_{i=1}^{r} (S_i^p \vee S_i^q)\right) & \xrightarrow{f_*} & \pi_{p+q-1}\left(\bigvee_{i=1}^{r} (S_i'^p \vee S_i'^q)\right) \end{array}$$

Since $f$ is of degree 1, $\bar{\bar{f}}_*\sigma = \sigma'$ and therefore $f_*(\partial\sigma) = \partial\sigma'$.

LEMMA 2.3. *Let $2p \geqq q + 1$, $q > 1$, and $p \neq q$. Then, there exists a representation $\partial W = \tilde{A}_1 \# \cdots \# \tilde{A}_r$ by $p$-sphere bundles over $q$-spheres which admit cross-sections such that*

(i) $$\begin{pmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_r \end{pmatrix} = L \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix},$$

*where $\alpha_i$, $\tilde{\alpha}_i$ are the characteristic elements of $A_i$, $\tilde{A}_i$ respectively, $i = 1, 2, \cdots, r$, and $L$ is a unimodular $(r \times r)$-matrix with integer components, and*

(ii) *the fibre and sectional bases $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$ of $H_p(\partial W)$ and $\{\tilde{f}_1, \cdots, \tilde{f}_r\}$ of $H_q(\partial W)$ associated with the representation satisfy $g_*(\tilde{e}_i) = e'_i$, $g_*(\tilde{f}_j) = f'_j$ for all $i, j = 1, 2, \cdots, r$.*

*So that, for the CW-complex $\tilde{X} = \{\bigvee_{i=1}^{r} (\tilde{S}_i^p \vee \tilde{S}_i^q)\} \cup_\varphi \tilde{e}^{p+q}$ associated with the representation $\partial W = \tilde{A}_1 \# \cdots \# \tilde{A}_r$, $f$ satisfies* (i), (ii), *and* (iii) *of Lemma* 2.2.

*Proof.* Let $\tilde{e}_i = g_*^{-1}(e'_i)$, $\tilde{f}_j = g_*^{-1}(f'_j)$ for $i, j = 1, 2, \cdots, r$. Since $e'_i \cdot f'_j = \delta_{ij}$, $i, j = 1, 2, \cdots, r$, and $g$ preserves the orientation of $\partial W$, $\tilde{e}_i \cdot \tilde{f}_j = \delta_{ij}$ for $i, j = 1, 2, \cdots, r$. Then, by Lemma 2.1, there is a representation $\partial W = \tilde{A}_1 \# \cdots \# \tilde{A}_r$ by $p$-sphere bundles over $q$-spheres which admit cross-sections about which the bases $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$ and $\{\tilde{f}_1, \cdots, \tilde{f}_r\}$ are fibre and sectional. Let $\tilde{g}_i = i_*(\tilde{f}_i)$, $i = 1, 2, \cdots, r$, and let $\tilde{\alpha}_i$ be the characteristic element of $\tilde{A}_i$. Then $\tilde{\alpha}_i = \alpha(\tilde{g}_i)$, where $\alpha: H_q(W) \to \pi_{q-1}(SO_{p+1})$ is the map assigning to each $x \in H_q(W) \cong \pi_q(W)$ the characteristic element of the normal bundle of the imbedded $q$-sphere which represents $x$. $\alpha$ is a homomorphism by Theorem 1 of Wall [18] since $\phi$ is trivial. Hence, the relation

$$\begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_r \end{pmatrix} = L \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}$$

induces the relation

$$\begin{pmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_r \end{pmatrix} = L \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix},$$

where $L$ is a unimodular $(r \times r)$-matrix of integer components. This completes the proof.

## 3. Proof of Theorem 1

Let $A_i$, $A_i'$, $i = 1, 2, \cdots, r$, be $p$-sphere bundles over $q$-spheres with cross-sections, where we assume that $2p > q + 1$, $q > 1$, and $p \neq q$. Put $W = \overline{A}_1 \natural \cdots \natural \overline{A}_r$, $W' = \overline{A}_1' \natural \cdots \natural \overline{A}_r'$, where $\overline{A}_i$, $\overline{A}_i'$ are the $(p+1)$-disk bundles associated with $A_i$, $A_i'$ respectively for $i = 1, 2, \cdots, r$. Then, $\partial W = \#_{i=1}^r A_i$ and $\partial W' = \#_{i=1}^r A_i'$. Let $\{e_1, \cdots, e_r\}$, $\{f_1, \cdots, f_r\}$ be the fibre and sectional bases of $H_p(\partial W)$, $H_q(\partial W)$ respectively, and similarly define $\{e_1', \cdots, e_r'\}$, $\{f_1', \cdots, f_r'\}$ for $\partial W'$.

ASSERTION 1. Suppose that there exists a homotopy equivalence $g: \partial W \to \partial W'$ of degree 1 which satisfies $g_*(e_i) = e_i'$, $g_*(f_j) = f_j'$ for $i, j = 1, 2, \cdots, r$. Then, $\lambda(A_i) = \lambda(A_i')$ for $i = 1, 2, \cdots, r$.

*Proof.* $\partial W$, $\partial W'$ have the homotopy type of $CW$-complexes $X = \{\bigvee_{i=1}^r (S_i^p \vee S_i^q)\} \cup_\varphi e^{p+q}$, $X' = \{\bigvee_{i=1}^r (S_i'^p \vee S_i'^q)\} \cup_{\varphi'} e'^{p+q}$ respectively, and $g$ induces a homotopy equivalence $f: X \to X'$ which may be assumed to be cellular. Let $p < q$. Then, by Lemma 2.2, $f_* \iota_p^i = \iota_p'^i$, $j_*'(f_* \iota_p^i) = j_*' \iota_q'^i$, $i = 1, 2, \cdots, r$, and $f_*(\partial\sigma) = \partial\sigma'$, where the notations are similar to those of the lemma, and note that $\partial\sigma = \{\varphi\}$ and $\partial\sigma' = \{\varphi'\}$.

So that, by Lemma 1.1,

$$(1) \qquad \begin{aligned} f_*(\partial\sigma) &= \sum_{i=1}^r (f_* \iota_p^i) \circ \eta^i + \sum_{i=1}^r [f_* \iota_q^i, f_* \iota_p^i] \\ &= \sum_{i=1}^r \iota_p'^i \circ \eta^i + \sum_{i=1}^r [f_* \iota_q^i, \iota_p'^i] \,. \end{aligned}$$

Since $j'_*(f_*\iota^i_q) = j'_*\iota'^i_q$, by the split exact sequence

$$0 \longrightarrow \pi_q\Big(\bigvee_{j=1}^{r} S'^p_j\Big) \longrightarrow \pi_q\Big(\bigvee_{j=1}^{r} (S'^p_j \vee S'^q_j)\Big) \xrightarrow{j'_*} \pi_q\Big(\bigvee_{j=1}^{r} (S'^p_j \vee S'^q_j), \bigvee_{j=1}^{r} S'^p_j\Big) \longrightarrow 0\ ,$$

we have $f_*\iota^i_q = \iota'^i_q + \theta'_i$ for some $\theta'_i \in \pi_q(\bigvee_{j=1}^{r} S'^p_j) \subset \pi_q(\bigvee_{j=1}^{r}(S'^p_j \vee S'^q_j))$, $i = 1, 2, \cdots, r$. And $\pi_q(\bigvee_{j=1}^{r} S'^p_j) \cong \sum_{j=1}^{r}\pi_q(S'^p_j)$ by the assumption $2p > q+1$, $q > 1$ (Hilton [4]). Therefore, $\theta'_i = \sum_{j=1}^{r}\theta'_{ij}$ for suitable $\theta'_{ij} \in \pi_q(S'^p_j) \subset \pi_q(\bigvee_{j=1}^{r} S'^p_j)$, $i, j = 1, 2, \cdots, r$. Let $\theta'_{ij} = \iota'^j_p \circ \theta_{ij}$, $\theta_{ij} \in \pi_q(S^p)$, $i, j = 1, 2, \cdots, r$. $\theta_{ij}$, $i, j = 1, 2, \cdots, r$, are the suspension elements since $\pi_q(S^p) = E\pi_{q-1}(S^{p-1})$ $(2p > q+1)$. So that,

$$(2)\qquad \begin{aligned}[f_*\iota^i_q, \iota'^i_p] &= [\iota'^i_q + \theta'_i, \iota'^i_p] = [\iota'^i_q, \iota'^i_p] + [\theta'_i, \iota'^i_p]\\ &= [\iota'^i_q, \iota'^i_p] + \sum_{j=1}^{r}[\theta'_{ij}, \iota'^i_p]\ ,\end{aligned}$$

and by Barcus-Barratt [2, (7.4)] or G. W. Whitehead [19, (3.59)],

$$(3)\qquad [\theta'_{ij}, \iota'^i_p] = [\iota'^j_p \circ \theta_{ij}, \iota'^i_p] = [\iota'^j_p, \iota'^i_p] \circ (-1)^{p+q}E^{p-1}\theta_{ij}\ .$$

Thus, combining (1), (2), and (3), we have

$$(4)\qquad f_*(\partial\sigma) = \sum_{i=1}^{r}(\iota'^i_p \circ \eta^i + [\theta'_{ii}, \iota'^i_p]) + \sum_{i=1}^{r}[\iota'^i_q, \iota'^i_p] + \sum_{i<j}[\iota'^i_p, \iota'^j_p] \circ \beta_{ij}\ ,$$

where $\beta_{ij} = (-1)^{p+q}E^{p-1}\theta_{ji} + (-1)^q E^{p-1}\theta_{ij}$. Since $f_*(\partial\sigma) = \partial\sigma'$ and by Lemma 1.1

$$(5)\qquad \partial\sigma' = \sum_{i=1}^{r}\iota'^i_p \circ \eta'^i + \sum_{i=1}^{r}[\iota'^i_q, \iota'^i_p]\ ,$$

comparing (4) with (5), we have

$$\iota'^i_p \circ \eta^i + [\theta'_{ii}, \iota'^i_p] = \iota'^i_p \circ \eta'^i\ , \qquad i = 1, 2, \cdots, r\ ,$$

by Hilton [4]. Hence,

$$\iota'^i_p \circ \eta^i + \iota'^i_p \circ [\theta_{ii}, \iota_p] = \iota'^i_p \circ \eta'^i\ , \qquad i = 1, 2, \cdots, r\ ,$$

that is,

$$\eta^i + [\theta_{ii}, \iota_p] = \eta'^i\ , \qquad i = 1, 2, \cdots, r\ .$$

Thus $\lambda(A_i) = \lambda(A'_i)$, $i = 1, 2, \cdots, r$ .

Let $p > q > 1$. Then, by Lemma 2.2, $f_*\iota_q^i = \iota_q'^i$, $j'_*(f_*\iota_p^i) = j'_*\iota_p'^i$, $i = 1, 2, \cdots, r$, and $f_*(\partial\sigma) = \partial\sigma'$. Similarly as above, $f_*\iota_p^i = \iota_p'^i + \theta_i'$ for some $\theta_i' \in \pi_p(\vee_{j=1}^r S_j'^q) \subset \pi_p(\vee_{j=1}^r (S_j'^p \vee S_j'^q))$, $i = 1, 2, \cdots, r$. Therefore, by Lemma 1.1,

$$\begin{aligned} f_*(\partial\sigma) &= \sum_{i=1}^r (f_*\iota_p^i) \circ \eta^i + \sum_{i=1}^r [f_*\iota_q^i, f_*\iota_p^i] \\ &= \sum_{i=1}^r (\iota_p'^i + \theta_i') \circ \eta^i + \sum_{i=1}^r [\iota_q'^i, \iota_p'^i + \theta_i'] \,. \end{aligned}$$

And, since $\eta^i \in \pi_{p+q-1}(S^p) = E\pi_{p+q-2}(S^{p-1})$,

$$f_*(\partial\sigma) = \sum_{i=1}^r \iota_p'^i \circ \eta^i + \sum_{i=1}^r (\theta_i' \circ \eta^i + [\iota_q'^i, \theta_i']) + \sum_{i=1}^r [\iota_q'^i, \iota_p'^i] \,, \tag{6}$$

where those elements of (6) belong to independent direct summands of $\pi_{p+q-1}(\vee_{j=1}^r (S_j'^p \vee S_j'^q))$ (Hilton [4]). Since $f_*(\partial\sigma) = \partial\sigma'$, comparing (6) with (5), we have

$$\iota_p'^i \circ \eta^i = \iota_p'^i \circ \eta'^i \,, \qquad i = 1, 2, \cdots, r \,,$$

that is,

$$\eta^i = \eta'^i \,, \qquad i = 1, 2, \cdots, r \,.$$

Thus, $\lambda(A_i) = \lambda(A_i')$ for $i = 1, 2, \cdots, r$.

This completes the proof of Assertion 1.

ASSERTION 2. *If $\partial W$ has the homotopy type of $\partial W'$, then*

$$\begin{pmatrix} \lambda(A_1') \\ \vdots \\ \lambda(A_r') \end{pmatrix} = L \begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix}$$

*for some unimodular $(r \times r)$-matrix $L$ with integer components.*

*Proof.* Let $g: \partial W \to \partial W'$ be a homotopy equivalence. Choosing suitable orientations, we may assume that $g$ preserves orientation. Note that exchange of orientations of fibres does not affect the characteristic elements of $A_i$, $A_i'$, $i = 1, 2, \cdots, r$, (cf. [10], p. 198). Then, by Lemma 2.3, there exists a representation $\partial W = \tilde{A}_1 \# \cdots \# \tilde{A}_r$ by $p$-sphere bundles over $q$-spheres which admit cross-sections such that $g_*(\tilde{e}_i) = e_i'$, $g_*(\tilde{f}_j) = f_j'$, $i, j = 1, 2, \cdots, r$, where $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$, $\{\tilde{f}_1, \cdots, \tilde{f}_r\}$ are the fibre and sectional bases associated with the representation of $\partial W$. So that, by Assertion 1,

$$\lambda(\tilde{A}_i) = \lambda(A'_i)\ , \qquad i = 1, 2, \cdots, r\ ,$$

and by (i) of Lemma 2.3,

$$\begin{pmatrix} \lambda(\tilde{A}_1) \\ \vdots \\ \lambda(\tilde{A}_r) \end{pmatrix} = L \begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix}$$

for some unimodular $(r \times r)$-matrix $L$. Thus, we have

$$\begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix} = L \begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix} .$$

ASSERTION 3. The converse of Assertion 1 is true.

*Proof.* The proof is a straight extension of that of James-Whitehead [10 (1.5), p. 210]. Let $\eta_i \in \pi_{p+q-1}(S^p)$ be a representative of $\lambda(A_i) = \lambda(A'_i)$, $i = 1, 2, \cdots, r$. Then, by (3.9) of [10], we may assume that $A_i = (S_i^p \cup S_i^q) \bigcup_{\varphi_i} e^{p+q}$, $A'_i = (S_i'^p \cup S_i'^q) \bigcup_{\varphi'_i} e'^{p+q}$ and $\{\varphi_i\} = \iota_p^i \circ \eta_i + [\iota_q^i, \iota_p^i]$, $\{\varphi'_i\} = \iota_p'^i \circ \eta_i + [\iota_q'^i, \iota_p'^i]$, $i = 1, 2, \cdots, r$. Then, a map $h\colon \bigvee_{i=1}^r (S_i^p \vee S_i^q) \to \bigvee_{i=1}^r (S_i'^p \vee S_i'^p)$ defined so that $h_* \iota_p^i = \iota_p'^i$, $h_* \iota_q^i = \iota_q'^i$, $i = 1, 2, \cdots, r$, satisfies $h_*(\partial\sigma) = \partial\sigma'$ by Lemma 1.1. Therefore, $h$ extends to a whole map $g\colon X \to X'$ of degree 1. Here, $X = \{\bigvee_{i=1}^r (S_i^p \vee S_i^q)\} \bigcup_\varphi e^{p+q}$, $X' = \{\bigvee_{i=1}^r (S_i'^p \vee S_i'^q)\} \bigcup_{\varphi'} e'^{p+q}$, $\#_{i=1}^r A_i \simeq X$, $\#_{i=1}^r A'_i \simeq X'$, and $\partial\sigma = \{\varphi\}$, $\partial\sigma' = \{\varphi'\}$. Hence, $g$ induces the isomorphisms between the homology groups in all dimensions. So, $g$ is a homotopy equivalence since $X$, $X'$ are simply connected, and $g_*(e_i) = e'_i$, $g_*(f_j) = f'_j$ for $i, j = 1, 2, \cdots, r$.

ASSERTION 4. The converse of Assertion 2 is also true.

*Proof.* Let $L$ be a unimodular $(r \times r)$-matrix of integer components and assume that

$$\begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix} = L \begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix} .$$

Let $\{\tilde{g}_1, \cdots, \tilde{g}_r\}$ be the basis of $H_q(W)$ such that

$$\begin{pmatrix} \tilde{g}_1 \\ \vdots \\ \tilde{g}_r \end{pmatrix} = L \begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix} ,$$

where $\{g_1, \cdots, g_r\}$ is the basis determined by zero cross-sections of $\bar{A}_i$, $i = 1, 2, \cdots, r$. Then, representing $W$ by the basis $\{\tilde{g}_1, \cdots, \tilde{g}_r\}$, we have $\partial W = \tilde{A}_1 \# \cdots \# \tilde{A}_r$ and

$$\begin{pmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_r \end{pmatrix} = L \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix},$$

where $\tilde{\alpha}_i, \alpha_i$ are the characteristic elements of $\tilde{A}_i, A_i$ respectively, $i = 1, 2, \cdots, r$. So that,

$$\begin{pmatrix} \lambda(\tilde{A}_1) \\ \vdots \\ \lambda(\tilde{A}_r) \end{pmatrix} = L \begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix} = \begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix},$$

and therefore by Assertion 3, $\partial W$ has the homotopy type of $\partial W'$. This completes the proof of Assertion 4.

Thus, in case $2p > q + 1$, we have proved Theorem 1 by Assertion 2 and Assertion 4.

## 4. Continued proof of Theorem 1

In the previous section, the condition $2p > q + 1$ $(q > 1)$ is needed only to prove Assertion 1. The other assertions hold under the condition that $2p \geqq q + 1$, $q > 1$, $p \neq q$. So, to complete the proof, it is sufficient to show Assertion 1 when $2p = q + 1$, $q > 1$. Note that then $1 < p < q$.

In the proof of Assertion 1, $f_* \iota^i_q = \iota'^i_q + \theta'_i$ for some $\theta'_i \in \pi_q(\vee^r_{j=1} S'^p_j) \subset \pi_q(\vee^r_{j=1}(S'^p_j \vee S'^q_j))$, $i = 1, 2, \cdots, r$. By Hilton [4],

$$\theta'_i = \sum_{j=1}^{r} \iota'^j_p \circ \theta^i_j + \sum_{j<k} [\iota'^j_p, \iota'^k_p] \circ \theta^i_{jk}$$

for some $\theta^i_j \in \pi_q(S^p)$, $\theta^i_{jk} \in \pi_q(S^{2p-1}) \cong Z$, $i, j, k = 1, 2, \cdots, r$. Here, $\theta^i_j$, $i, j = 1, 2, \cdots, r$, are also suspension elements if $p$ is odd. For, the sequence

$$\pi_{2p-2}(S^{p-1}) \xrightarrow{E} \pi_{2p-1}(S^p) \xrightarrow{H} Z$$

is exact and $\pi_{2p-1}(S^p)$ is finite if $p$ is odd. Now,

$$\begin{aligned}[f_* \iota^i_q, \iota'^i_p] &= [\iota'^i_q, \iota'^i_p] + [\theta'_i, \iota'^i_p] \\ &= [\iota'^i_q, \iota'^i_p] + \sum_{j=1}^{r} [\iota'^j_p \circ \theta^i_j, \iota'^i_p] + \sum_{j<k} [[\iota'^j_p, \iota'^k_p] \circ \theta^i_{jk}, \iota'^i_p],\end{aligned}$$

and

$$[\iota_p'^j \circ \theta_j^i, \iota_p'^i] = [\iota_p'^j, \iota_p'^i] \circ (-1)^{p-1} E^{p-1}\theta_j^i + [\iota_p'^j, [\iota_p'^j, \iota_p'^i]] \circ (-1)^p E^{p-1} H_0(\theta_j^i) ,$$
$$[[\iota_p'^j, \iota_p'^k] \circ \theta_{jk}^i, \iota_p'^i] = [[\iota_p'^j, \iota_p'^k], \iota_p'^i] \circ E^{p-1}\theta_{jk}^i , \qquad i, j, k = 1, 2, \cdots, r ,$$

by Barcus-Barratt [2, (7.4)]. Here, $H_0$ is the Hopf-Hilton homomorphism and $H_0(\theta_j^i)$ vanishes if $p$ is odd. Hence,

$$[f_*\iota_q^i, \iota_p'^i] = [\iota_q'^i, \iota_p'^i] + [\iota_p'^i \circ \theta_i^i, \iota_p'^i] + \sum_{j \neq i} [\iota_p'^j, \iota_p'^i] \circ (-1)^{p-1} E^{p-1}\theta_j^i$$
$$+ \sum_{j \neq i} [\iota_p'^j, [\iota_p'^j, \iota_p'^i]] \circ (-1)^p E^{p-1} H_0(\theta_j^i) + \sum_{j<k} [\iota_p'^i, [\iota_p'^j, \iota_p'^k]] \circ (-1)^p E^{p-1}\theta_{jk}^i .$$

Here, $[\iota_p'^j, \iota_p'^i] = (-1)^p [\iota_p'^i, \iota_p'^j]$, and by Jacobi identity (Hilton [4]),

$$[\iota_p'^i, [\iota_p'^j, \iota_p'^k]] = (-1)^{p+1} [\iota_p'^j, [\iota_p'^i, \iota_p'^k]] - [\iota_p'^k, [\iota_p'^i, \iota_p'^j]] \qquad (i < j < k) .$$

So, we have

$$\sum_i [f_*\iota_q'^i, \iota_p'^i] = \sum_i ([\iota_q'^i, \iota_p'^i] + [\iota_p'^i \circ \theta_i^i, \iota_p'^i]) + \sum_{i<j} [\iota_p'^i, \iota_p'^j] \circ \beta_j^i$$
$$(7) \qquad + \sum_{i \geqq j < k} [\iota_p'^i, [\iota_p'^j, \iota_p'^k]] \circ \gamma_{jk}^i ,$$

for certain $\beta_j^i \in \pi_{p+q-1}(S^{2p-1})$, $\gamma_{jk}^i \in \pi_{p+q-1}(S^{3p-2})$, $i, j, k = 1, 2, \cdots, r$. Then, by (7) and (1) of the section 3,

$$f_*(\partial\sigma) = \sum_i (\iota_p'^i \circ \eta^i + \iota_p'^i \circ [\theta_i^i, \iota_p]) + \sum_i [\iota_q'^i, \iota_p'^i]$$
$$(8) \qquad + \sum_{i<j} [\iota_p'^i, \iota_p'^j] \circ \beta_j^i + \sum_{i \geqq j < k} [\iota_p'^i, [\iota_p'^j, \iota_p'^k]] \circ \gamma_{jk}^i ,$$

where $\beta_j^i \in \pi_{p+q-1}(S^{2p-1})$, $\gamma_{jk}^i \in \pi_{p+q-1}(S^{3p-2})$, $i, j, k = 1, 2, \cdots, r$. Since $f_*(\partial\sigma) = \partial\sigma'$, thus comparing (8) with $\partial\sigma'$ given by (5) of the section 3, we have

$$\iota_p'^i \circ \eta^i + \iota_p'^i \circ [\theta_i^i, \iota_p] = \iota_p'^i \circ \eta'^i , \qquad i = 1, 2, \cdots, r ,$$

that is,

$$\eta^i + [\theta_i^i, \iota_p] = \eta'^i , \quad \theta_i^i \in \pi_q(S^p) , \quad i = 1, 2, \cdots, r .$$

Hence $\lambda(A_i) = \lambda(A_i')$, $i = 1, 2, \cdots, r$. Thus, Assertion 1 holds when $2p = q + 1$ and $q > 1$.

This completes the proof of Theorem 1.

## 5. Proof of Theorem 2

Let $M$ be a $(p-1)$-connected $2p$-dimensional closed differentiable manifold $(p > 2)$. Let $H = H_p(M)$, $\phi: H \times H \to Z$ be the intersection number pairing, and $\nu: H \to \pi_{p-1}(SO_p)$ be the map assigning to each $x \in H \cong \pi_p(M)$ the characteristic element $\nu(x)$ of the normal bundle of the imbedded $p$-sphere which represents $x$. $\nu$ satisfies the following relation

$$\nu(x + y) = \nu(x) + \nu(y) + \partial\phi(x, y) ,$$

where $\partial: \pi_p(S^p) \cong Z \to \pi_{p-1}(SO_p)$ belongs to the homotopy exact sequence of the fibering $SO_p \to SO_{p+1} \to S^p$ (Wall [18]). Put $\mu = J \circ \nu$, where $J$ denotes the $J$-homomorphism.

Let $M'$ be another $(p-1)$-connected $2p$-dimensional closed differentiable manifold $(p > 2)$. Then, $H'$, $\phi'$, $\nu'$, and $\mu'$ are similarly defined. The following proposition is essentially due to Lemma 8 of Wall [17], and the proof is analogous.[2)]

PROPOSITION 5.1. *M has the oriented homotopy type of M′ if and only if there exists an isomorphism $h: H \to H'$ such that the following diagrams commute:*

$$\begin{array}{ccc} H \times H & \xrightarrow{h \times h} & H' \times H' \\ & {}_{\phi}\searrow \quad \swarrow_{\phi'} & \\ & Z & \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{h} & H' \\ & {}_{\mu}\searrow \quad \swarrow_{\mu'} & \\ & \pi_{2p-1}(S^p) & \end{array}$$

Let $A_i$, $i = 1, 2, \cdots, r$, be $p$-sphere bundles over $p$-spheres $(p > 2)$ and consider the connected sum of those.

LEMMA 5.2. *Let $M = \#_{i=1}^r A_i$, $p > 2$, and $p \neq 4, 8$. Then, there exists a basis $\{e_1, \cdots, e_r; f_1, \cdots, f_r\}$ of $H_p(M)$ symplectic in the sense that $e_i \cdot e_j = f_i \cdot f_j = 0$, $e_i \cdot f_j = \delta_{ij}$, $i, j = 1, 2, \cdots, r$, such that $\mu(e_i) = 0$ and $\mu(f_j)$ is a representative of $\lambda(A_j)$ for $i, j = 1, 2, \cdots, r$.*

*Proof.* Let $\{e_1, \cdots, e_r; f_1, \cdots, f_r\}$ be a fibre and sectional basis of $H_p(M)$. We may assume that $e_i \cdot e_j = 0$, $f_i \cdot f_j = 0$ $(i \neq j)$, and $e_i \cdot f_j = \delta_{ij}$ for $i, j = 1, 2, \cdots, r$. Note that $\nu(e_i) = 0$, $i_*\nu(f_j) = \alpha(A_j)$, where $i_*: \pi_{p-1}(SO_p) \to \pi_{p-1}(SO_{p+1})$ is induced from the inclusion. So that, if $p$ is odd, we have the lemma since $f_j \cdot f_j = 0$, $j = 1, 2, \cdots, r$.

Let $p$ be even and $p \neq 4, 8$. Then, the homomorphism $H: \pi_{2p-1}(S^p) \to Z$ which assigns the Hopf invariant to each element of $\pi_{2p-1}(S^p)$ takes always even numbers. So that $f_j \cdot f_j = \pi_*(\nu(f_j)) = HJ(\nu(f_j)) = H\mu(f_j)$ is even for $j = 1, 2, \cdots, r$, where $\pi_*: \pi_{p-1}(SO_p) \to \pi_{p-1}(S^{p-1}) \cong Z$ is the homomorphism

2) But, we need not use Eckmann-Hilton [3].

induced by the projection $\pi\colon SO_p \to SO_p/SO_{p-1} = S^{p-1}$. Put $2k_j = f_j \cdot f_j$ and $\tilde{f}_j = f_j - k_j e_j$, $j = 1, 2, \cdots, r$. Then, $\{e_1, \cdots, e_r; \tilde{f}_1, \cdots, \tilde{f}_r\}$ is a symplectic basis of $H_p(M)$. $\nu(\tilde{f}_j) = \nu(f_j - k_j e_j) = \nu(f_j) + \nu(-k_j e_j) + \partial\phi(f_j, -k_j e_j) = \nu(f_j) - k_j \partial(1)$. So that, $\mu(\tilde{f}_j) = \mu(f_j) - k_j J \circ \partial(1) = \mu(f_j) + k_j P(1)$. Thus, $\{\mu(\tilde{f}_j)\} = \{\mu(f_j)\} = \lambda(A_j)$, $j = 1, 2, \cdots, r$, and this completes the proof.

By Proposition 5.1 and Lemma 5.2, we have at once

PROPOSITION 5.3. *Let $A_i$, $A'_i$ be p-sphere bundles over p-spheres, $p > 2$, $p \neq 4, 8$, and let $M = \sharp_{i=1}^r A_i$, $M' = \sharp_{i=1}^r A'_i$. Then, $M$ and $M'$ are of the same oriented homotopy type if and only if there exist symplectic bases $\{e_1, \cdots, e_r; f_1, \cdots, f_r\}$, $\{e'_1, \cdots, e'_r; f'_1, \cdots, f'_r\}$ of $H_p(M)$, $H_p(M')$ respectively such that $\mu(e_i) = \mu'(e'_i)$, $\mu(f_j) = \mu'(f'_j)$ for $i, j = 1, 2, \cdots, r$.*

Let $p = 8s + 1$ ($s \geqq 1$). In the following exact sequence

$$\pi_p(S^p) \xrightarrow{\partial} \pi_{p-1}(SO_p) \xrightarrow{i_*} \pi_{p-1}(SO_{p+1}) \cong Z_2 \longrightarrow 0 ,$$

$\operatorname{Ker} i_* = \operatorname{Im} \partial$ is a direct summand of $\pi_{p-1}(SO_p)$ (Kervaire [13, p. 168]). So, $\pi_{8s}(SO_{8s+1}) = \operatorname{Im} \partial \oplus G$ ($s \geqq 1$) for some subgroup $G$. $\operatorname{Im} \partial = \operatorname{Ker} i_* \cong Z_2$ is generated by $\partial \iota_p$ and $G \cong \pi_{8s}(SO_{8s+2}) \cong Z_2$ by $i_*$.

LEMMA 5.4. *Let $M = \sharp_{i=1}^r A_i$ and $p = 8s + 1$ ($s \geqq 1$). Then, there exists a symplectic basis $\{e_1, \cdots, e_r; f_1, \cdots, f_r\}$ of $H_p(M)$ such that $\mu(e_i) = 0$, $\mu(f_j) \in J(G)$ and $\mu(f_j)$ is a representative of $\lambda(A_j)$, $i, j = 1, 2, \cdots, r$.*

*Proof.* Let $\{e_1, \cdots, e_r; f_1, \cdots, f_r\}$ be a fibre and sectional basis of $H_p(M)$, which is symplectic since $p$ is odd. If $\nu(f_j) = a + b$, $a \in \operatorname{Im} \partial$, $b \in G$, and $a \neq 0$, then $a = \partial(1)$, and put $\tilde{f}_j = e_j + f_j$. Then, $\nu(\tilde{f}_j) = \nu(f_j) + \partial(1) = 2a + b = b \in G$, and $i_* \nu(\tilde{f}_j) = i_* \nu(f_j) = \alpha(A_j)$. So that $\mu(\tilde{f}_j) \in J(G)$ and $\mu(\tilde{f}_j)$ represents $\lambda(A_j)$. Repeating this for $j = 1, 2, \cdots, r$ if necessary, we have a required symplectic basis $\{e_1, \cdots, e_r; \tilde{f}_1, \cdots, \tilde{f}_r\}$ of $H_p(M)$.

Now, we prove Theorem 2. Let $A_i$, $A'_i$, $i = 1, 2, \cdots, r$, be $p$-sphere bundles over $p$-spheres, $p > 2$, $p \neq 4, 8$, and let $M = \sharp_{i=1}^r A_i$, $M' = \sharp_{i=1}^r A'_i$. Let $\{e_1, \cdots, e_r; f_1, \cdots, f_r\}$, $\{e'_1, \cdots, e'_r; f'_1, \cdots, f'_r\}$ be the symplectic bases of $H = H_p(M)$, $H' = H_p(M')$ respectively, given by Lemma 5.2 and Lemma 5.4.

Firstly, assume that $M$ has the homotopy type of $M'$. We may assume that $M$ has the oriented homotopy type of $M'$. For, the exchange of orientations of the fibres turns the orientation of $M$ but does not affect the characteristic elements of $A_i$, $i = 1, 2, \cdots, r$. So, by Proposition 5.1, there exists an isomorphism $h: H \to H'$ such that $\phi = \phi' \circ (h \times h)$ and $\mu = \mu' \circ h$. Let

$$
(1) \qquad \begin{pmatrix} h(e_1) \\ \vdots \\ h(e_r) \\ h(f_1) \\ \vdots \\ h(f_r) \end{pmatrix} = K \begin{pmatrix} e'_1 \\ \vdots \\ e'_r \\ f'_1 \\ \vdots \\ f'_r \end{pmatrix}, \qquad K = \begin{pmatrix} A & C \\ B & D \end{pmatrix},
$$

where $A$, $B$, $C$, and $D$ are $(r \times r)$-matrices of integer components. Then, since $h$ preserves intersection numbers, $KUK^t = U$, where

$$
U = \begin{pmatrix} 0 & I_r \\ (-1)^p I_r & 0 \end{pmatrix}
$$

and $I_r$ is the unit $(r \times r)$-matrix. Applying $\nu'$ to (1), we have

$$
\begin{pmatrix} \nu'(h(e_1)) \\ \vdots \\ \nu'(h(e_r)) \end{pmatrix} = C \begin{pmatrix} \nu'(f'_1) \\ \vdots \\ \nu'(f'_r) \end{pmatrix} + \begin{pmatrix} (AC^t)_{11} \\ \vdots \\ (AC^t)_{rr} \end{pmatrix} \partial(1),
$$

$$
\begin{pmatrix} \nu'(h(f_1)) \\ \vdots \\ \nu'(h(f_r)) \end{pmatrix} = D \begin{pmatrix} \nu'(f'_1) \\ \vdots \\ \nu'(f'_r) \end{pmatrix} + \begin{pmatrix} (BD^t)_{11} \\ \vdots \\ (BD^t)_{rr} \end{pmatrix} \partial(1),
$$

where $(AC^t)_{ii}$, $(BD^t)_{ii}$ mean the diagonal entries of $AC^t$, $BD^t$ respectively. Since $\mu' = J\nu'$ and $J \circ \partial = -P$,

$$
(2) \qquad \begin{pmatrix} \mu'(h(e_1)) \\ \vdots \\ \mu'(h(e_r)) \end{pmatrix} = C \begin{pmatrix} \mu'(f'_1) \\ \vdots \\ \mu'(f'_r) \end{pmatrix} - \begin{pmatrix} (AC^t)_{11} \\ \vdots \\ (AC^t)_{rr} \end{pmatrix} P(1),
$$

$$
(3) \qquad \begin{pmatrix} \mu'(h(f_1)) \\ \vdots \\ \mu'(h(f_r)) \end{pmatrix} = D \begin{pmatrix} \mu'(f'_1) \\ \vdots \\ \mu'(f'_r) \end{pmatrix} - \begin{pmatrix} (BD^t)_{11} \\ \vdots \\ (BD^t)_{rr} \end{pmatrix} P(1).
$$

Since $\mu(e_i) = 0$, and $\mu(f_j)$, $\mu'(f'_j)$ represent $\lambda(A_j)$, $\lambda(A'_j)$ respectively, by the relation $\mu = \mu' \circ h$, we have

$$
(4) \qquad C \begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix} = 0, \qquad D \begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix} = \begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix}.
$$

In the above, considering $h^{-1}$ in place of $h$, similarly we have

$$(5) \qquad C'\begin{pmatrix}\lambda(A_1)\\ \vdots \\ \lambda(A_r)\end{pmatrix} = 0\,, \qquad D'\begin{pmatrix}\lambda(A_1)\\ \vdots \\ \lambda(A_r)\end{pmatrix} = \begin{pmatrix}\lambda(A'_1)\\ \vdots \\ \lambda(A'_r)\end{pmatrix},$$

where $C'$, $D'$ belong to the symplectic $(2r \times 2r)$-matrix

$$K^{-1} = \begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix}.$$

$\lambda(A_i)$, $\lambda(A'_i)$, $i = 1, 2, \cdots, r$, belong to $J\pi_{p-1}(SO_p)/P\pi_p(S^p) \cong J\pi_{p-1}(SO) \cong Z/mZ$, where

$$m = \begin{cases} 1 & \text{if } p = 3,\ 5,\ 6,\ 7 \pmod 8, \\ 2 & \text{if } p = 1,\ 2 \pmod 8, \\ m(2s) & \text{if } p = 4s \ (s > 0), \end{cases}$$

and $m(2s)$ is the denominator of $B_s/4s$ (Adams [1]). Take representatives $\lambda_i$, $\lambda'_i$ of $\lambda(A_i)$, $\lambda(A'_i)$ respectively for $i = 1, 2, \cdots, r$ such that $0 \leqq \lambda_i$, $\lambda'_i \leqq m - 1$, $i = 1, 2, \cdots, r$. Then, it is not hard to see that there exist unimodular $(r \times r)$-matrices $Q$, $Q'$ $(r > 1)$ such that

$$(6) \qquad Q\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_r\end{pmatrix} = \begin{pmatrix}\lambda\\ 0\\ \vdots \\ 0\end{pmatrix} \pmod m\,, \qquad Q'\begin{pmatrix}\lambda'_1\\ \vdots \\ \lambda'_r\end{pmatrix} = \begin{pmatrix}\lambda'\\ 0\\ \vdots \\ 0\end{pmatrix} \pmod m\,,$$

where $0 \leqq \lambda$, $\lambda' < m$, and $\lambda = G.C.D.\,(\lambda_1, \cdots, \lambda_r, m)$, $\lambda' = G.C.D.\,(\lambda'_1, \cdots, \lambda'_r, m)$ if $\lambda, \lambda' > 0$. And, we know that $\lambda = \lambda'$. For, if $\lambda = 0$ then $\lambda_1 = \cdots = \lambda_r = 0$, and therefore $\lambda' = 0$ by (5). If $\lambda \neq 0$, then $\lambda' \neq 0$ and by (4), (5), and (6),

$$\begin{pmatrix}\lambda\\ 0\\ \vdots \\ 0\end{pmatrix} = QDQ'^{-1}\begin{pmatrix}\lambda'\\ 0\\ \vdots \\ 0\end{pmatrix} \pmod m\,, \qquad \begin{pmatrix}\lambda'\\ 0\\ \vdots \\ 0\end{pmatrix} = Q'D'Q^{-1}\begin{pmatrix}\lambda\\ 0\\ \vdots \\ 0\end{pmatrix} \pmod m\,.$$

So that, $\lambda' \mid \lambda$, $\lambda \mid \lambda'$ and therefore $\lambda = \lambda'$.
Let $L = Q^{-1}Q'$. Then, by (6), we have

$$(7)\qquad \begin{pmatrix} \lambda(A_1) \\ \vdots \\ \lambda(A_r) \end{pmatrix} = L \begin{pmatrix} \lambda(A'_1) \\ \vdots \\ \lambda(A'_r) \end{pmatrix}$$

for an unimodular $(r \times r)$-matrix $L$.

Let $r = 1$. If $p$ is even, then $C = \pm 1$, $D = 0$ or $D = \pm 1$ by the relation $KUK^t = U$. Hence, $\lambda(A_1) = \pm\lambda(A'_1)$ by (4). Let $p$ be odd. If $p = 3, 5, 7 \pmod 8$, (7) holds trivially. If $p = 1 \pmod 8$, then $\lambda(A_1)$, $\lambda(A'_1)$ are of order 2, and by (4), (5), $D\lambda(A'_1) = \lambda(A_1)$, $D'\lambda(A_1) = \lambda(A'_1)$. Therefore, $\lambda(A_1) = 0$ induces $\lambda(A'_1) = 0$. If $\lambda(A_1) \neq 0$, then $\lambda(A_1) = D\lambda(A'_1) = DD'\lambda(A_1) \neq 0$ and $D$, $D'$ must be odd. Hence $\lambda(A'_1) = D'\lambda(A_1) \neq 0$, and so $\lambda(A_1) = \lambda(A_2)$. Thus, in any case, (7) holds for $r = 1$.

Conversely, assume (7). We show that $M$ has the homotopy type of $M'$. Define the matrix $K$ of integer components by

$$K = \begin{pmatrix} (L^t)^{-1} & 0 \\ 0 & L \end{pmatrix}.$$

Then, $K$ is symplectic, that is, $KUK^t = U$. Let $h: H \to H'$ be the isomorphism defined by the formula of (1). Then, $h$ preserves intersection numbers and satisfies (2), (3). Put $h(e_i) = e''_i$, $h(f_j) = f''_j$ for $i, j = 1, 2, \cdots, r$. $\{e''_1, \cdots, e''_r; f''_1, \cdots, f''_r\}$ is a symplectic basis of $H'$. Since $B = C = 0$ and $D = L$, by (2), (3), we have

$$(8)\qquad \begin{pmatrix} \mu'(e''_1) \\ \vdots \\ \mu'(e''_r) \end{pmatrix} = 0, \qquad \begin{pmatrix} \mu'(f''_1) \\ \vdots \\ \mu'(f''_r) \end{pmatrix} = L \begin{pmatrix} \mu'(f'_1) \\ \vdots \\ \mu'(f'_r) \end{pmatrix},$$

and by (7),

$$(9)\qquad \begin{pmatrix} \mu(f_1) \\ \vdots \\ \mu(f_r) \end{pmatrix} = L \begin{pmatrix} \mu'(f'_1) \\ \vdots \\ \mu'(f'_r) \end{pmatrix} + \begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix} P(1)$$

for some integers $k_i$, $i = 1, 2, \cdots, r$. Hence, by (8), (9), we have

$$(10)\qquad \begin{pmatrix} \mu(f_1) \\ \vdots \\ \mu(f_r) \end{pmatrix} - \begin{pmatrix} \mu'(f''_1) \\ \vdots \\ \mu'(f''_r) \end{pmatrix} = \begin{pmatrix} k_1 \\ \vdots \\ k_r \end{pmatrix} [\iota_p, \iota_p].$$

If $p$ is even, apply the following commutative diagram to (10):

$$\begin{array}{ccc} \pi_{p-1}(SO_p) & \xrightarrow{\pi_*} & \pi_{p-1}(S^{p-1}) \\ \downarrow J & & \cong \downarrow E^p \\ \pi_{2p-1}(S^p) & \xrightarrow{H} & \pi_{2p-1}(S^{2p-1}) \, , \end{array}$$

where $\pi\colon SO_p \to S^{p-1}$ is the projection and $H$ is the Hopf homomorphism. Then, $H\mu(f_j) = HJ\nu(f_j) = E^p\pi_*(\nu(f_j)) = f_j \cdot f_j = 0$, similarly $H\mu'(f_j'') = 0, j = 1, 2, \cdots, r$, and $H[\iota_p, \iota_p] = \pm 2$. So, we know that $k_1 = k_2 = \cdots = k_r = 0$. Hence, $\mu(f_j) = \mu'(f_j'')$ for $j = 1, 2, \cdots, r$.

Let $p$ be odd. If $p = 3, 5, 7 \pmod 8$, the matter is trivial since $A_i$, $A_i'$, $i = 1, 2, \cdots, r$, are all product bundles. Let $p = 8s + 1$ $(s > 0)$. In the following commutative diagram

$$\begin{array}{ccccc} & \overset{\partial}{\nearrow} & \pi_{p-1}(SO_p) & \xrightarrow{\quad i_* \quad} & \pi_{p-1}(SO_{p+1}) \cong Z_2 \\ & & \| & & \\ \pi_p(S^p) & & \operatorname{Im}\partial \oplus G & & \downarrow J \\ & & \downarrow J & & \\ & \underset{P}{\searrow} & \pi_{2p-1}(S^p) & \xrightarrow{\quad E \quad} & \pi_{2p}(S^{p+1}) \, , \end{array}$$

$J\colon \pi_{p-1}(SO_{p+1}) \to \pi_{2p}(S^{p+1})$ is injective by Adams [1]. Therefore, by the fact that $[\iota_p, \iota_p] \neq 0$ if $p \neq 1, 3, 7$, the diagram induces that $J\colon \pi_{p-1}(SO_p) \to \pi_{2p-1}(S^p)$ is also injective. Hence, $J(\pi_{p-1}(SO_p)) = \{[\iota_p, \iota_p]\} \oplus J(G)$. On the other hand, by Lemma 5.4, we may assume that $\mu(f_j)$, $\mu'(f_j')$ belong to $J(G)$ for $j = 1, 2, \cdots, r$. So that, in (9), $k_1 = k_2 = \cdots = k_r = 0 \pmod 2$ since $P(1) = [\iota_p, \iota_p]$ is of order 2. Hence, $\mu(f_j) = \mu'(f_j'')$ for $j = 1, 2, \cdots, r$, by (10).

Thus, joining with (8), we have shown that

$$\mu(e_i) = \mu'(e_i'') = 0 \, , \quad \mu(f_j) = \mu'(f_j'') \, , \quad \text{for } i, j = 1, 2, \cdots, r \, , \tag{11}$$

where $p > 2$ and $p \neq 4, 8$. Hence, $M$ and $M'$ are of the same homotopy type by Proposition 5.3. This completes the proof of Theorem 2.

Let $p = 4, 8$, and let $\{e_1, \cdots, e_r; f_1, \cdots, f_r\}$ be a fibre and sectional basis of $H_p(M)$, where $M = \#_{i=1}^r A_i$. Then, $i_*\nu(f_j) + HJ(\nu(f_j)) = \alpha(A_j) + f_j \cdot f_j$ is even (cf. Wall [17, p. 171]), where $\pi_{p-1}(SO_{p+1}) \cong Z$ $(p = 4, 8)$. Therefore $f_j \cdot f_j$ is even (odd) if and only if $\alpha(A_j)$ is even (odd). Hence, $M$ is of type II (cf. Milnor [14]) if and only if $\alpha(A_j)$, $j = 1, 2, \cdots, r$, are all even. If $M$ is of type II, then Lemma 5.2 holds for $M$. So , in a quite similar way, we know, as mentioned in Remark 2, that Theorem 2 holds for $p=4, 8$ if the connected sums are of type II.

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